Lecture 10:

Sampling revisited

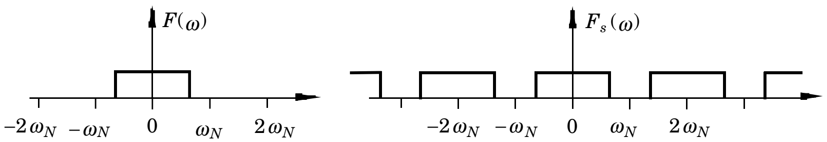
- The sampling theorem
- Reconstruction
- Aliasing and presampling filters
- Discrete time approximation of continuous time controller
  - Frequency domain
  - State space

The Sampling Theorem (Shannon 1949)

A continuous-time signal with Fourier transform $F(\omega) = 0$ for $|\omega| > \omega_0$, is uniquely defined by its sampled values, provided the sampling frequency satisfies $\omega_N = \omega_s/2 > \omega_0$
The signal can be reconstructed by the interpolation formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin \omega_s(t-kh)/2}{\omega_s(t-kh)/2}$$

Proof idea



Let $F(\omega)$ be the Fourier transform of f and expand the periodic function $F_s(\omega)$ as a Fourier series

$$F_s(\omega) = \frac{1}{h} \sum_{k=-\infty}^{\infty} F(\omega + k\omega_s) = \sum_{k=-\infty}^{\infty} C_k e^{-ikh\omega}$$

It is straightforward to verify that

$$C_k = f(kh) \qquad\qquad k = 0, \pm 1, \pm 2, \ldots$$

Hence $\{f(kh)\} \Rightarrow F_s(\omega) \Rightarrow F(\omega) \Rightarrow f(t)$

Reconstruction

Reconstruction

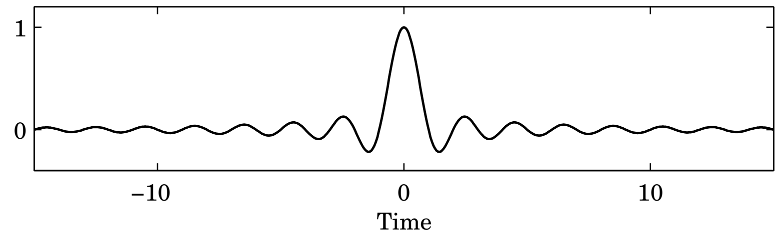
- Shannon

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(\omega_s(t-kh)/2)}{\omega_s(t-kh)/2}$$

- Zero order hold
- First order hold
- Predictive first order hold

Shannon reconstruction

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(\omega_s(t-kh)/2)}{\omega_s(t-kh)/2}$$

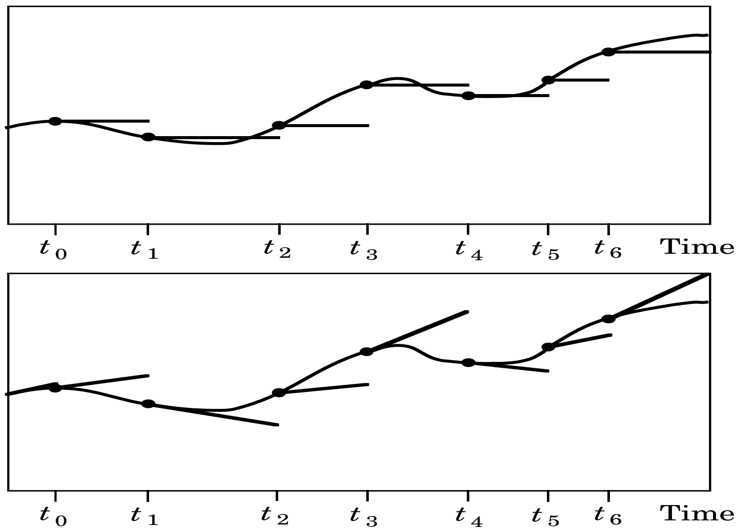


Properties?

$$\hat{f}(nh+\tau) = \sum_{k=-\infty}^{n+d} f(kh)h(nh+\tau-kh) \qquad h(\tau) = \frac{\sin \omega_s \tau/2}{\omega_s \tau/2}$$

Delay acceptable?

Zero and first order hold

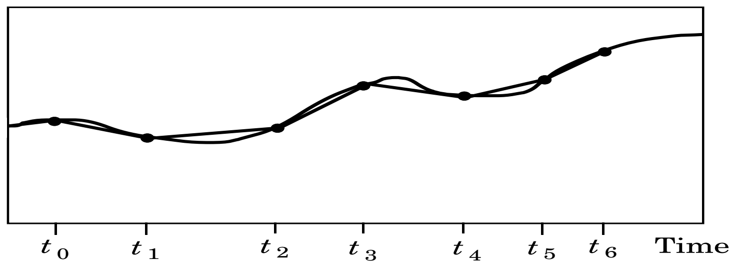


Predictive first order hold

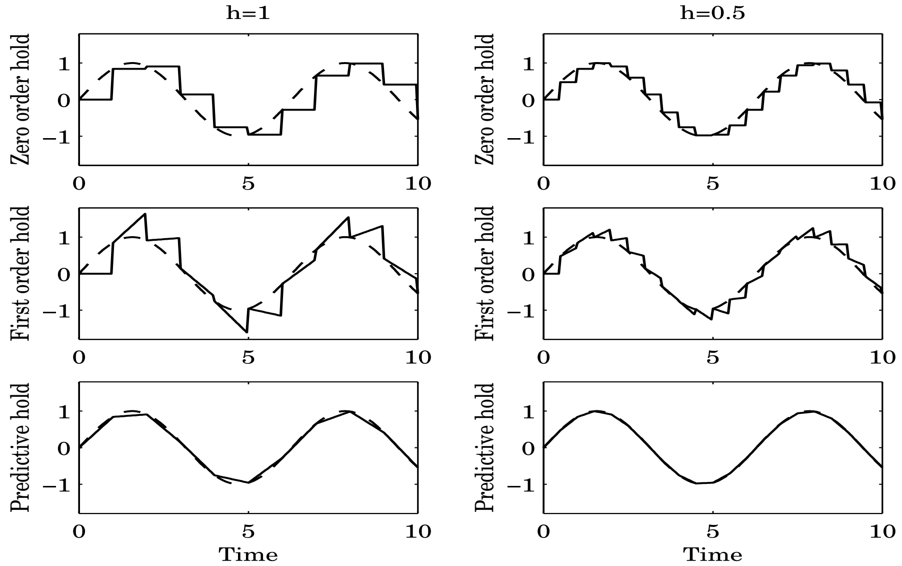
- Use forward difference instead of backward difference

$$u(kh+\tau) = u(kh) + \frac{\tau}{h}\big(u(kh+h) - u(kh)\big)$$

- Use model of the controller. New causality conditions.

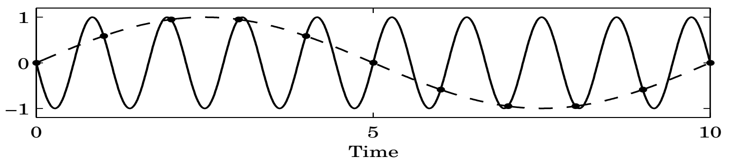


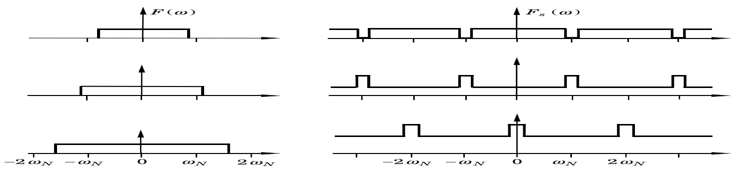
Sinusoidal signal with $h = 1$ and $h = 0.5$



Aliasing and presampling filters

Aliasing and frequency folding





$\omega_N = \omega_s/2$ Nyquist frequency

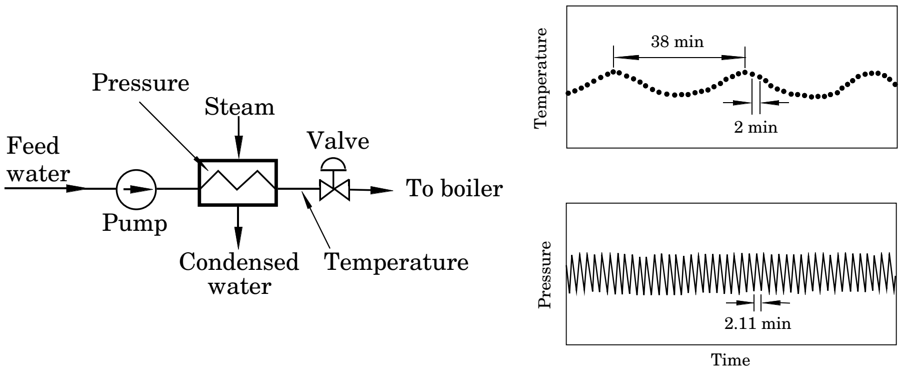
New frequencies since the system is NOT time-invariant

$$\omega_{sampled} = \omega + n\omega_s \qquad n = 0, \pm 1, \pm 2, \ldots$$

Mini-problem

A wheel, rotating 90 turns per second, is filmed using a camera with sampling time 10ms. What will the wheel rotation look like on the film?

Example — Feedwater heating in a ship boiler



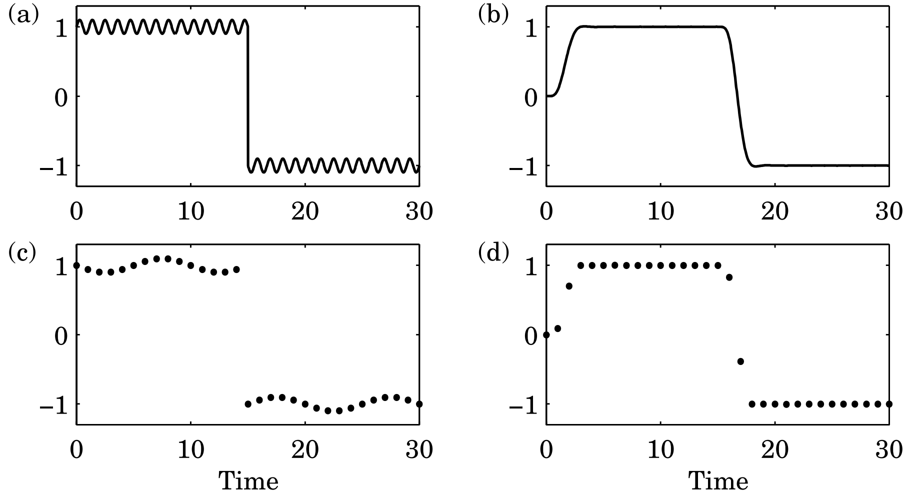
Pre- and postsampling filters

Typical control problem:

"Decrease the influence of a low-frequency process disturbance despite high-frequency measurement noise."

- Frequency separation
- Prefilter ω_N
  - Bessel, Butterworth, ITAE filters
- Postsampling filters
  - Avoid exciting mechanical resonances
  - Higher order hold

Example – Prefiltering

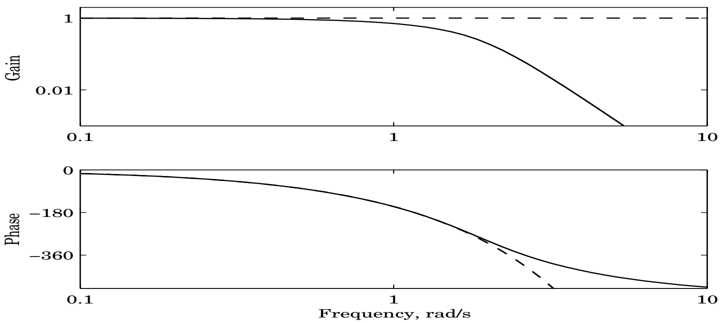


$\omega_d = 0.9$, $\omega_N = 0.5$, $\omega_{alias} = 0.1$ (6th order Bessel filter)

Consequence of using prefilter

Pre- and postfilter should be included in the process model
Exception: Fast sampling

A Besselfilter can be approximated with a delay:

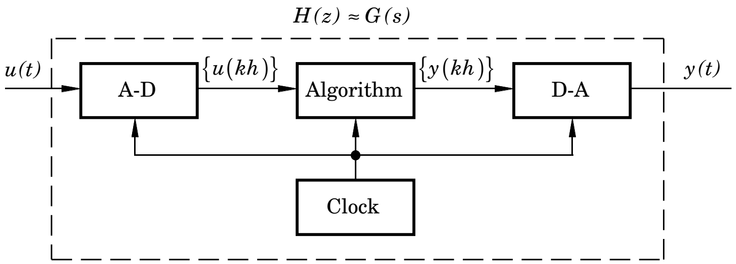


6th order Bessel (solid line), time delay (dashed line)

Discrete time approximation of continuous time controller

— Frequency domain

Implementing a controller using a computer



Want to get

$$\text{A/D + Algorithm + D/A} \approx G(s)$$

Methods:

- Approximate s i.e $G(s) \to H(z)$
- Make assumptions about $u(t)$
- Pole-zero matching

Matlab

```
SYSD = C2D(SYSC,TS,METHOD)  converts the continuous
system SYSC to a discrete-time system SYSD with
sample time TS.  The string METHOD selects the
discretization method among the following:
 'zoh'       Zero-order hold on the inputs.
 'foh'       Linear interpolation of inputs
             (triangle appx.)
 'tustin'    Bilinear (Tustin) approximation.
 'prewarp'   Tustin approximation with frequency
             prewarping.
             The critical frequency Wc is specified
             last as in C2D(SysC,Ts,'prewarp',Wc)
 'matched'   Matched pole-zero method
             (for SISO systems only).
```

Approximation methods

Forward difference (Euler's method)

$$\frac{dx(t)}{dt} \approx \frac{x(t+h) - x(t)}{h} = \frac{q-1}{h} x(t)$$

Backward difference

$$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t-h)}{h} = \frac{1-q^{-1}}{h} x(t)$$

Trapezoidal method (Tustin, bilinear)

$$\frac{\dot{x}(t+h) + \dot{x}(t)}{2} \approx \frac{x(t+h) - x(t)}{h}$$

Mini-problem

Compare the differential equation

$$\dot{y}(t) + 10y(t) = 0$$

to the Euler approximation

$$y(k+1) - y(k) + 10y(k) = 0$$

Is there any qualitative difference?

What is the source of the problem?

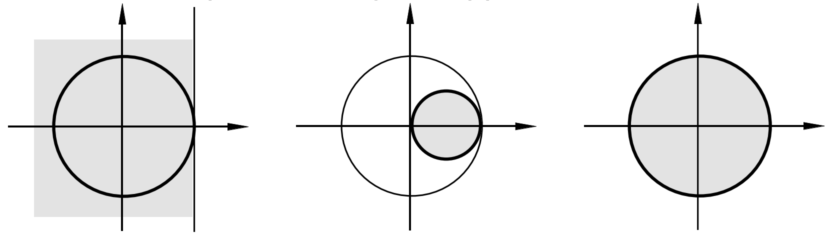
Properties of the approximation $H(z) \approx G(s)$

$$s = \frac{z-1}{h} \qquad \text{(Forward difference or Euler's method)}$$

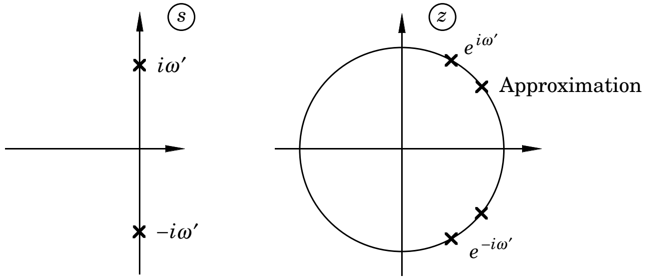
$$s = \frac{z-1}{zh} \qquad \text{(Backward difference)}$$

$$s = \frac{2}{h}\,\frac{z-1}{z+1} \qquad \text{(Tustin's or bilinear approximation)}$$

Where do stable poles of G get mapped?



Prewarping to reduce frequency distortion



Choose one fix-point ω_1

$$s = \frac{\omega_1}{\tan(\omega_1 h/2)} \cdot \frac{z-1}{z+1}$$

This implies that $H\left(e^{i\omega_1 h}\right) = G(i\omega_1)$ Tustin is obtained for $\omega_1 = 0$ since $\tan\left(\frac{\omega_1 h}{2}\right) \approx \frac{\omega_1 h}{2}$ for small ω.

Assumptions about the input

- u stepwise constant (Step invariance)
- u piecewise linear (Ramp invariance)

$$u(t) = \frac{t-kh}{h}\Big(\underline{u(kh+h)} - \underline{u(kh)}\Big) + \underline{u(kh)}$$

Integrating over a sampling period gives

$$x(kh+h) = \Phi x(kh) + \frac{1}{h}\Gamma_1 u(kh+h) + (\Gamma - \frac{1}{h}\Gamma_1)u(kh)$$

$$\Gamma_1 = \int_0^h e^{A\tau}(h-\tau)\,d\tau\,B$$

$$H(z) = D + C(zI - \Phi)^{-1}\left(\frac{z-1}{h}\Gamma_1 + \Gamma\right)$$

Direct term in the approximation

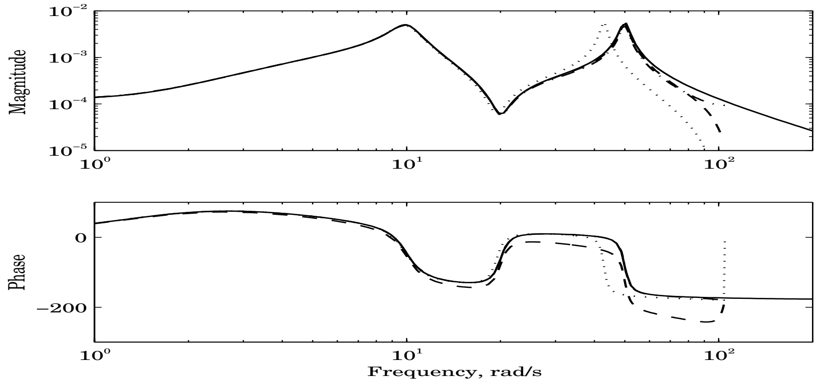
Pole-zero matching

1. All poles of $G(s)$ are mapped according to $z = e^{sh}$
2. All *finite* zeros are also mapped as $z = e^{sh}$
3. The zeros of $G(s)$ at $s = \infty$ are mapped into $z = -1$. One of the zeros of $G(s)$ at $s = \infty$ is mapped into $z = \infty$, i.e. $\deg A(z) - \deg B(z) = 1$
4. The gain of $H(z)$ is matched the gain of $G(s)$ at one frequency, center frequency or at steady-state, $s = 0$

Comparison of approximations

$$G(s) = \frac{(s+1)^2(s^2+2s+400)}{(s+5)^2(s^2+2s+100)(s^2+3s+2500)}$$

$h = 0.03$ gives $\omega_N = 105$ rad/s



$G(i\omega)$ (full), zoh (dashed), foh (dot-dashed), Tustin (dotted)

Selection of sampling interval

The Zero-Order-Hold has the transfer function

$$G_{ZoH}(s) = \frac{1 - e^{-sh}}{sh}$$

For small h

$$\frac{1 - e^{-sh}}{sh} \approx \frac{1 - 1 + sh - (sh)^2/2 + \cdots}{sh} = 1 - \frac{sh}{2} + \cdots$$

$$\exp(-sh/2) \approx 1 - \frac{sh}{2} + \cdots$$

$G_{ZoH}(s)$ can be approximated as a delay $h/2$

Assume the phase margin can be decreased by 5° to 15° then

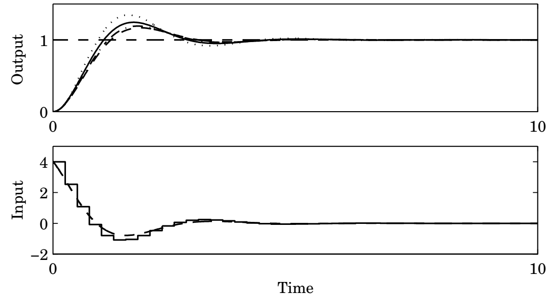
$$h\omega_c \approx 0.15 - 0.5$$

This leads to ω_N 5–20 larger than ω_c

Digital redesign of lead compensator

A lead compensator $G(s) = 4(s+1)/(s+2)$ for the process $P(s) = 1/(s^2+s)$ gives $\omega_c = 1.6$. Hence choose $h = 0.1 - -0.3$.

Euler's method: $$H_E(z) = 4\,\frac{\frac{z-1}{h}+1}{\frac{z-1}{h}+2} = 4\,\frac{z-(1-h)}{z-(1-2h)}$$



h = 0.1 (dash-dotted), 0.25 (full), 0.5 (dotted), $G(i\omega)$ (dashed)

Discrete time approximation of continuous time controller

— State space

State-feedback redesign

The system $\dot{x} = Ax + Bu$ with continuous-time state feedback

$$u(t) = Mu_c(t) - Lx(t)$$

gives $\dot{x} = \underbrace{(A - BL)}_{A_c} x + BMu_c = A_c x + BMu_c$ Sampling gives

$$x(kh + h) = \Phi_c x(kh) + \Gamma_c M u_c(kh)$$

Alternatively, sampling first, then applying discrete-time state feedback

$$u(kh) = \tilde{M} u_c(kh) - \tilde{L} x(kh)$$

gives

$$x(kh + h) = (\Phi - \Gamma\tilde{L})x(kh) + \Gamma\tilde{M}u_c(kh)$$

State feedback redesign cont'd

With $\tilde{L} = L_0 + L_1 h/2$ the two system matrices

$$\Phi_c = I + (A - BL)h + \left(A^2 - BLA - ABL + (BL)^2\right) h^2/2 + \cdots$$

$$\Phi - \Gamma\tilde{L} = I + (A - BL_0)h + \left(A^2 - ABL_0 - BL_1\right) h^2/2 + \cdots$$

are identical of order h^2 provided that $L_0 = L$, $L_1 = A - BL$.

To make the steady-state gain correct, let $\tilde{M} = M_0 + M_1 h/2$ and compare

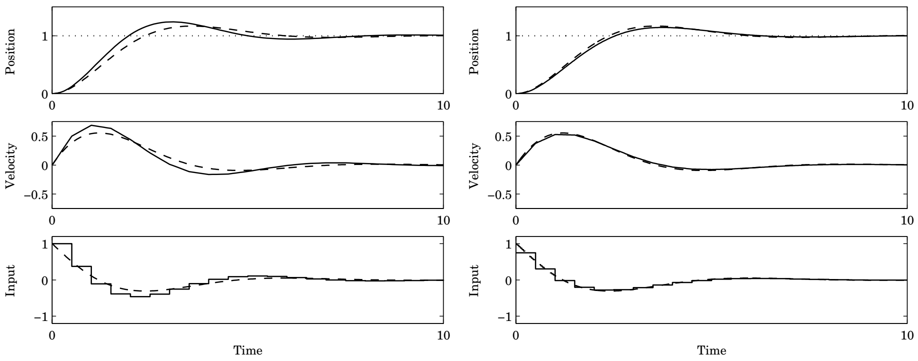
$$\Gamma_c M = BMh + (A - BL)BMh^2/2 + \cdots$$

$$\Gamma\tilde{M} = BM_0 h + (BM_1 + ABM_0)h^2/2 + \cdots$$

The expressions are identical of order h^2 provided that $M_0 = M$, $M_1 = -LBM$.

State-feedback redesign – Example

Double integrator with $h = 0.5$, $L = [1\ 1]$, and $M = 1$



$$\tilde{L} = \begin{pmatrix} 1 - 0.5h & 1 \end{pmatrix} \qquad \tilde{M} = 1 - 0.5h$$