1. X와 Y의 결합확률질량함수가 다음과 같다. '

$$f_{X,Y}(x,y)=\begin{cases} \frac{1}{9}, & (x,y)=(0,0),\ (0,2),\ (1,0),\ (1,2),\ (2,1) \\ \frac{2}{9}, & (x,y)=(0,1),\ (2,0) \\ 0, & \text{otherwise} \end{cases}$$

(1) X와 Y의 주변확률질량함수를 구하고, X와 Y의 독립성을 조사하여라.

(2) $X \le 1$일 때, Y의 조건부확률질량함수를 구하여라.

(sol)

(1) $R_{X\times Y}=\{(0,0),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}$, $R_X=\{0,1,2\}$, $R_Y=\{0,1,2\}$

$$f_X(0)=f_{X,Y}(0,0)+f_{X,Y}(0,1)+f_{X,Y}(0,2)=\frac{1}{9}+\frac{2}{9}+\frac{1}{9}=\frac{4}{9}$$

$$f_X(1)=f_{X,Y}(1,0)+f_{X,Y}(1,1)+f_{X,Y}(1,2)=\frac{1}{9}+0+\frac{1}{9}=\frac{2}{9}$$

$$f_X(2)=f_{X,Y}(2,0)+f_{X,Y}(2,1)+f_{X,Y}(2,2)=\frac{2}{9}+\frac{1}{9}+0=\frac{3}{9}$$

$$f_Y(0)=f_{X,Y}(0,0)+f_{X,Y}(1,0)+f_{X,Y}(2,0)=\frac{1}{9}+\frac{1}{9}+\frac{2}{9}=\frac{4}{9}$$

$$f_Y(1)=f_{X,Y}(0,1)+f_{X,Y}(1,1)+f_{X,Y}(2,1)=\frac{2}{9}+0+\frac{1}{9}=\frac{3}{9}$$

$$f_Y(2)=f_{X,Y}(0,2)+f_{X,Y}(1,2)+f_{X,Y}(2,2)=\frac{1}{9}+\frac{1}{9}+0=\frac{2}{9}$$

$$f_X(x)=\sum_{y\in R_Y} f_{X,Y}(x,y)=\begin{cases} \frac{4}{9}, & x=0 \\ \frac{2}{9}, & x=1 \\ \frac{3}{9}, & x=2 \\ 0, & \text{otherwise} \end{cases} \qquad f_Y(y)=\sum_{x\in R_X} f_{X,Y}(x,y)=\begin{cases} \frac{4}{9}, & y=0 \\ \frac{3}{9}, & y=1 \\ \frac{2}{9}, & y=2 \\ 0, & \text{otherwise} \end{cases}$$

독립성 :

$(x,y)=(1,1)$일 때, $f_X(1)\cdot f_Y(1)=\frac{2}{9}\cdot\frac{3}{9}\neq 0=f_{X,Y}(1,1)$이므로, X와 Y는 독립이 아니다.

(2) $f_{Y|X\le 1}(y)=P(Y=y\,|\,X\le 1)=\dfrac{P(Y=y,X\le 1)}{P(X\le 1)}=\dfrac{f_{X,Y}(y,0)+f_{X,Y}(y,1)}{f_X(0)+f_X(1)}$

$$f_{Y|X\le 1}(0)=\frac{f_{X,Y}(0,0)+f_{X,Y}(1,0)}{f_X(0)+f_X(1)}=\frac{\frac{1}{9}+\frac{1}{9}}{\frac{4}{9}+\frac{2}{9}}=\frac{2}{6}$$

$$f_{Y|X\le 1}(1)=\frac{f_{X,Y}(0,1)+f_{X,Y}(1,1)}{f_X(0)+f_X(1)}=\frac{\frac{1}{9}+0}{\frac{4}{9}+\frac{2}{9}}=\frac{2}{6}$$

$$f_{Y|X\le 1}(2)=\frac{f_{X,Y}(0,2)+f_{X,Y}(1,2)}{f_X(0)+f_X(1)}=\frac{\frac{1}{9}+\frac{1}{9}}{\frac{4}{9}+\frac{2}{9}}=\frac{2}{6}$$

$$f_{Y|X\le 1}(y)=\frac{P(Y=y,X\le 1)}{P(X\le 1)}=\frac{f_{X,Y}(y,0)+f_{X,Y}(y,1)}{f_X(0)+f_X(1)}=\begin{cases} \frac{1}{3}, & y=0 \\ \frac{1}{3}, & y=1 \\ \frac{1}{3}, & y=2 \\ 0, & \text{otherwise} \end{cases}$$

2. 두 확률변수 X와 Y의 결합확률질량함수가 다음과 같다.

$$f_{X,Y}(x,y)=k\left(\frac{1}{3}\right)^{x-1}\left(\frac{1}{4}\right)^{y-1},\quad x=1,2,3,\cdots,\ y=1,2,3,\cdots$$

(1) 상수 k를 구하여라.

(2) X와 Y의 주변확률질량함수를 구하고, X와 Y의 독립성을 조사하여라.

(3) $P(X+Y=4)$를 구하여라.

(4) $P(X\le 2\,|\,X+Y=4)$를 구하여라.

(5) $P(1\le X\le 3, 2\le Y\le 5)$를 구하여라.

(sol) $R_X=\{1,2,3,\cdots\}$, $R_Y=\{1,2,3,\cdots\}$

(1) $$\sum_{(x,y)\in R_{X\times Y}} f_{X,Y}(x,y)=\sum_{x\in R_X}\sum_{y\in R_Y} f_{X,Y}(x,y)=\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}k\left(\frac{1}{3}\right)^{x-1}\left(\frac{1}{4}\right)^{y-1}=k\sum_{x=1}^{\infty}\left(\frac{1}{3}\right)^{x-1}\sum_{y=1}^{\infty}\left(\frac{1}{4}\right)^{y-1}$$

$$=k\cdot\frac{1}{1-(1/3)}\cdot\frac{1}{1-(1/4)}=k\cdot\frac{3}{2}\cdot\frac{4}{3}=2k=1\ \Rightarrow\ k=\frac{1}{2}$$

(2) $$f_X(x)=\sum_{y\in R_Y} f_{X,Y}(x,y)=\sum_{y=1}^{\infty}\frac{1}{2}\left(\frac{1}{3}\right)^{x-1}\left(\frac{1}{4}\right)^{y-1}=\frac{1}{2}\left(\frac{1}{3}\right)^{x-1}\frac{1}{1-(1/4)}=\frac{2}{3}\left(\frac{1}{3}\right)^{x-1}$$

$$f_Y(y)=\sum_{x\in R_X} f_{X,Y}(x,y)=\sum_{x=1}^{\infty}\frac{1}{2}\left(\frac{1}{3}\right)^{x-1}\left(\frac{1}{4}\right)^{y-1}=\frac{1}{2}\left(\frac{1}{4}\right)^{y-1}\frac{1}{1-(1/3)}=\frac{3}{4}\left(\frac{1}{4}\right)^{y-1}$$

독립성 :

모든 $(x,y)\in R_{X,Y}=R_X\times R_Y$에 대하여

$$f_X(x)f_Y(y)=\frac{2}{3}\left(\frac{1}{3}\right)^{x-1}\frac{3}{4}\left(\frac{1}{4}\right)^{y-1}=\frac{1}{2}\left(\frac{1}{3}\right)^{x-1}\left(\frac{1}{4}\right)^{y-1}=f_{X,Y}(x,y)$$

이므로, X와 Y는 독립이다.

(3) $$P(X+Y=4)=\sum_{x+y=4} f_{X,Y}(x,y)$$

$$=f_{X,Y}(1,3)+f_{X,Y}(2,2)+f_{X,Y}(3,1)$$

$$=\frac{1}{2}\left(\frac{1}{3}\right)^{0}\left(\frac{1}{4}\right)^{2}+\frac{1}{2}\left(\frac{1}{3}\right)^{1}\left(\frac{1}{4}\right)^{1}+\frac{1}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{1}{4}\right)^{0}=\frac{1}{32}+\frac{1}{24}+\frac{1}{18}=\frac{37}{288}$$

(4) $$P(X\le 2\,|\,X+Y=4)=\frac{P(X\le 2, X+Y=4)}{P(X+Y=4)}=\frac{f_{X,Y}(1,3)+f_{X,Y}(2,2)}{f_{X,Y}(1,3)+f_{X,Y}(2,2)+f_{X,Y}(3,1)}=\frac{\frac{7}{96}}{\frac{37}{288}}=\frac{21}{37}$$

(5) $$P(1\le X\le 3, 2\le Y\le 5)=P(1\le X\le 3)P(2\le Y\le 5)$$

$$=\left[f_X(1)+f_X(2)+f_X(3)\right]\left[f_Y(2)+f_Y(3)+f_Y(4)+f_Y(5)\right]$$

$$=\left[\frac{2}{3}+\frac{2}{3}\left(\frac{1}{3}\right)^{1}+\frac{2}{3}\left(\frac{1}{3}\right)^{2}\right]\left[\frac{3}{4}\left(\frac{1}{4}\right)^{1}+\frac{3}{4}\left(\frac{1}{4}\right)^{2}+\frac{3}{4}\left(\frac{1}{4}\right)^{3}+\frac{3}{4}\left(\frac{1}{4}\right)^{4}\right]=\frac{26}{27}\,\frac{255}{1024}=\frac{1105}{4608}$$

3. 어떤 기계장치는 두 부품 중 어느 하나가 고장 날 때까지 작동한다. 그리고 두 부품의 수명에 대한 결합밀도함수는 다음과 같다. 단, 단위는 월이다.

$$f_{X,Y}(x,y)=\begin{cases} \frac{6(50-x-y)}{125000}, & 0<x<50-y<50 \\ 0 & , \text{otherwise} \end{cases}$$

(1) 기계장치가 10개월 안에 작동이 멈출 확률을 구하여라.
(2) 기계장치가 현재로부터 20개월 이상 작동할 확률을 구하여라.
(3) X와 Y의 주변확률밀도함수를 구하고, X와 Y의 독립성을 조사하여라.
(4) $X=20$일 때, $Y\le 20$인 조건부 확률을 구하여라.

(sol)

$R_{X\times Y}=\{(x,y)\,|\,0<x<50-y<50\}$, $R_X=\{x\,|\,0<x<50\}$, $R_Y=\{y\,|\,0<y<50\}$

(1) $P(X\le 10 \text{ or } Y\le 10)=1-P(X\ge 10, Y\ge 10)$

$$=1-\int_{10}^{40}\int_{10}^{50-x}\frac{6}{125000}(50-x-y)\,dy\,dx$$
$$=1-\frac{6}{125000}\int_{10}^{40}\left[(50-x)y-\frac{1}{2}y^2\right]_{10}^{50-x}dx$$
$$=1-\frac{6}{125000}\int_{10}^{40}\left[\frac{1}{2}(50-x)^2-10(50-x)+50\right]dx$$
$$=1-\frac{6}{125000}\int_{40}^{10}\left[\frac{1}{2}t^2-10t+50\right](-dt)$$
$$=1-\frac{6}{125000}\left[\frac{1}{6}t^3-5t^2+50t\right]_{10}^{40}$$
$$=1-\frac{6}{125000}\left[\frac{1}{6}(40^3-10^3)-5(40^2-10^2)+50(40-10)\right]=1-\frac{27}{125}=\frac{98}{125}$$

(2) $P(X\ge 20, Y\ge 20)=\int_{20}^{30}\int_{20}^{50-x}\frac{6}{125000}(50-x-y)\,dy\,dx$

$$=\frac{6}{125000}\int_{20}^{30}\left[(50-x)y-\frac{1}{2}y^2\right]_{20}^{50-x}dx$$
$$=\frac{6}{125000}\int_{20}^{30}\left[\frac{1}{2}(50-x)^2-20(50-x)+200\right]dx$$
$$=\frac{6}{125000}\int_{20}^{30}\left[\frac{1}{2}t^2-20t+200\right](-dt)$$
$$=\frac{6}{125000}\left[\frac{1}{6}t^3-10t^2+200t\right]_{20}^{30}=\frac{6}{125000}\frac{500}{3}=\frac{1}{125}$$

(3) $f_X(x)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dy=\int_0^{50-x}\frac{6}{125000}(50-x-y)\,dy=\frac{6}{125000}\left[(50-x)y-\frac{1}{2}y^2\right]_0^{50-x}=\frac{3}{125000}(50-x)^2$

$f_Y(y)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dx=\int_0^{50-y}\frac{6}{125000}(50-x-y)\,dx=\frac{6}{125000}\left[(50-y)x-\frac{1}{2}x^2\right]_0^{50-y}=\frac{3}{125000}(50-y)^2$

독립성 :

$f_X(x)f_Y(y)=\frac{3}{125000}(50-x)^2\frac{3}{125000}(50-y)^2\neq\frac{6(50-x-y)}{125000}=f_{X,Y}(x,y)$ 인 $(x,y)\in R_{X\times Y}$가 존재하므로, X와 Y는 독립이 아니다.

(4) $P(Y=y\,|\,X=20)=f_{Y|X=20}(y)=\frac{f_{X,Y}(20,y)}{f_X(20)}=\frac{\frac{6}{125000}(30-y)}{\frac{27}{1250}}=\frac{2}{900}(30-y)$, $0\le y\le 30$

$P(Y\le 20\,|\,X=20)=F_{Y|X=20}(20)=\int_0^{20}f_{Y|X=20}(y)\,dy=\int_0^{20}\frac{2}{900}(30-y)\,dy=\frac{2}{900}\left[30y-\frac{1}{2}y^2\right]_0^{20}=\frac{8}{9}$

4. 두 확률변수 X와 Y의 결합확률밀도함수가 다음과 같다.

$$f_{X,Y}(x,y)=\begin{cases} ke^{x+y} & , \ 0\le x\le 1,\ 0\le y\le 1 \\ 0 & , \ \text{otherwise}\end{cases}$$

(1) 상수 k를 구하여라.

(2) X와 Y의 주변확률밀도함수를 구하고, X와 Y가 i.i.d. 확률변수인지 조사하여라.

(3) $P(0.2\le X\le 0.8, 0.2\le Y\le 0.8$을 구하여라.

(4) $Y=\frac{1}{2}$ 일 때, X의 조건부확률밀도함수를 구하여라.

(sol)

$R_{X\times Y}=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$, $R_X=\{x\,|\,0\le x\le 1\}$, $R_Y=\{y\,|\,0\le y\le 1\}$

(1) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx=\int_0^1\int_0^1 ke^{x+y}\,dy\,dx$

$$=\int_0^1\left[ke^{x+y}\right]_0^1 dx=\int_0^1 ke^x(e-1)\,dx=\left[k(e-1)e^x\right]_0^1=k(e-1)^2=1 \Rightarrow k=\frac{1}{(e-1)^2}$$

(2) $\displaystyle f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy=\int_0^1\frac{e^{x+y}}{(e-1)^2}dy=\left[\frac{1}{(e-1)^2}e^{x+y}\right]_0^1=\frac{e^x}{e-1}$

$\displaystyle f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx=\int_0^1\frac{e^{x+y}}{(e-1)^2}dx=\left[\frac{1}{(e-1)^2}e^{x+y}\right]_0^1=\frac{e^y}{e-1}$

$\displaystyle f_X(x)f_Y(y)=\frac{e^x}{e-1}\,\frac{e^y}{e-1}=\frac{e^{x+y}}{(e-1)^2}$이고. $f_X(x)=f_Y(x)$이므로, X와 Y는 독립인 항등분포를 이룬다.

(3) $P(0.2\le X\le 0.8, 0.2\le Y\le 0.8)=P(0.2\le X\le 0.8)P(0.2\le Y\le 0.8)$

$$=\int_{0.2}^{0.8} f_X(x)\,dx\int_{0.2}^{0.8} f_Y(y)\,dy=\int_{0.2}^{0.8}\frac{e^x}{e-1}dx\int_{0.2}^{0.8}\frac{e^y}{e-1}dy=\left(\frac{e^{0.8}-e^{0.2}}{e-1}\right)^2$$

(4) $\displaystyle P(X=2\,|\,Y=\frac{1}{2})=\frac{f_{X,Y}(x,0.5)}{f_Y(0.5)}=\frac{\frac{e^{x+0.5}}{(e-1)^2}}{\frac{e^{0.5}}{e-1}}=\frac{e^x}{e-1}$

5. X와 Y의 결합확률밀도함수는 네 점 $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$을 꼭짓점으로 갖는 영역 D에서

$$f_{X,Y}(x,y)=k, \quad (x,y)\in D$$

로 주어진다.

(1) 상수 k를 구하여라.

(2) X와 Y의 주변확률밀도함수를 구하여라.

(3) X와 Y의 기댓값과 분산을 구하여라.

(4) $E[XY]$와 $\mathrm{Cov}(X,Y)$를 구하여라.

(sol) $R_{X\times Y}=\{(x,y)\mid |x|+|y|\le 1\}$, $R_X=\{x\mid -1\le x\le 1\}$, $R_Y=\{y\mid 0\le y\le 1\}$

(1) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx=4\int_0^1\int_0^{1-x} k\,dy\,dx$$

$$=4\int_0^1\Big[k\Big]_0^{1-x}dx=4\int_0^1 k(1-x)\,dx=4\Big[k(x-\frac{1}{2}x^2)\Big]_0^1=2k=1 \Rightarrow k=\frac{1}{2}$$

(2) $0\le x\le 1$일 때, $f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy=\int_{x-1}^{1-x}\frac{1}{2}\,dy=\Big[\frac{1}{2}\Big]_{x-1}^{1-x}=1-x$

$-1\le x<0$일 때, $f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy=\int_{-x-1}^{x+1}\frac{1}{2}\,dy=\Big[\frac{1}{2}\Big]_{-x-1}^{x+1}=x+1$

$0\le y\le 1$일 때, $f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx=\int_{y-1}^{1-y}\frac{1}{2}\,dx=\Big[\frac{1}{2}\Big]_{y-1}^{1-y}=1-y$

$-1\le y<0$일 때, $f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx=\int_{-y-1}^{y+1}\frac{1}{2}\,dx=\Big[\frac{1}{2}\Big]_{-x-1}^{x+1}=y+1$

(3) $E[X]=\int_{-\infty}^{\infty} x f_X(x)\,dx=\int_{-1}^{0} x(x+1)\,dx+\int_0^1 x(1-x)\,dx=\Big[\frac{1}{3}x^3+\frac{1}{2}x^2\Big]_{-1}^{0}+\Big[\frac{1}{2}x^2-\frac{1}{3}x^3\Big]_0^1=-\frac{1}{6}+\frac{1}{6}=0$

$E[Y]=\int_{-\infty}^{\infty} y f_Y(y)\,dy=\int_{-1}^{0} y(y+1)\,dy+\int_0^1 y(1-y)\,dy=\Big[\frac{1}{3}y^3+\frac{1}{2}y^2\Big]_{-1}^{0}+\Big[\frac{1}{2}y^2-\frac{1}{3}y^3\Big]_0^1=-\frac{1}{6}+\frac{1}{6}=0$

$E[X^2]=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx=\int_{-1}^{0} x^2(x+1)\,dx+\int_0^1 x^2(1-x)\,dx=\Big[\frac{1}{4}x^4+\frac{1}{3}x^3\Big]_{-1}^{0}+\Big[\frac{1}{3}x^3-\frac{1}{4}x^4\Big]_0^1=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$

$E[Y^2]=\int_{-\infty}^{\infty} y^2 f_Y(y)\,dy=\int_{-1}^{0} y^2(y+1)\,dy+\int_0^1 y^2(1-y)\,dy=\Big[\frac{1}{4}y^4+\frac{1}{3}y^3\Big]_{-1}^{0}+\Big[\frac{1}{3}y^3-\frac{1}{4}y^4\Big]_0^1=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$

$Var[X]=E[X^2]-(E[X])^2=\frac{1}{6}$, $Var[Y]=E[Y^2]-(E[Y])^2=\frac{1}{6}$,

(4) $E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dy\,dx=\int_{-1}^{0}\int_{-x-1}^{x+1} xy(\frac{1}{2})\,dy\,dx+\int_{-1}^{0}\int_{x-1}^{1-x} xy(\frac{1}{2})\,dy\,dx=0+0=0$

(4) $E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dy\,dx=\int_{-1}^{0}\int_{-x-1}^{x+1} xy(\frac{1}{2})\,dy\,dx+\int_{-1}^{0}\int_{x-1}^{1-x} xy(\frac{1}{2})\,dy\,dx=0+0=0$

6. X와 Y의 결합확률밀도함수가 다음과 같다.

$$f_{X,Y}(x,y)=\begin{cases} x+y, & 0\le x\le 1,\ 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$$

(1) X와 Y의 공분산과 상관계수를 구하여라.
(2) 기댓값 $E[X-2Y]$와 분산 $Var[X-2Y]$를 구하여라.
(3) 공분산 $\mathrm{Cov}(X-Y, X+Y)$를 구하여라.

(sol)

(1) $f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy=\int_0^1 (x+y)\,dy=\left[xy+\frac{1}{2}y^2\right]_0^1=x+\frac{1}{2}$

$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx=\int_0^1 (x+y)\,dx=\left[\frac{1}{2}x^2+xy\right]_0^1=\frac{1}{2}+y$

$E[X]=\int_{-\infty}^{\infty} x f_X(x)\,dx=\int_0^1 x\left(x+\frac{1}{2}\right)dx=\left[\frac{1}{3}x^3+\frac{1}{4}x^2\right]_0^1=\frac{7}{12}$

$E[X^2]=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx=\int_{-1}^{0} x^2\left(x+\frac{1}{2}\right)dx=\left[\frac{1}{4}x^4+\frac{1}{6}x^3\right]_0^1=\frac{5}{12}$

$Var[X]=E[X^2]-(E[X])^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144},$

$E[Y]=\frac{7}{12},\quad E[Y^2]=\frac{5}{12},\quad Var[Y]=\frac{11}{144}$

$E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dy\,dx=\int_0^1\int_0^1 xy(x+y)\,dy\,dx=\int_0^1\left(\frac{1}{2}x^2+\frac{1}{3}x\right)dx=\left[\frac{1}{6}x^3+\frac{1}{6}x^2\right]_0^1=\frac{1}{3}$

$\mathrm{Cov}(X,Y)=E[XY]-E[X]E[Y]=\frac{1}{3}-\left(\frac{7}{12}\right)\left(\frac{7}{12}\right)=-\frac{1}{144}$

$$\rho(X,Y)=\frac{\mathrm{Cov}(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}=\frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}}=-\frac{1}{11}$$

(2) $E[X-2Y]=E[X]-2E[Y]=\frac{7}{12}-2\frac{7}{12}=-\frac{7}{12}$

$Var[X-2Y]=Var[X]+Var[2Y]-2\mathrm{Cov}(X,2Y)=Var[X]+2^2Var[Y]-2\cdot 2\mathrm{Cov}(X,Y)=\frac{11}{144}+\frac{44}{144}+\frac{4}{144}=\frac{59}{144}$

(3) $\mathrm{Cov}(X-Y,X+Y)=E[(X-Y)(X+Y)]-E[X-Y]E[X+Y]$

$\qquad =(E[X^2]-E[Y^2])-(E[X]-E[Y])(E[X]+E[Y])=0-0\cdot\frac{7}{6}=0$

7. 하나의 동전을 3번 던졌을 때 나오는 앞면의 수를 X, 처음 2번의 시행에서 뒷면의 수를 Y라 하자. 두 확률변수 X와 Y의 공분산과 상관계수를 구하여라.

(sol) $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$X \backslash Y$	0	1	2	$f_X(x)$
0	0	0	$\frac{1}{8}$	$\frac{1}{8}$
1	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$
2	$\frac{1}{8}$	$\frac{1}{4}$	0	$\frac{3}{8}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

$$E[X] = \sum_{x \in R_X} x f_X(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2}$$

$$E[Y] = \sum_{y \in R_Y} y f_X(y) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} = 1$$

$$E[XY] = \sum_{(x,y) \in R_{X \times Y}} xy f_{X,Y}(x,y) = 0 \cdot 2 \cdot \frac{1}{8} + 1 \cdot 1 \cdot \frac{1}{4} + 1 \cdot 2 \cdot \frac{1}{8} + 2 \cdot 0 \cdot \frac{1}{8} + 2 \cdot 1 \cdot \frac{1}{4} + 3 \cdot 0 \cdot \frac{1}{8} = 1$$

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y] = 1 - \left(\frac{3}{2}\right)1 = -\frac{1}{2}$$

$$E[X^2] = \sum_{x \in R_X} x^2 f_X(x) = 0 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3$$

$$E[Y^2] = \sum_{y \in R_Y} y^2 f_Y(y) = 0 \cdot \frac{1}{4} + 1^2 \cdot \frac{2}{4} + 2^2 \cdot \frac{1}{4} = \frac{3}{2}$$

$$Var[X] = E[X^2] - (E[X])^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$$Var[Y] = E[Y^2] - (E[Y])^2 = \frac{3}{2} - (1)^2 = \frac{1}{2}$$

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{-\frac{1}{2}}{\sqrt{\frac{3}{4}}\sqrt{\frac{1}{2}}} = -\sqrt{\frac{2}{3}}$$