Lecture 2 on Linear Algebra

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- Matrices:

(i) By an m *by* n (or $m \times n$) *matrix*, we mean a rectangular array of mn numbers a_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$, arranged in m rows and n columns:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If $m = n$, then A is called a *square matrix* of order n.

(ii) The number a_{ij} in the ith row and jth column of a matrix A is called the (i, j) *entry* of A and often denoted by $A(i, j)$.

(iii) An m by 1 matrix, which has only one column, is called a *column vector*. A 1 by n matrix is called a *row vector*.

1.3. Matrices

• Multiplication of vectors by matrices: Let $A = [a_{ij}]$ be an m by n matrix.

(i) If $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-dimensional vector, we define

$$A\mathbf{x} = \begin{bmatrix} (a_{11}, \ldots, a_{1n}) \cdot \mathbf{x} \\ \vdots \\ (a_{m1}, \ldots, a_{mn}) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}.$$

(ii) Therefore, $A\mathbf{x}$ is an m-dimensional vector whose ith component is the (inner) product of the ith row vector of A and the n-dimensional vector $\mathbf{x}$:

$$(A\mathbf{x})_i = (a_{i1}, \ldots, a_{in}) \cdot \mathbf{x} = [a_{i1} \ \cdots \ a_{in}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

(iii) (*An extremely important formula*) $A\mathbf{x}$ is a linear combination of the columns of A; more precisely,

$$\begin{aligned} A\mathbf{x} &= \sum_{j=1}^{n} x_j \,(\text{the } j\text{th column of } A) \\ &= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}. \end{aligned}$$

Proof.

Example. Compute $A\mathbf{x}$ in two ways (by rows and by columns), where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Solution.

- The matrix equation $A\mathbf{x} = \mathbf{b}$: Let $A = [a_{ij}]$ be an m by n matrix, $\mathbf{x} = (x_1, \ldots, x_n)$ an n-dimensional vector, and $\mathbf{b} = (b_1, \ldots, b_m)$ an m-dimensional vector.

(i) $\mathbf{x}$ is a *solution* of the (matrix) equation $A\mathbf{x} = \mathbf{b}$, that is, it satisfies

$$A\mathbf{x} = \mathbf{b}$$

if and only if $(x_1, \ldots, x_n)$ is a solution of the system of linear equations

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_n. \end{gathered}$$

Proof.

(ii) Recall that

$$A\mathbf{x} = \sum_{j=1}^{n} x_j \,(\text{the } j\text{th column of } A)\,.$$

Therefore, *the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the columns of A.*

Example. Consider the following 3 by 3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Given $\mathbf{b} = (b_1, b_2, b_3)$, find a solution of $A\mathbf{x} = \mathbf{b}$.

Solution.

2.1 Vectors and linear equations

Example. Consider the following 3 by 3 matrix

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Given $\mathbf{b} = (b_1, b_2, b_3)$, find a solution of $C\mathbf{x} = \mathbf{b}$ if possible.

Solution.

2.1 Vectors and linear equations

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- Dependence and independence: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be m-dimensional vectors.

(i) We say that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are *(linearly) dependent* if there exists a nonzero vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ in $\mathbb{R}^n$ such that

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

(ii) Note that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are dependent if and only if some $\mathbf{v}_j$ is a linear combination of the other vectors:

$$\mathbf{v}_j = \sum_{k \neq j} d_k \mathbf{v}_k.$$

Proof.

(iii) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are not dependent, they are said to be *(linearly) independent*.

(iv) Let A be the m by n matrix whose columns are the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are dependent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Proof.

- Identity matrices:

(i) The n by n *identity matrix* is the square matrix $I = I_n$ of order n whose (i, j) entry is one if $i = j$ and zero if $i \neq j$:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(ii) The (i, j) entry of the identity matrix I will be denoted by δ_{ij}:

$$\delta_{ij} = I(i, j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

The function $(i, j) \mapsto \delta_{ij}$ is called the *Kronecker delta (function)*.

(iii) Let I be the n by n identity matrix. Then

$$I\mathbf{x} = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

(iv) The columns of the n by n identity matrix are the *standard unit vectors* in $\mathbb{R}^n$ and denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

(v) The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are independent.