Lecture 3 on Linear Algebra

김현석

Department of Mathematics, Sogang University (R1420, kimh@sogang.ac.kr)

March 10 (Thursday) 15:00-16:15

TA: 정선우 (kyungho2142@gmail.com)

2.2. The idea of elimination

• Solving linear equations by (forward) elimination and back substitution: For instance, consider the following 3 by 3 system of linear equations

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10. \end{aligned}$$

(i) Subtract 2 times equation 1 from equation 2. Then

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ -2x - 3y + 7z &= 10. \end{aligned}$$

(ii) Subtract -1 times equation 1 from equation 3. Then

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ y + 5z &= 12. \end{aligned}$$

(iii) Finally, subtracting 1 times equation 2 from equation 3, we obtain a *upper triangular* system

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ 4z &= 8. \end{aligned}$$

(iv) The last system can be easily solved by *back substitution* as follows:

$$4z = 8 \quad \Rightarrow \quad z = 2,$$

$$z = 2, \quad y + z = 4 \quad \Rightarrow \quad y = 2,$$

$$z = 2, \quad y = 2, \quad 2x + 4y - 2z = 2 \quad \Rightarrow \quad x = -1.$$

2.2. The idea of elimination

Remark. It should be clearly understood that

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ 4z &= 8 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x &= -1 \\ y &= 2 \\ z &= 2; \end{aligned}$$

or in matrix form,

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad U\mathbf{x} = \mathbf{c} \quad \Leftrightarrow \quad \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix},$$

$$U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}.$$

2.3 Elimination using matrices

- Augmented matrices:

(i) If A is an m by n matrix and $\mathbf{b}$ is an m-dimensional column vector, then the m by $(n+1)$ matrix

$$[A \ \ \mathbf{b}]$$

is called an *augmented matrix*.

(ii) By elimination we reduce a linear system $A\mathbf{x} = \mathbf{b}$ to an upper triangular system $U\mathbf{x} = \mathbf{c}$:

$$A\mathbf{x} = \mathbf{b} \quad \longrightarrow \quad U\mathbf{x} = \mathbf{c}.$$

This can be equivalently expressed by using augmented matrices as

$$[A \ \ \mathbf{b}] \quad \longrightarrow \quad [U \ \ \mathbf{c}]\,.$$

2.3 Elimination using matrices

Example. The augmented matrix for the linear system

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

is

$$[A \;\; \mathbf{b}] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}.$$

Use elimination to reduce $[A \;\; \mathbf{b}]$ to an upper triangular $[U \;\; \mathbf{c}]$.

Solution.

2.3 Elimination using matrices

2.3 Elimination using matrices

• Matrix multiplication: Let A be an m by n matrix and let B be an n by p matrix.

(i) The n by p matrix B has p columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$:

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}.$$

(ii) Note that

$$\mathbf{b}_k \in \mathbb{R}^n \quad \text{and} \quad A\mathbf{b}_k \in \mathbb{R}^m \quad (k = 1, 2, \dots, p).$$

(iii) The product AB of A and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$ is defined as the m by p matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_p$:

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

(iv) Recall again that

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

for all $\mathbf{x} \in \mathbb{R}^p$.

(v) (*Linearity*) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$ and $\alpha, \beta \in \mathbb{R}$, we have

$$B\left(\alpha\mathbf{x} + \beta\mathbf{y}\right) = \alpha B\mathbf{x} + \beta B\mathbf{y}.$$

Proof.

2.3 Elimination using matrices

(v) (*Associativity*) For all $\mathbf{x} \in \mathbb{R}^p$, we have

$$(AB)\mathbf{x} = A(B\mathbf{x}).$$

Proof.

- Elimination matrices:

(i) For $1 \leq i \neq j \leq n$, an *elimination matrix* E_{ij} is an n by n matrix that subtracts a multiple (say l) of row j from row i. More precisely, if A is any n by p matrix whose rows are $\mathbf{r}_1, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_n$, then

$$E_{ij}A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i - l\mathbf{r}_j \\ \vdots \\ \mathbf{r}_n \end{bmatrix}.$$

Alternatively, E_{ij} adds $(-l)$ times row j from row i.

(ii) In particular, if $1 = j < i \leq n$, then

$$E_{i1} \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i - lb_1 \\ \vdots \\ b_n \end{bmatrix} \text{ for all } \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n.$$

(iii) For the case when $n = 3$, we have

$$E_{21} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 - l\mathbf{r}_1 \\ \mathbf{r}_3 \end{bmatrix} \text{ for all } \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathbb{R}^p$$

if and only if

$$E_{21}\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 - lb_1 \\ b_3 \end{bmatrix} \text{ for all } \mathbf{b} \in \mathbb{R}^3$$

2.3 Elimination using matrices

if and only if

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly,

$$E_{31} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 - l\mathbf{r}_1 \end{bmatrix} \quad \Leftrightarrow \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix},$$

$$E_{32} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 - l\mathbf{r}_2 \end{bmatrix} \quad \Leftrightarrow \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l & 1 \end{bmatrix}.$$

Proof.

2.3 Elimination using matrices

2.3 Elimination using matrices

(iv) In general, E_{ij} *has the same entries as the identity matrix* I *except for the extra nonzero entry* $-l$ *in the* (i, j) *position*. For instance,

$$E_{21} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -l & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\vdots$$

$$E_{n1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l & 0 & \cdots & 1 \end{bmatrix}.$$

- Permutation matrices:

(i) For $1 \leq i \neq j \leq n$, a *permutation matrix* (or *row exchange matrix*) P_{ij} is an n by n matrix that exchanges row i and row j:

$$P_{ij} \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix}$$

for all row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_n$ of the same dimension.

(ii) For the case when $n = 3$,

$$P_{12}\begin{bmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{bmatrix}=\begin{bmatrix}\mathbf{r}_2\\\mathbf{r}_1\\\mathbf{r}_3\end{bmatrix}\quad\Leftrightarrow\quad P_{12}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},$$

$$P_{13}\begin{bmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{bmatrix}=\begin{bmatrix}\mathbf{r}_3\\\mathbf{r}_2\\\mathbf{r}_1\end{bmatrix}\quad\Leftrightarrow\quad P_{13}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},$$

$$P_{23}\begin{bmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{bmatrix}=\begin{bmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{bmatrix}\quad\Leftrightarrow\quad P_{23}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}.$$

Proof.

(iii) In general, P_{ij} is obtained by exchanging rows i and j of the identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

2.3 Elimination using matrices

- Elimination by multiplying elementary matrices:

Example. Consider the following augmented matrix:

$$[\,A \ \ \mathbf{b}\,] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}.$$

Then

$$E_{31}E_{21}\,[\,A \ \ \mathbf{b}\,] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

and

$$[U \ \ \mathbf{c}] = E_{32}E_{31}E_{21}\,[\,A \ \ \mathbf{b}\,] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}.$$

2.3 Elimination using matrices

Example. To solve the linear system

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

by elimination, we consider the following augmented matrix:

$$[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}.$$

Then

$$E_{21}\,[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix},$$

$$[U \ \ \mathbf{c}] = P_{23}E_{21}\,[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$