Lecture 4 on Linear Algebra

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March 15 (Tuesday) 15:00-16:15

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2.4. Rules for matrix operations

- Addition and scalar multiplication:

(i) If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, say m by n matrices, then $A + B$ is an m by n matrix defined by

$$A + B = [a_{ij} + b_{ij}] = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

(ii) If $A = [a_{ij}]$ is an m by n matrices and c is a scalar, then cA is an m by n matrix defined by

$$cA = [ca_{ij}] = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}.$$

(iii) We write $-A = (-1)A$ and $A - B = A + (-B)$.

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- Matrix multiplication II: Let $A = [a_{ij}]$ be an m by n matrix and let $B = [b_{ij}]$ be an n by p matrix.

(i) Recall that if $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-dimensional vector, then

$$A\mathbf{x} = \begin{bmatrix} (\text{row } 1) \cdot \mathbf{x} \\ \vdots \\ (\text{row } m) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$
$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \sum_{j=1}^{n} x_j \, (\text{column } j) \, .$$

(ii) (*Columns of* AB) Recall also that the kth column of AB is A times the kth column of B: if $B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_p]$, then

$$AB = [A\mathbf{b}_1 \ \ A\mathbf{b}_2 \ \ \cdots \ \ A\mathbf{b}_p] \, .$$

(iii) Note that each column of AB is a combination of the columns of A.

(iv) (*Entries of AB*) The (i, j) entry of $C = AB$ is the inner product of row i of A and column j of B:

$$C(i, j) = (\text{row } i) \cdot (\text{column } j) = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Proof.

Example. Let I denote the n by n identity matrix:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If A is any m by n matrix and B is any n by p matrix, then

$$AI = A \quad \text{and} \quad IB = B.$$

Proof.

Example. Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ be an m-dimensional column vector (m by 1 matrix) and $B = \begin{bmatrix} b_1 & \cdots & b_p \end{bmatrix}$ be a p-dimensional row vector (1 by p matrix). Then

$$AB = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} a_1 b_1 & \cdots & a_1 b_p \\ \vdots & \ddots & \vdots \\ a_m b_1 & \cdots & a_m b_p \end{bmatrix}.$$

In addition, if $m = p$, then

$$BA = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \sum_{k=1}^{m} a_k b_k.$$

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- Matrix multiplication III: Let $A = [a_{ij}]$ be an m by n matrix and let $B = [b_{ij}]$ be an n by p matrix.

(i) If $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ is an n-dimensional row vector, then

$$\begin{aligned} \mathbf{x}B &= \begin{bmatrix} \mathbf{x}\cdot(\text{column } 1) & \cdots & \mathbf{x}\cdot(\text{column } p) \end{bmatrix} \\ &= \begin{bmatrix} x_1 b_{11} + \cdots + x_n b_{n1} & \cdots & x_1 b_{1p} + \cdots + x_n b_{np} \end{bmatrix} \\ &= x_1 \begin{bmatrix} b_{11} & \cdots & b_{1p} \end{bmatrix} + \cdots + x_n \begin{bmatrix} b_{n1} & \cdots & b_{np} \end{bmatrix} \\ &= \sum_{j=1}^{n} x_j \,(\text{row } j)\,. \end{aligned}$$

(ii) (*Rows of AB*) The ith row of AB is the ith row of A multiplied by B:

$$AB = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} B = \begin{bmatrix} (\text{row } 1)\, B \\ \vdots \\ (\text{row } m)\, B \end{bmatrix}.$$

Hence each row of AB is a combination of the rows of B.

Proof.

(iii) The product AB is the sum of the kth column of A times the kth row of B over $k = 1, \ldots, n$:

$$\begin{aligned} AB &= \sum_{k=1}^{n} (\text{column } k \text{ of } A)\,(\text{row } k \text{ of } B) \\ &= \sum_{k=1}^{n} \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \cdots & b_{kp} \end{bmatrix}. \end{aligned}$$

Proof.

Example. Compute AB in four ways, where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution.

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Theorem. The following holds for any matrices A, B, and C of suitable sizes:

1. (Distributive law from the right) $(A+B)C = AC + BC$.
2. (Distributive law from the left) $A(B+C) = AB + AC$.
3. (Associative law) $(AB)C = A(BC)$.

Proof.

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- Powers of a matrix: Let A be a square matrix.

(i) The *powers* of A is defined by

$$A^1 = A, \quad A^2 = AA, \quad A^3 = A^2A = AA^2, \ldots.$$

(ii) The following laws hold for any $m, n \in \mathbb{N}$:

$$A^mA^n = A^{m+n}, \qquad (A^n)^m = A^{mn}.$$

- Block matrices and block multiplication:

(i) If A is m by n, B_1 is n by p_1, and B_2 is n by p_2, then

$$A\,[\,B_1 \;\; B_2\,] = [\,AB_1 \;\; AB_2\,]\,.$$

Proof.

(ii) If A_1 is m_1 by n, A_2 is m_1 by n, and B is n by p, then

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}.$$

Proof.

(iii) If A_1 is m_1 by n, A_2 is m_1 by n, B_1 is n by p_1, and B_2 is n by p_2, then

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix}.$$

Proof.

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(iv) If A_1 is m by n_1, A_2 is m by n_2, B_1 is n_1 by p, and B_2 is n_2 by p, then

$$[\,A_1 \;\; A_2\,] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1 + A_2B_2.$$

Proof.

(v) If A_{ij} is m_i by n_j and B_{ij} is n_i by p_j for all i, j, then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Proof.