Lecture 1 on Linear Algebra

김현석

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March 3 (Thursday) 15:00-16:15

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1.1. Vectors and linear combinations

- Vectors:

(i) By an n-*dimensional vector*, we mean an n-tuple

$$\mathbf{v} = (v_1, \ldots, v_n)$$

of (real or complex) numbers $v_1, \ldots, v_n$. The numbers $v_1, \ldots, v_n$ are called the *components* of $\mathbf{v}$. The vector $\mathbf{v} = (v_1, \ldots, v_n)$ will be also written as a *column vector* of the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

(ii) The set of all n-dimensional vectors of real numbers is denoted by $\mathbb{R}^n$.

(iii) The vector whose components are all zero is called the *zero vector* and denoted by $\mathbf{0}$.

- Basic operations:

(i) (*Vector addition and subtraction*) Two vectors $\mathbf{v}$ and $\mathbf{w}$ of the same dimension can be added and subtracted as follows:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} \pm \mathbf{w} = \begin{bmatrix} v_1 \pm w_1 \\ \vdots \\ v_n \pm w_n \end{bmatrix}.$$

(ii) (*Scalar multiplication*) A vector $\mathbf{v}$ can be multiplied by any number c as follows:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \Rightarrow \quad c\mathbf{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}.$$

A number c is called a *scalar*.

(iii) We define $-\mathbf{v} = (-1)\mathbf{v}$. Note that

$$\mathbf{v} + (-\mathbf{v}) = 0\mathbf{v} = \mathbf{0}$$

and

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$

(iv) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be n-dimensional vectors. Then by a *(linear) combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_m$, we mean a vector of the form

$$c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m,$$

where $c_1, \ldots, c_m$ are some scalars.

- Lengths: Let $\mathbf{v} = (v_1, \ldots, v_n)$ be a vector.

(i) The *length* of $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

(ii) $\mathbf{v}$ is called a *unit vector* if its length equals one, that is, $\|\mathbf{v}\| = 1$.

(iii) If $\mathbf{v}$ is nonzero, then

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} := \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is a unit vector in the same direction as $\mathbf{v}$.

1.2 Lengths and dot products

Example. The standard unit vectors in $\mathbb{R}^2$:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example. The standard unit vectors in $\mathbb{R}^3$:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1.2 Lengths and dot products

• Dot product on $\mathbb{R}^3$: Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be nonzero vectors in $\mathbb{R}^3$.

(i) The *dot product* of $\mathbf{v}$ and $\mathbf{w}$ is defined by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta,$$

where $0 \leq \theta \leq \pi$ is the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$.

(ii) Note that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

(iii) Recall the law of cosines:

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.$$

(iv) By a simple algebra,

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Proof.

(v) Since $|\theta| \leq 1$, we derive the *Cauchy-Schwarz inequality* in $\mathbb{R}^3$:

$$(v_1 w_1 + v_2 w_2 + v_3 w_3)^2 \leq \left(v_1^2 + v_2^2 + v_3^2\right)\left(w_1^2 + w_2^2 + w_3^2\right).$$

Proof.

1.2 Lengths and dot products

• Dot products on $\mathbb{R}^n$: Let $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be vectors in $\mathbb{R}^n$.

(i) The *dot product* or *inner product* of $\mathbf{v}$ and $\mathbf{w}$ is defined by

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i = v_1 w_1 + \cdots + v_n w_n.$$

(ii) The length of $\mathbf{v}$ can be written as

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

(iii) Two important inequalities:

1. (The Cauchy-Schwarz inequality) $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.
2. (The triangle inequality) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

1.2 Lengths and dot products

• Angles: Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^n$.

(i) By the Cauchy-Schwarz inequality, there exists a unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}.$$

The number θ is called the *angle* between the two vectors $\mathbf{v}$ and $\mathbf{w}$.

(ii) It can be easily proved that

1. $\theta = \pi/2$ if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.
2. $\theta = 0$ if and only if $\mathbf{v} = c\mathbf{w}$ for some $c > 0$.
3. $\theta = \pi$ if and only if $\mathbf{v} = c\mathbf{w}$ for some $c < 0$.

• Orthogonality: Let $\mathbf{v}$ and $\mathbf{w}$ be *any* vectors in $\mathbb{R}^n$. Then $\mathbf{v}$ and $\mathbf{w}$ are said to be *orthogonal* or *perpendicular* if $\mathbf{v} \cdot \mathbf{w} = 0$. Therefore, any vector is orthogonal to the zero vector.