Chapter 5 : TREES

Data Structures Lecture Note
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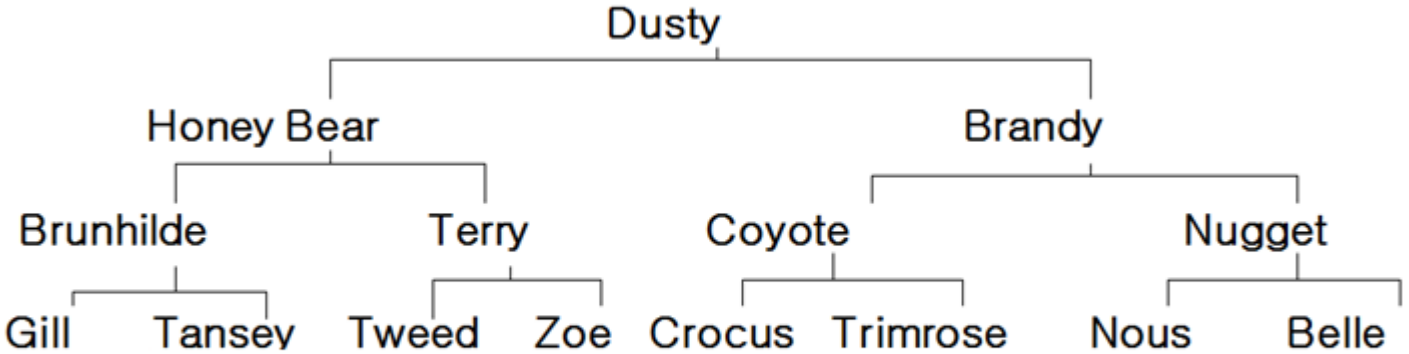
5.1 INTRODUCTION

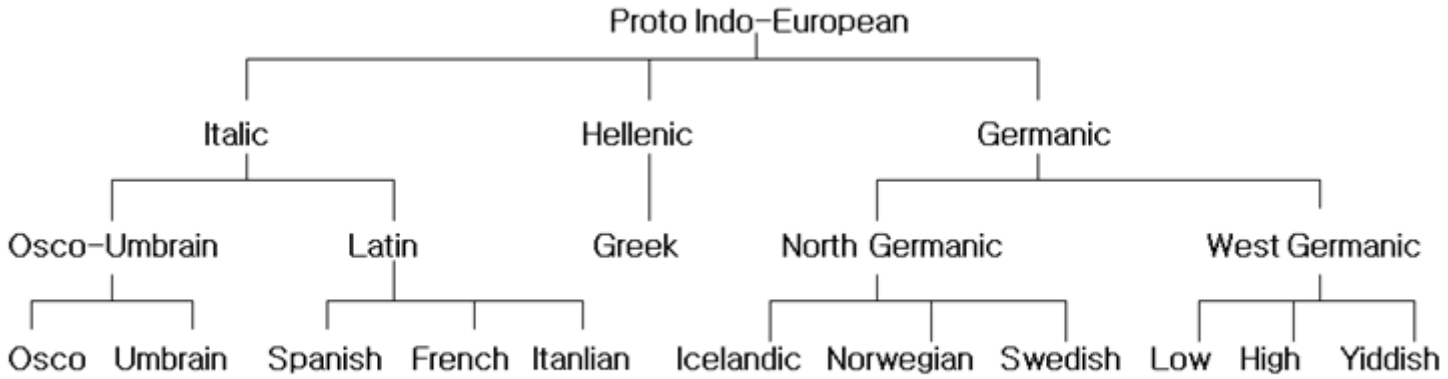
5.1.1 Terminology

The intuitive concept of a tree implies that we organize the data

[Figure 5.1] Two types of genealogical charts



(a) Pedigree



(b) Lineal

Definition : A tree is a finite set of one or more nodes such that:

(1) There is a specially designated node called the *root*.

(2) The remaining nodes are partitioned into n≥0 disjoint sets T_1, T_2, …, T_n where each of these sets is a tree

We call $T_1, \ldots, T_n$ the subtrees of the root.

Terms used when referring to trees:

- A ***node*** stands for the item of information and the branches to other nodes.
- The ***degree*** of a node is the number of subtrees of the node.
- The ***degree of a tree*** is the maximum degree of the nodes in the tree.
- A node with degree zero is a ***leaf*** or ***terminal*** node.
- A node that has subtrees is the ***parent*** of the roots of the subtrees, and the roots of the subtrees are the ***children*** of the node.
- Children of the same parent are ***siblings***.
- The ***ancestors*** of a node are all the nodes along the path from the root to the node. Conversely, the ***descendants*** of a node are all the nodes that are in its subtrees.

Terms used when referring to trees:

- The ***level*** of a node is defined by :

  Initially letting the root be at level one.

  For all subsequent nodes, the level is the level of the node's parent plus one.

- The ***height*** or ***depth*** of a tree is the maximum level of any node in the tree.

5.1.2 Representation Of Trees

List Representation

Representing a tree as a list in which each of the subtrees is also a list.

For example, the tree of Figure 5.2 is written as :

(A(B(E(K,L),F),C(G),D(H(M),I,J)))

If we wish to use linked lists, then a node must have a varying number of fields depending on the number of branches.

[Figure 5.3] a possible representation for trees

data	*link 1*	*link 2*	. . .	*link n*

It is often easier to work with nodes of a fixed size.

5.1.2 Representation Of Trees

Left Child-Right Sibling Representation

The representations we consider require exactly two link or pointer fields per node.

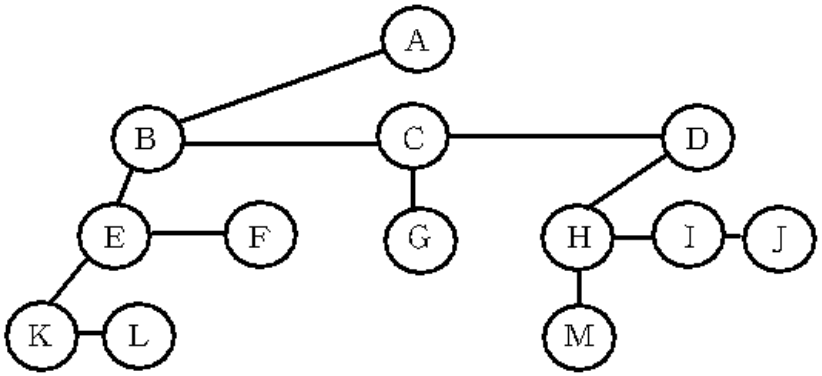
Note that every node has only one leftmost child and one closest right sibling.

(* Strictly speaking, the order of children in a tree is not important. *)

[Figure 5.4]

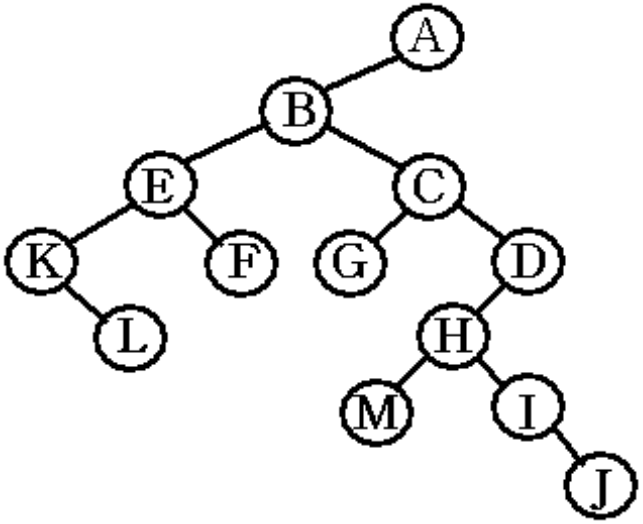
data	
left child	right sibling

[Figure 5.5]

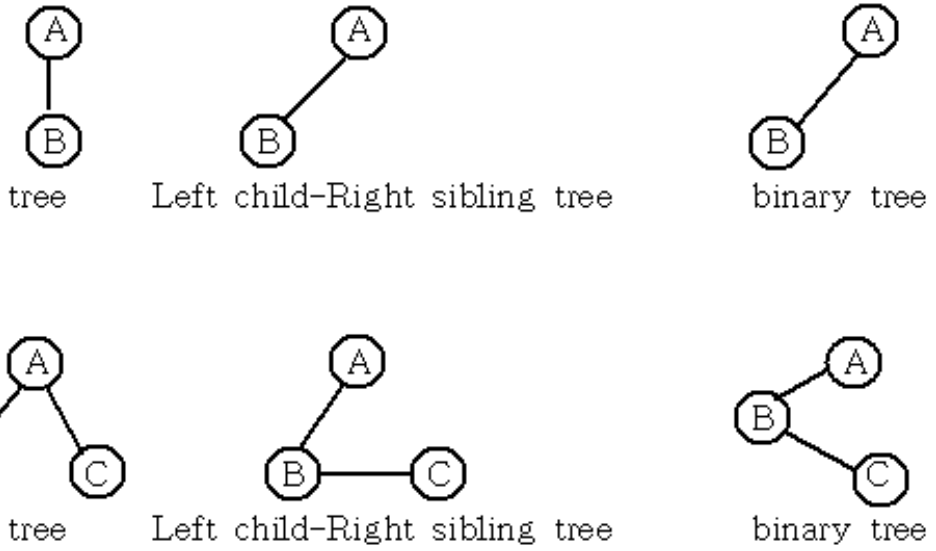


Representation As A Degree Two Tree

[Figure 5.6]



[Figure 5.7]



5.2 BINARY TREES

5.2.1 The Abstract Data Type

- The chief characteristic of a binary tree is the stipulation that the degree of any given node must not exceed two.
- For binary trees, we distinguish between the left subtree and the right subtree, while for trees the order of the subtrees is irrelevant.
- Definition : A binary tree is a finite set of nodes that is either empty or consists of a root and two disjoint binary trees called the left subtree and the right subtree.

[Distinction between a binary tree and a tree]

(1) There is an empty binary tree.

(2) In a binary tree, we distinguish between the order of the children while in a tree we do not.

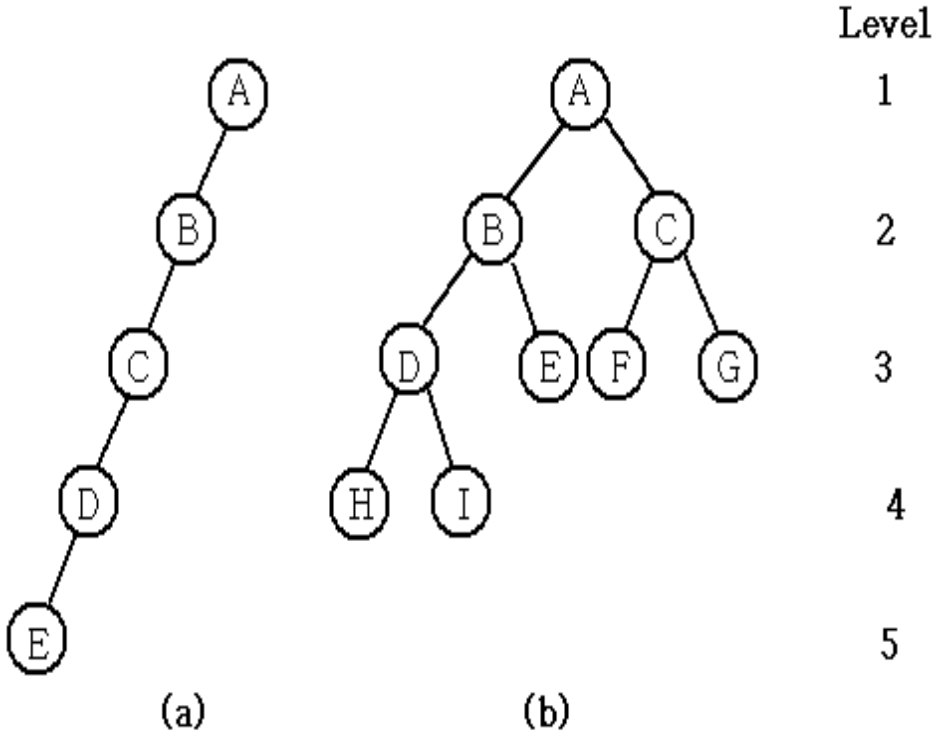
Example : [Figure 5.8] Two different binary trees

Special Types of binary trees

Skewed tree, Complete binary tree

(Figure 5.9)

[Figure 5.6]



The same terminology we used to describe trees applies to binary trees.

- node,
- degree of a node,degree of a tree,
- leaf or terminal,
- parent, children (left child, right child), sibling,
- ancestor, descendant,
- level of a node, height or depth,

Structure 5.1 : Abstract data type Binary_Tree.

- Structure Binary_Tree (abbreviated BinTree) is
- objects : a finite set of nodes either empty or consisting of a root node, left Binary_Tree, and right Binary_Tree.
- functions :

for all bt, bt1, bt2 ∈ BinTree, item ∈ element

BinTree Create() ::= creates an empty binary tree

Boolean IsEmpty(*bt*) ::= **if** (*bt* == empty binary tree) **return** *TRUE* **else return** *FALSE*

BinTree MakeBT(*bt*1, *item*, *bt*2) ::= **return** a binary tree whose left subtree is *bt*1, whose right subtree is *bt*2, and whose root node contains the data *item*.

BinTree Lchild(*bt*) ::= **if** (IsEmpty(*bt*)) **return** error **else return** the left subtree of *bt*.

element Data(bt) ::= **if** (IsEmpty(*bt*)) **return** error **else return** the data in the root node of *bt*.

BinTree Rchild(*bt*) ::= **if** (IsEmpty(*bt*)) **return** error **else return** the right subtree of *bt*.

5.2.2 Properties Of Binary Trees

Lemma 5.1 [Maximum number of nodes]:

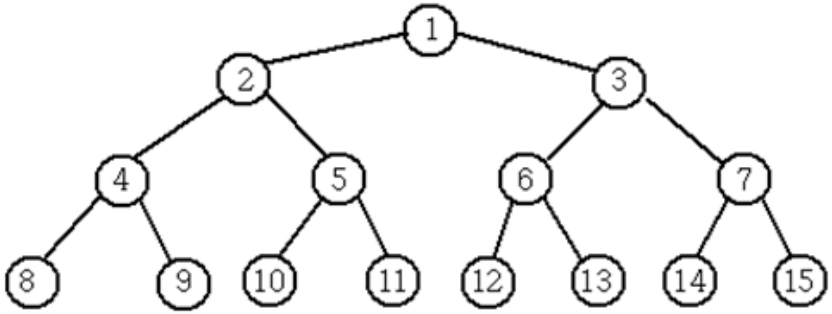
(1) The maximum number of nodes on level *i* of a binary tree is 2^{i-1}, $i \geq 1$.

(2) The maximum number of nodes in a binary tree of depth k is $2^k - 1$, $k \geq 1$.

Definition : A full binary tree of depth k is a binary tree of depth k having $2k - 1$ nodes, $k \geq 0$.

We can number the nodes of a full binary tree, starting with the root on level 1, continuing with the nodes on level 2, and so on. Nodes on any level are numbered from left to right.

[Figure 5.10] Full binary tree of depth 4 with sequential node numbers



Definition : A binary tree with n nodes and depth k is complete iff its nodes correspond to the nodes numbered from 1 to n in the full binary tree of depth k.

5.2.3 Binary Tree Representations

Array Representation

By using the numbering scheme shown in Figure 5.10,

we can use a one-dimensional array to store the nodes in a binary tree. (We do not use the 0-th position of the array.)

Lemma 5.3 : If a complete binary tree with n nodes (depth = $\lfloor \log_2 n+1 \rfloor$) is represented sequentially, then for any node with index i, $1 \le i \le n$, we have :

(1) parent(i) is at $\lfloor i/2 \rfloor$ if i≠1.

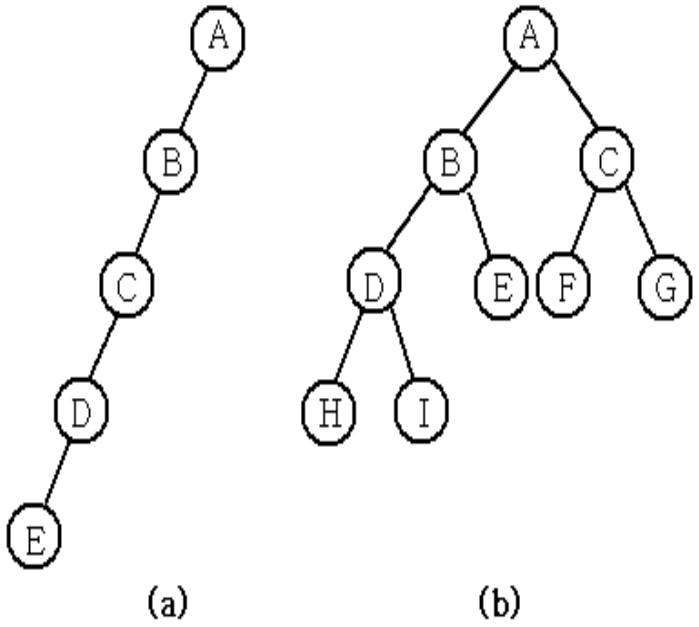
If i =1, i is at the root and has no parent.

(2) left_child(i) is at 2i if 2i≤n. If 2i>n, then i has no left child.

(3) right_child(i) is at 2i+1 if 2i+1≤n.

If 2i+1>n, then i has no right child.

[Figure 5.9]



[Figure 5.11]

[1]	A
[2]	B
[3]	–
[4]	C
[5]	–
[6]	–
[7]	–
[8]	D
[9]	–
.	.
.	.
.	.
[16]	E

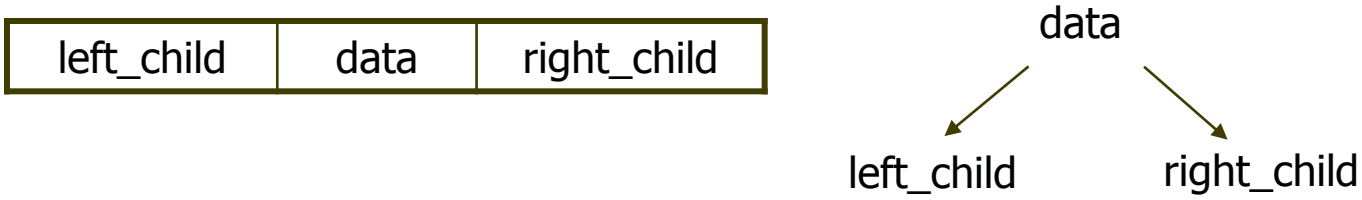
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

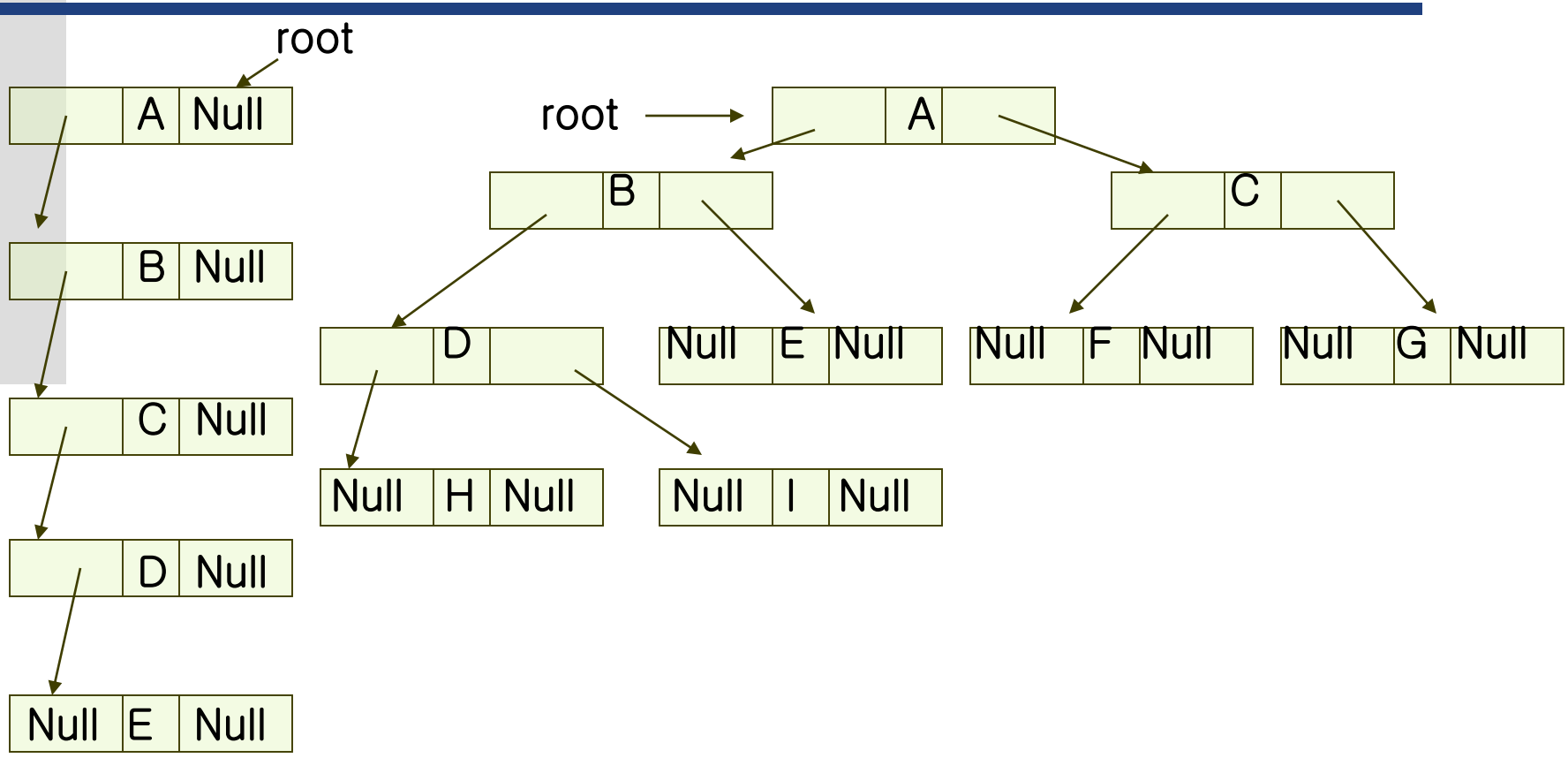
Linked Representation

- Node structure :

```
typedef struct node *tree_pointer;
typrdef struct node {
        int data;
        tree_pointer left_child, right_child;
 };
```

[Figure 5.12] Node representation for binary trees





[Figure 5.13] The linked representation for the trees in Figure 5.9.

5.3 BINARY TREE TRAVERSALS

- One of the operations that arises frequently is traversing a tree, that is, visiting each node in the tree exactly once.
- A full traversal produces a linear order for the information in a tree.
- When traversing a tree we want to treat each node and its subtrees in the same way.
- Let, for each node in a tree,
  L stands for moving left,
  V stands for visiting the node (e.g., printing out the data field),
  R stands for moving right.

- Six possible combinations of traversal :

  LVR : inorder traversal

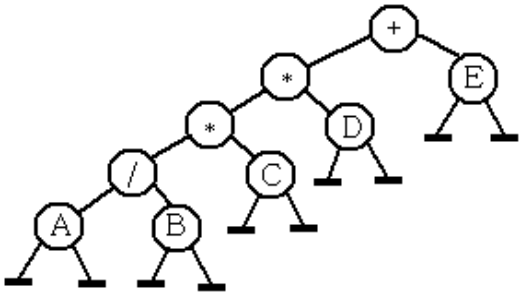
  LRV : postorder traversal

  VLR : preorder traversal

  VRL, RVL, RLV.

- There is a natural correspondence between these traversals and producing the infix, postfix, and prefix forms of an expression.

- [Figure 5.15] Binary tree with arithmetic expression



- **Inorder Traversal**

```
void  inorder (tree_pointer  ptr)
      /* inorder tree traversal */
      {
        if  (ptr)  {
                inorder (ptr -> left_child);
                printf ("%d", ptr -> data);
                inorder (ptr -> right_child);
           }
      }
```

- The data fields of Figure 5.15 are output in the order :

  A / B * C * D + E

[Figure 5.16]

Call of inorder	Value in root	Action	inorder	in root	Value Action
1	+		11	C	
2	*		12	NULL	
3	*		11	C	printf
4	/		13	NULL	
5	A		2	*	printf
6	NULL		14	D	
5	A	printf	15	NULL	
7	NULL		14	D	printf
4	/	printf	16	NULL	
8	B		1	+	printf
9	NULL		17	E	
8	B	printf	18	NULL	
10	NULL		17	E	printf
3	*	printf	19	NULL	

- **Preorder Traversal**

```
void  preorder (tree_pointer  ptr)
      /* preorder tree traversal */
      {
        if  (ptr)  {
                printf ("%d", ptr -> data);
                preorder (ptr -> left_child);
                preorder (ptr -> right_child);
            }
      }
```

- The data fields of Figure 5.15 are output in the order :

  + * * / A B C D E

- **Postorder Traversal**

```
void  postorder (tree_pointer  ptr)
/* postorder tree traversal */
{
  if  (ptr)  {
          postorder (ptr -> left_child);
          postorder (ptr -> right_child);
          printf ("%d", ptr -> data);
      }
}
```

- The data fields of Figure 5.15 are output in the order :

  A B / C * D * E +

- **Iterative Inorder Traversal**

Figure 5.16 implicitly shows the stacking and unstacking of Program 5.1.

- a node that has no action indicates that the node is added to the stack,
- while a node that has a printf action indicates that the node is removed from the stack.

Notice that :

- the left nodes are stacked until a null node is reached,
- the node is then removed from the stack, and
- the node's right child is stacked.

```
void iter_inorder(tree_pointer node)
     {
      int  top = -1;  /* initialize stack */
      tree_pointer  stack[MAX_STACK_SIZE];
      for ( ; ; )  {
         for ( ; node; node = node -> left_child)
                    add(&top, node);   /* add to stack */
         node = delete(&top);    /* delete from stack */
         if (!node)  break;       /* empty stack  */
         printf ("%d", node -> data);
         node = node -> right_child;
        }
     }
```

Analysis of iter_inorder : Let *n* be the number of nodes in the tree.

Note that every node of the tree is placed on and removed from the stack exactly once.

The time complexity is $\Theta(n)$.

The space complexity is equal to the depth of the tree which is $O(n)$.

- **Level Order Traversal**

A traversal that requires a queue.

Level order traversal visits the nodes
using the ordering scheme suggested in Figure 5.10.

The data fields of Figure 5.15 are output in the order :
*+ * E * D / C A B*

```
void level_order(tree_pointer ptr)
      /* level order tree traversal */
      {
        int  front = rear = 0;
        tree_pointer  queue[MAX_QUEUE_SIZE];
        if (!ptr)  return;   /* empty tree */
        addq(front, &rear, ptr);
        for ( ; ; )  {
                  ptr = deleteq(&front, rear);     /*empty list returns NULL*/
                  if (ptr)  {
                            printf("%d", ptr->data);
                            if (ptr->left_child)
                                      addq(front, &rear, ptr->left_child);
                            if (ptr->right_child)
                                      addq(front, &rear, ptr->right_child);
                     }
                  else  break;
           }
      }
```

5.4 ADDITIONAL BINARY TREE OPERATIONS

- By using the definition of a binary tree and

the recursive versions of inorder, preorder, and post order traversals, we can easily create C functions for other binary tree operations.

- Copying Binary Trees

One practical operation is copying a binary tree. (Program 5.6)
Note that this function is only a slightly modified version
of postorder (Program 5.3)

- [Program 5.6]

```
tree_pointer  copy(tree_pointer original)
  /* this function returns a tree_pointer to an exact copy
      of the original tree */
  {
     tree_pointer  temp;
     if (original)  {
                    temp = (tree_pointer) malloc(sizeof(node));
                    if (IS_FULL(temp)) {
                                fprintf(stderr, "The memory is full□n");
                                exit(1);
                          }
                    temp->left_child = copy(original->left_child);
                    temp->right_child = copy(original->right_child);
                    temp->data = original->data;
                    return  temp;
          }
        return  NULL;
      }
```

- **Testing For Equality Of Binary Trees**

Equivalent trees have the same structure and the same information in the corresponding nodes

```
int equal(tree_pointer first,  tree_pointer second)
/* function returns FALSE if the binary trees first and
          second are not equal, otherwise it returns TRUE */
{
  return ((!first && !second) || (first && second &&
                    (first->data == second->data) &&
                equal(first->left_child, second->left_child) &&
                equal(first->right_child, second->right_child))
}
```

The Satisfiability Problem

Consider the formulas constructed by taking variables
x1, x2, . . ., xn and operators $\wedge$ (and), $\vee$ (or), and $\neg$ (not).
The variables can hold only one of two possible values, true or false.

The expressions are defined by the following rules :

(1) A variable is an expression.
(2) If x and y are expressions, then $\neg x$, $x \wedge y$, $x \vee y$ are expressions.
(3) Parentheses can be used to alter the normal order of evaluation, which is $\neg$ before $\wedge$ before $\vee$.

These rules comprise the formulas in the propositional calculus since other operations, such as implication, can be expressed using $\neg$, $\wedge$, and $\vee$.

Consider an expression :

$$x_1 \vee (x_2 \wedge \neg x_3)$$

If x_1 and x_3 are false and x_2 is true,

the value of this expression is true.

false $\vee$(true$\wedge$ $\neg$false)

= false $\vee$ true

= true

The *satisfiability* problem for formulas of the propositional calculus asks if there is an assignment of values to the variables that cause the value of the expression to be true.

A first version of satisfiability algorithm:

[Program 5.8]

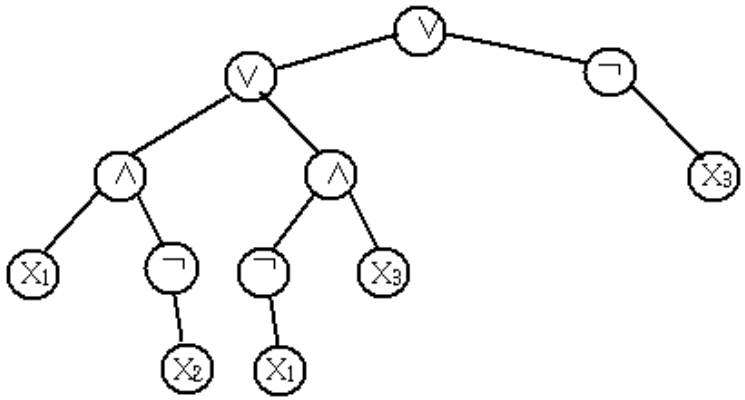
```
for (all 2n possible combinations) {
        generate the next combination;
        replace the variables by their values;
        evaluate the expression;
        if (its value is true) {
                printf(<combination>);
                return;
                }
      }
printf("No satisfiable combination□n");
```

<Evaluating an propositional formula>

Assume that our formula is already in a binary tree.

For a formula :

$$(x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_3) \vee \neg x_3$$



[Figure 5.18] Corresponding binary tree.

To evaluate an expression we can traverse the tree in postorder,
evaluating the subtrees until entire expression
is reduced to a single value.

This corresponds to
the postfix evaluation of an arithmetic expression.

[Figure 5.19]


```
typedef enum {not, and, or, true, false} logical;
typedef struct node *tree_pointer;
typedef struct node {
        tree_pointer  left_child;
        logical       data;
        short int     value;
        tree_pointer  right_child;
        };
```

[Program 5.9]

```
void post_order_eval(tree_pointer node)
{
  /* modified postorder traversal to evaluate
   a propositional calculus tree          */
     if (node) {
          post_order_eval(node->left_child);
          post_order_eval(node->right_child);
         switch(node->data) {
           case not : node->value =
                    !node->right_child->value;
                    break;
           case and : node->value =
                   node->right_child->value &&  node->left_child->value;
                   break;
          case or : node->value =
                    node->right_child->value ||  node->left_child->value;
                    break;
           case true : node->value = TRUE;
                    break;
           case false : node->value = FALSE;
         }
     }
}
```

5.5 THREADED BINARY TREES

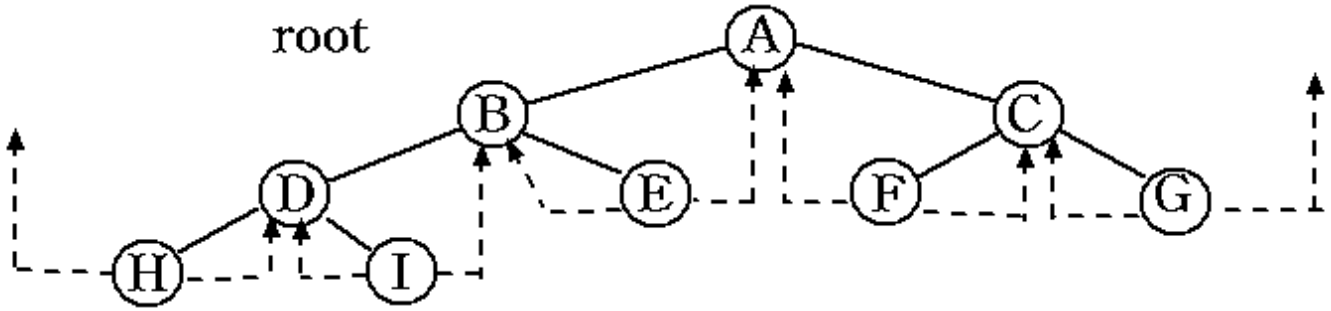
A binary tree T with n nodes has 2n links and
among them, (n+1) are NULL links.
A.J. Perlis and C. Thornton have devised a clever way
to make use of these null links.

They replace the null links by pointers, called *threads*, to other nodes in the tree by using the following rules (assume that ptr represents a node) :

(1) If *ptr->left_child* is null, replace *ptr->left_child* with a pointer to the node that would be visited before ptr in an inorder traversal. That is we replace the null link with a pointer to the *inorder predecessor* of ptr.

(2) If *ptr->right_child* is null, replace *ptr->right_child* with a pointer to the node that would be visited after ptr in an inorder traversal. That is we replace the null link with a pointer to the *inorder successor* of ptr.

[Figure 5.21]



When we represent the tree in memory,

we must be able to distinguish between threads and normal pointers.

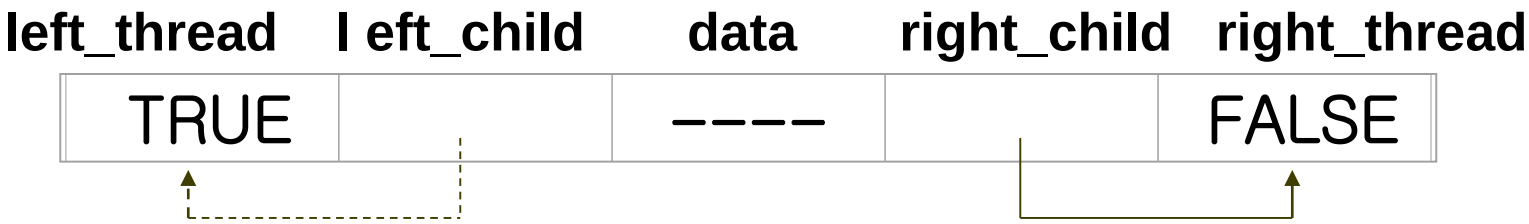
This is done by adding two additional fields

to the node structure, *left_thread* and *right_thread*.

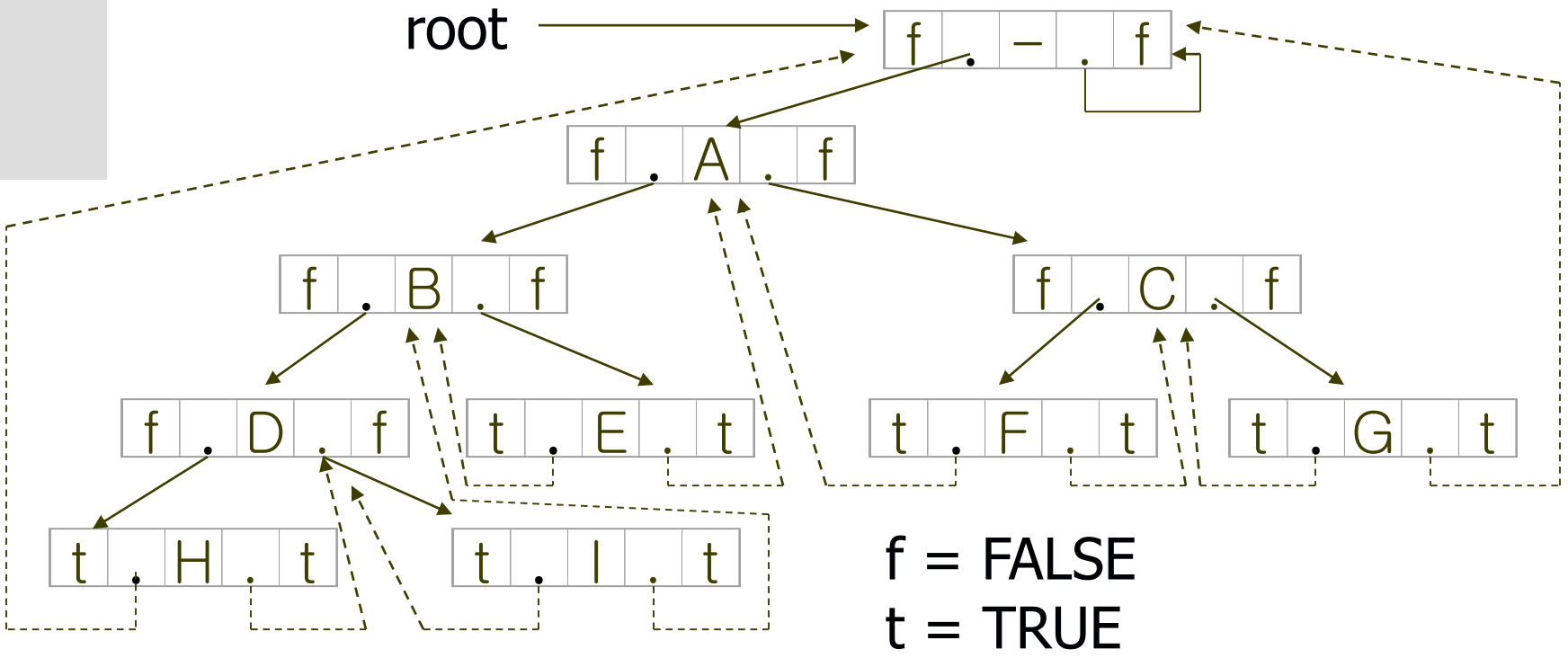
```
typedef struct threaded_tree *threaded_pointer;
typedef struct threaded_tree {
        short int left_thread;
        threaded_pointer left_child;
        char  data;
        threaded_pointer right_child;
        short int right_thread;
};
```

We assume that all threaded binary trees have a head node.

[Figure 5.22] An empty threaded tree.



[Figure 5.23]



Inorder Traversal of a Threaded Binary Tree

Determining the inorder successor of a node.

[Program 5.10] Finding the inorder successor of a node.

```
threaded_pointer insucc(threaded_pointer tree)
{
 /* find the inorder successor of tree
                              in a threaded binary tree */
      threaded_pointer temp;
      temp = tree->right_child;
      if (!tree->right_thread)
                  while (!temp->left_thread)
                  temp = temp->left_child;
      return temp;
 }
```

To perform an inorder traversal we make repeated calls to insucc.

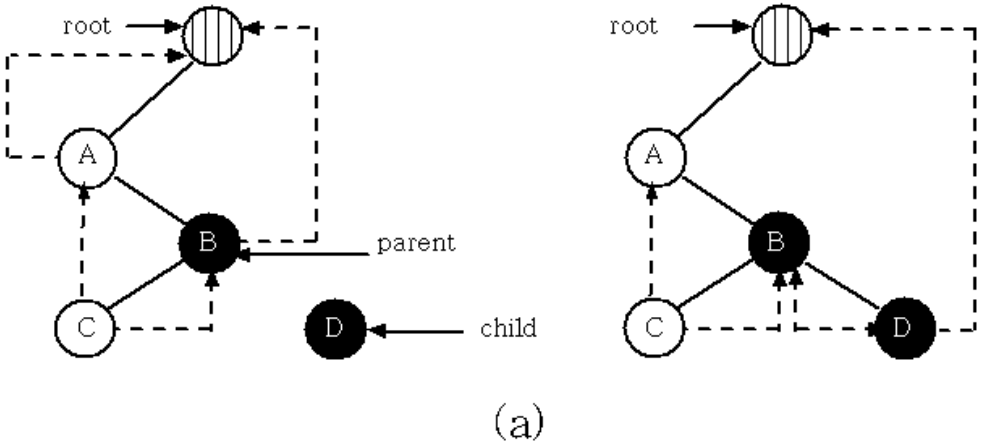
[Program 5.11] : Inorder traversal of a threaded binary tree.

```
void tinorder(threaded_pointer tree)
{
 /* traverse the threaded binary tree inorder */
    threaded_pointer temp = tree;
    for ( ; ; ) {
            temp = insucc(temp);
            if (temp == tree) break;
            printf("%3c", temp->data);
     }
 }
```

Inserting A Node Into A Threaded Binary Tree

Assume that we have a node, *parent*, that has an empty right subtree. We wish to insert *child* as the right child of parent.

[Figure 5.24] (a)



(a)

To do this we must :

(1) change *parent-> right_thread* to FALSE

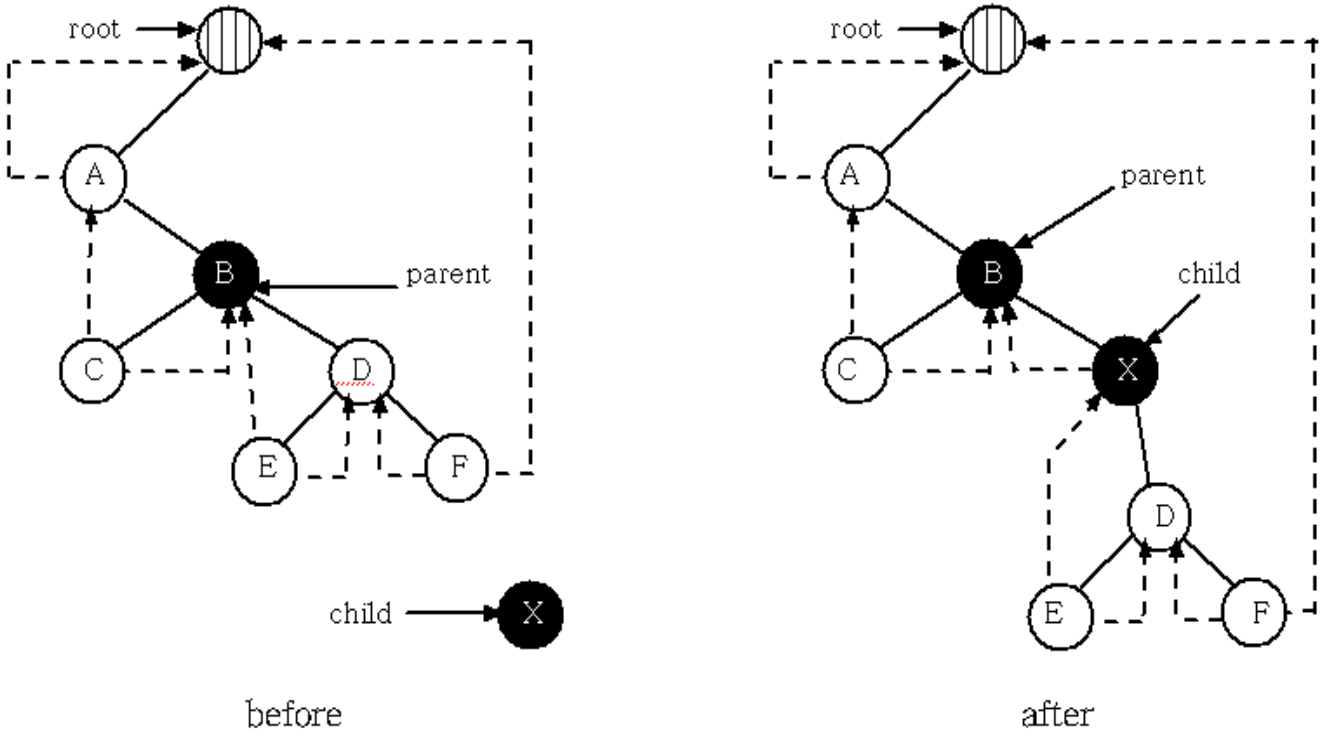
(2) set *child->left_thread* and *child->right_thread* to TRUE

(3) set *child->left_child* to point to *parent*

(4) set *child->right_child* to *parent->right_child*

(5) change *parent->right_child* to point to *child*

For the case that *parent* has a nonempty right subtree,
[Figure 5.24](b)



(b)

C code which handles both cases.

[Program 5.12] : Right insertion in a threaded binary tree

```
void insert_right(threaded_pointer parent,  threaded_pointer child)  {
 /* insert child as the right child of parent  in a threaded binary tree */
      threaded_pointer temp;
      child->right_child = parent->right_child;
      child->right_thread = parent->right_thread;
      child->left_child = parent;
      child->left_thread = TRUE;
      parent->right_child = child;
      parent->right_thread = FALSE;
      if (!child->right_thread) {
            temp = insucc(child);
            temp->left_child = child;
        }
 }
```

5.6 HEAPS

5.6.1 The Heap Abstract Data Type

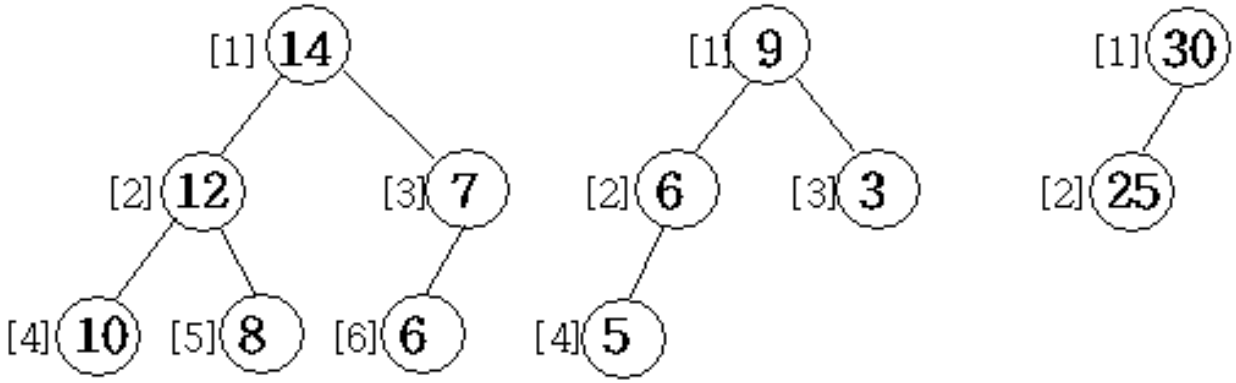
Definition : A *max tree* is a tree in which the key value in each node is no smaller than the key values in its children (if any).

A *max heap* is a complete binary tree that is also a max tree.

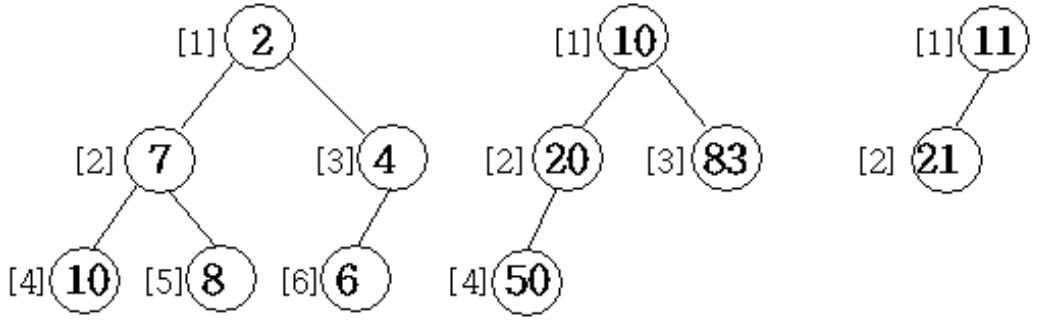
Definition : A *min tree* is a tree in which the key value in each node is no larger than the key values in its children (if any).

A *min heap* is a complete binary tree that is also a min tree.

[Figure 5.25] sample max heaps



[Figure 5.26] sample min heaps



Notice that we represent a heap as an array, although we do not use position 0.

From the heap definitions it follows that

- the root of a min tree contains the smallest key in the tree.
- the root of a max tree contains the largest key in the tree.

Basic operations on a max heap :

(1) Creation of an empty heap

(2) Insertion of a new element into the heap

(3) Deletion of the largest element from the heap

[Structure 5.2] : Abstract data type MaxHeap.

```
Structure MaxHeap is
 object: a complete binary tree of n>=0 elements organized so that
         the value in each node is at least as large as those in its children
 functions:
       for all heap   MaxHeap, item   Element, n, max_size   integer
       MaxHeap Create(max_size) ::= create an empty heap that can hold
                                                    a maximum of max_size elements.
       Boolean HeapFull(heap, n)  ::= if (n == max_size) return TRUE
                                                    else return FALSE
       MaxHeap Insert(heap, item, n) ::= if (!HeapFull(heap, n))
                                                    insert an item into heap and
                                                    return the resulting heap
                                                    else return error.
       Boolean HeapEmpty(heap, n) ::= if (n<=0) return TRUE
                                                    else return FALSE
       MaxHeap Delete(heap, n) ::= if (!HeapEmpty(heap, n)) return
                                                    one of the largest element in the
                                                    heap and remove it from the heap
                                                    else return error.
```

5.6.2 Priority Queues

Heaps are frequently used to implement priority queues.

Unlike the queues, FIFO lists, a priority queue deletes
the element with the highest (or the lowest) priority.

At any time
an element with arbitrary priority can be inserted
into a priority queue.

Implementing priority queues :

Heaps are used as an efficient implementation of the priority queues.

To examine some of the other representations see Figure 5.27.

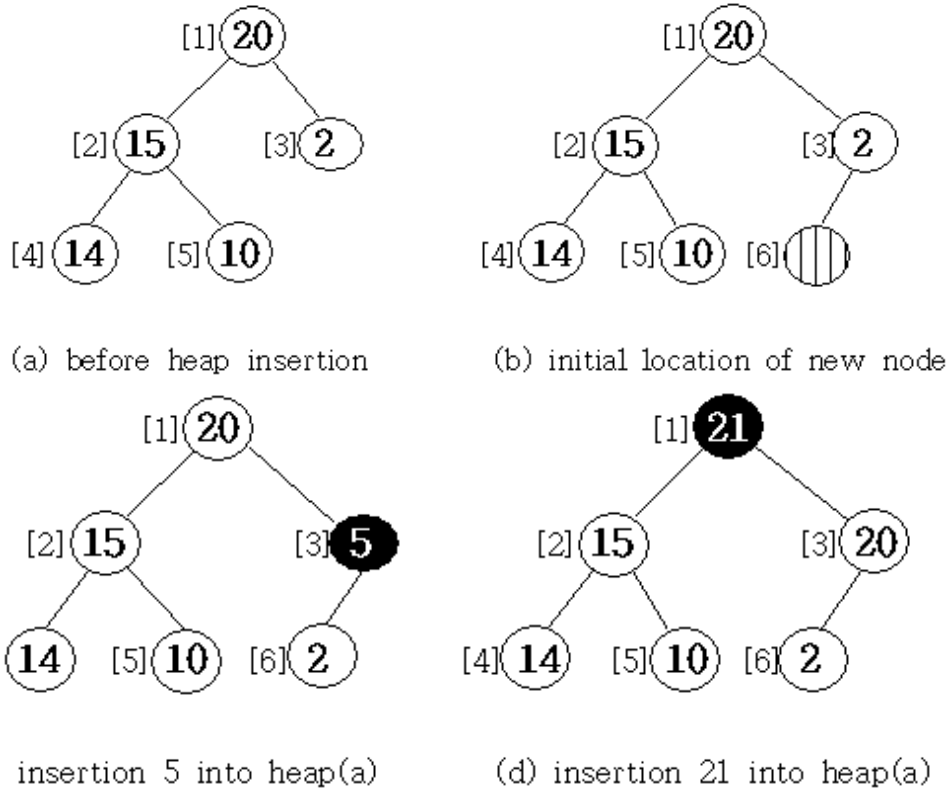
Representation	Insertion	Deletion
Unordered array	$\Theta(1)$	$\Theta(n)$
Unordered linked list	$\Theta(1)$	$\Theta(n)$
Sorted array	$O(n)$	$\Theta(1)$
Sorted linked list	$O(n)$	$\Theta(1)$
Max heap	$O(\log_2 n)$	$O(\log_2 n)$

[Figure 5.27]

5.6.3 Insertion Into A Max Heap

To illustrate the insertion operation, See Figure 5.28

[Figure 5.28]



We use the array representation discussed in Section 5.2.3.

C declaration:

```
#define MAX_ELEMENTS 200    /*maximum heap size+1 */
#define HEAP_FULL(n) (n == MAX_ELEMENTS-1)
#define HEAP_EMPTY(n) (!n)
typedef struct {
                int key;
                /* other fields */
                } element;
  element heap[MAX_ELEMENTS];
  int n = 0;
```

We can insert a new element in a heap with n elements
by following the steps below :

(1) place the element in the new node (i.e., n+1 th position)
(2) move along the path from the new node to the root,
if the element of the current node is larger than the one of its parent
then interchange them and repeat.

[Program 5.13] Insertion into a max heap

```
void insert_max_heap(element item, int *n)
{
 /* insert item into a max heap of current size *n */
      int i;
      if (HEAP_FULL(*n)) {
            fprintf(stderr, "The heap is full. □n");
            exit(1);
         }
      i = ++(*n);
      while ((i != 1) && (item.key > heap[i/2].key)) {
              heap[i] = heap[i/2];
              i /= 2;
         }
      heap[i] = item;
 }
```

Analysis of *insert_max_heap* :

The function first checks for a full heap.
If not, set *i* to the size of the new heap (n+1).
Then determines the correct position of item in the heap
by using the while loop.
This while loop is iterated O(log2 n) times.

Hence the time complexity is O(log2 n).

5.6.4 Delete From A Max Heap

When we delete an element from a max heap,

we always take it from the root of the heap.

If the heap had n elements, after deleting the element in the root,

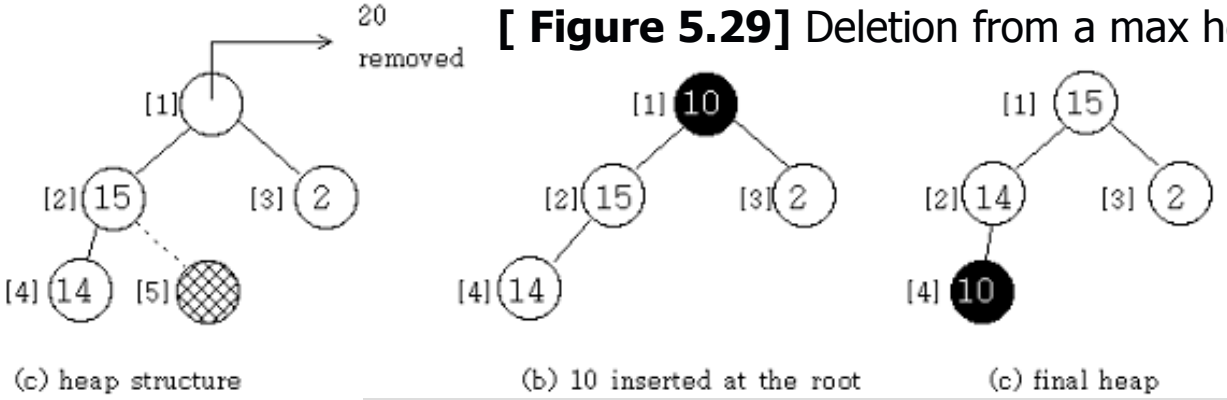
the heap must become a complete binary tree with one less nodes,

i.e., (n-1) elements.

We place the element in the node at position n in the root node

and to establish the heap we move down the heap,

comparing the parent node with its children and exchanging out-of-order elements until the heap is reestablished.



[Figure 5.29] Deletion from a max heap

[Program 5.14] : Deletion from a max_heap

```
element delete_max_heap(int *n)
{
 /* delete element with the highest key from the heap */
    int  parent, child;
    element  item, temp;
    if (HEAP_EMPTY(*n))  {
                fprintf(stderr, "The heap is empty");
                exit(1);
            }
     /* save value of the element with the largest key */
    item = heap[1];
    /* use last element in heap to adjust heap  */
    temp = heap[(*n)--];
    parent = 1;   child = 2;
```

```
    while (child <= *n)  {
        /* find the larger child of the current parent  */
        if ((child < *n) &&(heap[child].key< heap[child+1].key))
                                child++;
        if (temp.key >= heap[child].key)   break;
         /* move to the next lower level  */
         heap[parent] = heap[child];
         parent = child;
         child *= 2;
    }
    heap[parent] = temp;
    return  item;
}
```

Analysis of *delete_max_heap* [Program 5.14]:

The function delete_max_heap operates
by moving down the heap,
comparing and exchanging parent and child nodes
until the heap definition is re-established.

Since the height of a heap with n elements is $\lceil \log_2(n+1) \rceil$,
the while loop is iterated O(log2 n) times.

Hence the time complexity is O(log2 n).

5.7 BINARY SEARCH TREES

5.7.1 Introduction

While a heap is well suited for applications that require priority queues, it is not well suited for applications in which we delete and search arbitrary elements.

A *binary search tree* has a better performance than any of the data structures studied so far for operations, insertion, deletion, and searching of arbitrary element.

In fact, with a binary search tree we can perform these operations by both key value (e.g., delete the element with key x) and by rank (e.g., delete the fifth smallest element).

Definition: A *binary search tree* is a binary tree. It may be empty. If it is not empty, it satisfies the following properties :

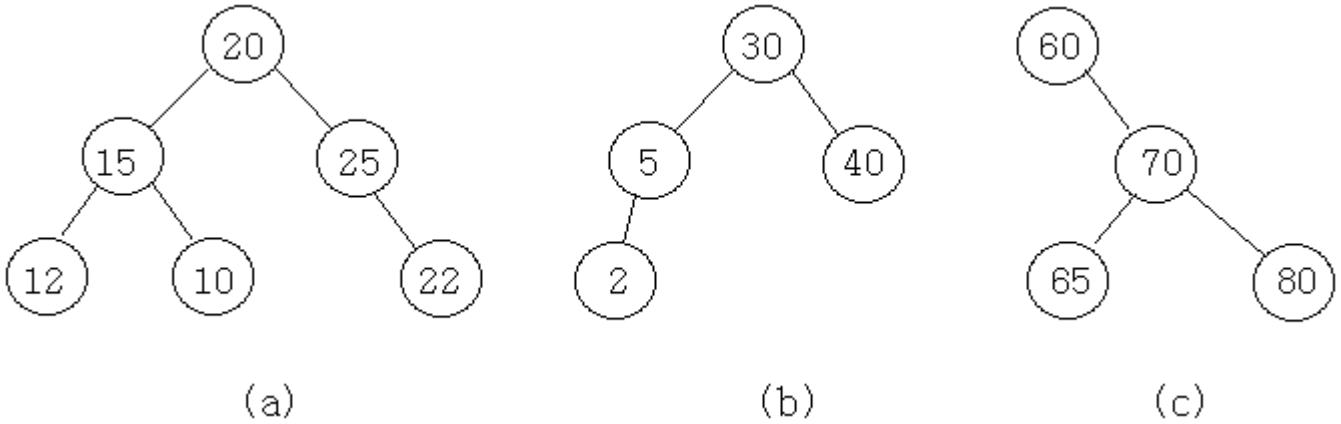
(1) Every element has a key, and no two elements have the same key, that is, the keys are unique.

(2) The keys in a nonempty left subtree must be smaller than the keys in the root.

(3) The keys in a nonempty right subtree must be larger than the keys in the root.

(4) The left and right subtrees are also binary search trees. □

[Figure 5.30] some sample binary trees



If we traverse a binary search tree in inorder
and print the data of the nodes in the order visited,
what would be the order of data printed?

5.7.2 Searching A Binary Search Tree

[Program 5.15] : Recursive search for a binary search tree

```
tree_pointer search(tree_pointer root, int key)
{
 /* return a pointer to the node that contains key.
    If there is no such node, return NULL.   */
    if (!root)  return NULL;
    if (key == root->data)  return root;
    if (key < root->data)
                return  search(root->left_child, key);
    return  search(root->right_child, key);
}
```

[Program 5.16] Iterative search for a binary search tree

```
tree_pointer search2(tree_pointer tree, int key)
{
/* return a pointer to the node that contains key.
   If there is no such node, return NULL.   */
   while (tree)  {
      if (key == tree->data)  return tree;
      if (key < tree->data)
               tree = tree->left_child;
      else
               tree = tree->right_child;
         }
   return NULL;
}
```

Analysis of *search* and *search2* :

If h is the height of the binary search tree,

then the time complexity of both search and search2 is O(h).

However, search has an additional stack space requirement which is O(h).

Searching a binary tree is

similar to the binary search of a sorted list.

5.7.3 Inserting Into A Binary Search Tree

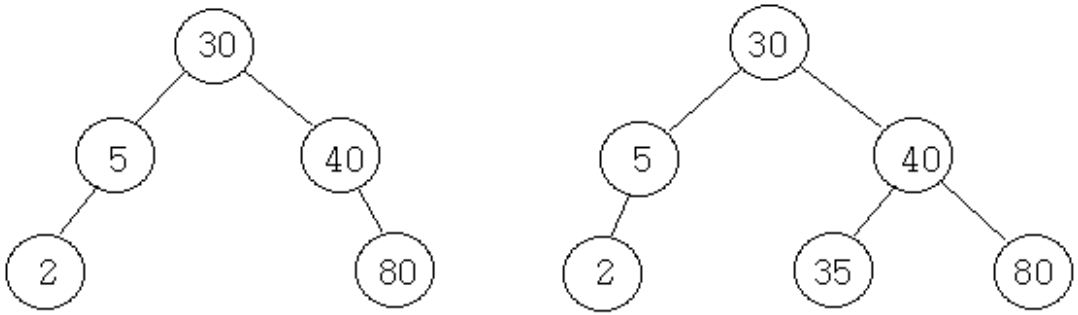
To insert a new element, key :

First, we verify that the key is different from those of existing elements by searching the tree.

If the search is unsuccessful,

then we insert the element at the point the search terminated.

[Figure 5.31]



(a) Insert 80 (b) Insert 35

[Program 5.17] : Inserting an element into a binary search tree

```
void insert_node(tree_pointer *node, int num)
{
  /* If num is in the tree pointed at by node do nothing;
     otherwise add a new node with data = num  */
     tree_pointer  ptr, temp = modified_search(*node, num);
     if (temp || !(*node))  {
        /* num is not in the tree  */
              ptr = (tree_pointer) malloc(sizeof(node));
              if (IS_FULL(ptr)) {
                        fprintf(stderr, "The memory is full");
                        exit(1);
              }
```

```
    ptr->data = num;
    ptr->left_child = ptr->right_child = NULL;
    if (*node)          /* insert as child of temp  */
        if (num < temp->data)
              temp->left_child = ptr;
        else  temp->right_child = ptr;
    else  *node = ptr;
  }
}
```

function *modified_search* searches the binary search tree *node for the key num.
If the tree is empty or if num is presented, it returns NULL.
Otherwise, it returns a pointer to the last node of the tree
that was encountered during the search.

Analysis of *insert_node* :

Let h be the height of the binary search tree.

Since the search requires O(h) time and

the remainder of the algorithm takes Θ(1) time.

So overall time needed by insert_node is O(h).

5.7.4 Deletion From A Binary Search Tree

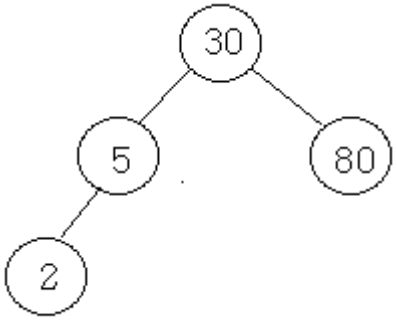
· Deletion of a leaf node :

Set the corresponding child field of its parent to NULL
and free the node.

· Deletion of a nonleaf node with single child :

Erase the node and
then place the single child in the place of the erased node.

[Figure 5.32] Deletion from a binary tree



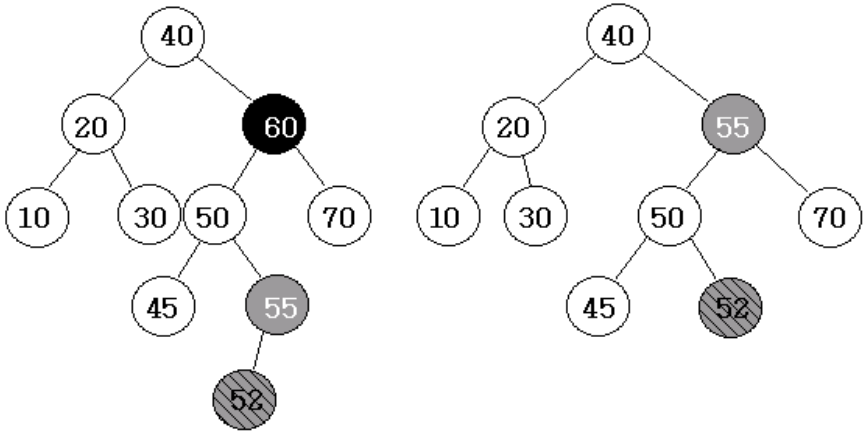
· **Deletion of a nonleaf node with two children** :

Replace the node with either the largest element in its left subtree
or the smallest element in its right subtree.

Then delete this replacing element from the subtree from which it was taken.

Note that the largest and smallest elements in a subtree
are always in a node of degree zero or one.

[Figure 5.33.]



(a) tree before deletion of 60 (b) tree after deletion of 60

It is easy to see that a deletion can be performed in O(h) time,
where h is the height of the binary search tree.

5.7.5 Height Of A Binary Search Tree

Unless care is taken,

the height of a binary search tree with n elements can become as large as n.

However,

when insertion and deletions are made at random,

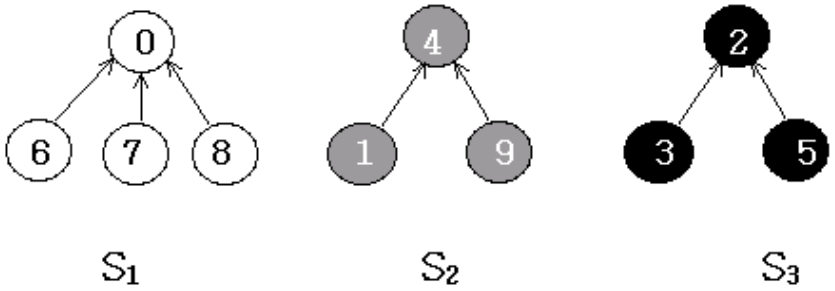
the height of the binary search tree is O(log2 n), on the average.

Search trees with a worst case height of O(log2 n) are called balanced search trees.

5.10 SET REPRESENTATION

We study the use of trees in the representation of sets.
For simplicity, we assume that the elements of the sets
are the numbers 0, 1, 2, . . ., n-1.
We also assume that the sets being represented are pairwise disjoint.

[Figure 5.39] for a possible representation.



Notice that for each set the nodes are linked from the children to the parent.
The operations to perform on these sets are:

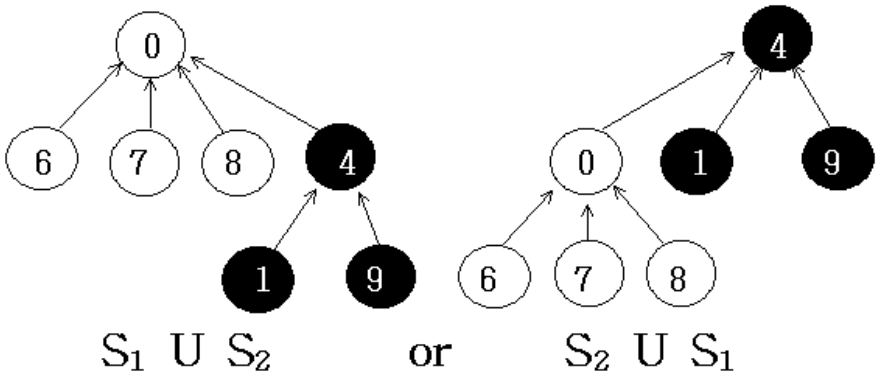
(1) *Disjoint set union*. If we wish to get the union of two disjoint sets Si and Sj, replace Si and Sj by Si∪Sj.

(2) *Find(i)*. Find the set containing the element, *i*.

5.10.1 Union and Find Operations

Suppose that we wish to obtain the union of S1 and S2.
We simply make one of the trees a subtree of the other.
S1∪S2 could have either of the representations of Figure 5.40

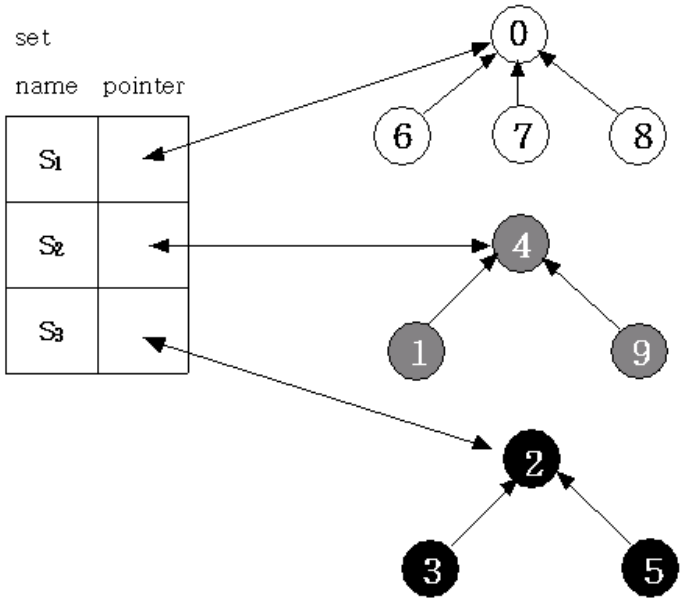
[Figure 5.40] Possible representation of S1∪S2



To implement the set union operation, we simply set
the parent field of one of the roots to the other root.

Figure 5.41 shows how to name the sets.

[Figure 5.41] Data representation of S1,S2 and S3



To simplify the discussion of the union and find algorithms,
we will ignore the set names and identify the sets
by the roots of the trees representing them.

Since the nodes are in the trees are numbered 0 through n-1,
we can use the node's number as an index.

[Figure 5.42] : Array representation of the trees in Figure 5.39.

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

Notice that root nodes have a parent of –1.

We can implement find(i) by simply following
the indices starting at i and continuing
until we reach a negative parent index.

[Program 5.18] : Initial attempt at union-find functions.

```
int find(int i)
{
    for ( ; parent[i] >= 0 ; i = parent[i])
                ;
     return  i;
 }

 void union1(int i, int j)
{
      parent[i] = j;
}
```

Analysis of *union*1 and *find*1 :

Let us process the following sequence of union-find operations:

union(0, 1), find(0)
union(1, 2), find(0)
.
.
.
union(n-2, n-1), find(0)

This sequence produces the degenerate tree of Figure 5.43.

[Figure 5.43] Degenerate tree

Since the time taken for a union is constant,

all the n-1 unions can be processed in time O(n).

For each *find*, if the element is at level *i*,

then the time required to find its root is O(*i*).

Hence the total time needed to process the n-1 finds is :

$$\sum_{i=1}^{n-1} i = O(n^2)$$

By avoiding the creation of degenerate trees,

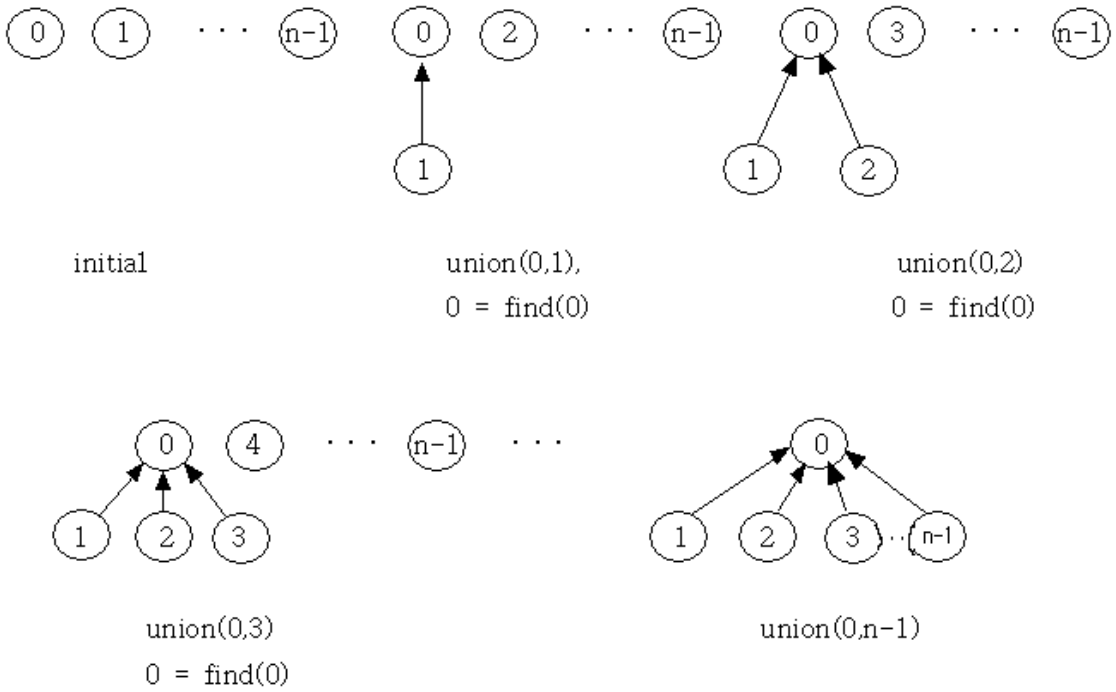
we can attain far more efficient implementations

of the union and find operations.

Definition : Weighting rule for *union*(i, j). If the number of nodes in tree i is less than the number in tree j then make j the parent of i; otherwise make i the parent of j. □

When we use this rule on the sequence of set unions described above, we obtain the trees of Figure 5.44.

[Figure 5.44] Trees obtained using the weighting rule



To implement the weighting rule,

we need to know how many nodes there are in every tree.

That is, we need to maintain a count field in the root of every tree.

We can maintain the count in the parent field of the roots as a negative number.

[Program 5.19] : Union operation incorporating the weighting rule.

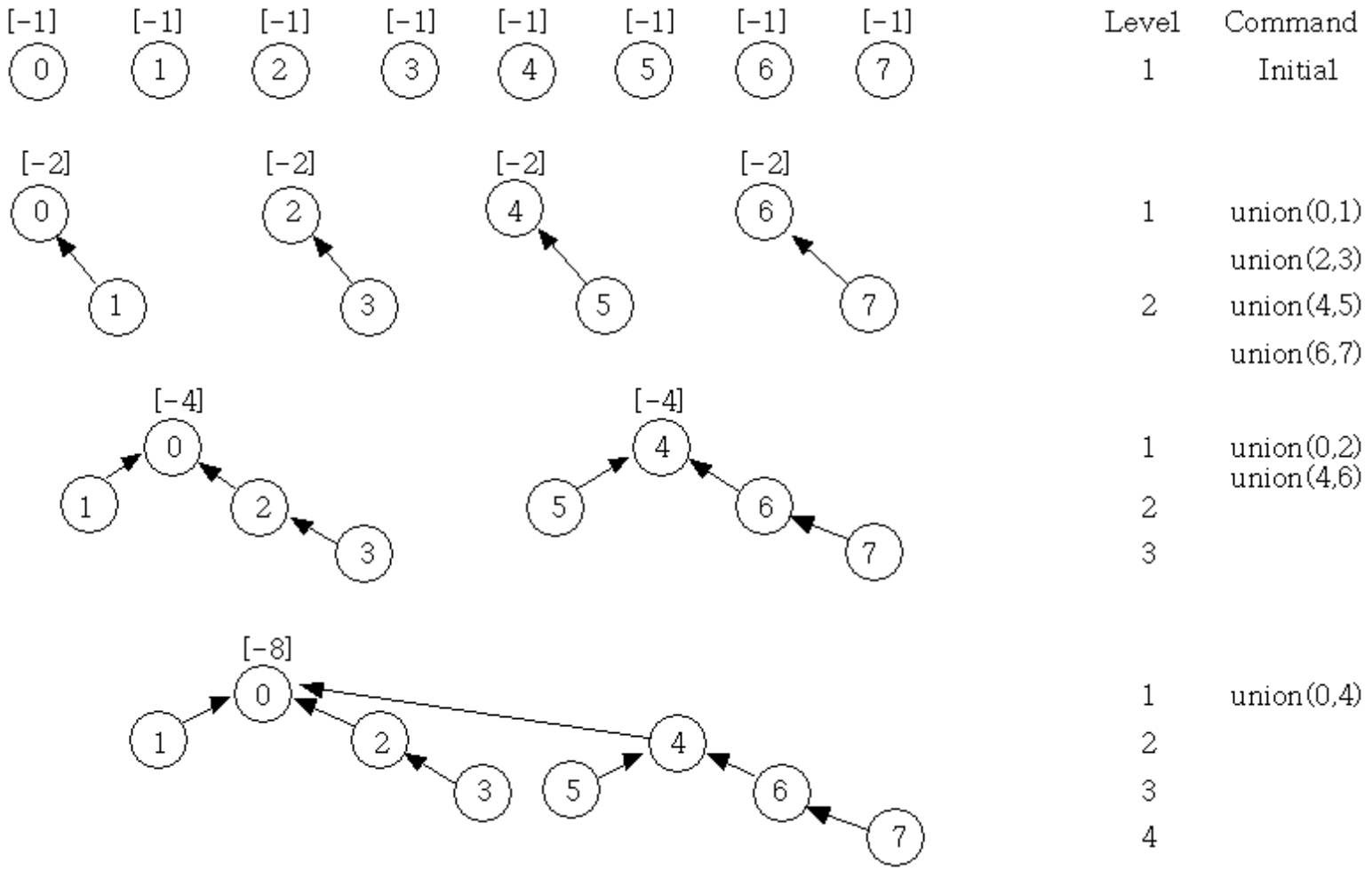
```
void union2(int i, int j)
{
  /*  parent[i] = -count[i] and parent[j] = -count[j]  */
    int temp = parent[i] + parent[j];
    if (parent[i] > parent[j])   {
            parent[i] = j;   /* make j the new root  */
            parent[j] = temp;
    }
    else  {
            parent[j] = i;   /* make i the new root  */
            parent[i] = temp;
    }
}
```

Lemma 5.4 : Let T be a tree with n nodes created as a result of *union*2. Then the depth of T ≤ $\lfloor \log_2 n \rfloor + 1$

Example 5.1 : Consider the behavior of union2 on the following sequence of unions starting from the initial configuration :

union(*0*, *1*) *union*(*2*, *3*) *union*(*4*, *5*) *union*(*6*, *7*)
union(*0*, *2*) *union*(*4*, *6*) *union*(*0*, *4*)

Figure 5.43 shows the result.



As is evident from this example, in the general case, the depth can be $\lfloor \log_2 n \rfloor + 1$ if the tree has *n* nodes. □

As a result of Lemma 5.4,

the time to process a *find* in an *n* element tree is $O(\log_2 n)$.

If we process an intermixed sequence

of n-1 *union* and m *find* operations,

then the time becomes $O(n+m\log_2 n)$.

5.10.2 Equivalence Classes

[Figure 5.44] Trees for equivalence example