Lecture 5 on Linear Algebra

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- Invertible matrices and inverses: Let A be a square matrix.

(i) A is said to be *invertible* if there is a matrix B such that

$$AB = BA = I.$$

Such a matrix B, if it exists, is called an *inverse (matrix)* of A. If A is not invertible, it is said to be *noninvertible* or *singular*.

(ii) If B and C are square matrices such that $AB = CA = I$, then $B = C$. Therefore, the inverse of A, if it exists, is unique and denoted by A^{-1}.

Proof.

(iii) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

Example. A 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. In this case,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof.

Example. Every elimination matrix is invertible. In fact, if E_{ij} is the elimination matrix that subtracts l times row j from row i, then E_{ij}^{-1} is the elimination matrix that subtracts $-l$ times row j from row i. For instance, for $n = 3$,

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l & 0 & 1 \end{bmatrix},$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l & 1 \end{bmatrix} \quad \Rightarrow \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l & 1 \end{bmatrix}.$$

Example. A permutation matrix P_{ij} is invertible and the inverse is P_{ij} itself:

$$P_{ij}^{-1} = P_{ij}.$$

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2.5. Inverse matrices

- Linear systems and inverse matrices: Let A be an n by n matrix.

(i) Suppose that A is invertible. Then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution in $\mathbb{R}^n$, given by $\mathbf{x} = A^{-1}\mathbf{b}$; that is,

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x} = A^{-1}\mathbf{b}.$$

Proof.

(ii) Suppose that A has a *right-inverse*, that is, there exists a matrix B such that $AB = I$. Then for every $\mathbf{b} \in \mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x} = B\mathbf{b}$.

Proof.

(iii) Suppose that for each $j = 1, 2, \ldots, n$, there exists $\mathbf{b}_j \in \mathbb{R}^n$ such that $A\mathbf{b}_j = \mathbf{e}_j$. Let B be the matrix whose columns are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$:

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = I.$$

Hence B is a right-inverse of A.

(iv) Suppose that A has a *left-inverse* B (i.e., $BA = I$). Then the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b} \in \mathbb{R}^n$. In particular, $A\mathbf{x} = \mathbf{0}$ can only have the trivial solution $\mathbf{x} = \mathbf{0}$.

Proof.

(v) If A has a zero column or a zero row, then A is singular.

Proof.

Example. A diagonal matrix

$$D = \operatorname{diag}(d_1, d_2, \ldots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

is invertible if and only if its diagonal entries d_j are all nonzero. In this case,

$$D^{-1} = \operatorname{diag}\left(d_1^{-1}, d_2^{-2}, \ldots, d_n^{-1}\right).$$

Note also that

$$D \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } n \end{bmatrix} = \begin{bmatrix} d_1(\text{row } 1) \\ d_2(\text{row } 2) \\ \vdots \\ d_n(\text{row } n) \end{bmatrix}.$$

2.5. Inverse matrices

Theorem. *If A and B are invertible, then AB is also invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

Corollary.

1. *If A and AB are invertible, then B is also invertible.*
2. *If B and AB are invertible, then A is also invertible.*

Proof.

2.5. Inverse matrices

- Finding inverse matrices by elimination:

(i) Let $A = [a_{ij}]$ be a 4 by 4 matrix. Suppose that $a_{i1} \neq 0$ for some $i = 1, 2, 3, 4$. Then using a permutation matrix P_1 if necessary, we have

$$P_1 A = \begin{bmatrix} p_1 & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix},$$

where p_1 is a nonzero number, called a *pivot*.

(ii) If $E_1 = E_{41}E_{31}E_{21}$ for some elimination matrices E_{i1}, then

$$A' = E_1 P_1 A = \begin{bmatrix} p_1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix},$$

which holds trivially with $p_1 = 0$ even when $a_{i1} = 0$ for all i.

(iii) Suppose that $a'_{i2} \neq 0$ for some $i = 2, 3, 4$. Then using a permutation matrix P_2 if necessary, we have

$$P_2 A' = \begin{bmatrix} p_1 & * & * & * \\ 0 & p_2 & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix},$$

where p_2 is a nonzero number.

(iv) If $E_2 = E_{42}E_{32}$ for some elimination matrices E_{i2}, then

$$A'' = E_2 P_2 A' = \begin{bmatrix} p_1 & * & * & * \\ 0 & p_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix},$$

which holds trivially with $p_2 = 0$ even when $a'_{i2} = 0$ for all $i = 2, 3, 4$.

(v) Similarly, using a permutation matrix P_3 and an elimination matrix E_3 if necessary, we have

$$U = E_3 P_3 A'' = \begin{bmatrix} p_1 & * & * & * \\ 0 & p_2 & * & * \\ 0 & 0 & p_3 & * \\ 0 & 0 & 0 & p_4 \end{bmatrix}.$$

If some p_i is nonzero, it is called a *pivot*.

(vi) If $p_4 = 0$, then U is singular (and so A is singular). If $p_4 \neq 0$, then using $E_4 = E_{14}E_{24}E_{34}$, we have

$$E_4 U = \begin{bmatrix} p_1 & * & * & 0 \\ 0 & p_2 & * & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix}.$$

(vii) If $p_4 \neq 0$ but $p_3 = 0$, then L is singular. If $p_3 p_4 \neq 0$, then using $E_5 = E_{13}E_{23}$, we have

$$E_5 E_4 U = \begin{bmatrix} p_1 & * & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix}.$$

(viii) If $p_3 p_4 \neq 0$ but $p_1 p_2 = 0$, then L is singular. If $p_1 p_2 p_3 p_4 \neq 0$, then using $E_5 = E_{12}$, we have

$$D = E_6 E_5 E_4 U = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix},$$

which is an invertible diagonal matrix.

2.5. Inverse matrices

Theorem. *Let A be a square matrix.*

1. *There are finitely many elementary matrices $E_1, E_2, \ldots, E_k$ such that*
$$U = E_k \cdots E_2 E_1 A$$
is an upper triangular matrix.
2. *A is invertible if and only if U is invertible if and only if the diagonal entries of U are all nonzero.*
3. *If U is invertible, then there are finitely many elimination matrices $E_{k+1}, E_{k+2}, \ldots, E_{k+l}$ such that*
$$D = E_{k+l} \cdots E_{k+2} E_{k+1} U$$
is an invertible diagonal matrix. Moreover, we have
$$A^{-1} = D^{-1} E_{k+l} \cdots E_{k+2} E_{k+1} E_k \cdots E_2 E_1.$$

2.5. Inverse matrices

Theorem. *Let A and B be n by n matrices. Then AB is invertible if and only if both A and B are invertible.*

Proof.

Theorem. *Let A and B be n by n matrices. Then $AB = I$ if and only if $BA = I$ if and only if $B = A^{-1}$.*

Example. Let A_{11} and A_{22} be square matrices. Then the upper triangular 2 by 2 block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$$

is invertible if and only if A_{11} and A_{22} are both invertible. In this case,

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{bmatrix}.$$

Proof.

2.5. Inverse matrices

- The Gauss-Jordan elimination method: Let A be an n by n matrix.

(i) Consider the augmented matrix

$$[\,A \quad I\,],$$

where I is the n by n identity matrix.

(ii) There are finitely many elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 [\,A \quad I\,] = [\,U \quad E_k \cdots E_2 E_1\,],$$

where $U = E_k \cdots E_2 E_1 A$ is an upper triangular matrix.

(iii) A is invertible if and only if U is invertible.

(iv) If U is invertible, then we can find finitely many elimination matrices $E_{k+1}, E_{k+2}, \ldots, E_{k+l}$ and an invertible diagonal matrix D such that

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\begin{aligned}
&D^{-1}E_{k+l}\cdots E_{k+2}E_{k+1}E_k\cdots E_2E_1\left[\, A \quad I \,\right] \\
&\qquad = \left[\, I \quad D^{-1}E_{k+l}\cdots E_{k+2}E_{k+1}E_k\cdots E_2E_1 \,\right].
\end{aligned}
$$

(v)The inverse of A is

$$A^{-1} = D^{-1}E_{k+l}\cdots E_{k+2}E_{k+1}E_k\cdots E_2E_1.$$

(vi) Applying the Gauss-Jordan method, we can do:

1. determine whether A is invertible or not;
2. find the inverse of A if A is invertible.

Example. Compute the inverse of K by Gauss-Jordan elimination, where K is the following symmetric, tridiagonal matrix:

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution.

2.5. Inverse matrices