# Lecture 6 on Linear Algebra

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# 2.6 Elimination = Factorization: $A = LU$

- The $A = LU$ or $A = LDU$ factorization: Let $A$ be an $n$ by $n$ matrix.

(i) Assume that elimination is possible without row exchanges, that is, there are finitely many elimination matrices $E_1, E_2, \ldots, E_k$ of the form $E_{ij}, i > j$, such that

$$E_k \cdots E_2 E_1 A = \begin{bmatrix} d_1 & * & \cdots & * \\ 0 & d_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = U.$$

Then

$$A = LU,$$

where

$$L = (E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

(ii) Recall that if $E_{ij}$ subtracts $l_{ij}$ times row $j$ from row $i$, then $E_{ij}^{-1}$ subtracts $-l_{ij}$ times row $j$ from row $i$.

# 2.6 Elimination = Factorization: $A = LU$

(iii) The matrix $L$ is lower triangular and easily computable. To illustrate this, we assume that $n = 4$. Then the most general form of $L$ is

$$L = E_{21}^{-1}E_{31}^{-1}E_{41}^{-1}E_{32}^{-1}E_{42}^{-1}E_{43}^{-1}.$$

Note that

$$E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix},$$

$$E_{32}^{-1}E_{42}^{-1}E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & l_{42} & l_{43} & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}.$$

(v) If $d_1 d_2 \cdots d_n \neq 0$, then

$$\begin{bmatrix} d_1 & * & \cdots & * \\ 0 & d_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix};$$

hence $A$ can be written as

$$A = LDU,$$

where

1. $L, U$ are triangular matrices with $1$'s on the diagonal and
2. $D$ is a diagonal matrix with pivots on the diagonal.

(vi) Such an $LDU$ decomposition is unique:

$$L_1 D_1 U_1 = L_2 D_2 U_2 \quad \Rightarrow \quad L_1 = L_2,\ D_1 = D_2,\ U_1 = U_2.$$

*Proof.* Assume that $n = 4$. Then

$$L_1 D_1 U_1 = L_2 D_2 U_2 \quad \Rightarrow \quad L_2^{-1} L_1 D_1 = D_2 U_2 U_1^{-1},$$

$$L_2^{-1} L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix},$$

and

$$U_2 U_1^{-1} = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

# 2.7. Transposes and permutations

- Transposes:

(i) If $A = [a_{ij}]$ is an $m$ by $n$ matrix, then the *transpose* $A^T$ of $A$ is the $n$ by $m$ matrix whose columns are the rows of $A$. For instance,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}.$$

(ii) Note that the $(i, j)$ entry of $A^T$ is the $(j, i)$ entry of $A$:

$$A^T(i, j) = a_{ji} \quad (i = 1, \ldots, n; j = 1, \ldots, m).$$

(iii) The transpose of $A^T$ is $A$:

$$\left(A^T\right)^T = A.$$

(iv) If $A$ and $B$ are of the same size, then

$$(A+B)^T = A^T + B^T.$$

(v) (*Important!*) If $A$ is $m$ by $n$ and $B$ is $n$ by $p$, then

$$(AB)^T = B^T A^T.$$

*Proof.*

(vi) If $A$ is invertible, then $A^T$ is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

*Proof.*

(vii) If $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are any (column) vectors, then

$$\mathbf{x}\,\mathbf{y}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix}.$$

# 2.7. Transposes and permutations

(viii) For all $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$,

$$\begin{aligned}\mathbf{y}^T\mathbf{x} &= \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= \sum_{j=1}^{n} x_j y_j = \mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}.\end{aligned}$$

(ix) If $A$ is an $m$ by $n$ matrix, then

$$(A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A^T\mathbf{y}) \quad \text{for all } \mathbf{x}\in\mathbb{R}^n,\ \mathbf{y}\in\mathbb{R}^m.$$

*Proof.*

# 2.7. Transposes and permutations

- Symmetric matrices:

(i) A square matrix $A$ is said to be *symmetric* if

$$A^T = A \quad \text{or equivalently} \quad a_{ij} = a_{ji} \quad \text{for all } i,\ j.$$

(ii) Let $A$ be an invertible matrix. If $A$ is symmetric, so is $A^{-1}$.

*Proof.*

(iii) Let $A$ be invertible and symmetric. If $A$ is factored into $LDU$ without row exchanges, then $U = L^T$ and so

$$A = LDL^T.$$

*Proof.*

**Example**. If $R$ is any matrix, then both $RR^T$ and $R^TR$ are symmetric.

*Proof*.

**Example**. Find the $LDU$ decomposition of a symmetric matrix

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

*Solution*.

# 2.7. Transposes and permutations

- Permutation matrices:

(i) A *permutation matrix* is a matrix $P$ that has the rows of the identity matrix in any order. In other words, any permutation matrix can be obtained from the identity matrix $I$ by reordering the rows of $I$. For instance, if $n = 3$, then there are 6 permutation matrices:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{32},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_{21}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P_{32}P_{21},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P_{21}P_{32}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = P_{31}.$$

(ii) A square matrix is a permutation matrix if and only if it has a single "$1$" in every row and every column. Hence the transpose of a permutation matrix is also a permutation matrix.

(iii) There are exactly $n! = n(n-1)\cdots 2 \cdot 1$ permutation matrices.

*Proof.*

(iv) Every permutation matrix $P$ can be written as a product of row exchanges matrices:

$$P = P_{i_1 j_1} \cdots P_{i_k j_k},$$

where $P_{i_l, j_l}$ are the permutation matrix that exchanges row $i_l$ and row $j_l$.

(v) The product of permutation matrices is a permutation matrix.

(vi) (*Important!*) If $P$ is any permutation matrix, then

$$P^T P = PP^T = I.$$

*Proof.*

(vii) (**Fact**) If $A$ is invertible, then there exists a permutation matrix $P$ such that $PA$ admits an $LU$ decomposition:

$$PA = LU.$$

**Example**. Find the $PA = LU$ decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 7 & 9 \end{bmatrix}.$$

*Solution*.