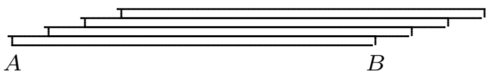
GEH1036 TUTORIAL 1 (WEEK 3)

1. A configuration of tiles is formed from n identical uniform square tiles of side $2x$ and thickness y. The tiles are rigidly glued together one tile over another in such a way that two parallel sides of the tiles are aligned and the other two parallel sides of each tile protrude over those directly below by a distance of r. The whole configuration is placed with the lowest slab on a horizontal floor.



The figure above is a horizontal cross-section of the configuration that cuts the tiles lines parallel to the sides. Let A and B be the lowest corners of the bottom slab. With A as the origin, denote by x_k the horizontal distance from A of the centre of the kth tile:

$$x_1 = x, x_2 = x + r, x_3 = x + 2r, \ldots.$$

The centre of gravity of the configuration is then at a horizontal distance x_c from A where $x_c = (x_1 + x_2 + \cdots + x_n)/n$. The configuration will topple over if $x_c > |AB| = 2x$.

(a) Write down an explicit expression for x_c in terms of x, r and n.;

(b) Find the maximum of slabs that can be so placed without toppling over in the following cases. (i) $x = 50$, $r = 5$; (ii) $x = 50$, $r = 8$.

Solution. (a) Since $x_i = x + (i-1)r$, we have

$$\begin{aligned}
x_1 + \cdots + x_n &= \sum_{i=1}^{n}(x + (i-1)r) \\
&= \sum_{i=1}^{n} x + r\left(\sum_{i=1}^{n}(i-1)\right) \\
&= xn + r\left(\sum_{i=1}^{n} i - n\right) \\
&= xn + r\left(\frac{n(n+1)}{2} - n\right) \\
&= \frac{n(2x + (n-1)r)}{2}.
\end{aligned}$$

Therefore $x_c = (2x + (n-1)r)/2$.

(b) Thus the configuration topples if $(2x + (n-1)r)/2 > 2x$, i.e., $(n-1)r > 2x$.

(i) Will topple iff $5(n-1)) > 100$, i.e., $n > 21$. Thus the maximum number of slabs (without toppling) is 21.

(ii) Will topple iff $8(n-1)) > 100$, i.e., $n > 13.5$. Thus the maximum number of slabs (without toppling) is 13.

A problem with the question. The sentence "The configuration will topple over if $x_c > |AB| = 2x$" is meant to mean "The configuration will topple over if and only if $x_c > |AB| = 2x$" which is used in the solution of part (b). This is a problem because for example if $n = 2$, then from part (a) we have $x_c = (2x + r)/2$ which can never be larger than $2x$ for all $0 \le r \le 2x$. Thus the configuration consisting of two slabs will never topple no matter what. This is not true in real life. The correct criteria is to consider the centre of gravity of all the slabs excluding the lowest slab. Do you see why? Can you write down the correct criteria for toppling?

2. Find the total number of (i) squares, (ii) rectangles that can be formed from the gridlines in an 8×9 chequered board (Figure 1).

(Think of the "best" way of specifying a rectangle.)

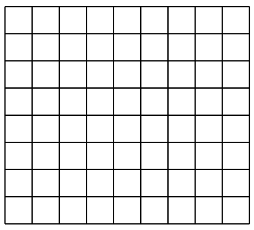


Figure 1

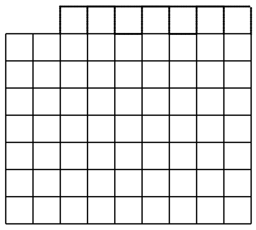


Figure 2

Solution. (a) To count the squares, start with a square at the top left corner of the board and move it vertically down and then horizontally across to account for all squares of that shape. Count the number of such squares. Starting with a 1×1 square and ending up with an 8×8 square, we obtain the total number of squares as

$$(8 \times 9) + (7 \times 8) + (6 \times 7) + (5 \times 6) + (4 \times 5) + (3 \times 4) + (2 \times 3) + (1 \times 2) = 240.$$

Alternative soln: A square, when projected to the sides of the grid, yield two segments of equal length. Conversely, a horizontal and a vertical segment of equal length,corresponds to a square. There are 8 vertical segments and 9 horizontal segments of length 1. Thus there are 8×9 1×1 squares. Likewise, there are 7×8 2×2 squares, etc.

(b) (Note that rectangles include squares). As before, each rectangle corresponds to a vertical segment and a horizontal segment. If r and s are, respectively, the number of horizontal and vertical segments of various lengths, then the number of rectangles is rs. Now $r = 9+8+\cdots+1 = 45$ and $s = 8+7+\cdots+1 = 36$ and the answer is $45 \times 36 = 1620$.

Alternatively, r and s can be calculated as follows. Each horizontal segment is determined by its two end points. Thus $r = \binom{10}{2} = 45$. Similarly, $s = \binom{9}{2} = 36$.

Appendix. Refer to the sum

$$(8 \times 9) + (7 \times 8) + (6 \times 7) + (5 \times 6) + (4 \times 5) + (3 \times 4) + (2 \times 3) + (1 \times 2) = 240$$

in (a). In general, is there a way to evaluate the sum $\sum_{k=1}^{n} k\,(k+1)$?

First, observe that $k\,(k+1) = k^2 + k$. It has been known how to evaluate the sum $\sum_{k=1}^{n} k$, which is equal to $\frac{n\,(n+1)}{2}$. So, it suffices for us to deal with $\sum_{k=1}^{n} k^2$. Refer to the following simple algebraic identity:

$$(k+1)^3 - k^3 = 3\,k^2 + 3\,k + 1.$$

We have

$$(n+1)^3 - 1 = \sum_{k=1}^{n} [(k+1)^3 - k^3] = 3 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} k + n.$$

Thus, we derive a formula for the sum $\sum_{k=1}^{n} k^2$, which is given by

$$\sum_{k=1}^{n} k^2 = \frac{n\,(n+1)\,(2n+1)}{6}.$$

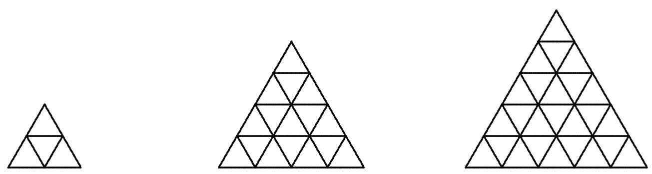
Now, complete the evaluation of the sum $\sum_{k=1}^{n} k\,(k+1)$. $\square$

3. Find the number of squares in an 8×9 chequered with two squares removed from the top row of the board (Figure 2).

(Consider the squares of the original (complete) chessboard that include at least one of the squares removed.)

Solution. Label the first deleted square (top left square) by A and the other deleted square by B. There are 8 squares that contain A and 8 squares that contain B but not A. Thus the required number of squares is $240 - (8 + 8) = 224$.

4. Find the number of triangles in each of the following figures.



Solution. Two types of triangles can be found embedded in the figures, namely the type Δ or the type ∇. Assume the smallest triangle has each side of unit length. We need to compute Δ_k which is the number of Δ of side length k, and ∇_k which is the number ∇ of side length k. For the first figure, the answer is 5. For the second figure

$$(\Delta_1 + \cdots + \Delta_4) + (\nabla_1 + \cdots + \nabla_3) = (10 + 6 + 3 + 1) + (6 + 1 + 0) = 27.$$

For the third figure

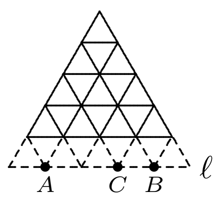
$$(\Delta_1 + \cdots + \Delta_5) + (\nabla_1 + \cdots + \nabla_3) = (15 + 10 + 6 + 3 + 1) + (10 + 3 + 0) = 48.$$

Note: There's in fact a formula for the number of triangles with n levels:

$$\frac{n^3}{4} + \frac{5n^2}{8} + \frac{n}{4} - \frac{1}{16} + \frac{(-1)^n}{16}.$$

For example, when $n = 4$ this formula gives the answer for the second triangle in the figure above. However, this is complicated to obtain.

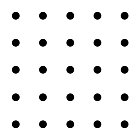
Nevertheless we can find the number of type Δ triangles given n levels. Add one line ℓ beneath the lowest line. Intersect the non horizontal lines with ℓ giving $n + 2$ points. For each type Δ triangle, extend the two non horizontal lines to intersect ℓ at two points, say A, B, with A on the left. Then take another point C on ℓ so that AC is equal to the length of the side of the triangle. Thus the given triangle corresponds to 3 points on ℓ. Conversely, every collection of three points on ℓ gives rise to such a triangle. Thus the number of type Δ triangles is $\binom{n+2}{3}$.



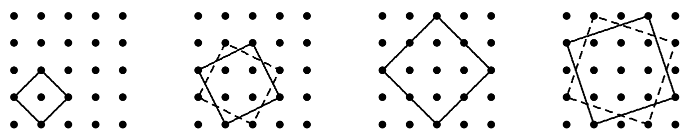
Try to figure out which triangle corresponds to the indicated points in the diagram above.

The number of type ∇ triangles given n levels is more tedious to find, and it will not be inside our syllabus. However, in the following a very brief suggestion is given for your reference. Set a_n to denote the number of type ∇ triangles given n levels. We would like to discover the increment $a_{n+1} - a_n$. For illustration's sake, take $n = 4$. Refer to the above figure with a newly added line ℓ. Take note that, with this line ℓ, it gives us the case with $5 = 4 + 1$ levels. Now, all one needs to do is count extra number of triangles of type ∇. Continue this scheme and be careful that the way to deal with the case when n is even (and hence $n+1$ becomes odd) and the case when n is odd (and hence $n+1$ becomes even) would be slightly different. (Try it yourself if you want to and have fun.)

5. Find the number of squares that can be formed by joining dots in the following diagram.



Solution. There are $4^2+3^2+2^2+1^2=30$ squares whose sides are vertical or horizontal. There are other types of squares as shown in the figures. There are 9 of type 2 (fig. 1), 8 of type 3 (fig. 2), 1 of type 4 (fig. 3) and 2 of type 5 (fig. 4). The total is $30+9+8+1+2=50$.



(1) (2) (3) (4)

6. When a triangle is drawn on the plane, the plane is divided into two regions. When two triangles are drawn in the plane, what is the maximum number of regions that the plane is divided into? What is the answer if there are n triangles, where $n \geq 1$?

Solution. It is not hard to see that when $n=2$, there are 8 regions. It is a little harder to see that there are 20 when $n=3$. The following table shows that answers for various n:

$n=$	0	1	2	3	$\cdots$	n
Regions	1	2	8	20	$\cdots$	?
Plus	–	1	6	12	$\cdots$	$6(n-1)$

It seems when the n^{th} is drawn, the number of (maximum) additional regions is $6(n-1)$. This can be seen as follows. The new triangle intersects an existing triangle in 6 points, dividing the perimeter of the triangle into 6 portions. Each portion divides an existing region into 2 and therefore we get 6 additional regions. Hence the total number of additional regions is $6(n-1)$ since there are $n-1$ existing triangles. Hence then total number of regions is given by:

$$1+1+6+12+18+\cdots+6(n-1) = 2+6(1+2+3+\cdots(n-1)) = 2+6\left(\frac{(n-1)(n)}{2}\right) = 2+3n(n-1).$$