Optimisation and Decision Models Homework 4

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Question 3

Note: An Excel file named "Ex4_Qn3.xlsx" has been submitted along with this report.

(a) The weekly "Adjusted Close" price data of the below 10 stocks are tabulated in the "Data" tab of the Excel file.

Company Name	Ticker	Sector	Remarks
Royal Dutch Shell	RDSA	Oil & Gas	
HSBC	HSBA	Banking	
BP	BP	Oil & Gas	
Vodafone Group	VOD	Telecomms	
GlaxoSmithKline	GSK	Pharmaceuticals	
British American Tobacco	BATS	Tobacco	The originally mentioned ticker 'BTI' is not found in London Stock Exchange, hence the correct ticker 'BATS' for British American Tobacco is used instead
Anheuser-Busch InBev SA/NV	BUD	Beverages	Replaced as BUD as per instructed
Diageo	DGE	Beverages	
Exxon Mobil Corporation	XOM	Oil & Gas	Replaced as XOM as per instructed
Rio Tinto Group	RIO	Mining	

Table 1 List of Companies Used

The weekly price data is downloaded from Yahoo Finance for the period between 3rd Nov 2014 and 4th Nov 2016.

Note the following:

- The assignment question indicated the ticker of British American Tobacco as “BTI”. However, this ticker is an ADR listed in NYSE and denominated in US dollars. In London Stock Exchange, there is a ticker “BATS” for British American Tobacco. This homework uses BATS price data from London Stock Exchange.
- Anheuser-Busch InBev SA/NV (Ticker: BUD) price from NYSE is used since SABMiller is no longer a listed company.
- Exxon Mobil Corporation (Ticker: XOM) price from NYSE is used since BG Group is no longer a listed company.
- As instructed, the currency difference between USD and GBP is ignored for this homework.

(b) The mean weekly return for each stock and the covariances of the weekly returns between all pairs of stocks are tabulated in the “Data” tab of the Excel file.

The mean weekly return for each stock is copied here for easy reference:

	RDSA	HSBA	BP	VOD	GSK	BATS	BUD	DGE	XOM	RIO
Mean Weekly Return	0.001148	0.001331	0.002086	0.001644	0.002116	0.003417	0.001225	0.002142	-0.00044	0.00114

Table 2 Mean Weekly Returns of the 10 Stocks

The covariance table is also copied below for easy reference. Please note that the population covariance formula is used for calculating the below values.

Covariances	RDSA	HSBA	BP	VOD	GSK	BATS	BUD	DGE	XOM	RIO
RDSA	0.001618	0.00068	0.001365	0.000456	0.000586	0.000704	0.000404	0.000493	0.000736	0.00128
HSBA	0.00068	0.001011	0.000731	0.00048	0.000448	0.000422	0.000296	0.000405	0.00029	0.000741
BP	0.001365	0.000731	0.001623	0.000516	0.000626	0.000735	0.000501	0.000636	0.000705	0.001182
VOD	0.000456	0.00048	0.000516	0.000782	0.000469	0.000479	0.000272	0.000459	0.000148	0.000409
GSK	0.000586	0.000448	0.000626	0.000469	0.000699	0.000562	0.000297	0.000503	0.000223	0.000423
BATS	0.000704	0.000422	0.000735	0.000479	0.000562	0.000873	0.000474	0.000628	0.000252	0.000595
BUD	0.000404	0.000296	0.000501	0.000272	0.000297	0.000474	0.000824	0.000468	0.00024	0.000412
DGE	0.000493	0.000405	0.000636	0.000459	0.000503	0.000628	0.000468	0.000809	0.000148	0.000497
XOM	0.000736	0.00029	0.000705	0.000148	0.000223	0.000252	0.00024	0.000148	0.000748	0.000547
RIO	0.00128	0.000741	0.001182	0.000409	0.000423	0.000595	0.000412	0.000497	0.000547	0.002343

Table 3 Covariance of the Weekly Returns between All Pairs of Stocks

(c) The Markowitz problem can be formulated as an optimisation problem (using Return Maximisation Formulation) as below. This model is also implemented in the "Qn3c – Markowitz Problem" tab of the Excel file.

<u>Decision Variables</u>

x_{RDSA} = Fraction of budget to invest to RDSA stock
x_{HSBA} = Fraction of budget to invest to HSBA stock
x_{BP} = Fraction of budget to invest to BP stock
x_{VOD} = Fraction of budget to invest to VOD stock
x_{GSK} = Fraction of budget to invest to GSK stock
x_{BATS} = Fraction of budget to invest to BATS stock
x_{BUD} = Fraction of budget to invest to BUD stock
x_{DGE} = Fraction of budget to invest to DGE stock
x_{XOM} = Fraction of budget to invest to XOM stock
x_{RIO} = Fraction of budget to invest to RIO stock

<u>Other Notations</u>

Let $\boldsymbol{x}$ be the vector containing all the 10 fractions, $\boldsymbol{x} = \begin{pmatrix} x_{RDSA} \\ x_{HSBA} \\ \vdots \\ x_{RIO} \end{pmatrix}$

Let $\boldsymbol{\mu}$ be the vector containing the mean weekly return of all the 10 stocks, $\boldsymbol{\mu} = \begin{pmatrix} \mu_{RDSA} \\ \mu_{HSBA} \\ \vdots \\ \mu_{RIO} \end{pmatrix}$

Note: The values of all the 10 mean weekly returns can be found in Table 2.

Let $\sum$ be the covariance matrix (of the weekly returns between all pairs of stocks),

$$\Sigma = \begin{pmatrix} \boldsymbol{VAR_{RDSA}} & \boldsymbol{COV_{HSBA,RDSA}} & \cdots & \boldsymbol{COV_{RIO,RDSA}} \\ \boldsymbol{COV_{RDSA,HSBA}} & \boldsymbol{VAR_{HSBA}} & \cdots & \boldsymbol{COV_{RIO,HSBA}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{COV_{RDSA,RIO}} & \boldsymbol{COV_{HSBA,RIO}} & \cdots & \boldsymbol{VAR_{RIO}} \end{pmatrix}$$

Note: $\boldsymbol{VAR}$ indicates the variance of a stock and $\boldsymbol{COV}$ indicates covariance between a pair of stocks. The values of this covariance matrix can be found in Table 3.

Expected Portfolio Return = $\boldsymbol{\mu}^{\boldsymbol{T}}\boldsymbol{x}$
Portfolio Variance (i.e. Portfolio Risk) = $\boldsymbol{x}^{\boldsymbol{T}} \sum \boldsymbol{x}$
$\bar{R}$ is the maximum acceptable risk, measured using portfolio variance.

Objective Function
Maximise expected portfolio return: $\boldsymbol{\mu}^{\boldsymbol{T}}\boldsymbol{x}$

Constraints
subject to $x_{RDSA} + x_{HSBA} + x_{BP} + x_{VOD} + x_{GSK} + x_{BATS} + x_{BUD} + x_{DGE} + x_{XOM} + x_{RIO} = 1$ (Budget Constraint)
$\boldsymbol{x}^{\boldsymbol{T}} \sum \boldsymbol{x} \leq \bar{R}$ (Maximum Acceptable Risk)
$x_{RDSA}, x_{HSBA}, x_{BP}, x_{VOD}, x_{GSK}, x_{BATS}, x_{BUD}, x_{DGE}, x_{XOM}, x_{RIO} \geq 0$ (Nonnegative Weight: No Short Sales)

The Markowitz problems, with twelve different values of $\bar{R}$, have been solved and the respective optimal portfolios are tabulated in the "Optimal Portfolio Composition" tab of the Excel file.

The expected returns and risks of these twelve optimal portfolios are then used to plot the efficient frontier. The expected return and risk (i.e. variance) of the 10 individual stocks are also plotted on the same graph.

The graph that shows the efficient frontier and the 10 individual assets is in the “Graph” tab of the Excel file. The same plot is copied below for easy reference.

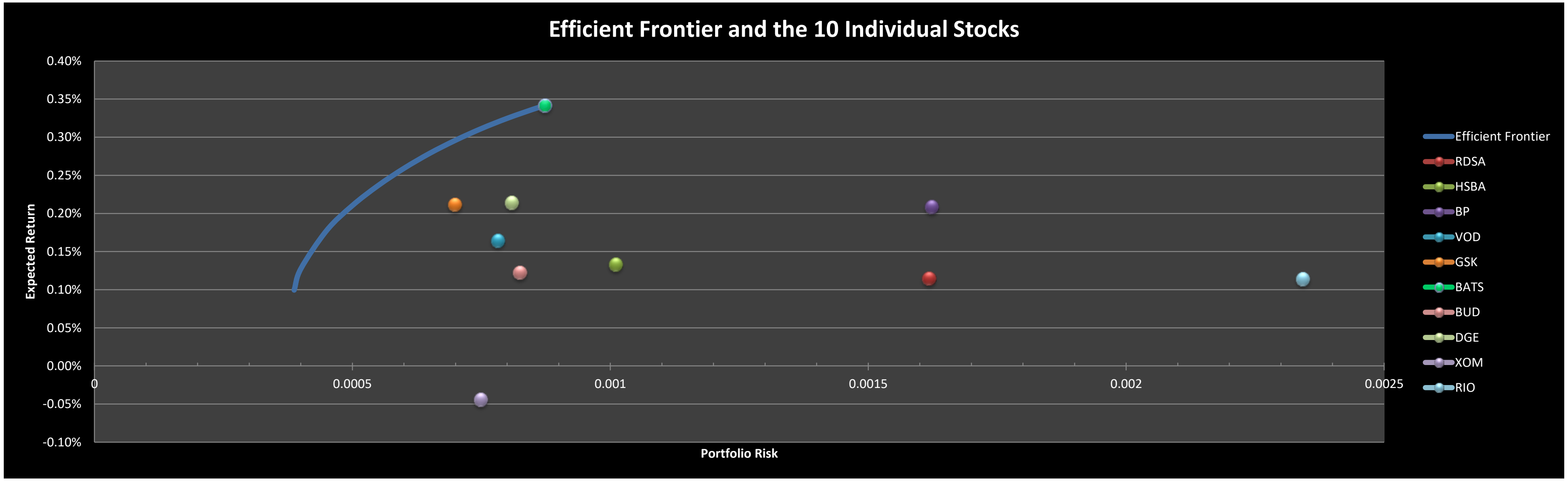

Figure 1 Return vs Risk for the Efficient Frontier and 10 Individual Stocks

The X-axis is the portfolio risk measured using the portfolio variance, $\boldsymbol{x}^T \sum \boldsymbol{x}$, where $\boldsymbol{x}$ is a vector containing investment fractions of each of the 10 assets and $\sum$ is the covariance matrix (of the weekly returns between all pairs of stocks). The Y-axis is the Expected Portfolio Return, $\boldsymbol{\mu}^T \boldsymbol{x}$, where $\boldsymbol{\mu}$ is the vector containing the mean weekly returns of the 10 stocks.

Three of the computed optimal portfolios (High Risk: $\bar{R} = 0.0009$, Medium Risk: $\bar{R} = 0.00065$, Low Risk: $\bar{R} = 0.0004$) are selected to be displayed below:

1) **High Return / High Risk ($\bar{R} = 0.0009$)**

Expected Portfolio Return = 0.3417%
Portfolio Risk (i.e. Portfolio Variance) = 0.000873

Portfolio Weights									
RDSA	**HSBA**	**BP**	**VOD**	**GSK**	**BATS**	**BUD**	**DGE**	**XOM**	**RIO**
0	0	0	0	0	1	0	0	0	0

Table 4 Portfolio Weights of a High Return / High Risk Portfolio

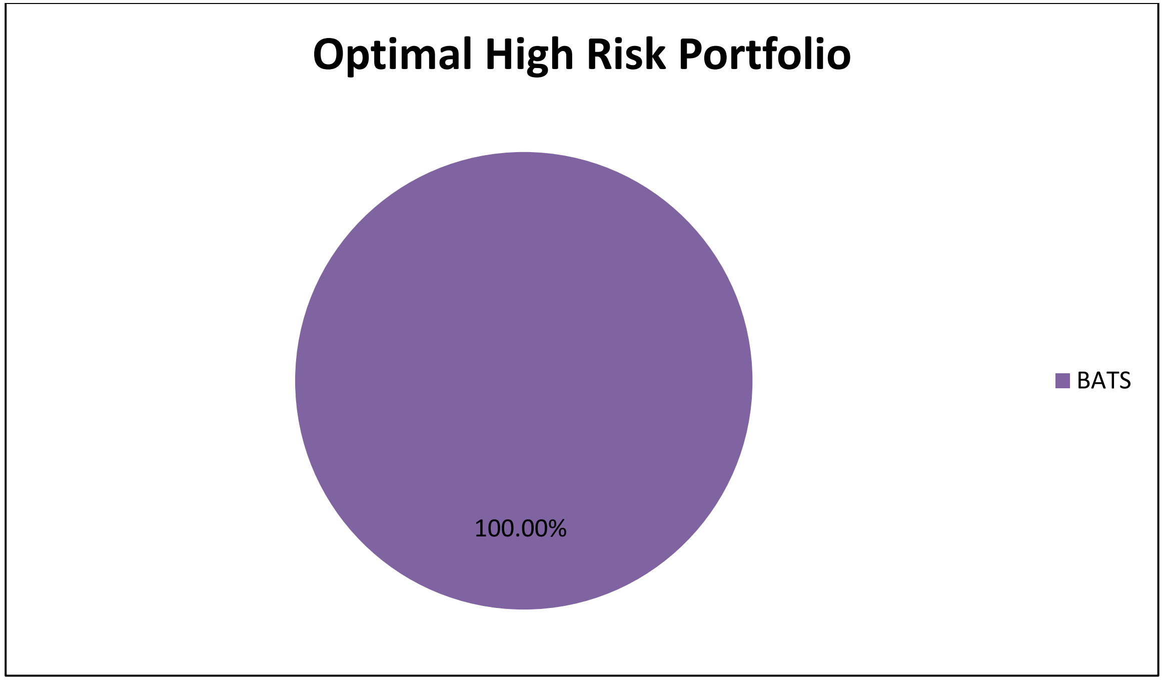

Figure 2 Portfolio Composition of an Optimal High Risk Portfolio

Amongst the 10 stocks, BATS has the highest expected weekly return. If we have a very high risk tolerance (higher than the variance of BATS), the optimal strategy is to invest 100% in BATS in order to achieve highest possible expected return.

2) **Medium Return / Medium Risk ($\bar{R} = 0.00065$)**

Expected Portfolio Return = 0.2791%
Portfolio Risk (i.e. Portfolio Variance) = 0.00065

Portfolio Weights									
RDSA	HSBA	BP	VOD	GSK	BATS	BUD	DGE	XOM	RIO
0	0.044082298	0	0.092935649	0.183055094	0.619991436	0.059935523	0	0	0

Table 5 Portfolio Weights of a Medium Return / Medium Risk Portfolio

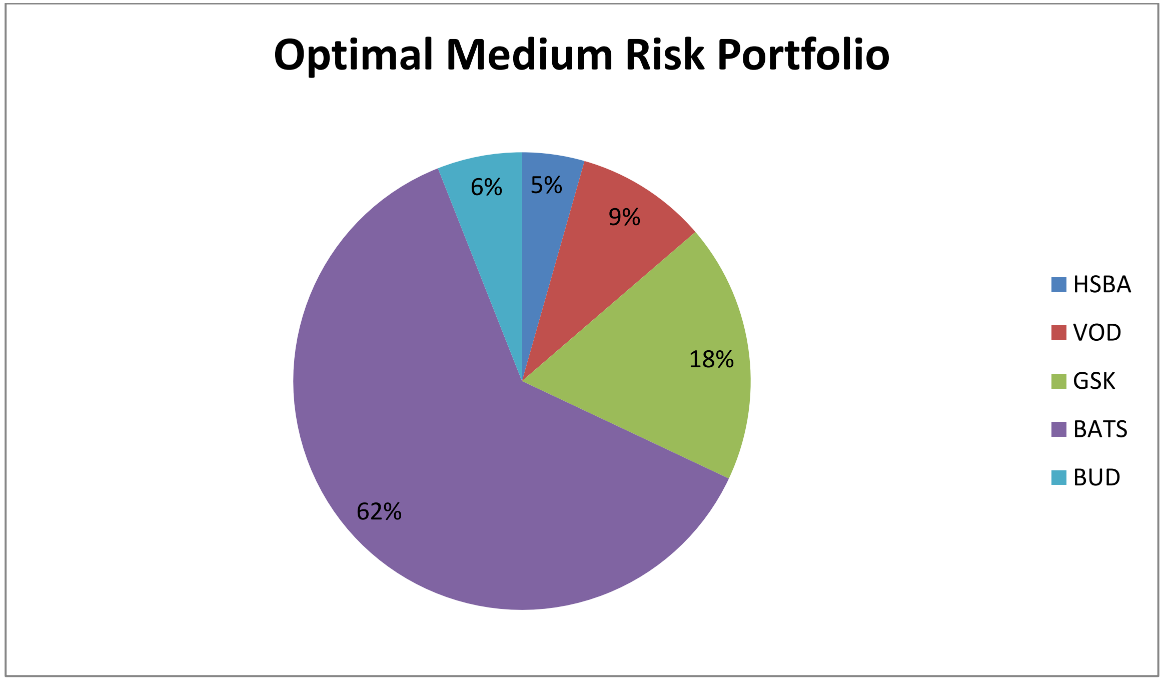

Figure 3 Portfolio Composition of an Optimal Medium Risk Portfolio

In addition to BATS, the medium risk portfolio diversifies into four other stocks to reduce the portfolio risk. A large portion is still invested with BATS in order to achieve maximum possible return given the medium risk tolerance.

3) **Low Return / Low Risk ($\bar{R} = 0.0004$)**

Expected Portfolio Return = 0.1276%

Portfolio Risk (i.e. Portfolio Variance) = 0.0004

Portfolio Weights									
RDSA	**HSBA**	**BP**	**VOD**	**GSK**	**BATS**	**BUD**	**DGE**	**XOM**	**RIO**
0	0.035765647	0	0.190994675	0.19360155	0.062627826	0.179400441	0.085171801	0.25243806	0

Table 6 Portfolio Weights of a Low Return / Low Risk Portfolio

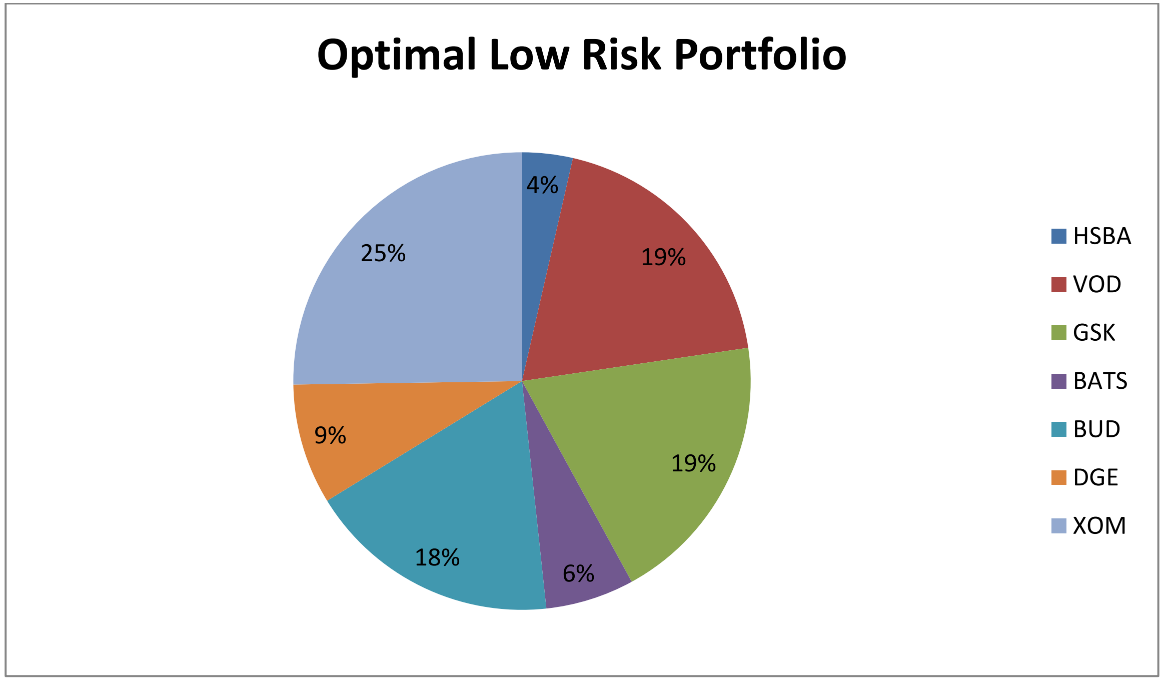

Figure 4 Portfolio Composition of an Optimal Low Risk Portfolio

Due to the low risk tolerance, the low risk portfolio diversifies into many different stocks so that the portfolio risk is reduced to an acceptable level.

(d) If we are only allowed to invest in at most one company per sector, we will need to use new binary variable to construct the relevant constraints.

The new optimisation problem becomes (this optimisation problem is also implemented in the “Qn3d – One Company Per Sector” tab of the Excel file):

<u>Decision Variables</u>

x_{RDSA} = Fraction of budget to invest to RDSA stock
x_{HSBA} = Fraction of budget to invest to HSBA stock
x_{BP} = Fraction of budget to invest to BP stock
x_{VOD} = Fraction of budget to invest to VOD stock
x_{GSK} = Fraction of budget to invest to GSK stock
x_{BATS} = Fraction of budget to invest to BATS stock
x_{BUD} = Fraction of budget to invest to BUD stock
x_{DGE} = Fraction of budget to invest to DGE stock
x_{XOM} = Fraction of budget to invest to XOM stock
x_{RIO} = Fraction of budget to invest to RIO stock
y_{RDSA} = Binary Variable: 1 if RDSA is invested, 0 otherwise
y_{HSBA} = Binary Variable: 1 if HSBA is invested, 0 otherwise
y_{BP} = Binary Variable: 1 if BP is invested, 0 otherwise
y_{VOD} = Binary Variable: 1 if VOD is invested, 0 otherwise
y_{GSK} = Binary Variable: 1 if GSK is invested, 0 otherwise
y_{BATS} = Binary Variable: 1 if BATS is invested, 0 otherwise
y_{BUD} = Binary Variable: 1 if BUD is invested, 0 otherwise
y_{DGE} = Binary Variable: 1 if DGE is invested, 0 otherwise
y_{XOM} = Binary Variable: 1 if XOM is invested, 0 otherwise
y_{RIO} = Binary Variable: 1 if RIO is invested, 0 otherwise

<u>Objective Function</u>

Maximise expected portfolio return: $\boldsymbol{\mu}^{\boldsymbol{T}}\boldsymbol{x}$

Constraints

subject to $x_{RDSA} + x_{HSBA} + x_{BP} + x_{VOD} + x_{GSK} + x_{BATS} + x_{BUD} + x_{DGE} + x_{XOM} + x_{RIO} = 1$ (Budget Constraint)

$\boldsymbol{x^T} \sum \boldsymbol{x} \le \bar{R}$ (Maximum Acceptable Risk)

$y_{RDSA} + y_{BP} + y_{XOM} \le 1$ (Oil & Gas Sector)

$y_{HSBA} \le 1$ (Banking Sector)

$y_{VOD} \le 1$ (Telecomms Sector)

$y_{GSK} \le 1$ (Pharmaceuticals Sector)

$y_{BATS} \le 1$ (Tobacco Sector)

$y_{BUD} + y_{DGE} \le 1$ (Beverages Sector)

$y_{RIO} \le 1$ (Mining Sector)

$x_{RDSA} \le y_{RDSA}$ (RDSA Invested?)

$x_{HSBA} \le y_{HSBA}$ (HSBA Invested?)

$x_{BP} \le y_{BP}$ (BP Invested?)

$x_{VOD} \le y_{VOD}$ (VOD Invested?)

$x_{GSK} \le y_{GSK}$ (GSK Invested?)

$x_{BATS} \le y_{BATS}$ (BATS Invested?)

$x_{BUD} \le y_{BUD}$ (BUD Invested?)

$x_{DGE} \le y_{DGE}$ (DGE Invested?)

$x_{XOM} \le y_{XOM}$ (XOM Invested?)

$x_{RIO} \le y_{RIO}$ (RIO Invested?)

$x_{RDSA}, x_{HSBA}, x_{BP}, x_{VOD}, x_{GSK}, x_{BATS}, x_{BUD}, x_{DGE}, x_{XOM}, x_{RIO} \ge 0$ (Nonnegative Weight: No Short Sales)

$y_{RDSA}, y_{HSBA}, y_{BP}, y_{VOD}, y_{GSK}, y_{BATS}, y_{BUD}, y_{DGE}, y_{XOM}, y_{RIO} \in \{0, 1\}$ (Binary Variable)