Optimisation and Decision Models Exercise 2

Author: Siow Meng Low

Question 3

(a) The original Linear Program is as below and we would like to find the dual:

$$\begin{aligned} &\boldsymbol{minimise}\ 3x_1 + 5x_2 - x_3 \\ &\boldsymbol{subject\ to}\ \ x_1 + x_3 = 4 \\ &\qquad x_2 - 2x_3 \le 2 \\ &\qquad x_1, x_2 \ge 0, x_3\ \boldsymbol{unrestricted} \end{aligned}$$

Firstly, change all the constraints into "greater than or equal to" constraints:

$$\begin{aligned} &\boldsymbol{minimise}\ 3x_1 + 5x_2 - x_3 \\ &\boldsymbol{subject\ to}\ \ x_1 + x_3 \ge 4 \\ &\qquad -x_1 - x_3 \ge -4 \\ &\qquad -x_2 + 2x_3 \ge -2 \\ &\qquad x_1, x_2 \ge 0, x_3\ \boldsymbol{unrestricted} \end{aligned}$$

Replace unrestricted variable x_3 with the difference of two non-negative variables x_3^+ and x_3^-:

$$\begin{aligned} &\boldsymbol{minimise}\ 3x_1 + 5x_2 - x_3^+ + x_3^- \\ &\boldsymbol{subject\ to}\ \ x_1 + x_3^+ - x_3^- \ge 4 \\ &\qquad -x_1 - x_3^+ + x_3^- \ge -4 \\ &\qquad -x_2 + 2x_3^+ - 2x_3^- \ge -2 \\ &\qquad x_1, x_2, x_3^+, x_3^- \ge 0 \end{aligned}$$

With the above standard form of minimisation problem, the corresponding vectors and matrix are:

$$\boldsymbol{b} = \begin{pmatrix} 3 \\ 5 \\ -1 \\ 1 \end{pmatrix}, A^T = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -2 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$$

Now the linear program is in standard form, we can then find the dual of this linear program using the indirect method:

$$\begin{aligned} &\boldsymbol{maximise}\ \ 4y_1 - 4y_2 - 2y_3 \\ &\boldsymbol{subject\ to}\ \ y_1 - y_2 \le 3 \\ &\qquad -y_3 \le 5 \\ &\qquad y_1 - y_2 + 2y_3 \le -1 \\ &\qquad -y_1 + y_2 - 2y_3 \le 1 \\ &\qquad y_1, y_2, y_3 \ge 0 \end{aligned}$$

We can further simplify this dual. This maximisation problem contains the difference between the two non-negative variables y_1 and y_2:

$$\begin{aligned} &\boldsymbol{maximise}\ \ 4(y_1 - y_2) - 2y_3 \\ &\boldsymbol{subject\ to}\ \ (y_1 - y_2) \le 3 \\ &\qquad -y_3 \le 5 \\ &\qquad (y_1 - y_2) + 2y_3 \le -1 \\ &\qquad -(y_1 - y_2) - 2y_3 \le 1 \\ &\qquad y_1, y_2, y_3 \ge 0 \end{aligned}$$

Using another unrestricted variable, $y_0 = y_1 - y_2$, we can simplify this problem into:

$$\begin{aligned} &\textbf{maximise } 4y_0 - 2y_3 \\ &\textbf{subject to } y_0 \le 3 \\ &\quad -y_3 \le 5 \\ &\quad y_0 + 2y_3 \le -1 \\ &\quad -y_0 - 2y_3 \le 1 \\ &\quad y_0 \textbf{ unrestricted}, y_3 \ge 0 \end{aligned}$$

The last two inequality constraints can be rewritten into a single equality constraint:

$$\begin{aligned} &\textbf{maximise } 4y_0 - 2y_3 \\ &\textbf{subject to } y_0 \le 3 \\ &\quad -y_3 \le 5 \\ &\quad y_0 + 2y_3 = -1 \\ &\quad y_0 \textbf{ unrestricted}, y_3 \ge 0 \end{aligned}$$

(b) The original Linear Program is as below and we would like to find the dual:

$$\begin{aligned} &\textbf{maximise } x_1 - x_3 \\ &\textbf{subject to } x_1 + x_2 = 4 \\ &\quad x_3 \le 2 \\ &\quad x_1, x_2 \le 0, x_3 \textbf{ unrestricted} \end{aligned}$$

The corresponding vectors and matrix of this maximisation problem are:

$$c = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Using direct method, the dual of the linear program is:

$$\begin{aligned} &\textbf{minimise } 4y_1 + 2y_2 \\ &\textbf{subject to } y_1 \le 1 \\ &\quad y_1 \le 0 \\ &\quad y_2 = -1 \\ &y_1 \textbf{ unrestricted}, y_2 \ge 0 \end{aligned}$$

The first two inequality constraints related to y_1 can be further simplified by combining into one inequality constraint:

$$\begin{aligned} &\textbf{minimise } 4y_1 + 2y_2 \\ &\textbf{subject to } y_1 \le 0 \\ &\quad y_2 = -1 \\ &y_1 \textbf{ unrestricted}, y_2 \ge 0 \end{aligned}$$

Question 4

(a) The objective of 1-norm regression problem is to: $\boldsymbol{minimise} \quad \|\boldsymbol{y} - \boldsymbol{X\beta}\|_1$

Using the definition of 1-norm and given that the data set has 50 records, the objective of the 1-norm regression problem can be written as: $\boldsymbol{minimise} \quad \sum|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}| \qquad , \forall\, \boldsymbol{i = 1, 2, \cdots, 50}$

$\boldsymbol{y}$ is the vector representing 50 total CEO compensations, whereas $\boldsymbol{X}$ is the matrix containing the data (years in current position, change in stock price, change in sales, whether CEO has MBA) of these 50 records. In matrix form, they are:

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{comp_1} \\ \boldsymbol{comp_2} \\ \vdots \\ \boldsymbol{comp_{50}} \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} \boldsymbol{years_1} & \boldsymbol{stock_1} & \boldsymbol{sales_1} & \boldsymbol{mba_1} \\ \boldsymbol{years_2} & \boldsymbol{stock_2} & \boldsymbol{sales_2} & \boldsymbol{mba_2} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{years_{50}} & \boldsymbol{stock_{50}} & \boldsymbol{sales_{50}} & \boldsymbol{mba_{50}} \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta_{years}} \\ \boldsymbol{\beta_{stock}} \\ \boldsymbol{\beta_{sales}} \\ \boldsymbol{\beta_{mba}} \end{pmatrix}$$

where $\boldsymbol{y_i}$ denotes the i^{th} element of vector $\boldsymbol{y}$ and $\boldsymbol{x_i^T}$ denotes the i^{th} row of matrix $\boldsymbol{X}$. The four variables within the $\boldsymbol{\beta}$ vector are what we would like to estimate in the regression problem.

Introducing auxiliary variables $\boldsymbol{\theta_1, \theta_2, \cdots, \theta_{50}}$, the minimisation problem becomes:

$$\begin{aligned} &\boldsymbol{minimise} \quad \boldsymbol{\theta_1 + \theta_2 + \cdots + \theta_{50}} \\ &\boldsymbol{subject\ to} \quad \boldsymbol{\theta_i} = |\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}| \qquad \forall\, \boldsymbol{i = 1, 2, \cdots, 50} \end{aligned}$$

Next, we relax the equality constraints to get the equivalent formulation:

$$\begin{aligned} &\boldsymbol{minimise} \quad \boldsymbol{\theta_1 + \theta_2 + \cdots + \theta_{50}} \\ &\boldsymbol{subject\ to} \quad \boldsymbol{\theta_i} \geq |\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}| \qquad \forall\, \boldsymbol{i = 1, 2, \cdots, 50} \end{aligned}$$

Lastly, we remove the absolute value operator using a pair of inequality constraints to get the final Linear Program formulation:

$$\begin{aligned} &\boldsymbol{minimise} \quad \boldsymbol{\theta_1 + \theta_2 + \cdots + \theta_{50}} \\ &\boldsymbol{subject\ to\ \theta_i} \geq \boldsymbol{y_i} - \boldsymbol{x_i^T\beta} \qquad \forall\, \boldsymbol{i = 1, 2, \cdots, 50} \\ &\boldsymbol{subject\ to\ \theta_i} \geq \boldsymbol{x_i^T\beta} - \boldsymbol{y_i} \qquad \forall\, \boldsymbol{i = 1, 2, \cdots, 50} \end{aligned}$$

(b) The AMPL model file is attached as “Qn4b.mod” and run file is “Qn4b.run”.

The optimal solution is:

- $\beta_{years} = 169.2738$
- $\beta_{stock} = 2.41$
- $\beta_{sales} = -0.1425$
- $\beta_{mba} = 382.5$

The optimal objective value (i.e. total residual) is 14652.83

(c) The objective of infinity-norm regression problem is to: $\boldsymbol{minimise} \quad \|\boldsymbol{y} - \boldsymbol{X\beta}\|_\infty$

Using the definition of infinity-norm and given that the data set has 50 records, the objective of the infinity-norm regression problem can be written as:

$$\boldsymbol{minimise}\ \ (\boldsymbol{max}\{|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}| \quad , \forall\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{50}\})$$

$\boldsymbol{y}$ is the vector representing 50 total CEO compensations, whereas $\boldsymbol{X}$ is the matrix containing the data (years in current position, change in stock price, change in sales, whether CEO has MBA) of these 50 records. In matrix form, they are:

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{comp_1} \\ \boldsymbol{comp_2} \\ \vdots \\ \boldsymbol{comp_{50}} \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} \boldsymbol{years_1} & \boldsymbol{stock_1} & \boldsymbol{sales_1} & \boldsymbol{mba_1} \\ \boldsymbol{years_2} & \boldsymbol{stock_2} & \boldsymbol{sales_2} & \boldsymbol{mba_2} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{years_{50}} & \boldsymbol{stock_{50}} & \boldsymbol{sales_{50}} & \boldsymbol{mba_{50}} \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta_{years}} \\ \boldsymbol{\beta_{stock}} \\ \boldsymbol{\beta_{sales}} \\ \boldsymbol{\beta_{mba}} \end{pmatrix}$$

where $\boldsymbol{y_i}$ denotes the i^{th} element of vector $\boldsymbol{y}$ and $\boldsymbol{x_i^T}$ denotes the i^{th} row of matrix $\boldsymbol{X}$. The four variables within the $\boldsymbol{\beta}$ vector are what we would like to estimate in the regression problem.

Firstly, we introduce an auxiliary variable, $\boldsymbol{t} = \boldsymbol{max}\{|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}| \quad , \forall\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{50}\}$. The minimisation problem becomes:

$$\begin{aligned} &\boldsymbol{minimise}\ \ \boldsymbol{t} \\ &\boldsymbol{subject\ to\ t} = \boldsymbol{max}\{|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}|,\ \ \forall\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{50}\} \end{aligned}$$

With $\boldsymbol{t}$ being defined as the maximum of all the $|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}|$ terms, $\boldsymbol{t}$ must be greater than or equal to all the individual $|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}|$ terms. We can hence relax the equality constraint and rewrite the minimisation problem as:

$$\begin{aligned} &\boldsymbol{minimise}\ \ \boldsymbol{t} \\ &\boldsymbol{subject\ to\ t} \geq |\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}| \qquad \forall\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{50} \end{aligned}$$

The optimal objective value of $\boldsymbol{t}$ will be the maximum of all the $|\boldsymbol{y_i} - \boldsymbol{x_i^T\beta}|$ terms.

Lastly, we remove the absolute value operator using a pair of inequality constraints to get the final Linear Program formulation:

$$\begin{aligned} &\boldsymbol{minimise}\ \ \boldsymbol{t} \\ &\boldsymbol{subject\ to\ t} \geq \boldsymbol{y_i} - \boldsymbol{x_i^T\beta} \qquad \forall\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{50} \\ &\boldsymbol{subject\ to\ t} \geq \boldsymbol{x_i^T\beta} - \boldsymbol{y_i} \qquad \forall\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{50} \end{aligned}$$

(d) The AMPL model file is attached as “Qn4d.mod” and run file is “Qn4d.run”.

The optimal solution is:

- $\beta_{years} = 221.052$
- $\beta_{stock} = 4.9406$
- $\beta_{sales} = 0.1015$
- $\beta_{mba} = 261.9554$

The optimal objective value (i.e. maximum residual) is 805.5371