$$1+\frac{99^2}{10{,}000}+\frac{99}{k}+\frac{99}{10{,}000}k = f(k)$$

$$f'(k) = -\frac{99}{k^2}+\frac{99}{10{,}000} = 0$$

$$\frac{99}{k^2} = \frac{99}{10{,}000} \qquad 99k^2 = 990{,}000$$

$$k^2 = 10{,}000$$

$$\underline{k=100}$$

Conclusion

① optimal deploy

$$p_1(x) = \frac{1}{c}\left(100\, I_{(0,\frac{1}{100})} + 1\cdot I_{(\frac{1}{100},1)}\right)$$

Note

$$\frac{p_1(x)}{h(x)} = \text{constant}$$

"Stratification"

② $\frac{p(x)}{p_1(x)}$ = "likelyhood ratio" or "price kernel", or "Randon–Nikodym density"

Rk. suppose python provides sampling for $X \sim p(x)$ then
it is also possible for $Y \sim h(X)$ sampling for any function h.

ex ① If $X \sim N(0,1)$ then. numpy.random.normal(·) gives sampling

② $Y = (X-K)^+$, then

$X =$

$Y = h(X)$ where $h(x) = (x-K)^+$.

Back to integral

$$\alpha = \int_0^1 h(x)\,dx = \mathbb{E}[h(X)] \quad \text{where } X \sim U(0,1)$$

{ Importance sampling.

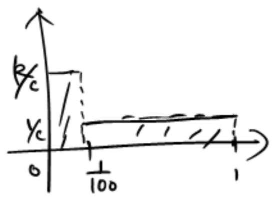
$$\alpha = \int_0^1 h(x) \frac{p(x)}{p_1(x)} \cdot p_1(x)\,dx \longleftarrow$$

$$p_1(x) = \frac{1}{c}\left(k\, I_{(0,\frac{1}{100})} + 1 \cdot I_{(\frac{1}{100},1)}\right)$$

① $\int_0^1 p_1(x)\,dx = 1 \implies c = ?$

$$\frac{k}{c}\cdot\frac{1}{100} + \frac{1}{c}\cdot\frac{99}{100} = 1$$

$$c = \frac{k+99}{100}$$



② Find $MSE(\hat{\alpha}_N) = \frac{1}{N} Var(\alpha_1)$

where $\alpha_1 = h(X)\frac{p(X)}{p_1(X)}$ where $X \sim p_1$

$$Var(\alpha_1) = \mathbb{E}\alpha_1^2 - (\mathbb{E}[\alpha_1])^2$$

$$= ? - (1.99)^2$$

$$\mathbb{E}[\alpha_1^2] = \int_0^1 \left(\frac{h(x)p(x)}{p_1(x)}\right)^2 p_1(x)\,dx$$

$$= \int_0^{\frac{1}{100}} \left(\frac{100\cdot 1}{k/c}\right)^2 \cdot \frac{k}{c}\,dx + \int_{\frac{1}{100}}^1 \left(\frac{1\cdot 1}{1/c}\right)^2 \frac{1}{c}\,dx$$

$$= \frac{10000}{k/c}\cdot\frac{1}{100} + \frac{1}{1/c}\cdot\frac{99}{100}$$

$$= \frac{c}{k}\cdot 100 + c\cdot\frac{99}{100} = \left(\frac{100}{k} + \frac{99}{100}\right)\frac{(k+99)}{100} = \min ?$$

$$\geq 2\sqrt{\frac{c^2 99}{k}} \qquad \frac{k+99}{k} + \frac{99(k+99)}{10000}$$

$$1 + \frac{99}{k} + \frac{99}{10{,}000}k + \frac{99^2}{10{,}000}$$

HW 5-1-2

$$\beta_N = \frac{1}{N}\sum_{i=1}^N (\alpha_i - \overline{\alpha}_N)^2$$

$$\overline{\alpha}_N = \frac{1}{N}\sum_{i=1}^N \alpha_i$$

$\alpha_i \in L^4$, iid

We want. β_N is L^2-consistent to $Var(\alpha_1)$.

Pf

$$\beta_N = \frac{1}{N}\sum_{i=1}^N (\alpha_i^2 - 2\alpha_i\overline{\alpha}_N + \overline{\alpha}_N^2)$$

$$= \frac{1}{N}\left(\sum_{i=1}^N \alpha_i^2 - 2\overline{\alpha}_N \sum_{i=1}^N \alpha_i + N\cdot\overline{\alpha}_N^2\right)$$

$$= \frac{1}{N}\sum_{i=1}^N \alpha_i^2 - 2(\overline{\alpha}_N)^2 + \overline{\alpha}_N^2$$

$$= \underbrace{\frac{1}{N}\sum_{i=1}^N \alpha_i^2}_{} - \underbrace{\overline{\alpha}_N^2}_{}$$

LLN. (L^2)

If $(X_i, i=1\cdots N\cdots)$ iid. L^2 r.v. then

$$\frac{1}{N}\sum_{i=1}^N X_i \longrightarrow \mathbb{E}[X] \quad \text{in } L^2.$$

① $\frac{1}{N}\sum_{i=1}^N \alpha_i^2 \longrightarrow \mathbb{E}[\alpha_1^2]$ in L^2

② $\left(\frac{1}{N}\sum_{i=1}^N \alpha_i\right)^2 \longrightarrow \mathbb{E}[\alpha_1]^2$ in L^2.

Thus $\beta_N \longrightarrow \mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2 = Var(\alpha_1)$ in L^2.

In the above, we used that.

$$\boxed{\begin{array}{ll} \text{if} & X_i \to X \text{ in } L^2 \\ & Y_i \to Y \text{ in } L^2 \\ \text{then} & \\ & X_i + Y_i \to X+Y \text{ in } L^2 \end{array}}$$

§ OMC and definite integral.

Goal

$$\alpha = \int_0^1 h(x)\,dx = 1.99$$

where $h(x) = 100\, I_{(0,\frac{1}{100})}(x) + I_{(\frac{1}{100},1)}(x)$