1 Abstract

- derive Crank-Nicolson scheme
- prove unconditional stability

2 Problem

We have seen that FTCS scheme is stable for the heat equation

$$u_t = u_{xx}, \quad t > 0, x \in \mathbb{R}$$

with initial data

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

when $s = \frac{\theta}{h^2} < 1/2$ holds. Next, we are going to present Crank-Nicolson scheme and investigate its sability.

3 Analysis

3.1 Solution

We recall that FFD in time is

$$u_t(x, t) \simeq \frac{u(x, t+\theta) - u(x, t)}{\theta} := \delta_\theta^t u(x, t)$$

and CFD2 in state is

$$u_{xx}(x, t) \simeq \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2} := \delta_h^{xx} u(x, t),$$

where h and θ are some positive mesh size in space h and in time, respectively. Again, we set

$$s = \frac{\theta}{h^2}.$$

Discrete domain is accordingly a grid of

$$\{(jh, n\theta) : j + 1 \in \mathbb{N}, j \in \mathbb{Z}\}.$$

Recall that FTCS is to find numerical values u_j^n at a grid point $(jh, n\theta)$, such that

$$\delta_\theta^t u(jh, n\theta) \simeq \frac{u_j^{n+1} - u_j^n}{\theta} := (\delta_\theta^t u)_j^n, \quad \delta_h^{xx} u(jh, n\theta) \simeq \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} := (\delta_h^{xx} u)_j^n.$$

Plug it into the heat equation, we obtain FTCS discrete heat equation

$$u_j^{n+1} = s u_{j+1}^n + (1-2s) u_j^n + s u_{j-1}^n, \quad \forall j \in \mathbb{Z}, n+1 \in \mathbb{N}. \tag{1}$$

The Crank-Nicolson with parameter λ (denoted by CS-λ) is to find numerical values u_j^n at each grid point $(jh, n\theta)$, such that

$$\delta_\theta^t u(jh, n\theta) \simeq (\delta_\theta^t u)_j^n, \quad \delta_h^{xx} u(jh, n\theta) \simeq (1-\lambda)(\delta_h^{xx} u)_j^n + \lambda(\delta^{xx} u)_j^{n+1}.$$

Note that, if $\lambda = 0$, then CS-λ is FTCS discussed earlier. If $\lambda = 1$, then we call CS-λ as BTCS (Backward in time and central in state). In this below, we only discuss for $\lambda \in (0, 1]$. Plug it into the heat equation, we obtain CS-λ discrete heat equation

$$(\delta_\theta^t u)_j^n = (1-\lambda)(\delta_h^{xx} u)_j^n + \lambda(\delta^{xx} u)_j^{n+1},$$

which is equivalent to

$$-s\lambda u_{j+1}^{n+1} + (1+2s\lambda)u_j^{n+1} - s\lambda u_{j-1}^{n+1} = s(1-\lambda)u_{j+1}^n + (1-2s+2s\lambda)u_j^n + s(1-\lambda)u_{j-1}^n. \tag{2}$$

The numerical solution is to find (u_j^n) satisfying (2) ogether with the initial condition

$$u_j^0 = \phi(jh), \quad \forall j \in \mathbb{Z}. \tag{3}$$

3.2 Stability

Again by using separation in variables, we deduce that

$$T_n = \xi^n(k), X_j = e^{ijkh}$$

where

$$\xi(k) = \frac{1 - 2(1-\lambda)s(1-\cos kh)}{1 + 2\lambda s(1-\cos kh)}. \tag{4}$$

Proposition 1 *If $\lambda \in [1/2, 1]$, then CS-λ is always stable. In other words, there is no restriction on the selection of (h, θ).*

ex. Write stencil and pseudocode with $\lambda = 1/2$. Explain why it is an implicit scheme while FTCS is explicit scheme.

ex. Prove the above Proposition 1.