Discussion

① Most options in US stock markets are American style.

② We treat American as European, do calibration

③ (HW) Find conditions for

American call = European call

④ $BSM(S_0, r, \sigma, \delta, K, T, \text{otype}) = BSM(S_0, r-\delta, \sigma, 0, K, T, \text{otype})$

Dividend yield

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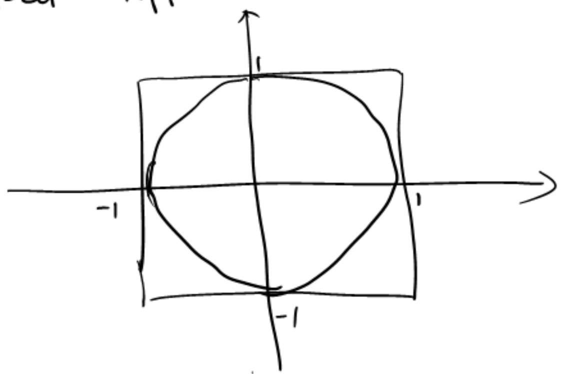
MC for π

Goal: Approximate π.



Q Suppose (X, Y) is the random coordinate of your bullet, which is uniformly distributed in $(-1, 1) \times (-1, 1)$. then what's the value of

$$\mathbb{P}\{(X, Y) \in B_1\} = ?$$

B_r stands for a circle with radius r, center as origin

A

$$\mathbb{P}((X, Y) \in B_1) = \frac{\text{Area of } B_1}{\text{Area of Square}} = \frac{\pi}{2 \times 2} = \frac{\pi}{4}.$$

Algo

① Shoot N times uniformly on sqr. $(-1, 1) \times (-1, 1)$

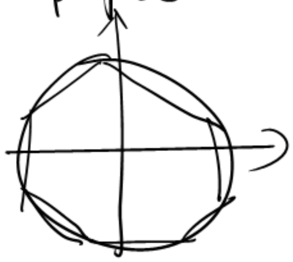
② Count successful shots, say n.
(a shot hit B_1)

③ $\hat{\pi} = 4 \cdot \frac{n}{N}$

Monte-carlo is an estimation algo which produce random result. i.e. if you run twice, two results might be different.

Q Can you propose a deterministic algo for π estimation?

A



π = Area of B_1

$\approx$ Area of inscribed n-polygon

Δ $\text{Bias} = \mathbb{E}[\hat{\alpha}] - \alpha$

Δ $\text{MSE} = \mathbb{E}[(\hat{\alpha} - \alpha)^2]$

Δ $\text{RMSE} = (\text{MSE})^{\frac{1}{2}}$ ...

Think smaller MSE always results in smaller RMSE.
then. why ppl. introduce RMSE?

Δ Prop1: $\text{MSE}(\hat{\alpha}) = |\text{Bias}(\hat{\alpha})|^2 + \text{Var}(\hat{\alpha})$

Pf We shall prove.

$$\mathbb{E}[(\hat{\alpha} - \alpha)^2] \stackrel{?}{=} |\mathbb{E}(\hat{\alpha} - \alpha)|^2 + \text{Var}(\hat{\alpha} - \alpha)$$

Let $X = \hat{\alpha} - \alpha$

Then. it's always true that.

$$\mathbb{E}[X^2] = |\mathbb{E}(X)|^2 + \text{Var}(X) \qquad \square$$

ex Let

$$\hat{\alpha} = 4 \cdot I(X_1^2 + Y_1^2 < 1), \text{ where, } X_1, Y_1 \sim U(-1, 1)$$

find Bias and MSE.

Def.

$$I(x<1) = \begin{cases} 1 & x<1 \\ 0 & x \geq 1 \end{cases}$$

$$I^2(x<1) = I(x<1)$$

Soln

$$\mathbb{E}\hat{\alpha} = 4 \cdot \mathbb{E}[I(X_1^2 + Y_1^2 < 1)]$$

$$= 4 \cdot \mathbb{P}(X_1^2 + Y_1^2 < 1)$$

$$= 4 \cdot \frac{\pi}{4} = \pi = \alpha$$

Bias $= 0$

$$\text{MSE} = \mathbb{E}[(\hat{\alpha} - \alpha)^2] = \text{Var}(\hat{\alpha})$$

$$= \mathbb{E}|\hat{\alpha}|^2 - (\mathbb{E}\hat{\alpha})^2 = \mathbb{E}[16 \cdot I(X_1^2 + Y_1^2 < 1)] - \pi^2$$

$$= 16 \cdot \mathbb{P}(X_1^2 + Y_1^2 < 1) - \pi^2$$

$$= 16 \cdot \frac{\pi}{4} - \pi^2 = 4\pi - \pi^2$$

ex We change $N=1$ to $N=100$,
then what's Bias and MSE.?

Soln
$$\text{Bias}=0$$
$$\text{MSE}=\frac{4\pi-\pi^2}{100}$$

Def $X\in L^2$, or say X is square integrable, if
$$\mathbb{E}[X^2]<\infty.$$

Q Find X s.t. $X\notin L^2$.

ex Suppose $\{\hat{\alpha}_i : i=1,2\cdots N\}$ is the collection of the same unbiased estimators for α. then

① Show that $\beta_N \triangleq \frac{\hat{\alpha}_1+\cdots+\hat{\alpha}_N}{N}$ is unbiased estimator.

② show that
$$MSE(\beta_N)=\frac{1}{N}MSE(\hat{\alpha}_1)$$

Def Let $(\beta_N)_{N\in\mathbb{N}}$ be a sequence of MC's.
We say (β_N) is consistent in some sense, if
$$\beta_N\to\alpha \quad \text{in some sense (could be } L^2\text{, a.s. in Prob)}$$

Recall

① If (β_N) are deterministic, then
$$\beta_n\to\alpha \quad \underbrace{\text{iff}}_{\text{if and only if}} \quad |\beta_n-\alpha|\to 0$$

② If β_N are random, then
$$\beta_n\to\alpha \text{ in } L^2 \quad \text{iff} \quad \mathbb{E}|\beta_N-\alpha|^2\to 0$$

③ If β_n are random then.
$$\beta_n\to\alpha \text{ a.s.} \quad \text{iff} \quad \mathbb{P}(\lim_n \beta_n=\alpha)=1$$

④ If β_n are random. then
$\beta_n\to\alpha$ in Prob iff
$$\forall\varepsilon>0,\ \lim_n \mathbb{P}(|\beta_n-\alpha|<\varepsilon)=1$$

ex Prove that L^2-consistency implies consistency-in-prob.

Pf Let $X_n=\beta_n-\alpha$
Then

L^2- consistency. $\quad \mathbb{E}\, X_n^2 \to 0$ (known)

consistency-in-Prob $\quad \forall \varepsilon>0, \quad \mathbb{P}(|X_n|>\varepsilon) \overset{?}{\to} 0$ $\Downarrow$

$$\mathbb{P}(|X_n|>\varepsilon) = \mathbb{E}\left[I(|X_n|>\varepsilon)\right]$$

$$= \mathbb{E}\left[I\left(\frac{|X_n|^2}{\varepsilon^2}>1\right)\right] \quad \xleftrightarrow{b/c} \quad I(|Y|>1) = I(Y^2>1)$$

$$\le \mathbb{E}\left[\frac{X_n^2}{\varepsilon^2}\right] \quad \xleftarrow{b/c}$$

Chebyshev Inequality

$$\mathbb{E}\left[I(Y^2>1)\right] \overset{?}{\le} \mathbb{E}[Y^2]$$

b/c $I(Y^2>1) \le Y^2$

$$= \frac{1}{\varepsilon^2}\,\mathbb{E}[X_n^2] \xrightarrow[n\to\infty]{} 0$$

$\boxed{S}$

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GAC

- Asset : $S = GBM(S_0, r, \sigma)$
- payoff:

$$V_T = (A_T - K)^+$$

where $A_T = (S(t_1) \cdots S(t_n))^{1/n}$

$0 \le t_1 \le \cdots \le t_n \le T$

prop Set $t_0 = 0$

$$C_0 = e^{(\hat{r}-r)T} BSM(S_0, \hat{r}, \hat{\sigma}, K, T, call)$$

where $\hat{r}, \hat{\sigma}$ are from

$$\mu = r - \frac{1}{2}\sigma^2$$

$$\hat{\mu} T = \frac{\mu}{n} \sum_{i=1}^{n} t_i$$

$$\hat{\sigma}^2 T = \frac{\sigma^2}{n^2} \sum_{i=0}^{n-1} (n-i)^2 (t_{i+1} - t_i)$$

$$\hat{r} = \hat{\mu} + \frac{1}{2}\hat{\sigma}^2$$

Prelims

- $S = GBM(S_0, r, \sigma)$, $\forall\ 0 = t_0 \le t_1 \le t_2$

$$\ln \frac{S(t_1)}{S(t_0)} = \mu (t_1 - t_0) + \sigma \sqrt{t_1 - t_0}\, Z_1$$

$$\ln \frac{S(t_2)}{S(t_1)} = \mu (t_2 - t_1) + \sigma \sqrt{t_2 - t_1}\, Z_2$$

where $Z_1, Z_2 \sim N(0,1)$, indep

- Sum of indep normal r.vs is again normal, i.e. $\forall$ iid $N(0,1)$ r.v. $(Z_i; i = 1, 2 \cdots n)$, we have

$$\sigma_1 Z_1 + \sigma_2 Z_2 + \cdots + \sigma_n Z_n = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)^{\frac{1}{2}} Z$$

- If $S = GBM(S_0, r, \sigma)$, then

$$e^{-rT} E[(S_T - K)^+] = BSM(S_0, r, \sigma, K, T, call)$$

$$E[(S_T - K)^+] = e^{rT} BSM(S_0, r, \cdots)$$
$$= FwdBSM(S_0, r \cdots)$$

Pf $\ln A_T = \frac{1}{n}\Big(\underbrace{\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_n)}_{\mathrm{I}}\Big)$

$$\mathrm{I} = \begin{cases} \ln S(t_0) + \ln \frac{S(t_1)}{S(t_0)} + \\ \ln S(t_0) + \ln \frac{S(t_1)}{S(t_0)} + \ln \frac{S(t_2)}{S(t_1)} + \cdot \\ \vdots \\ \ln S(t_0) + \ln \frac{S(t_1)}{S(t_0)} + \ln \frac{S(t_2)}{S(t_1)} + \cdots + \ln \frac{S(t_n)}{S(t_{n-1})} \end{cases}$$

$$= n \ln S(t_0) + \left(n \ln \frac{S(t_1)}{S(t_0)} + (n-1) \ln \frac{S(t_2)}{S(t_1)} + \cdots + 1 \cdot \ln \frac{S(t_n)}{S(t_{n-1})}\right)$$

$$\ln A_T = \ln S(t_0) + \frac{1}{n}\left(\quad - \quad - \quad - \quad - \quad - \quad \right)$$

$$\left.\begin{cases} \ln \frac{S(t_1)}{S(t_0)} = \mu(t_1 - t_0) + \sigma\sqrt{t_1 - t_0}\, Z_1 \\ \ln \frac{S(t_2)}{S(t_1)} = \mu(t_2 - t_1) + \sigma\sqrt{t_2 - t_1}\, Z_2 \\ \vdots \\ \ln \frac{S(t_n)}{S(t_{n-1})} = \mu(t_n - t_{n-1}) + \sigma\sqrt{t_n - t_{n-1}}\, Z_n \end{cases}\right\} \quad Z_1, Z_2 \cdots Z_n \overset{iid}{\sim} N(0,1)$$

$$\ln A_T = \ln S(t_0) + \frac{1}{n}\left(\mathrm{II}_1 + \mathrm{II}_2\right)$$

$$\mathrm{II}_1 = n \cdot \mu(t_1 - t_0) + (n-1)\mu \cdot (t_2 - t_1) + \cdots + 1 \cdot \mu(t_n - t_{n-1})$$

$$= \mu\left(n(t_1 - t_0) + (n-1)(t_2 - t_1) + \cdots + 1 \cdot (t_n - t_{n-1})\right)$$

$$= \mu \sum_{i=1}^{n} t_i = n \hat{\mu} T.$$

$$\mathrm{II}_2 = n \cdot \sigma\sqrt{t_1 - t_0}\, Z_1 + (n-1)\sigma\sqrt{t_2 - t_1}\, Z_2 + \cdots + 1 \cdot \sigma\sqrt{t_n - t_{n-1}}\, Z_n$$

$$= \bar{\sigma} \bar{Z}$$

where $\bar{\sigma}^2 = \sigma^2 \sum_{i=0}^{n-1} (n-i)^2 (t_{i+1} - t_i)$

$$= n^2 \hat{\sigma}^2 T$$

So.

$$\ln A_T = \ln S_0 + \hat{\mu} T + \hat{\sigma}\sqrt{T}\, \bar{Z}$$

$$A_T = GBM(S_0, \underbrace{\hat{r}}_{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2}, \hat{\sigma})(T)$$

$$V_0 = e^{-rT}\, \mathbb{E}(A_T - K)^+ = e^{-rT}\, \mathrm{FwdBSM}(S_0, \hat{r}, \hat{\sigma}, K, T, \mathrm{call})$$