bm_1d

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1 Exact sampling of Brownian path and Geometric Brownian path

1.1 Abstract

You will learn eact sampling of BM path and GBM path

Reference:

[1] Section 3.1 of [Gla03]: Random walk construction

1.2 Analysis

To have exact simulation of Brownian path by random walk, one can follow (3.2) of [1], i.e.

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}.$$

1.2.1 A function to generate a BM path using uniform mesh

Given an interval $[0, T]$, we can generate exact simulation with n uniform mesh by

$$\hat{W}(t_{i+1}) = \hat{W}(t_i) + \frac{1}{\sqrt{n}} Z_{i+1}, \text{ for } i = 0, 1, \ldots, n-1.$$

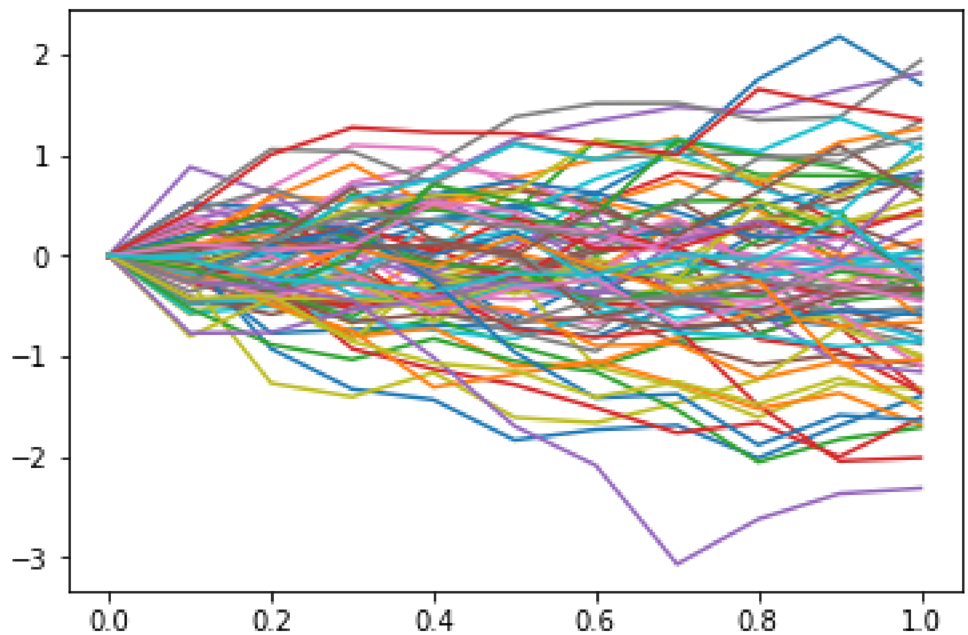
Demo Generate multiple paths and plot

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
```

```
In [2]: #define a function of BM path generator
        def BM_gen(T1, T2, n):  #para: start time, end time, and the mesh number
            t = np.linspace(T1, T2, num = n+1) #init mesh
            W = np.zeros(n+1) #init BM
            #Run (3.2)
            for i in range(n):
                W[i+1] = W[i] + 1./np.sqrt(n) * np.random.normal()

            return t, W
```

```
In [3]: #test BM_gen and plot
        #Also compute average and variance of $W(n)$

        MeshN = 10; SimN = 80; #Mesh number and simulation number
        SumW = 0; SqsumW = 0 #init sum of $W(n)$ and sum of $W^2(n)$
        for i in range(SimN):
            [t,W] = BM_gen(0., 1., MeshN)
            SumW += W[MeshN]; SqsumW += np.square(W[MeshN])
            plt.plot(t, W);

        MeanW = SumW/SimN; VarW = SqsumW/SimN - MeanW**2
        print("mean is", MeanW, "Variance is", VarW)
```

```
mean is -0.20913269439465204 Variance is 0.8564578613469325
```



Generating GBM path
GBM is given by

$$X(t) = x_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)\}.$$

We can replace $W(t)$ by its exact simulation $\hat{W}(t)$ to get exact simulation of $X(t)$, i.e.

$$\hat{X}(t) = x_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma \hat{W}(t)\}.$$

Application to Arithmetic asian option price
Arithmetic asian call option with maturity T and strick K has its pay off as

$$C(T) = (A(T) - K)^+,$$

where $A(T)$ is arithmetic average of the stock price at times $0 \le t_1 < t_2, \ldots, < t_n = T$, i.e.

$$A(T) = \frac{1}{n} \sum_{i=1}^{n} S(t_i).$$

The call price can be thus written by

$$C_0 = \mathbb{E}[e^{-rT}(A(T) - K)^+].$$

Unlike the geometric asian option, arithmetic counterpart does not have explicit formula for its price. In this below, we shall use MC. In practice, an arithmetic asian option with a given number n of time steps takes the price average at $n+1$ points

$$t_i = (i-1)\frac{T}{n}, \quad i = 1, 2, \ldots, (n+1).$$

Pseudocode bsm_arithmetic_asian_price(otype, strike, maturity, num_step, num_path):

- generate (num_path) many paths by exact sampling;
- compute discounted payoff for each path;
- Take the average for the option price.