1 Abstract

- Analyze BTCS (Backward in time and central in state) to log transformed BS PDE;
- Recover CRR model;
- This can be deemed as Markov chain approximation and
- readily generalized to American option pricing;

2 Problem

BSM model assumes stock price under EMM follows a GBM

$$dS_t = S_t(rdt + \sigma dW_t)$$

and the forward option price with payoff f is

$$v(s,t) = \mathbb{E}^{s,t}[f(S_t)].$$

If we take its log price as $X_t = \ln S_t$, then it follows ABM of, with $\hat{r} = r - \frac{1}{2}\sigma^2$

$$dX_t = \hat{r}dt + \sigma dW_t$$

and with $\hat{f}(x) = f(e^x)$

$$u(x,t) = \mathbb{E}^{x,t}[\hat{f}(X_t)].$$

Then, computing u is equivalent to computing v, i.e the forward option price is $v(s,t) = u(\ln x, t)$.

To compute u, we will use BTCS on its backward PDE

$$u_t + \hat{r}u_x + \frac{1}{2}\sigma^2 u_{xx} = 0, \ (x,t) \in \mathbb{R} \times (0,T)$$

with terminal condition

$$u(x,T) = \hat{f}(x).$$

3 Analysis

We use Backward in time, i.e.

$$u_t \sim \frac{u_j^n - u_j^{n-1}}{\Delta t}$$

and central scheme in state, i.e.

$$u_x = \frac{u_{j+1}^n - u_{j-1}^n}{\Delta x}, \quad u_{xx} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}.$$

This leads us to an explicit iteration. For simplicity, we set $\Delta x = \sigma\sqrt{\Delta t}$, then the iteration is

$$u_j^{n-1} = pu_{j+1}^n + (1-p)u_{j-1}^n,$$

where

$$p = \frac{1 + \hat{r}\sqrt{\Delta t}}{2\sigma}.$$

ex. Prove that the above scheme is stable.

Indeed, this suggests an equivalent Markov chain approximation:

- Let X be a random walk with transition probability

$$p(X_n = (j+1)\Delta x | X_{n-1} = j\Delta x) = p, \quad p(X_n = (j-1)\Delta x | X_{n-1} = j\Delta x) = 1-p.$$

  Then, $u_j^n = \mathbb{E}_j^n[\hat{f}(X_N)]$.

Indeed, this is CRR model.

ex. In the original CRR model, the random walk X follows transition probability with upper leg probability

$$\hat{p} = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

Prove that

$$\hat{p} - p = o(\sqrt{\Delta t}).$$