fourier_method

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1 Fourier transform in european option pricing

1.1 Abstract

Our goal is

- Use Fourier transform method for european call pricing whenever characteristic function is available for its log price

1.2 Problem

In the **Heston** model, the dynamic involves two processes (S_t, ν_t). More precisely, the asset price S follows generalized geometric Brownian motion with random volatility process $\sqrt{\nu_t}$, i.e.

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_{1,t},$$

while squared of volatility process ν follows CIR process

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t}(\rho dW_{1,t} + \bar{\rho} dW_{2,t})$$

with $\rho^2 + \bar{\rho}^2 = 1$. Feller condition for its existence of the solution is

$$2\kappa\theta > \xi^2.$$

Our goal is to use Fourier transform to evaluate $\text{Call}(T = 1, K = 100)$ underlying Heston model with the following parameters:

$$S_0 = 100, \nu(0) = .04, r = .05, \kappa = 1.2, \theta = .04, \xi = .3, \rho = .5.$$

There is no explicit formula for both call price and pdf of S_T. However, the explicit formula for the **characteristic function** of $\log S_T$ is available. Therefore, it is perfectly suitable for Fourier transform method.

1.3 Analysis

1.3.1 Main result

We assume the characteristic function of a random variable X is defined by

$$\phi(u) = \mathbb{E} \exp(iuX).$$

ex Prove that, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then its characteristic function is

$$\log \phi(u) = iu\mu - \frac{1}{2}u^2\sigma^2.$$

Propostion

Let interest rate be r and the characteristic function of $\ln(S_T)$ be ϕ. The price of Call(T, K) is

$$C = S_0 I_1 - Ke^{-rT} I_2,$$

where

$$I_1(\phi, \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big(\frac{e^{-iu\ln(K)}\phi(u-i)}{iu\phi(-i)}\Big)du$$

and

$$I_2(\phi, \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big(\frac{e^{-iu\ln(K)}\phi(u)}{iu}\Big)du.$$

The above presentation actually gives straightforward evaluation for python as long as the characteristic function is available. One may use scipy.integrate.quad for two integrations in the formula.

1.3.2 Proof

We recall that the characteristic function of a random variable X is defined by

$$\phi(u) = \mathbb{E}\exp(iuX).$$

- (hw) Prove

$$\int_0^\infty \frac{\sin t}{t} dt = \pi/2.$$

Lemma

$$I(X > H) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin(u(X-H))}{u} du.$$

Lemma

For $\phi = \phi_X$, we have

$$\mathbb{P}(X > H) = I_2(\phi, H).$$

Lemma

For $\phi = \phi_X$, we have

$$\frac{\mathbb{E}[e^X I(X > H)]}{\mathbb{E}[e^X]} = I_1(\phi, H)$$

Proof of proposition

$$C = \mathbb{E}[e^{-rT} S_T I(\ln S_T > \ln K)] - Ke^{-rT}\mathbb{E}[I(\ln S_T > \ln K)].$$

Note that

$$\mathbb{E}[e^{-rT} S_T] = S_0.$$

Therefore, we have

$$C = S_0 \frac{\mathbb{E}[S_T I(\ln S_T > \ln K)]}{\mathbb{E}[S_T]} - Ke^{-rT}\mathbb{E}[I(\ln S_T > \ln K)].$$

Now, we conclude the result by utilizing the above lemmas with $X = \ln S_T$.