A Model-Free Semi-Closed Form Pricing Formula

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Our goal here is to establish a well known model-free semi-closed form pricing formula for vanilla call options. The only assumption we are going to make is a constant risk-free interest rate r. The pricing formula is convenient for any model for the underlying asset where the characteristic function of the log return is known analytically, a constrain satisfied by many common option pricing models, including the Heston model, the Black-Scholes model, and all Lévy models.

1 The Formula

Assume that the risk-free interest rate is a constant r. The price of a vanilla call option with strike K, maturity T and its underlying asset price modeled by a stochastic process S_t is

$$c = S_0\Pi_1 - Ke^{-rT}\Pi_2,$$

where

$$\begin{aligned} \Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\log\left(\frac{K}{S_0}\right)}\phi_T(u-i)}{iu\phi_T(-i)}\right] du, \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\log\left(\frac{K}{S_0}\right)}\phi_T(u)}{iu}\right] du, \end{aligned}$$

and $\phi_T(u) = E[e^{iu\log(S_T/S_0)}]$ is the characteristic function of $\log(S_T/S_0)$. For most models the ugly looking integrands actually behave very well, no singularities. Thus evaluating the option price with numerical integration rules is straightforward.

To derive the formula, we start with the universal pricing formula $c = e^{-rT}E[(S_T - K)^+]$, where the expectation is taken under the risk-neutral probability. Note that

$$\begin{aligned} c &= e^{-rT}E[(S_T-K)^+] \\ &= e^{-rT}E\left[(S_T-K)1_{\{S_T>K\}}\right] \\ &= e^{-rT}E\left[S_T 1_{\{S_T>K\}}\right] - Ke^{-rT}E\left[1_{\{S_T>K\}}\right] \\ &= S_0 E\left[e^{-rT}\left(\frac{S_T}{S_0}\right)1_{\{S_T>K\}}\right] - Ke^{-rT}P(S_T>K). \end{aligned}$$

In terms of log return, we rewrite

$$c = S_0\underbrace{E\left[e^{-rT}\left(\frac{S_T}{S_0}\right)1_{\left\{\log\frac{S_T}{S_0}>\log\frac{K}{S_0}\right\}}\right]}_{\Pi_1} - Ke^{-rT}\underbrace{P\left(\log\frac{S_T}{S_0}>\log\frac{K}{S_0}\right)}_{\Pi_2}. \tag{1}$$

The majority of the derivation of the pricing formula is the following two lemmas:

Lemma 1.1 *Let X be a random variable and $\phi(u) = E[e^{iuX}]$ its characteristic function. Then*

$$P(X > H) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-iuH}\phi(u)}{iu}\right] du.$$

Lemma 1.2 *With the same notation as Lemma 1.1, we have*

$$\frac{E\left[e^X 1_{\{X>H\}}\right]}{E[e^X]} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-iuH}\phi(u-i)}{iu\phi(-i)}\right] du.$$

Once these lemmas are proved, the only step it takes to derive Eq. (1) is to apply Lemma 1.2 on the first term and Lemma 1.1 on the second.

Proof of Lemma 1.1: Plugging in the definition of the characteristic function, the right hand side is

$$\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iuH}E[e^{iuX}]}{iu}\right] du,$$

where the integrand is

$$\begin{aligned}
\mathrm{Re}\left[E\left[\frac{e^{iu(X-H)}}{iu}\right]\right] &= E\left[\mathrm{Re}\left[\frac{e^{iu(X-H)}}{iu}\right]\right] \\
&= E\left[\mathrm{Re}\left[\frac{\cos(u(X-H)) + i\sin(u(X-H))}{iu}\right]\right] \\
&= E\left[\frac{\sin(u(X-H))}{u}\right].
\end{aligned}$$

That the function inside the expectation is even in u is important in numerical point of view. Thus the right hand side of Lemma 1.1 can be rewritten as

$$\frac{1}{2} + \frac{1}{\pi}\int_0^\infty E\left[\frac{\sin(u(X-H))}{u}\right] du = E\left[\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin(u(X-H))}{u} du\right], \tag{2}$$

where Fubini's theorem is used to pass the integral inside the expectation. Substituting $t = u(X-H)$, we get

$$\int_0^\infty \frac{\sin(u(X-H))}{u} du = \begin{cases} \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} & \text{if } X > H \\ 0 & \text{if } X = H \\ \int_0^{-\infty} \frac{\sin t}{t} dt = -\frac{\pi}{2} & \text{if } X < H \end{cases}.$$

It is a well known fact that $\int_0^\infty \frac{\sin t}{t} = \pi/2$. We provide a derivation in Appendix. Plug this into Eq. (2) to get $E\left[1_{\{X>H\}}\right] = P(X > H)$. ■

Proof of Lemma 1.2: Plugging in the definition of the characteristic function, the right hand side is

$$\begin{aligned}
& \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iuH}E[e^{i(u-i)X}]}{iuE[e^X]}\right]du \\
= & \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{1}{E[e^X]}\mathrm{Re}\left[\frac{e^{-iuH}E[e^{iuX+X}]}{iu}\right]du \\
= & \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{1}{E[e^X]}\mathrm{Re}\left[E\left[\frac{e^{iu(X-H)+X}}{iu}\right]\right]du \\
= & \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{1}{E[e^X]}E\left[\mathrm{Re}\left[\frac{e^{iu(X-H)+X}}{iu}\right]\right]du \\
= & \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{1}{E[e^X]}E\left[e^X\mathrm{Re}\left[\frac{\cos(u(X-H))+i\sin(u(X-H))}{iu}\right]\right]du \\
= & \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{1}{E[e^X]}E\left[e^X\frac{\sin(u(X-H))}{u}\right]du \\
= & \frac{1}{E[e^X]}E\left[e^X\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{\sin(u(X-H))}{u}du\right)\right]=\frac{E\left[e^X 1_{\{X>H\}}\right]}{E[e^X]}.
\end{aligned}$$

■

2 Applied on the Black-Scholes Model

In the Black-Scholes model, the log return of the underlying asset is assumed to follow a drift Brownian motion in the risk-neutral probability:

$$\log\frac{S_T}{S_0}=\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T\sim N\left(\left(r-\frac{\sigma^2}{2}\right)T,\sigma^2 T\right).$$

Thus

$$\phi_T(u)=e^{i\left(r-\frac{\sigma^2}{2}\right)Tu-\frac{\sigma^2 T}{2}u^2},$$

and

$$\begin{aligned}
\Pi_2 &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\log\left(\frac{K}{S_0}\right)}\phi_T(u)}{iu}\right]du \\
&= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{iu\left(\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T\right)-\frac{\sigma^2 T}{2}u^2}}{iu}\right]du \\
&= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\cos\left(u\left(\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T\right)\right)+i\sin\left(u\left(\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T\right)\right)}{iu}\right]e^{-\frac{\sigma^2 T}{2}u^2}du \\
&= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{\sin\left(u\left(\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T\right)\right)}{u}e^{-\frac{\sigma^2 T}{2}u^2}du \\
&= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{\sin(d_2 t)}{t}e^{-\frac{t^2}{2}}dt,
\end{aligned}$$

where a substitution $t = \sigma\sqrt{T}u$ is applied and we set

$$d_2 = \frac{\log \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

for simplicity. Since d_2 is just a constant parameter, the integrand is a well-behaved smooth function of t.

To find Π_1, we could have used Lemma 1.2, but let us apply the Girsanov Theorem and rewrite

$$\begin{aligned}
\Pi_1 &= E\left[e^{-rT}\left(\frac{S_T}{S_0}\right) 1_{\left\{\log \frac{S_T}{S_0} > \log \frac{K}{S_0}\right\}}\right] \\
&= \tilde{E}\left[1_{\left\{\log \frac{S_T}{S_0} > \log \frac{K}{S_0}\right\}}\right] \\
&= \tilde{P}\left(\log \frac{S_T}{S_0} > \log \frac{K}{S_0}\right),
\end{aligned} \tag{3}$$

where the expectation $\tilde{E}$ is taken under another probability measure $\tilde{P}$, and the Radon-Nikodym derivative of $\tilde{P}$ with respect to the risk-neutral probability measure is

$$e^{-rT}\left(\frac{S_T}{S_0}\right) = e^{-\frac{\sigma^2}{2}T + \sigma W_T}.$$

With this Radon-Nikodym derivative, the Girsanov Theorem tells us that $\tilde{W}_t = W_t - \sigma t$ is a Brownian motion under $\tilde{P}$ measure. Thus we can rewrite the log return as

$$\begin{aligned}
\log \frac{S_T}{S_0} &= \left(r - \frac{\sigma^2}{2}\right) T + \sigma\left(\tilde{W}_T + \sigma T\right) \\
&= \left(r + \frac{\sigma^2}{2}\right) T + \sigma\tilde{W}_T,
\end{aligned}$$

which has distribution $N\left(\left(r + \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$ under $\tilde{P}$ measure. Now apply Lemma 1.1 again on Eq. (3) like how we got Π_2 above. We obtain

$$\Pi_1 = \tilde{P}\left(\log \frac{S_T}{S_0} > \log \frac{K}{S_0}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin(d_1 t)}{t} e^{-\frac{t^2}{2}}\, dt$$

with

$$d_1 = \frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

With such a simple model, we can actually get a series solution of the price by integrating the series expansions of $\sin(d_1 t)/t$ and $\sin(d_2 t)/t$ multiplied by $e^{-t^2/2}$. [1] The obtained series will match that of $\mathrm{erf}(x)$. If the result is further simplified, we will get the celebrated Black-Scholes formula.

[1] Integrating term by term is possible in this example because of the dominated convergence theorem.

A The Improper Integral of $\sin t/t$

It is well known that $\int_0^\infty \sin t/t \, dt = \pi/2$. The argument we provide here is through double integral. Note that

$$\int_{-\infty}^0 e^{xt} \, dx = \frac{1}{t}.$$

Thus

$$\begin{aligned} \int_0^\infty \frac{\sin t}{t} \, dt &= \int_0^\infty \int_{-\infty}^0 e^{xt} \sin t \, dx dt \\ &= \int_{-\infty}^0 \int_0^\infty e^{xt} \sin t \, dt dx. \end{aligned} \tag{4}$$

The domain of integration is set up in a way that xt is negative, making sure of the convergence of the integral. Integration by part yields

$$\int e^{xt} \sin t \, dt = \frac{1}{x} e^{xt} \sin t - \frac{1}{x^2} e^{xt} \cos t - \int \frac{1}{x^2} e^{xt} \sin t \, dt.$$

Due to the fact that $xt < 0$, we have

$$\begin{aligned} \int_0^\infty e^{xt} \sin t \, dt &= \left[\frac{\frac{1}{x} e^{xt} \sin t - \frac{1}{x^2} e^{xt} \cos t}{1 + \frac{1}{x^2}} \right]_{t=0}^{t=\infty} \\ &= \left[e^{xt} \, \frac{x \sin t - \cos t}{1 + x^2} \right]_{t=0}^{t=\infty} \\ &= \frac{1}{1 + x^2}. \end{aligned}$$

Plug this into Eq. (4) to get

$$\int_0^\infty \frac{\sin t}{t} \, dt = \int_{-\infty}^0 \frac{1}{1 + x^2} \, dx = \frac{\pi}{2}.$$