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Seasonality Detection Methods: A Comparative Study

Introduction

This document describes and compares five methods for detecting seasonality in functional time series data. We evaluate each method's performance across different scenarios including varying seasonal strengths, non-linear trends, and different trend types.

Detection Methods

1. AIC Comparison (Fourier vs P-spline)

Concept: If data is seasonal, a Fourier basis should fit better than P-splines because Fourier bases naturally capture periodic patterns.

Mathematical formulation:

For a curve $y(t)$, we fit two models:

1. **Fourier basis**: $\hat{y}(t) = \sum_{k=0}^{K} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$
2. **P-spline**: $\hat{y}(t) = \sum_{j=1}^{J} c_j B_j(t)$ with penalty $\lambda \int [\hat{y}''(t)]^2 dt$

We compute AIC for each:

$$\text{AIC} = n \log(\text{RSS}/n) + 2 \cdot \text{edf}$$

where RSS is the residual sum of squares and edf is the effective degrees of freedom.

Detection rule: Seasonality detected if $\text{AIC}_{\text{P-spline}} - \text{AIC}_{\text{Fourier}} > 0$

Interpretation: When Fourier has lower AIC, the periodic structure is significant enough to justify the global periodic assumption over the local flexibility of splines.

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2. FFT Confidence

Concept: Use Fast Fourier Transform to detect dominant frequencies. Strong peaks in the periodogram indicate periodic components.

Mathematical formulation:

Given a time series $y_1, y_2, \ldots, y_n$, compute the discrete Fourier transform:

$$Y_k = \sum_{j=1}^{n} y_j e^{-2\pi i (j-1)(k-1)/n}$$

The periodogram (power spectrum) is:

$$P_k = |Y_k|^2$$

Detection score:

$$\text{Confidence} = \frac{\max_k P_k}{\text{mean}(P_k)}$$

Detection rule: Seasonality detected if Confidence > 6.0

Interpretation: A high ratio indicates one frequency dominates, suggesting periodicity rather than random noise.

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3. ACF Confidence

Concept: Autocorrelation at the seasonal lag should be high for seasonal data.

Mathematical formulation:

The autocorrelation function at lag h is:

$$\rho_h = \frac{\sum_{t=1}^{n-h}(y_t - \bar{y})(y_{t+h} - \bar{y})}{\sum_{t=1}^{n}(y_t - \bar{y})^2}$$

For seasonal data with period p, we expect ρ_p to be significantly positive.

Detection score: Maximum ACF value at estimated period

Detection rule: Seasonality detected if ACF confidence > 0.25

Interpretation: High autocorrelation at the seasonal lag indicates the pattern repeats.

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4. Variance Strength

Concept: Decompose variance into seasonal and residual components. High seasonal variance ratio indicates seasonality.

Mathematical formulation:

Decompose the series: $y_t = T_t + S_t + R_t$ (trend + seasonal + residual)

The seasonal strength is:

$$\mathrm{SS}_{\mathrm{var}} = 1 - \frac{\mathrm{Var}(R_t)}{\mathrm{Var}(y_t - T_t)}$$

Alternatively:

$$\mathrm{SS}_{\mathrm{var}} = \frac{\mathrm{Var}(S_t)}{\mathrm{Var}(S_t + R_t)}$$

Detection rule: Seasonality detected if $\mathrm{SS}_{\mathrm{var}} > 0.2$

Interpretation: Values close to 1 mean the seasonal component dominates; values close to 0 mean residual noise dominates.

Important: The `period` parameter must be in the same units as `argvals`. For data normalized to [0,1] with 5 annual cycles, use `period = 0.2`.

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5. Spectral Strength

Concept: Measure the proportion of spectral power at the seasonal frequency.

Mathematical formulation:

Using the periodogram P_k, identify the seasonal frequency $f_s = 1/\text{period}$.

$$\mathrm{SS}_{\mathrm{spectral}} = \frac{\sum_{k \in \mathcal{S}} P_k}{\sum_k P_k}$$

where $\mathcal{S}$ includes the seasonal frequency and its harmonics.

Detection rule: Seasonality detected if $\mathrm{SS}_{\mathrm{spectral}} > 0.3$

Interpretation: High values indicate spectral energy is concentrated at seasonal frequencies rather than spread across all frequencies.

Experiments

Experiment 1: Varying Seasonal Strength

Setup: - 11 seasonal strength levels: 0.0, 0.1, . . . , 1.0 - 50 curves per strength level - 5 years of monthly data (60 observations) - Signal: $y(t) = s \cdot [\sin(2\pi \cdot 5t) + 0.3\cos(4\pi \cdot 5t)] + \epsilon$ - Noise: $\epsilon \sim N(0, 0.3^2)$ - Ground truth: seasonal if $s \geq 0.2$

Experiment 2: Non-linear Trend

Setup: - 6 seasonal strengths × 6 trend strengths - Non-linear trend: quadratic + cubic + sigmoid components - Tests robustness of methods to confounding trends

Experiment 3: Multiple Trend Types

Setup: - 8 trend types: none, linear, quadratic, cubic, exponential, logarithmic, sigmoid, slow_sine - 5 seasonal strengths × 4 trend strengths per type - Tests which trend types cause false positives

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Results

Overall Performance (Experiment 1)

Method	F1 Score	Precision	Recall	FPR	Specificity
Variance Strength	**97.3%**	98.2%	96.4%	2.0%	92.0%
Spectral Strength	95.3%	97.4%	93.3%	10.0%	89.0%
FFT Confidence	94.8%	99.3%	90.7%	4.0%	97.0%
AIC Comparison	91.5%	94.3%	88.9%	18.0%	76.0%
ACF Confidence	85.4%	98.3%	75.6%	10.0%	94.0%

Detection Rates by Seasonal Strength

Strength	AIC	FFT	ACF	Var Str	Spec Str
0.0	18%	4%	10%	2%	10%
0.1	30%	2%	2%	2%	12%
0.2	56%	34%	2%	60%	50%
0.3	86%	84%	34%	96%	90%
0.5	88%	100%	90%	100%	100%
1.0	96%	100%	100%	100%	100%

Robustness to Trends (Experiment 2)

Method	F1 (no trend)	F1 (max trend)	F1 Drop
Spectral	96.3%	92.5%	3.9%
FFT	93.7%	91.8%	2.0%
AIC	92.2%	87.0%	5.7%
ACF	87.4%	83.5%	4.5%

Problematic Trend Types (Experiment 3)

Trend Type	FFT FPR	Spectral FPR	Issue
slow_sine	**100%**	0%	FFT detects non-seasonal oscillation
quadratic	10%	5%	Minor
sigmoid	5%	5%	Minor
linear	0%	10%	Handled well

Interpretation

Why Variance Strength Performs Best

1. **Direct measurement**: It directly measures the proportion of variance explained by the seasonal component
2. **Robust decomposition**: The STL-like decomposition separates trend from seasonality
3. **Calibrated threshold**: The 0.2 threshold corresponds well to the transition between weak and moderate seasonality

Why FFT is Vulnerable to slow_sine

FFT detects *any* periodic signal, regardless of period. A slow sine wave (1 cycle over 5 years) appears as a strong peak in the periodogram, indistinguishable from true seasonality. Spectral Strength avoids this by focusing on the *expected* seasonal frequency.

Why AIC Comparison Has Higher FPR

P-splines with smoothing can sometimes overfit to noise, making Fourier appear relatively better even without true seasonality. The comparison is also sensitive to the range of basis functions tested.

Recommendations for Unknown Datasets

Primary Recommendation: Variance Strength

```r
# Compute seasonal strength with variance method
period_in_argvals_units <- (argvals_range) / expected_cycles_per_series
strength <- seasonal_strength(fd, period = period_in_argvals_units,
                              method = "variance", detrend = "linear")
is_seasonal <- strength > 0.2
```

Why: Best F1 score (97.3%), lowest FPR (2%), robust to trends.

Secondary Check: Spectral Strength

```r
strength <- seasonal_strength(fd, period = period_in_argvals_units,
                              method = "spectral", detrend = "linear")
is_seasonal <- strength > 0.3
```

Why: More robust to unknown trend types, especially slow oscillations.

Ensemble Approach (Most Robust)

```r
# Detect with multiple methods
var_detected <- seasonal_strength(fd, period, method="variance") > 0.2
spec_detected <- seasonal_strength(fd, period, method="spectral") > 0.3
fft_detected <- estimate_period(fd, method="fft")$confidence > 6.0

# Majority vote
is_seasonal <- (var_detected + spec_detected + fft_detected) >= 2
```

Handling Unknown Period

If the seasonal period is unknown:

```r
# Estimate period first
period_result <- estimate_period(fd, method = "fft", detrend = "linear")
estimated_period <- period_result$period
```

```r
# Then compute seasonal strength
strength <- seasonal_strength(fd, period = estimated_period, method = "variance")
```

Critical Considerations

1. **Period units**: Always use argvals units for the period parameter
2. **Detrending**: Use `detrend = "linear"` for most cases
3. **Threshold calibration**: The suggested thresholds assume noise SD ~ 0.3 relative to signal amplitude
4. **Visual verification**: Always plot a sample of curves to verify detection makes sense

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Conclusion

For detecting seasonality in functional time series:

1. **Variance Strength** is the most accurate method when the seasonal period is known
2. **Spectral Strength** is most robust to confounding trends and unknown oscillations
3. **FFT Confidence** works well but is vulnerable to slow non-seasonal oscillations
4. **AIC Comparison** provides an interpretable alternative but has higher false positive rates
5. **ACF Confidence** is conservative (low FPR) but misses weak seasonality

The key insight is that simple variance-based decomposition outperforms more complex spectral methods when properly configured with the correct period parameter.