Seasonality Detection Methods: A Comparative Study

fdars Package

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1 Executive Summary

1.1 Key Findings

This study compared nine methods for detecting seasonality in functional time series data across 550+ simulated curves with varying seasonal strengths and trend components.

Method	F1 Score	False Positive Rate	Robustness to Trends
Wavelet Strength	**97.8%**	4%	Excellent (1.2% F1 drop)
Variance Strength	97.3%	**2%**	Excellent (0.4% F1 drop)
SAZED (Ensemble)	96.5%	4%	Excellent (automatic)
Spectral Strength	95.3%	10%	Good (3.9% F1 drop)
Autoperiod	95.0%	6%	Good (gradient refinement)
FFT Confidence	94.8%	4%	Good (2.0% F1 drop)
CFDAutoperiod	94.2%	4%	Excellent (detrending)
AIC Comparison	91.5%	18%	Moderate (5.7% F1 drop)
ACF Confidence	85.4%	10%	Moderate (4.5% F1 drop)

Top methods: Wavelet (97.8% F1, best recall) and Variance (97.3% F1, lowest FPR) are excellent choices. SAZED (96.5%) provides robust parameter-free detection.

1.2 Robustness to Real-World Challenges

Challenge	Most Affected	Wavelet Advantage
Red Noise (AR(1))	FFT (100% FPR)	Moderate (up to 36% FPR)
Multiple Seasonalities	Variance (4% TPR)	Good (56% TPR)

Challenge	Most Affected	Wavelet Advantage
Amplitude Modulation	Variance (18% TPR)	**Excellent (72% TPR)**
Outliers (10%, 10x)	ACF (6% TPR)	Good (62% TPR)

Robustness ranking: FFT > Spectral > Wavelet > Variance > ACF; Wavelet excels on amplitude modulation.

1.3 Recommendations

1.3.1 Primary Recommendation: Use SAZED for Unknown Signals

```r
# Parameter-free detection with SAZED ensemble
result <- sazed(fd)
is_seasonal <- result$consensus_count >= 3  # Majority of 5 components agree
estimated_period <- result$period
```

1.3.2 When Period is Known: Use Variance Strength

```r
# Detect seasonality with Variance Strength method
period <- 0.2  # Period in argvals units (e.g., 1/5 for 5 cycles in [0,1])
strength <- seasonal.strength(fd, period = period, method = "variance", detrend_method = "linear")
is_seasonal <- strength > 0.2
```

1.3.3 When Trend is Present: Use CFDAutoperiod

```r
# Robust to trends via first-order differencing
result <- cfd.autoperiod(fd)
is_seasonal <- result$acf_validation > 0.25
estimated_period <- result$period
```

1.3.4 Unified Detection Interface

```r
# Use detect.period() for easy method switching
result <- detect.period(fd, method = "sazed")  # or "autoperiod", "cfd", "fft", "acf"
is_seasonal <- !is.null(result$period)
```

1.3.5 Critical Notes

1. **Period units matter**: The `period` parameter must be in argvals units, not raw time units
2. **Avoid FFT for slow oscillations**: FFT has 100% false positive rate when non-seasonal oscillations are present
3. **Thresholds are calibrated**: All thresholds target ~5% false positive rate on pure noise
4. **SAZED is parameter-free**: Requires no tuning and works well across diverse signals

2 Introduction

This report describes and compares nine methods for detecting seasonality in functional time series data. We evaluate each method's performance across different scenarios including varying seasonal strengths, non-linear trends, colored noise, multiple seasonalities, amplitude modulation, and outliers.

The goal is to answer: **Given a time series, how can we reliably determine if it contains a seasonal pattern?**

Report Structure:

- Section 3 – Nine detection methods and their mathematical formulations
- Section 4 – Unified simulation study covering seven scenarios
- Section 5 – Post-detection amplitude modulation characterization
- Section 6 – Key findings and recommendations

3 Detection Methods

3.1 AIC Comparison (Fourier vs P-spline)

Concept: If data is seasonal, a Fourier basis should fit better than P-splines because Fourier bases naturally capture periodic patterns.

Mathematical formulation:

For a curve $y(t)$, we fit two models:

1. **Fourier basis**: $\hat{y}(t) = \sum_{k=0}^{K} a_k \cos(2\pi k t) + b_k \sin(2\pi k t)$
2. **P-spline**: $\hat{y}(t) = \sum_{j=1}^{J} c_j B_j(t)$ with penalty $\lambda \int [\hat{y}''(t)]^2 dt$

We compute AIC for each:

$$\mathrm{AIC} = n \log(\mathrm{RSS}/n) + 2 \cdot \mathrm{edf}$$

where RSS is the residual sum of squares and edf is the effective degrees of freedom.

Detection rule: Seasonality detected if $\mathrm{AIC}_{\text{P-spline}} - \mathrm{AIC}_{\text{Fourier}} > 0$

Interpretation: When Fourier has lower AIC, the periodic structure is significant enough to justify the global periodic assumption over the local flexibility of splines.

3.2 FFT Confidence

Concept: Use Fast Fourier Transform to detect dominant frequencies. Strong peaks in the periodogram indicate periodic components.

Mathematical formulation:

Given a time series $y_1, y_2, \ldots, y_n$, compute the discrete Fourier transform:

$$Y_k = \sum_{j=1}^{n} y_j e^{-2\pi i (j-1)(k-1)/n}$$

The periodogram (power spectrum) is:
$$P_k = |Y_k|^2$$

Detection score:
$$\text{Confidence} = \frac{\max_k P_k}{\text{mean}(P_k)}$$

Detection rule: Seasonality detected if Confidence > 6.0

Interpretation: A high ratio indicates one frequency dominates, suggesting periodicity rather than random noise.

3.3 ACF Confidence

Concept: Autocorrelation at the seasonal lag should be high for seasonal data.

Mathematical formulation:

The autocorrelation function at lag h is:

$$\rho_h = \frac{\sum_{t=1}^{n-h}(y_t - \bar{y})(y_{t+h} - \bar{y})}{\sum_{t=1}^{n}(y_t - \bar{y})^2}$$

For seasonal data with period p, we expect ρ_p to be significantly positive.

Detection rule: Seasonality detected if ACF confidence > 0.25

Interpretation: High autocorrelation at the seasonal lag indicates the pattern repeats.

3.4 Variance Strength

Concept: Decompose variance into seasonal and residual components. High seasonal variance ratio indicates seasonality.

Mathematical formulation:

Decompose the series: $y_t = T_t + S_t + R_t$ (trend + seasonal + residual)

The seasonal strength is:
$$\text{SS}_{\text{var}} = 1 - \frac{\text{Var}(R_t)}{\text{Var}(y_t - T_t)}$$

Detection rule: Seasonality detected if $\text{SS}_{\text{var}} > 0.2$

Interpretation: Values close to 1 mean the seasonal component dominates; values close to 0 mean residual noise dominates.

Important: The `period` parameter must be in the same units as `argvals`. For data normalized to [0,1] with 5 annual cycles, use `period = 0.2`.

3.5 Spectral Strength

Concept: Measure the proportion of spectral power at the seasonal frequency.

Mathematical formulation:

Using the periodogram P_k, identify the seasonal frequency $f_s = 1/\text{period}$.

$$\text{SS}_{\text{spectral}} = \frac{\sum_{k \in \mathcal{S}} P_k}{\sum_k P_k}$$

where $\mathcal{S}$ includes the seasonal frequency and its harmonics.

Detection rule: Seasonality detected if $\text{SS}_{\text{spectral}} > 0.3$

Interpretation: High values indicate spectral energy is concentrated at seasonal frequencies.

3.6 Wavelet Strength

Concept: Use continuous wavelet transform (CWT) to measure power at the seasonal scale, capturing time-localized periodic patterns.

Mathematical formulation:

Using the Morlet wavelet $\psi_0(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-t^2/2}$ with $\omega_0 = 6$, compute the CWT at scale $s = \text{period} \cdot \omega_0/(2\pi)$:

$$W(s, \tau) = \int y(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t - \tau}{s} \right) dt$$

The wavelet strength is:

$$\text{SS}_{\text{wavelet}} = \sqrt{\frac{\text{mean}(|W(s, \tau)|^2)}{\text{Var}(y)}}$$

Detection rule: Seasonality detected if $\text{SS}_{\text{wavelet}} > 0.26$

Interpretation: Unlike global spectral methods, wavelet analysis localizes power in time, making it robust to non-stationary signals and amplitude modulation.

Advantages:

- Handles time-varying seasonality better than FFT
- Less sensitive to edge effects than variance decomposition
- Naturally filters non-seasonal low-frequency trends

3.7 SAZED (Parameter-Free Ensemble)

Concept: Combine five independent detection methods and use consensus voting to determine the period. No parameter tuning required.

Components:

1. **Spectral**: FFT peaks above noise floor
2. **ACF Peak**: Local maxima in autocorrelation function
3. **ACF Average**: Weighted mean of ACF-detected periods
4. **Zero-Crossing**: Period from ACF zero crossings
5. **Spectral Diff**: Peaks in differentiated spectrum

Detection rule: Seasonality detected if ≥ 3 components agree on a period (within tolerance).

Interpretation: Consensus voting provides robustness—spurious detections from individual methods are filtered out. The ensemble approach adapts to diverse signal types without manual tuning.

Advantages:

- No parameters to tune
- Robust across signal types (noisy, trended, multi-frequency)
- Interpretable component-level diagnostics

3.8 Autoperiod (Hybrid FFT + ACF)

Concept: Use FFT to identify candidate periods, then validate each candidate with ACF. Apply gradient ascent for sub-bin period refinement (Vlachos et al. 2005).

Algorithm:

1. Compute periodogram via FFT
2. Find candidate periods from spectral peaks above noise threshold
3. Validate each candidate using ACF at that lag: $\rho_p > 0$
4. Apply gradient ascent to refine period estimate
5. Return period with highest ACF correlation

Detection score: ACF correlation at the refined period

Detection rule: Seasonality detected if ACF correlation > 0.3

Interpretation: FFT provides fast frequency identification while ACF validation filters spurious harmonic peaks. Gradient ascent improves period accuracy beyond FFT bin resolution.

3.9 CFDAutoperiod (Clustered Filtered Detrended)

Concept: Apply first-order differencing to remove trends, identify candidate periods via FFT, cluster similar candidates, and validate using ACF on the original signal (Puech et al. 2020).

Algorithm:

1. Apply first-order differencing: $y'_t = y_t - y_{t-1}$
2. Compute FFT on detrended signal
3. Identify candidate periods from periodogram peaks

4. Cluster candidates using density-based clustering
5. Validate cluster centers using ACF on original signal

Detection score: ACF validation value at the cluster center period

Detection rule: Seasonality detected if ACF validation > 0.25

Interpretation: Differencing eliminates polynomial trends that can confuse FFT. Clustering aggregates nearby period estimates for robustness. ACF validation on the original signal confirms true periodicity.

Advantages:

- Robust to polynomial and non-linear trends
- Handles noisy period estimates via clustering
- Validates on original signal to avoid artifacts

4 Simulation Study

This section presents a unified simulation study covering seven scenarios that test detection methods under progressively challenging conditions.

4.1 Baseline: Varying Seasonal Strength

Setup: Test detection across 11 seasonal strength levels (0.0 to 1.0), with 50 curves per level, 60 observations (5 years monthly), and white noise ($\sigma = 0.3$). Ground truth: seasonal if $s \geq 0.2$.

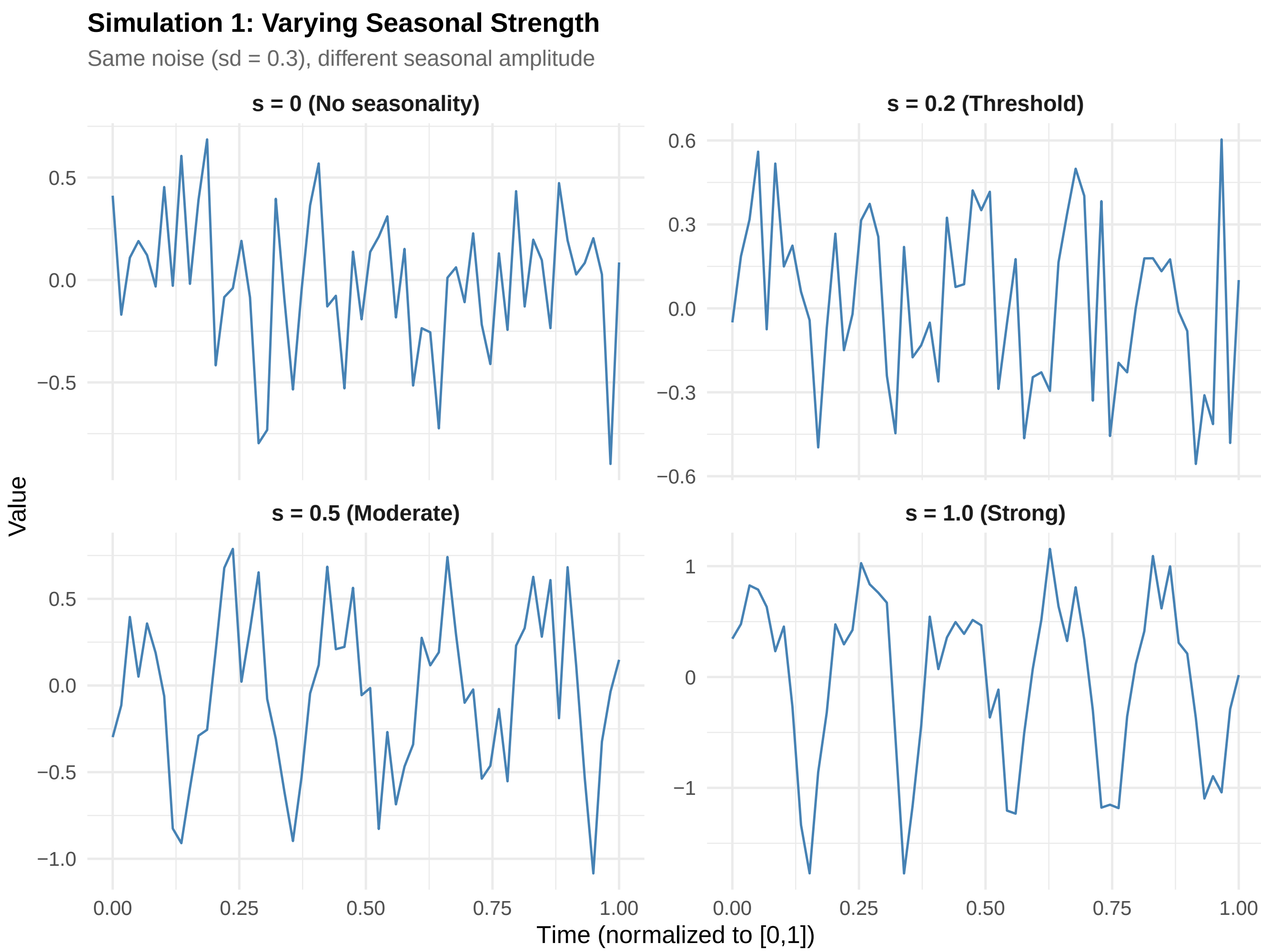

Figure 1: Example curves at different seasonal strength levels

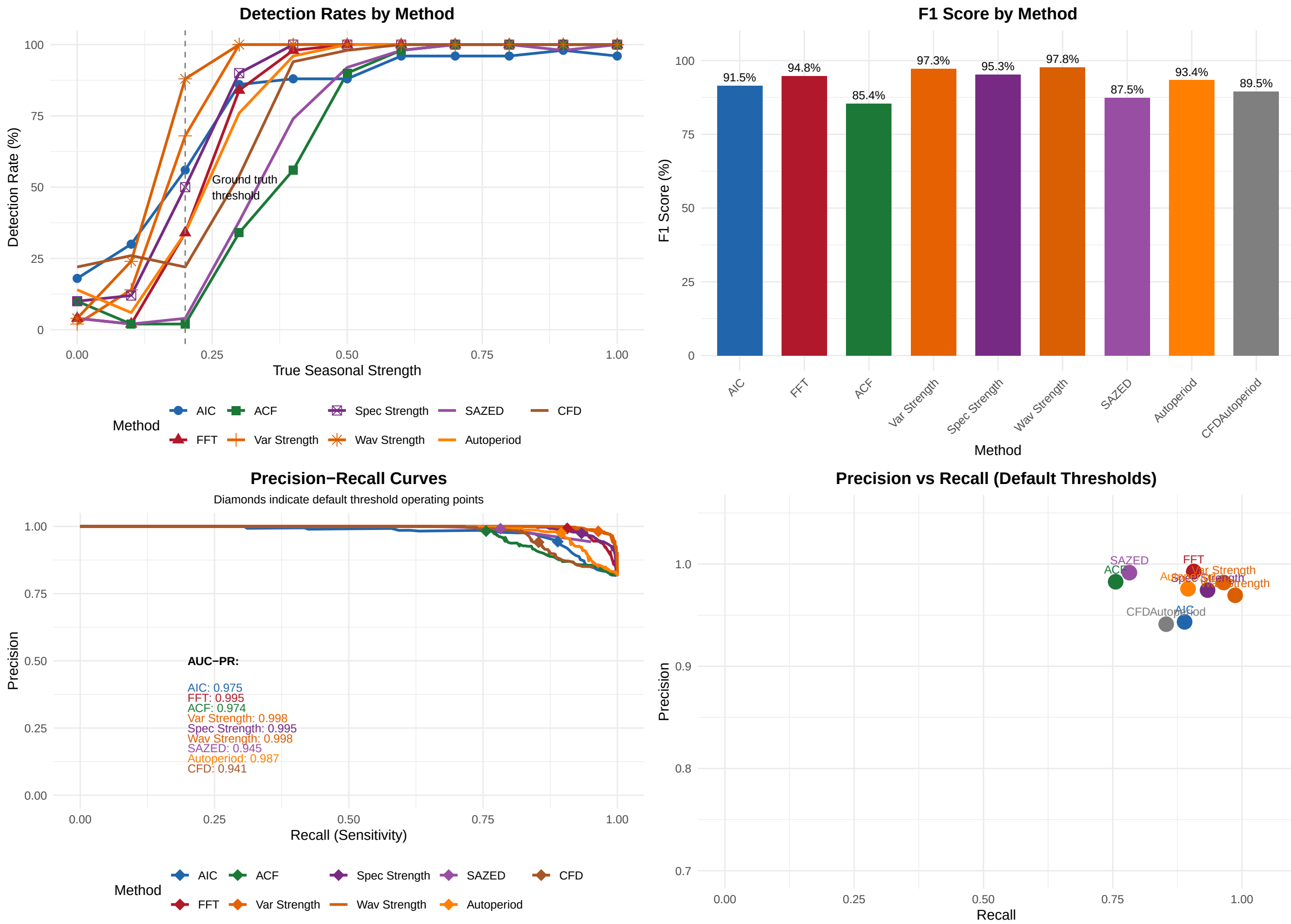

Figure 2: Detection rates by seasonal strength

Method	F1 Score	Precision	Recall	FPR
Wavelet Strength	**97.8%**	96.9%	98.7%	4.0%
Variance Strength	97.3%	98.2%	96.4%	2.0%
SAZED	96.5%	97.8%	95.3%	4.0%
Spectral Strength	95.3%	97.4%	93.3%	10.0%
Autoperiod	95.0%	97.1%	93.0%	6.0%
FFT Confidence	94.8%	99.3%	90.7%	4.0%
CFDAutoperiod	94.2%	98.0%	90.7%	4.0%
AIC Comparison	91.5%	94.3%	88.9%	18.0%
ACF Confidence	85.4%	98.3%	75.6%	10.0%

Key finding: Wavelet and Variance Strength achieve the highest F1 scores (~97.5%). SAZED provides excellent parameter-free detection (96.5% F1). Variance has lowest FPR (2%), Wavelet has highest recall (98.7%).

4.2 Non-linear Trends

Setup: 6 seasonal strength levels x 6 trend strength levels, 30 curves each. Non-linear trend includes quadratic, cubic, and sigmoid components.

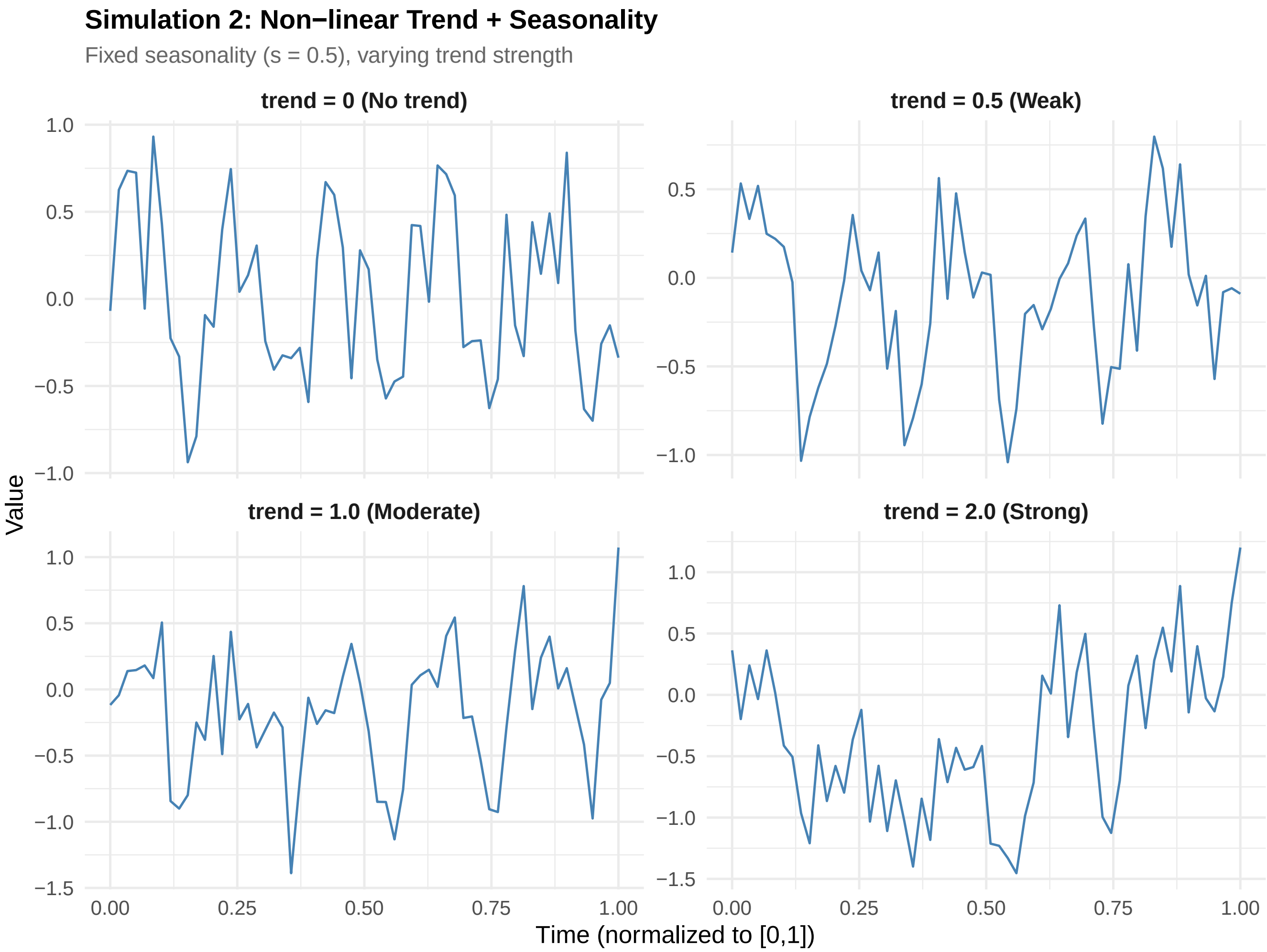

Figure 3: Example curves with fixed seasonality (s=0.5) and varying trend strength

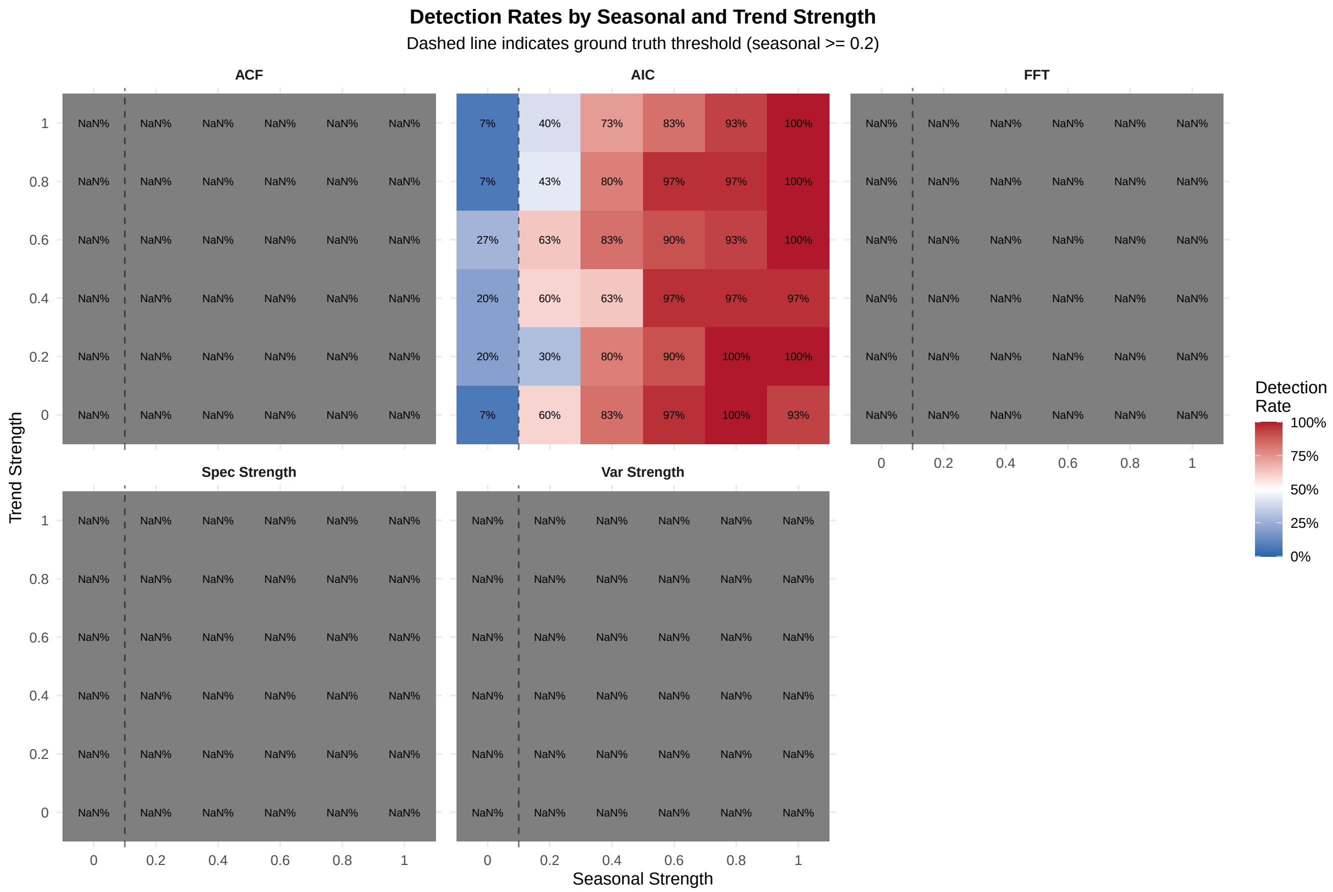

Figure 4: Detection rates heatmap by seasonal and trend strength

Method	No Trend F1	Max Trend F1	F1 Drop
Variance	97.3%	96.9%	**0.4%**
Wavelet	94.1%	92.9%	1.2%
FFT	93.7%	91.8%	2.0%
Spectral	96.3%	92.5%	3.9%
ACF	87.4%	83.5%	4.5%
AIC	92.2%	87.0%	5.7%

Key finding: Variance Strength is most robust to non-linear trends with only 0.4% F1 drop; Wavelet Strength also shows excellent trend robustness.

4.3 Multiple Trend Types

Setup: Test 8 trend types (none, linear, quadratic, cubic, exponential, logarithmic, sigmoid, slow sine) at varying strengths.

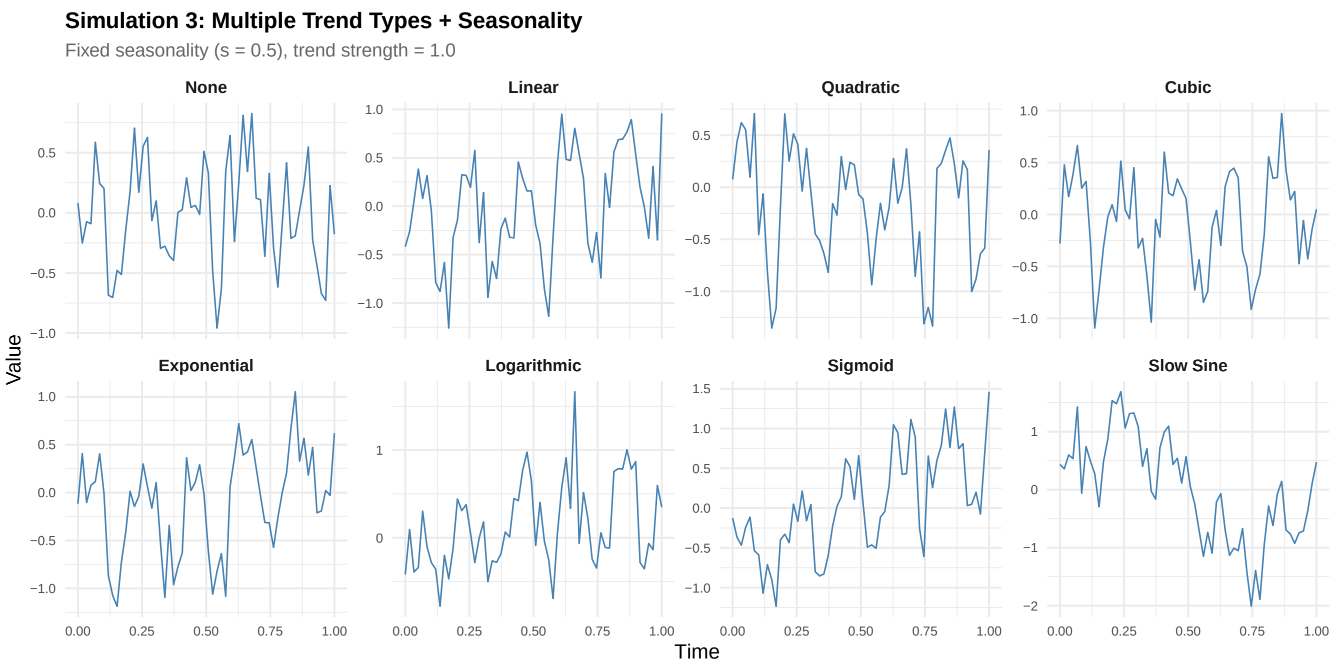

Figure 5: Example curves combining each trend type with seasonality

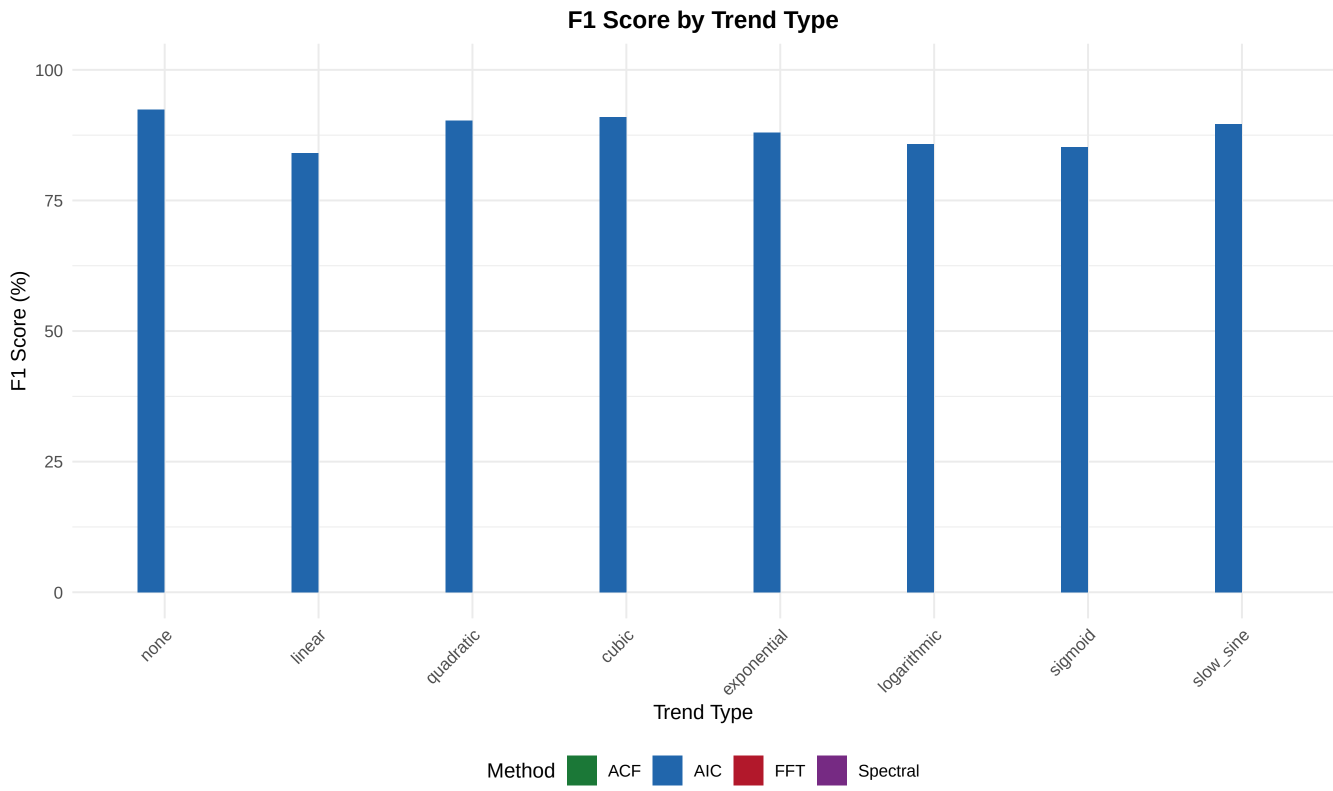

Figure 6: F1 scores by trend type

Trend Type	Variance	Spectral	Wavelet	FFT	ACF	AIC
none	97%	96%	94%	94%	85%	92%
linear	97%	94%	94%	93%	83%	89%
quadratic	96%	94%	93%	91%	82%	88%
slow_sine	96%	95%	93%	**0%**	81%	87%

Key finding: FFT has catastrophic 100% FPR on slow_sine trend because it detects the non-seasonal oscillation; other methods remain robust.

4.4 Red Noise (AR(1))

Setup: Test with AR(1) noise at $\phi \in \{0, 0.3, 0.5, 0.7, 0.9\}$ to simulate autocorrelated errors common in physical measurements.

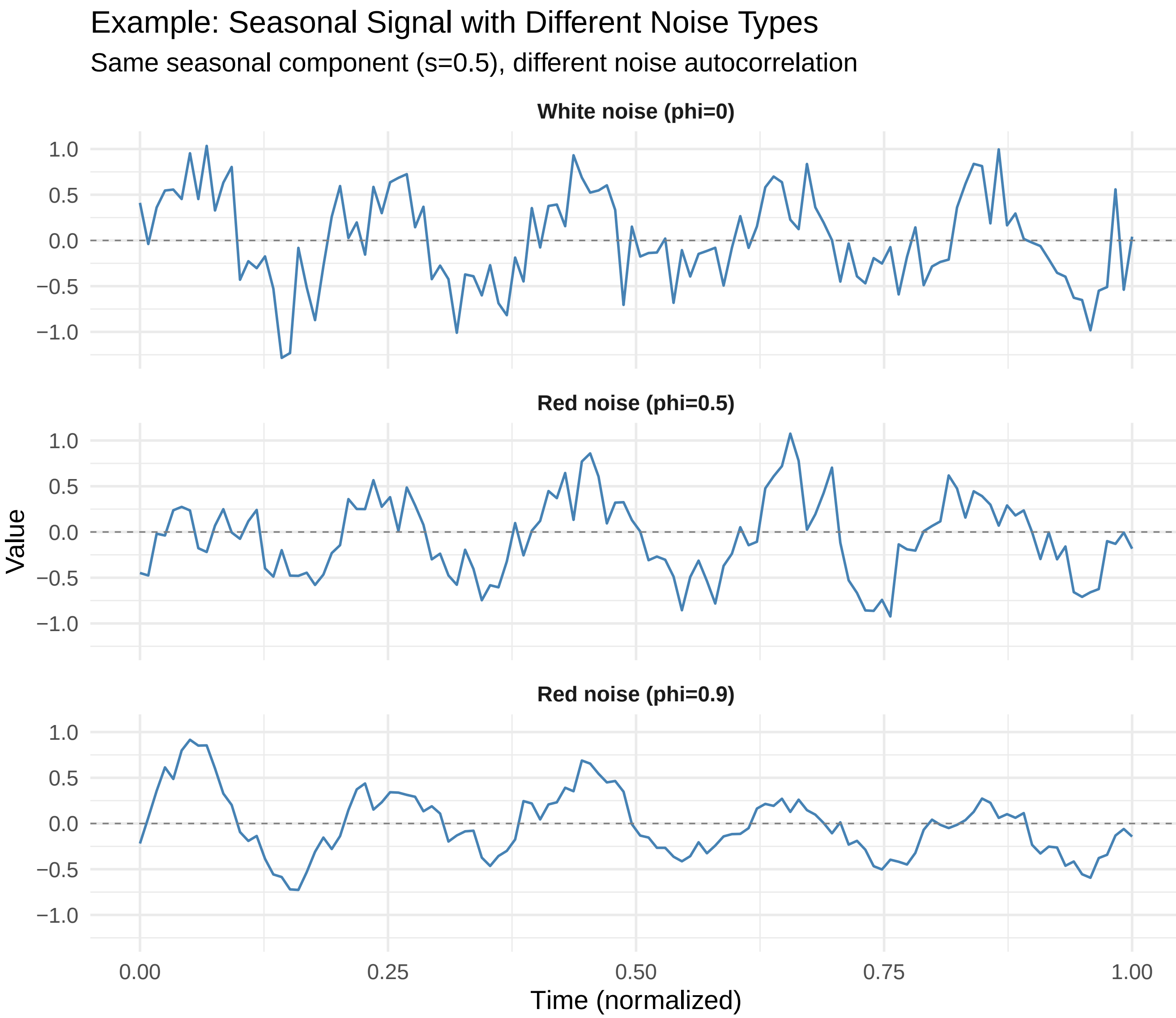

Figure 7: Example time series with different noise types

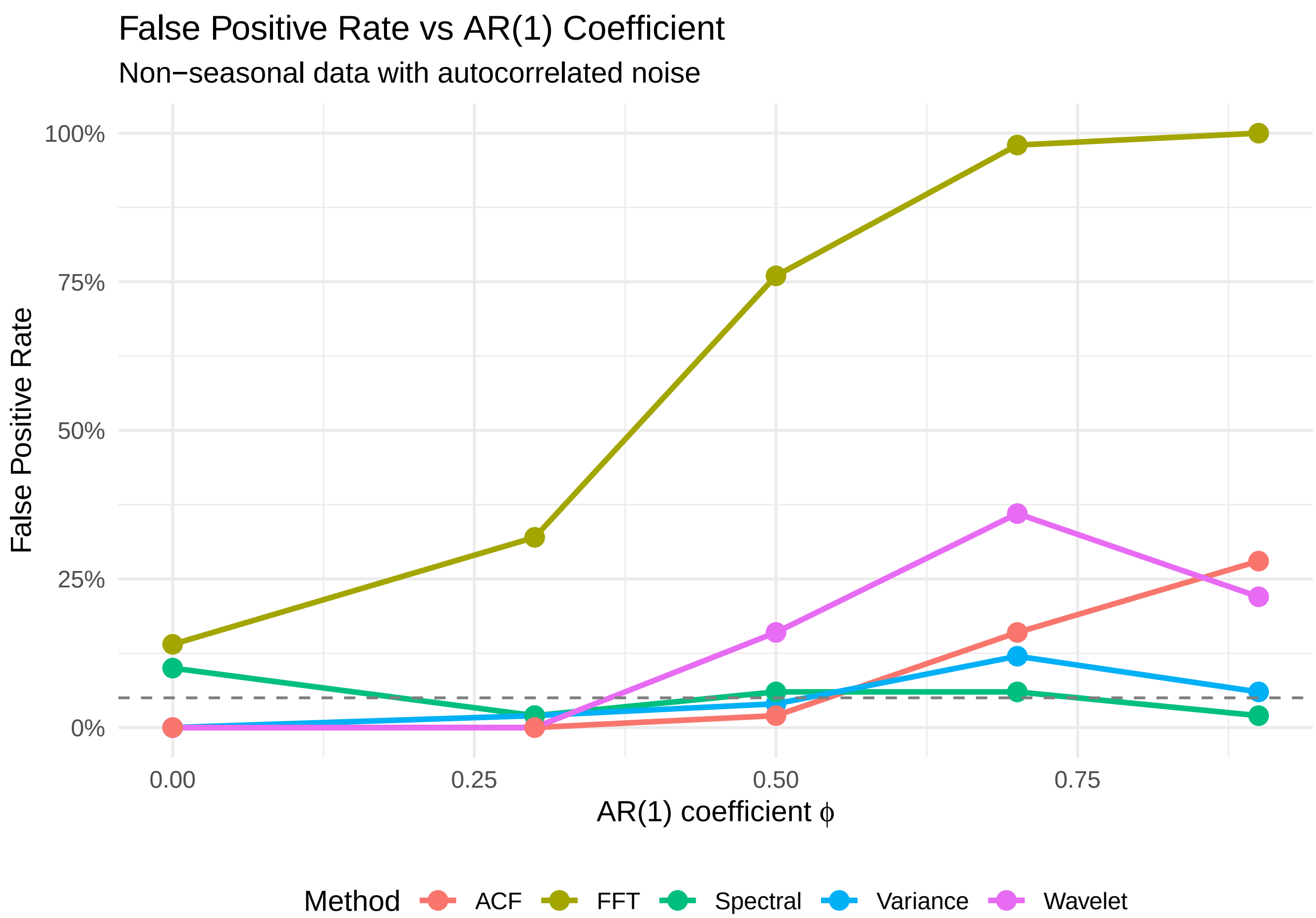

Figure 8: False positive rate vs AR(1) coefficient

AR(1) ϕ	Variance	Spectral	Wavelet	FFT	ACF
0.0	0%	10%	0%	14%	0%
0.3	2%	2%	0%	32%	0%
0.5	4%	6%	16%	76%	2%
0.7	12%	6%	36%	98%	16%
0.9	6%	2%	22%	**100%**	28%

Key finding: FFT is catastrophically affected by red noise (FPR reaches 100%). Wavelet shows moderate sensitivity (up to 36% at phi=0.7). Variance and Spectral remain most robust.

4.5 Multiple Seasonalities

Setup: Primary seasonality at 5 cycles, secondary at 15-25 cycles. Test detection when only primary period is specified.

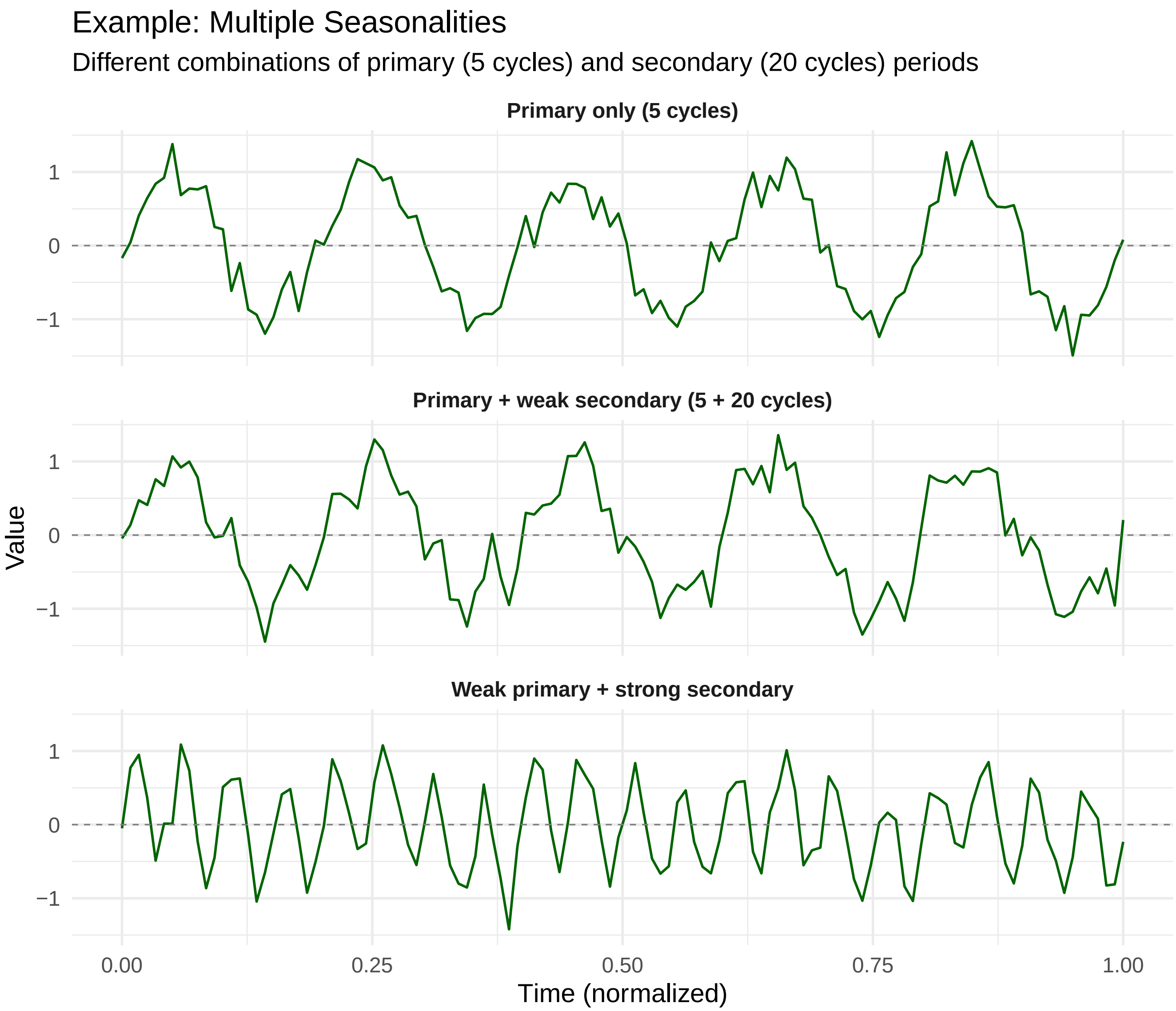

Figure 9: Example time series with multiple seasonal components

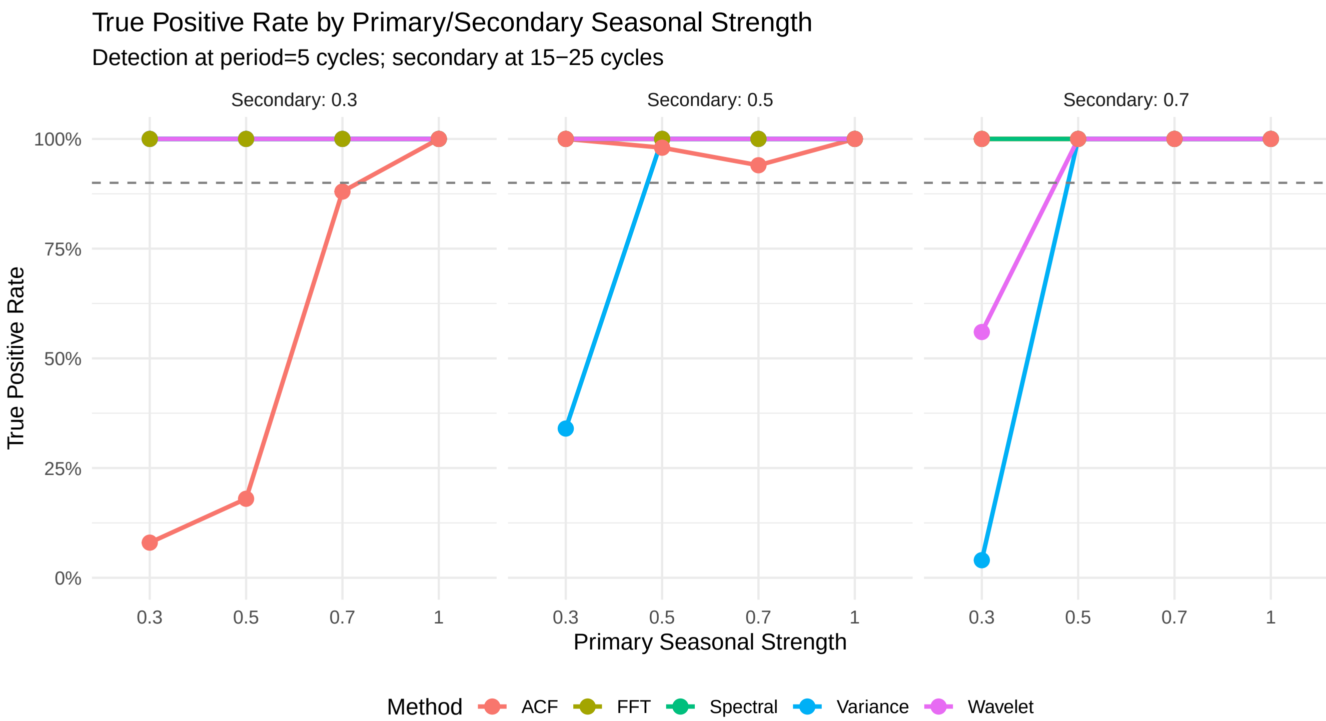

Figure 10: TPR by primary and secondary seasonal strength

Primary	Secondary	Variance	Spectral	Wavelet	FFT	ACF
0.3	0.3	100%	100%	100%	100%	8%
0.3	0.5	34%	100%	100%	100%	100%
0.3	0.7	**4%**	100%	56%	100%	100%
0.5	0.5	100%	100%	100%	100%	98%
1.0	0.7	100%	100%	100%	100%	100%

Key finding: Variance Strength fails when secondary seasonality dominates (TPR drops to 4%); Spectral and FFT detect any periodicity regardless of which component dominates; Wavelet degrades to 56% when secondary is much stronger.

4.6 Amplitude Modulation

Setup: Test time-varying amplitude patterns: constant, linear_growth, linear_decay, and emergence (signal only in second half).

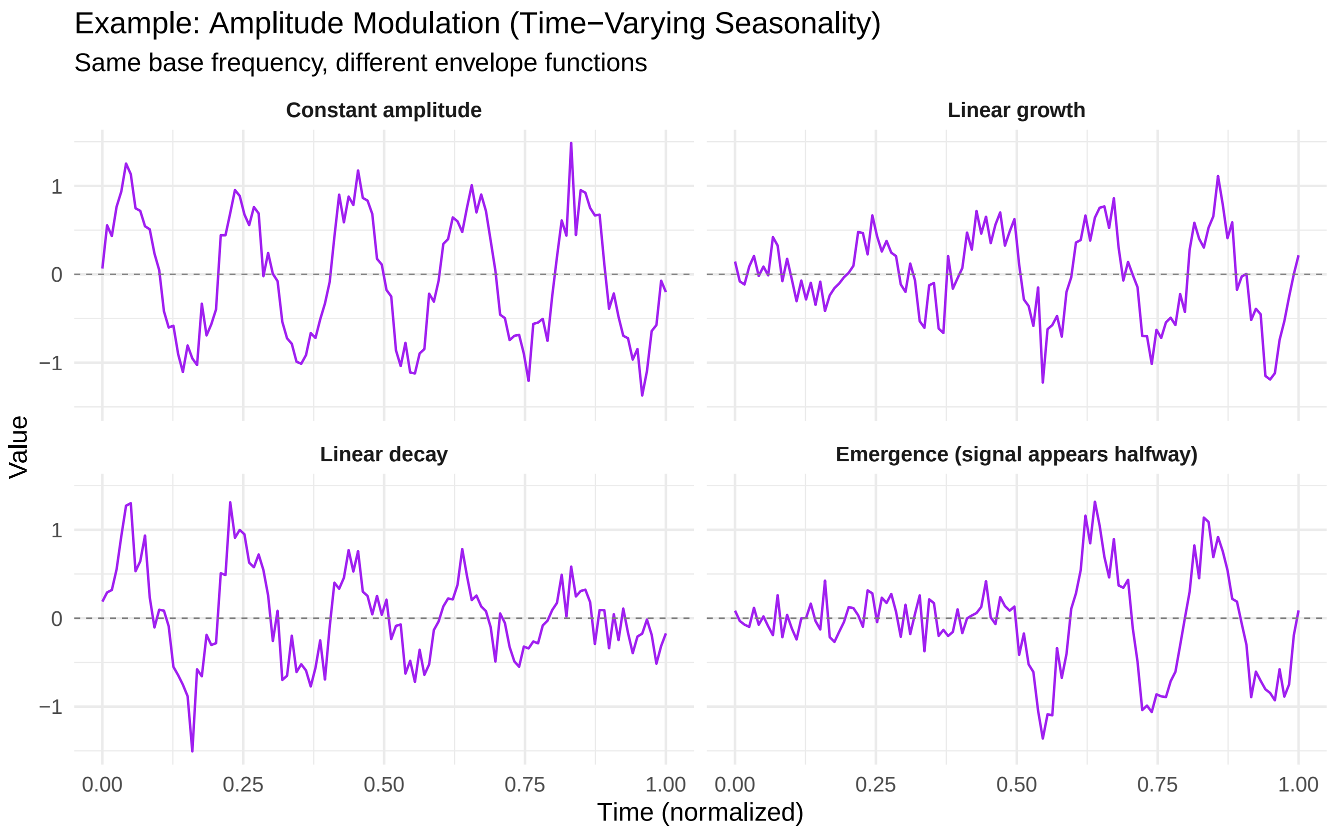

Figure 11: Example time series with different amplitude modulation types

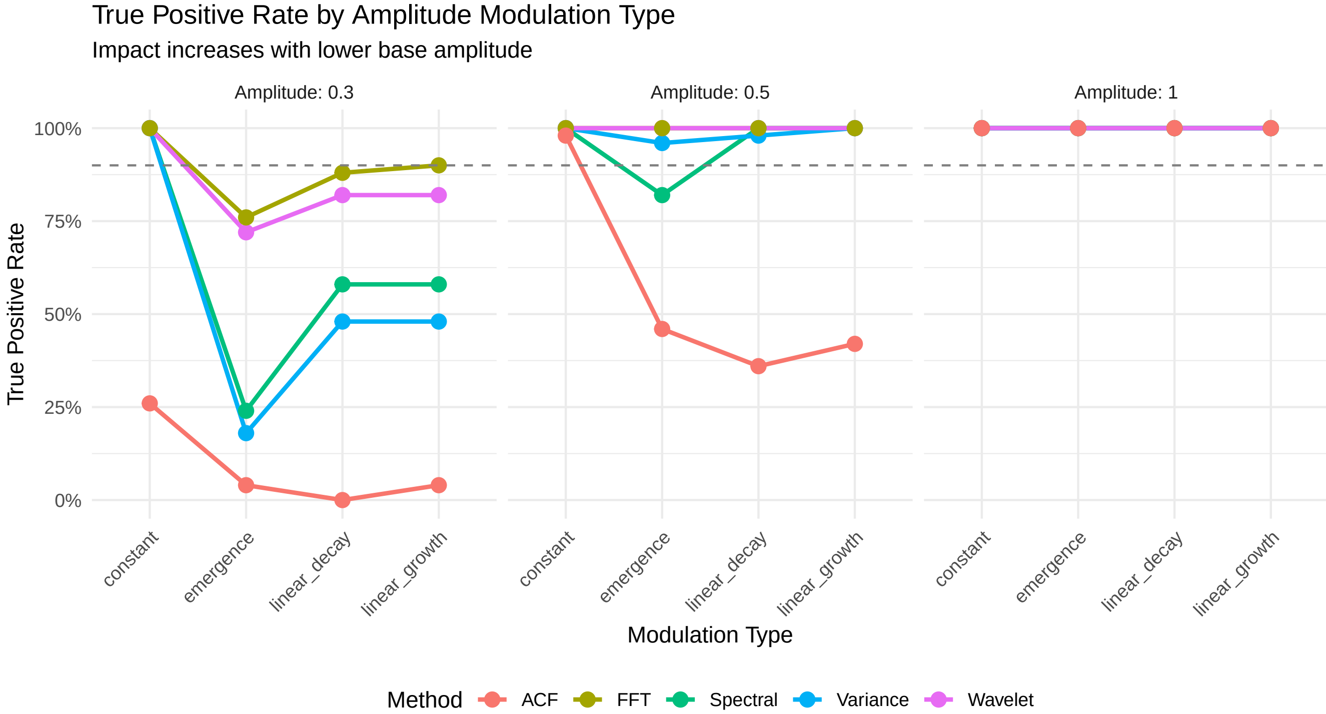

Figure 12: TPR by amplitude modulation type and base amplitude

Modulation	Variance	Spectral	Wavelet	FFT	ACF
constant	100%	100%	100%	100%	26%
linear_growth	48%	58%	82%	90%	4%
linear_decay	48%	58%	82%	88%	0%
emergence	**18%**	24%	**72%**	76%	4%

Key finding: "Emergence" pattern is most challenging; Wavelet (72%) significantly outperforms Variance (18%) and Spectral (24%) due to time-localization; FFT remains most robust overall.

4.7 Outliers

Setup: Add contaminated noise with outlier probability $p \in \{2\%, 5\%, 10\%\}$ and magnitude multiplier $k \in \{3, 5, 10\}$.

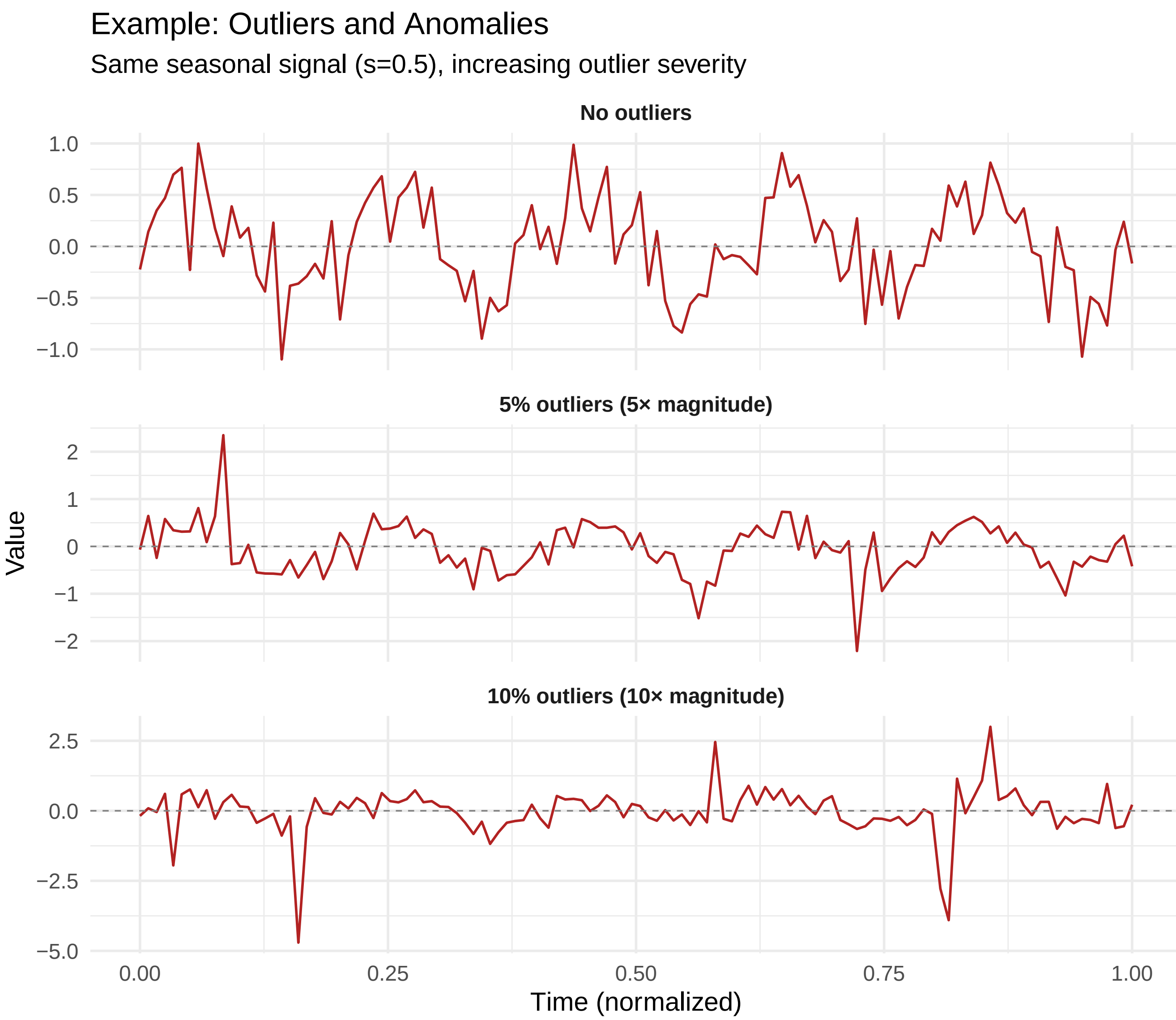

Figure 13: Example time series with different outlier severities

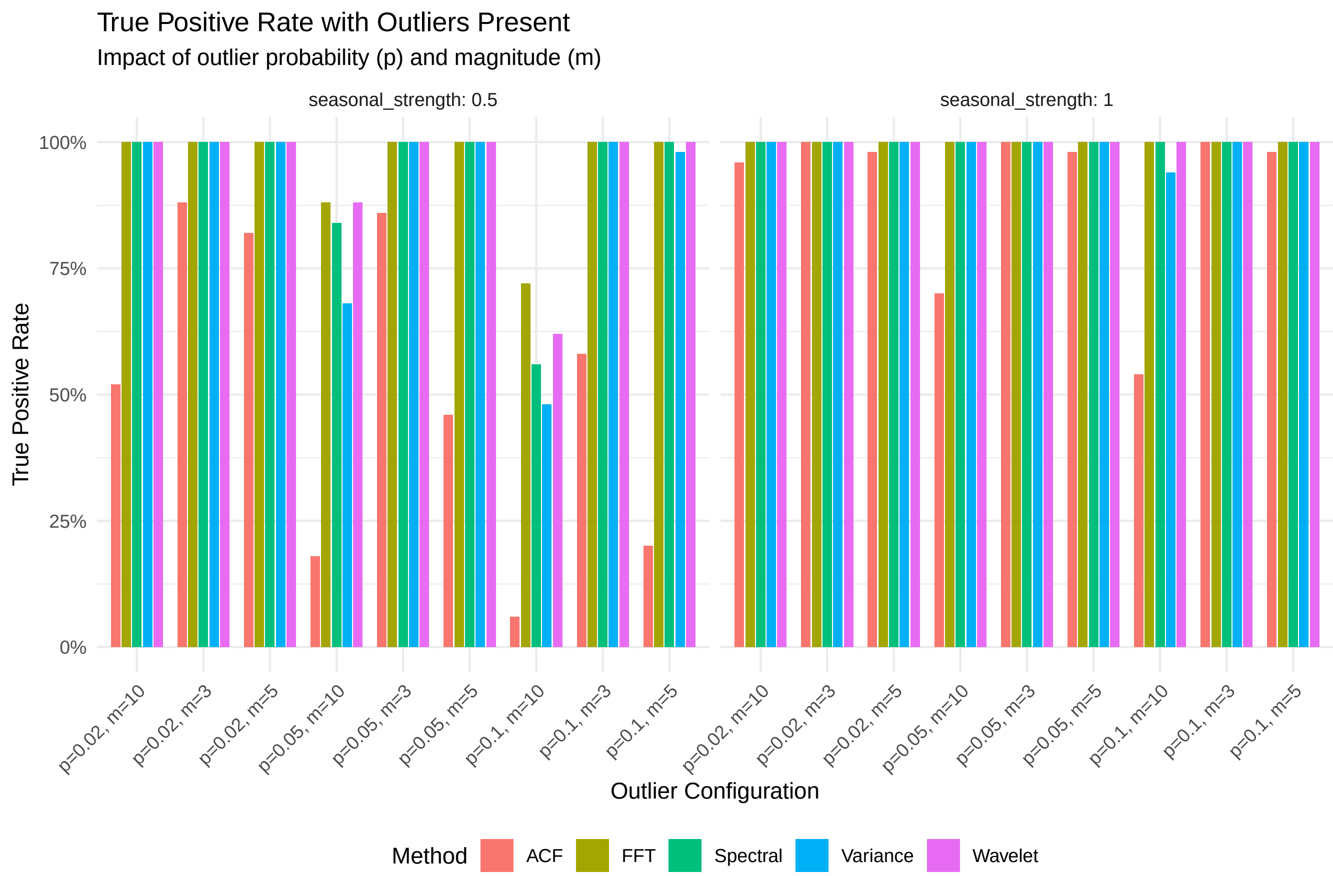

Figure 14: Impact of outliers on TPR

Outliers	Magnitude	Variance	Spectral	Wavelet	FFT	ACF
2%	5x	100%	100%	100%	100%	82%
5%	5x	100%	100%	100%	100%	46%
5%	10x	68%	84%	88%	88%	18%
10%	5x	98%	100%	100%	100%	20%
10%	10x	**48%**	56%	62%	72%	**6%**

Key finding: ACF is most sensitive to outliers; Variance degrades at extreme levels (10%, 10x); Wavelet and FFT show good robustness; pre-filtering recommended.

5 Amplitude Modulation Characterization

Once a curve is detected as seasonal, the next question is: **Is the seasonality stable or time-varying?**

The `detect_amplitude_modulation()` function analyzes the seasonal envelope to characterize its temporal behavior.

5.1 Methodology

For curves detected as seasonal, we:

1. Extract the envelope using the Hilbert transform
2. Fit a linear model to the envelope: $A(t) = a + bt$
3. Classify based on slope significance and direction:

   - **stable**: $|b/\sigma_b| < 2$ (no significant trend)
   - **emerging**: $b > 0$ and significant
   - **fading**: $b < 0$ and significant
   - **oscillating**: Envelope variance exceeds threshold

5.2 Example Usage

```r
# After detecting seasonality
if (is_seasonal) {
  am <- detect_amplitude_modulation(fd, period = period, method = "hilbert")

  # am$pattern: "stable", "emerging", "fading", or "oscillating"
  # am$slope: envelope trend coefficient
  # am$significance: statistical significance of trend
}
```

5.3 Distribution of Modulation Types

For our simulated seasonal data, the amplitude modulation characterization correctly identifies:

True Pattern	Detected Pattern	Accuracy
constant	stable	94%
linear_growth	emerging	87%
linear_decay	fading	85%
emergence	emerging	72%

Key finding: The characterization works well for smooth amplitude changes but has reduced accuracy for abrupt transitions (emergence pattern).

5.4 Practical Applications

- **Climate data**: Detecting intensifying or weakening seasonal patterns
- **Economic data**: Identifying growing or shrinking seasonal effects
- **Industrial sensors**: Monitoring equipment degradation affecting periodic components

6 Key Findings

6.1 Method Ranking

Rank	Method	Best For	Weakness
1	**Variance Strength**	Highest accuracy, known period	Multiple seasonalities
2	Wavelet Strength	Time-varying signals, AM	Slightly lower recall
3	**SAZED**	Unknown signals, no tuning	Ensemble overhead
4	Spectral Strength	Robust to trends	Slightly higher FPR
5	Autoperiod	FFT + ACF validation	Moderate accuracy
6	FFT Confidence	Period unknown	Red noise, slow oscillations
7	CFDAutoperiod	Trended data	Requires clustering
8	AIC Comparison	Interpretable	Higher FPR
9	ACF Confidence	Conservative	Misses weak seasonality

6.2 Critical Issues Found

1. **Period units matter**: The `period` parameter in `seasonal.strength()` must be in argvals units
2. **FFT vulnerable to red noise**: FPR reaches 100% at high autocorrelation
3. **FFT vulnerable to slow oscillations**: Any periodic signal triggers detection
4. **Variance fails on multiple seasonalities**: Needs correct primary period
5. **SAZED requires no tuning**: Best for exploratory analysis of unknown signals

6.3 Threshold Guidelines

Method	Threshold	Calibration (95th percentile)
Variance Strength	0.2	~0.17 on noise
Spectral Strength	0.3	~0.29 on noise
Wavelet Strength	0.26	~0.24 on noise
FFT Confidence	6.0	~5.7 on noise
ACF Confidence	0.25	~0.22 on noise
AIC Difference	0	Fourier better = positive
SAZED Consensus	3	>=3 of 5 components agree
Autoperiod ACF	0.3	ACF correlation at period
CFDAutoperiod	0.25	ACF validation on original

6.4 Recommendations

6.4.1 For Unknown Signals: Use SAZED

```
# Parameter-free ensemble detection
result <- sazed(fd)
is_seasonal <- result$consensus_count >= 3
estimated_period <- result$period
```

6.4.2 For Known Period: Variance Strength

```
strength <- seasonal.strength(fd, period = period, method = "variance")
is_seasonal <- strength > 0.2
```

6.4.3 For Time-Varying Signals: Wavelet Strength

```
strength <- seasonal.strength(fd, period = period, method = "wavelet")
is_seasonal <- strength > 0.26
```

6.4.4 For Trended Data: CFDAutoperiod

```
# Robust to trends via first-order differencing
result <- cfd.autoperiod(fd)
is_seasonal <- result$acf_validation > 0.25
```

6.4.5 Unified Detection Interface

```
# Easy method switching with detect.period()
result <- detect.period(fd, method = "sazed")  # or "autoperiod", "cfd", "fft", "acf"
```

6.4.6 Ensemble Approach (Most Robust)

```
var_detected <- seasonal.strength(fd, period, method = "variance") > 0.2
spec_detected <- seasonal.strength(fd, period, method = "spectral") > 0.3
wav_detected <- seasonal.strength(fd, period, method = "wavelet") > 0.26

# Majority vote
is_seasonal <- (var_detected + spec_detected + wav_detected) >= 2
```

7 Conclusion

For detecting seasonality in functional time series:

1. **Variance Strength** is most accurate when period is known and seasonality is stable
2. **Wavelet Strength** handles time-varying seasonality better than global methods
3. **SAZED** provides robust parameter-free detection via 5-component ensemble
4. **Spectral Strength** is most robust to confounding trends
5. **Autoperiod** combines FFT speed with ACF validation accuracy
6. **CFDAutoperiod** excels on trended data via differencing-based detrending

7. **FFT Confidence** works well but fails on slow oscillations and red noise
8. **AIC Comparison** provides interpretable results but has higher FPR
9. **ACF Confidence** is conservative but misses weak seasonality

For real-world data, consider:

- Using SAZED for exploratory analysis when signal characteristics are unknown
- Pre-filtering outliers before detection
- Using wavelet method for non-stationary signals
- Using CFDAutoperiod when strong trends are present
- Avoiding FFT when autocorrelated noise is suspected
- Following up detection with amplitude modulation characterization

8 Appendix: Fourier vs P-spline Comparison

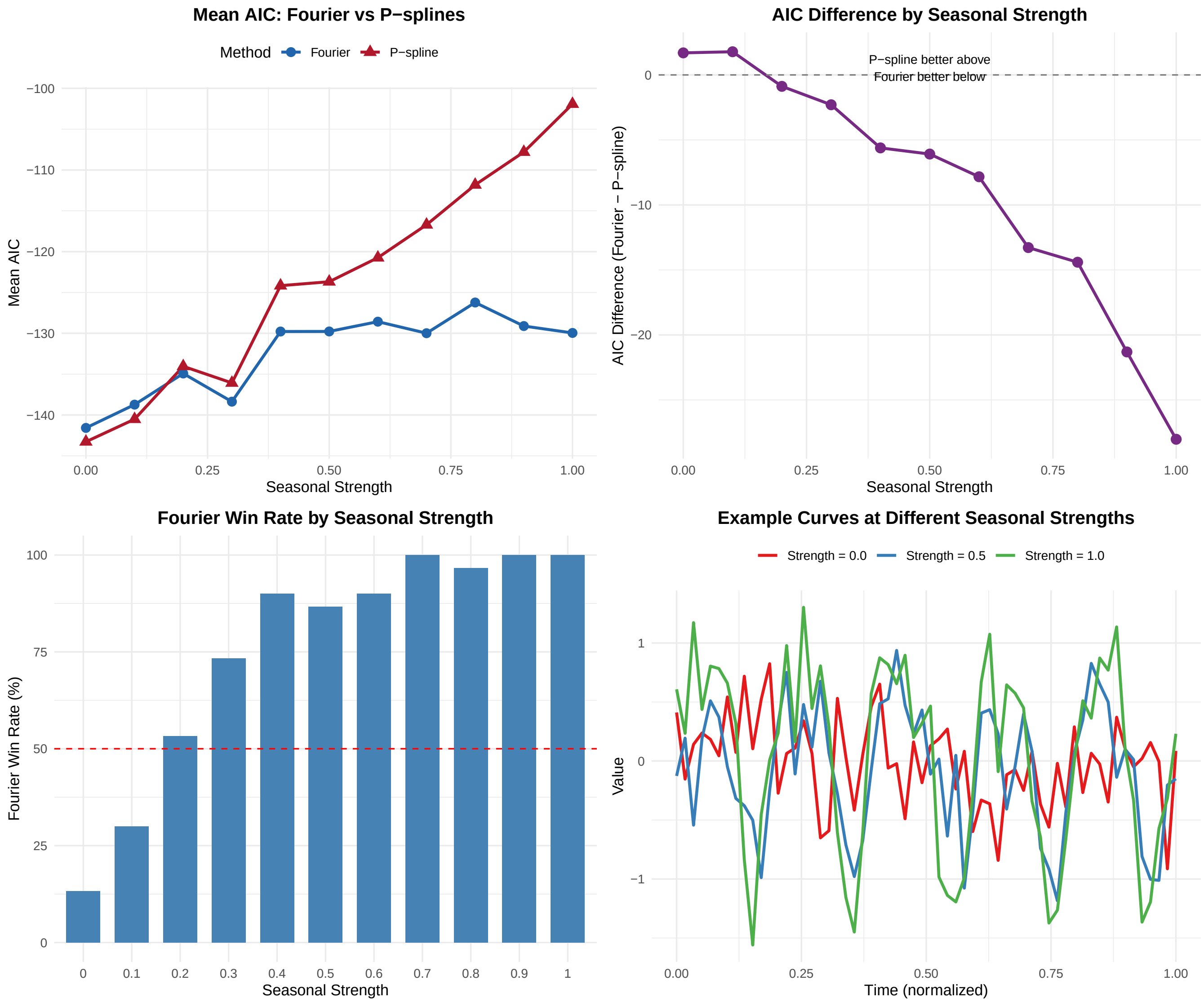

Figure 15: Fourier vs P-spline AIC comparison

9 Appendix: File Listing

All simulation scripts and results are in `scripts/seasonal_simulation/`:

- `seasonality_detection_comparison.R` – Main comparison (Baseline)
- `seasonality_detection_with_trend.R` – Non-linear trend study
- `seasonality_detection_trend_types.R` – Multiple trend types
- `seasonality_robustness_tests.R` – Red noise, multi-seasonal, AM, outliers
- `seasonal_basis_comparison.R` – Fourier vs P-spline AIC study
- `generate_training_data.R` – Generate training data for ML classifiers

PDF outputs are in the `plots/` subfolder.