Executive Summary

fdars Package

2026-01-01

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1 Key Findings

This study compared six methods for detecting seasonality in functional time series data across 550+ simulated curves with varying seasonal strengths and trend components.

Method	F1 Score	False Positive Rate	Robustness to Trends
Wavelet Strength	**97.8%**	4%	Excellent (1.2% F1 drop)
Variance Strength	97.3%	**2%**	Excellent (0.4% F1 drop)
Spectral Strength	95.3%	10%	Good (3.9% F1 drop)
FFT Confidence	94.8%	4%	Good (2.0% F1 drop)
AIC Comparison	91.5%	18%	Moderate (5.7% F1 drop)
ACF Confidence	85.4%	10%	Moderate (4.5% F1 drop)

Top methods: Wavelet (97.8% F1, best recall) and Variance (97.3% F1, lowest FPR) are both excellent choices for seasonality detection.

2 Robustness to Real-World Challenges

Challenge	Most Affected	Wavelet Advantage
Red Noise (AR(1))	FFT (100% FPR)	Moderate (up to 36% FPR)
Multiple Seasonalities	Variance (4% TPR)	Good (56% TPR)
Amplitude Modulation	Variance (18% TPR)	**Excellent (72% TPR)**
Outliers (10%, 10x)	ACF (6% TPR)	Good (62% TPR)

Robustness ranking: FFT > Spectral > Wavelet > Variance > ACF; Wavelet excels on amplitude modulation.

3 Recommendations

3.1 Primary Recommendation: Use Variance Strength

```r
# Detect seasonality with Variance Strength method
period <- 0.2  # Period in argvals units (e.g., 1/5 for 5 cycles in [0,1])
strength <- seasonal_strength(fd, period = period, method = "variance", detrend = "linear")
is_seasonal <- strength > 0.2
```

3.2 When Period is Unknown: Two-Step Approach

```r
# Step 1: Estimate period using FFT (no period required)
result <- estimate_period(fd, method = "fft", detrend = "linear")
estimated_period <- result$period

# Step 2: Measure strength with estimated period
strength <- seasonal_strength(fd, period = estimated_period, method = "variance")
is_seasonal <- strength > 0.2
```

3.3 Critical Notes

1. **Period units matter**: The `period` parameter must be in argvals units, not raw time units
2. **Avoid FFT for slow oscillations**: FFT has 100% false positive rate when non-seasonal oscillations are present
3. **Thresholds are calibrated**: All thresholds target ~5% false positive rate on pure noise

Introduction

This report describes and compares six methods for detecting seasonality in functional time series data. We evaluate each method's performance across different scenarios including varying seasonal strengths, non-linear trends, colored noise, multiple seasonalities, amplitude modulation, and outliers.

The goal is to answer: **Given a time series, how can we reliably determine if it contains a seasonal pattern?**

Report Structure:

- Section **??** – Six detection methods and their mathematical formulations
- Section **??** – Unified simulation study covering seven scenarios
- Section **??** – Post-detection amplitude modulation characterization
- Section **??** – Key findings and recommendations

Detection Methods

4 AIC Comparison (Fourier vs P-spline)

Concept: If data is seasonal, a Fourier basis should fit better than P-splines because Fourier bases naturally capture periodic patterns.

Mathematical formulation:

For a curve $y(t)$, we fit two models:

1. **Fourier basis**: $\hat{y}(t) = \sum_{k=0}^{K} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$
2. **P-spline**: $\hat{y}(t) = \sum_{j=1}^{J} c_j B_j(t)$ with penalty $\lambda \int [\hat{y}''(t)]^2 dt$

We compute AIC for each:

$$\text{AIC} = n \log(\text{RSS}/n) + 2 \cdot \text{edf}$$

where RSS is the residual sum of squares and edf is the effective degrees of freedom.

Detection rule: Seasonality detected if $\text{AIC}_{\text{P-spline}} - \text{AIC}_{\text{Fourier}} > 0$

Interpretation: When Fourier has lower AIC, the periodic structure is significant enough to justify the global periodic assumption over the local flexibility of splines.

5 FFT Confidence

Concept: Use Fast Fourier Transform to detect dominant frequencies. Strong peaks in the periodogram indicate periodic components.

Mathematical formulation:

Given a time series $y_1, y_2, \ldots, y_n$, compute the discrete Fourier transform:

$$Y_k = \sum_{j=1}^{n} y_j e^{-2\pi i(j-1)(k-1)/n}$$

The periodogram (power spectrum) is:

$$P_k = |Y_k|^2$$

Detection score:

$$\text{Confidence} = \frac{\max_k P_k}{\text{mean}(P_k)}$$

Detection rule: Seasonality detected if Confidence > 6.0

Interpretation: A high ratio indicates one frequency dominates, suggesting periodicity rather than random noise.

6 ACF Confidence

Concept: Autocorrelation at the seasonal lag should be high for seasonal data.

Mathematical formulation:

The autocorrelation function at lag h is:

$$\rho_h = \frac{\sum_{t=1}^{n-h}(y_t - \bar{y})(y_{t+h} - \bar{y})}{\sum_{t=1}^{n}(y_t - \bar{y})^2}$$

For seasonal data with period p, we expect ρ_p to be significantly positive.

Detection rule: Seasonality detected if ACF confidence > 0.25

Interpretation: High autocorrelation at the seasonal lag indicates the pattern repeats.

7 Variance Strength

Concept: Decompose variance into seasonal and residual components. High seasonal variance ratio indicates seasonality.

Mathematical formulation:

Decompose the series: $y_t = T_t + S_t + R_t$ (trend + seasonal + residual)

The seasonal strength is:

$$\text{SS}_{\text{var}} = 1 - \frac{\text{Var}(R_t)}{\text{Var}(y_t - T_t)}$$

Detection rule: Seasonality detected if $\text{SS}_{\text{var}} > 0.2$

Interpretation: Values close to 1 mean the seasonal component dominates; values close to 0 mean residual noise dominates.

Important: The `period` parameter must be in the same units as `argvals`. For data normalized to [0,1] with 5 annual cycles, use `period = 0.2`.

8 Spectral Strength

Concept: Measure the proportion of spectral power at the seasonal frequency.

Mathematical formulation:

Using the periodogram P_k, identify the seasonal frequency $f_s = 1/\text{period}$.

$$\text{SS}_{\text{spectral}} = \frac{\sum_{k \in \mathcal{S}} P_k}{\sum_k P_k}$$

where $\mathcal{S}$ includes the seasonal frequency and its harmonics.

Detection rule: Seasonality detected if $\text{SS}_{\text{spectral}} > 0.3$

Interpretation: High values indicate spectral energy is concentrated at seasonal frequencies.

9 Wavelet Strength

Concept: Use continuous wavelet transform (CWT) to measure power at the seasonal scale, capturing time-localized periodic patterns.

Mathematical formulation:

Using the Morlet wavelet $\psi_0(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-t^2/2}$ with $\omega_0 = 6$, compute the CWT at scale $s = \text{period} \cdot \omega_0/(2\pi)$:

$$W(s, \tau) = \int y(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t - \tau}{s} \right) dt$$

The wavelet strength is:

$$\text{SS}_{\text{wavelet}} = \sqrt{\frac{\text{mean}(|W(s,\tau)|^2)}{\text{Var}(y)}}$$

Detection rule: Seasonality detected if $\text{SS}_{\text{wavelet}} > 0.26$

Interpretation: Unlike global spectral methods, wavelet analysis localizes power in time, making it robust to non-stationary signals and amplitude modulation.

Advantages:

- Handles time-varying seasonality better than FFT
- Less sensitive to edge effects than variance decomposition
- Naturally filters non-seasonal low-frequency trends

Simulation Study

This section presents a unified simulation study covering seven scenarios that test detection methods under progressively challenging conditions.

10 Baseline: Varying Seasonal Strength

Setup: Test detection across 11 seasonal strength levels (0.0 to 1.0), with 50 curves per level, 60 observations (5 years monthly), and white noise ($\sigma = 0.3$). Ground truth: seasonal if $s \geq 0.2$.

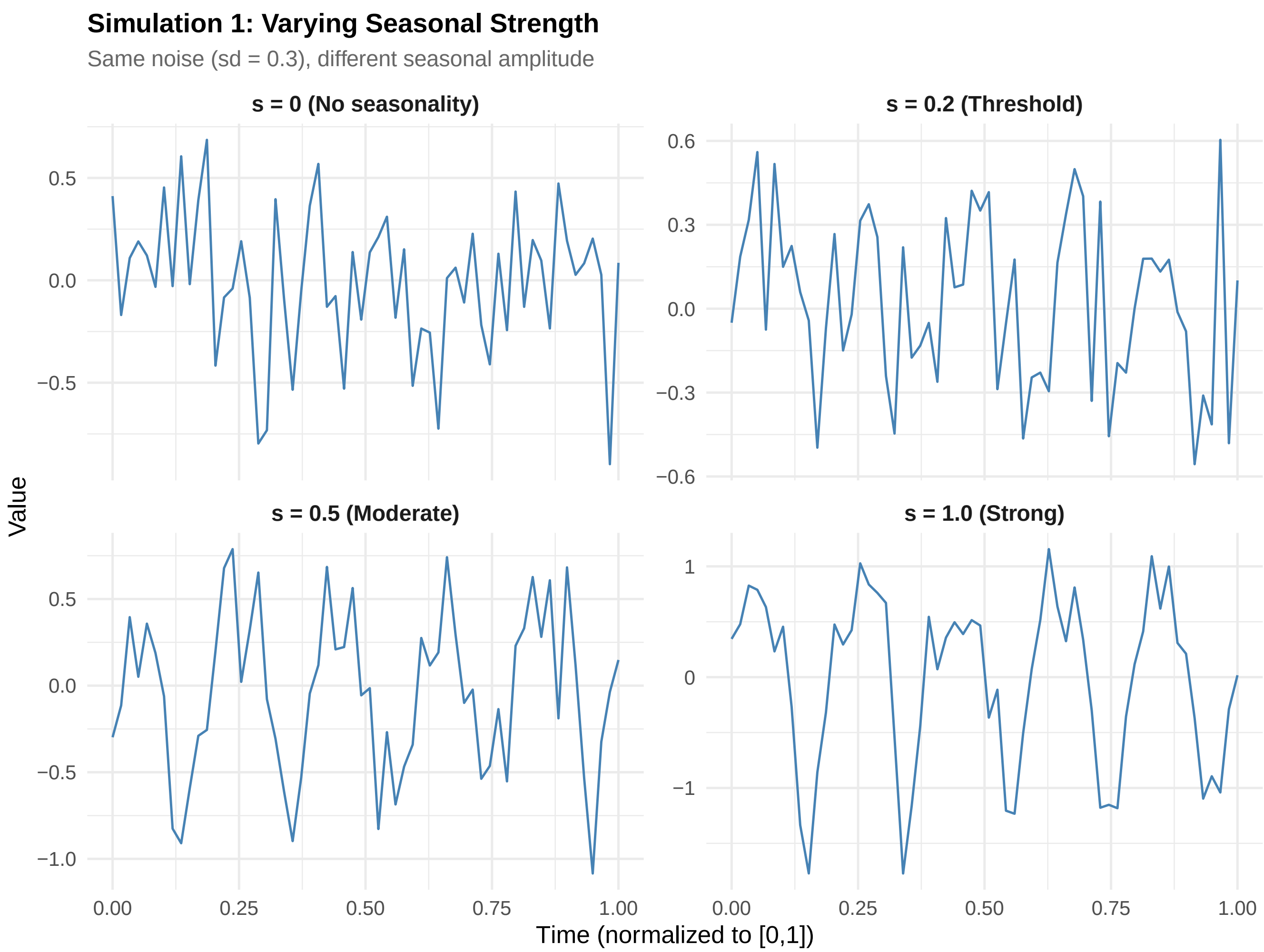

Figure 1: Example curves at different seasonal strength levels

Method	F1 Score	Precision	Recall	FPR
Wavelet Strength	**97.8%**	96.9%	98.7%	4.0%
Variance Strength	97.3%	98.2%	96.4%	2.0%
Spectral Strength	95.3%	97.4%	93.3%	10.0%
FFT Confidence	94.8%	99.3%	90.7%	4.0%
AIC Comparison	91.5%	94.3%	88.9%	18.0%
ACF Confidence	85.4%	98.3%	75.6%	10.0%

Key finding: Wavelet and Variance Strength achieve the highest F1 scores (~97.5%). Variance has lowest FPR (2%), Wavelet has highest recall (98.7%).

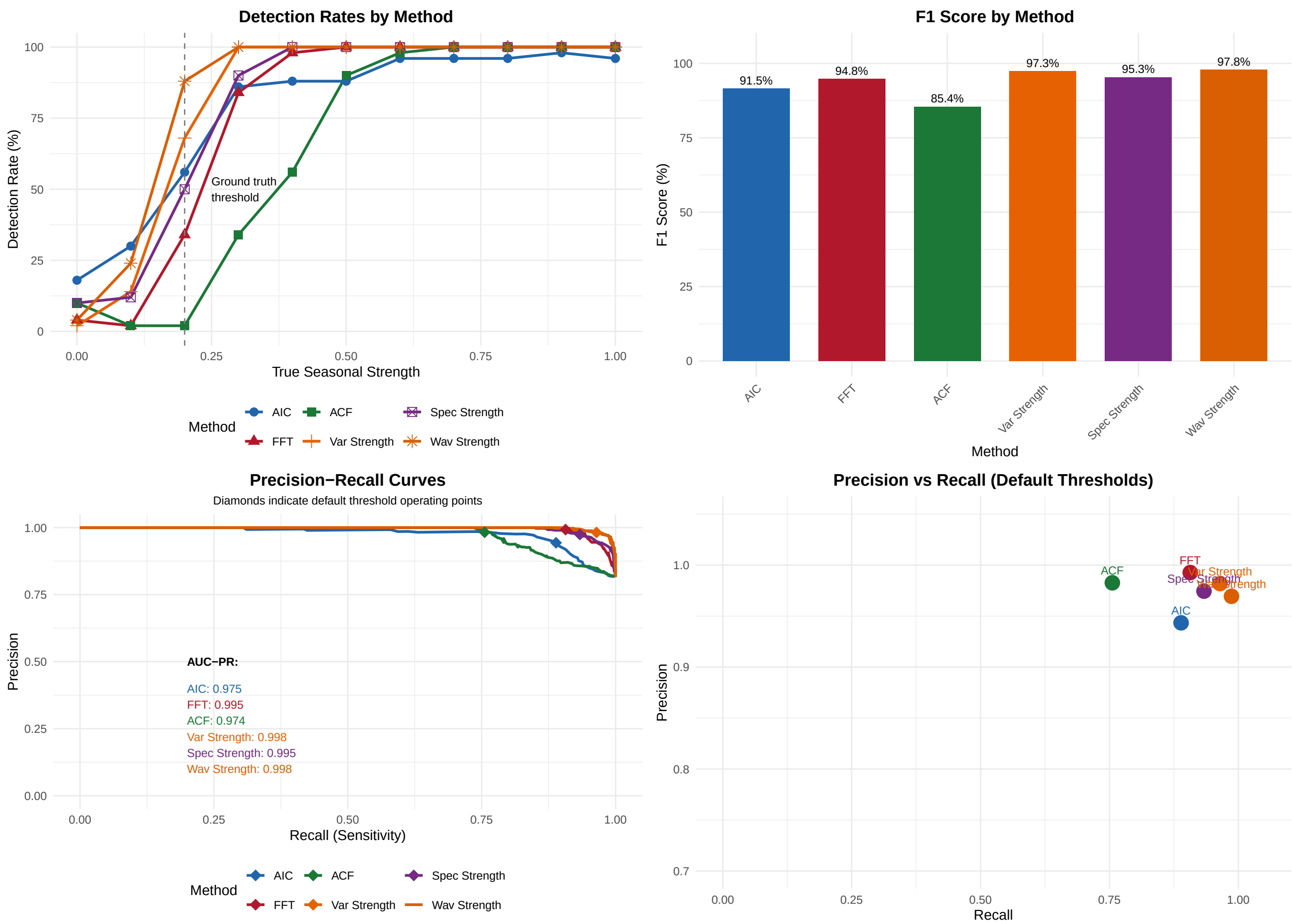

Figure 2: Detection rates by seasonal strength

11 Non-linear Trends

Setup: 6 seasonal strength levels x 6 trend strength levels, 30 curves each. Non-linear trend includes quadratic, cubic, and sigmoid components.

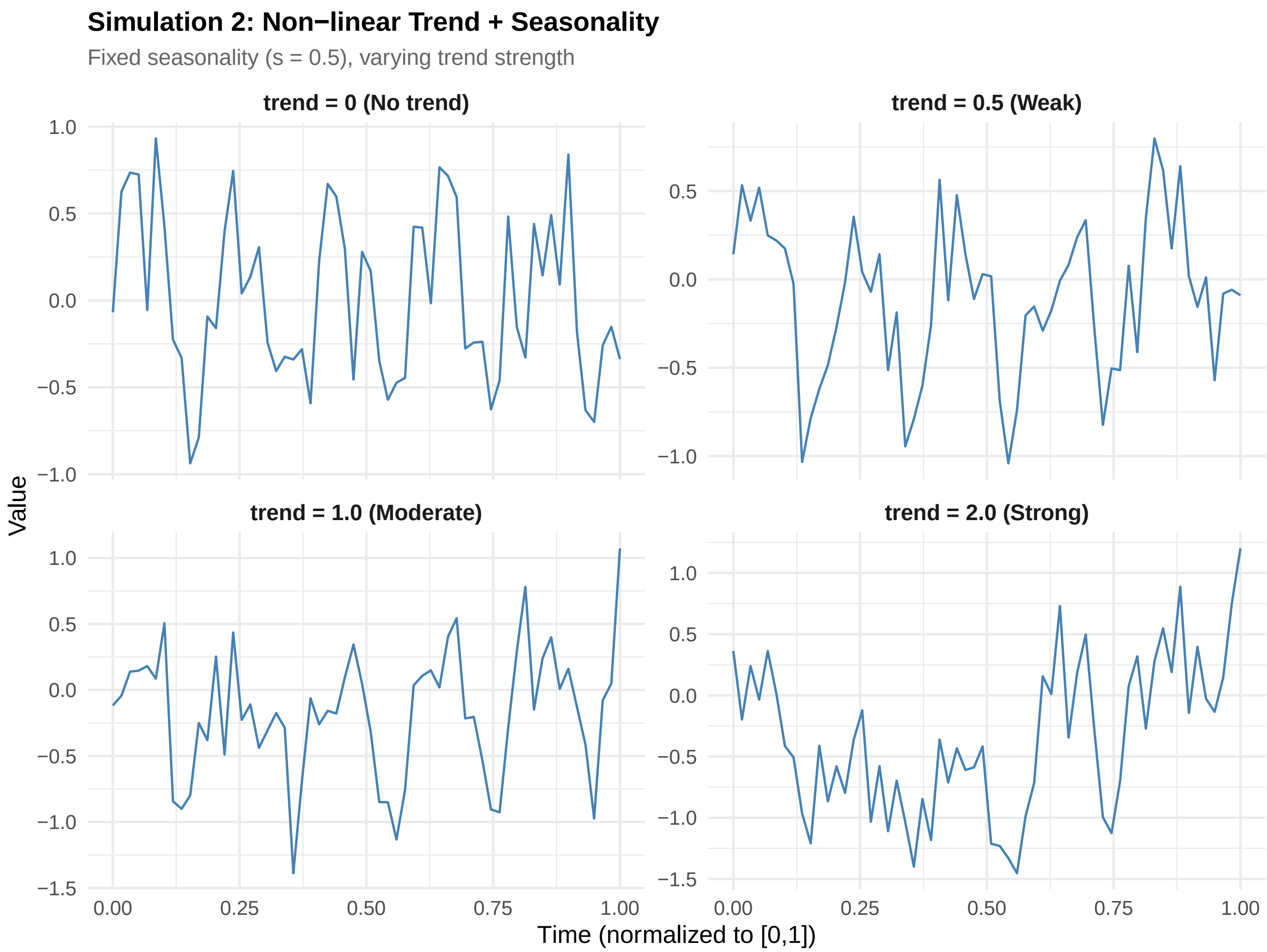

Figure 3: Example curves with fixed seasonality (s=0.5) and varying trend strength

Method	No Trend F1	Max Trend F1	F1 Drop
Variance	97.3%	96.9%	**0.4%**
Wavelet	94.1%	92.9%	1.2%
FFT	93.7%	91.8%	2.0%
Spectral	96.3%	92.5%	3.9%
ACF	87.4%	83.5%	4.5%
AIC	92.2%	87.0%	5.7%

Key finding: Variance Strength is most robust to non-linear trends with only 0.4% F1 drop; Wavelet Strength also shows excellent trend robustness.

Figure 4: Detection rates heatmap by seasonal and trend strength

12 Multiple Trend Types

Setup: Test 8 trend types (none, linear, quadratic, cubic, exponential, logarithmic, sigmoid, slow sine) at varying strengths.

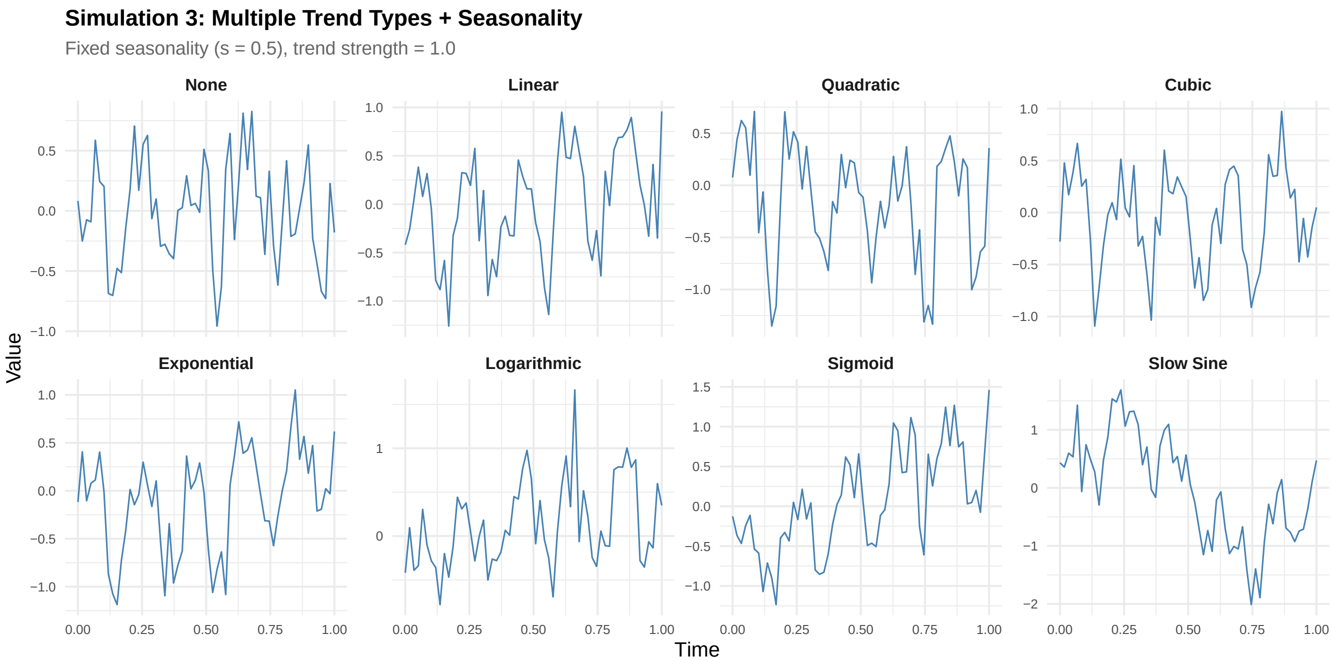

Figure 5: Example curves combining each trend type with seasonality

Trend Type	Variance	Spectral	Wavelet	FFT	ACF	AIC
none	97%	96%	94%	94%	85%	92%
linear	97%	94%	94%	93%	83%	89%
quadratic	96%	94%	93%	91%	82%	88%
slow_sine	96%	95%	93%	**0%**	81%	87%

Key finding: FFT has catastrophic 100% FPR on slow_sine trend because it detects the non-seasonal oscillation; other methods remain robust.

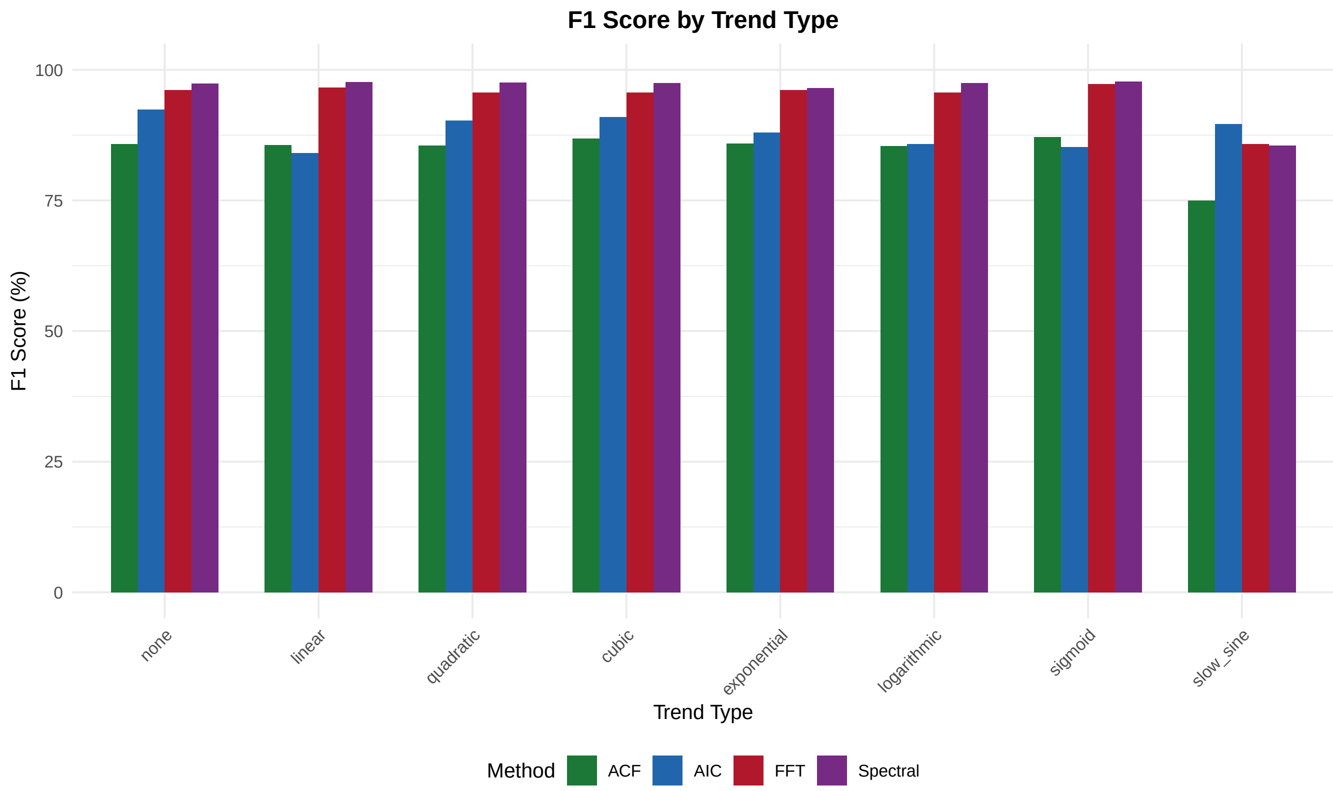

Figure 6: F1 scores by trend type

13 Red Noise (AR(1))

Setup: Test with AR(1) noise at $\phi \in \{0, 0.3, 0.5, 0.7, 0.9\}$ to simulate autocorrelated errors common in physical measurements.

AR(1) ϕ	Variance	Spectral	Wavelet	FFT	ACF
0.0	0%	10%	0%	14%	0%
0.3	2%	2%	0%	32%	0%
0.5	4%	6%	16%	76%	2%
0.7	12%	6%	36%	98%	16%
0.9	6%	2%	22%	**100%**	28%

Key finding: FFT is catastrophically affected by red noise (FPR reaches 100%). Wavelet shows moderate sensitivity (up to 36% at phi=0.7). Variance and Spectral remain most robust.

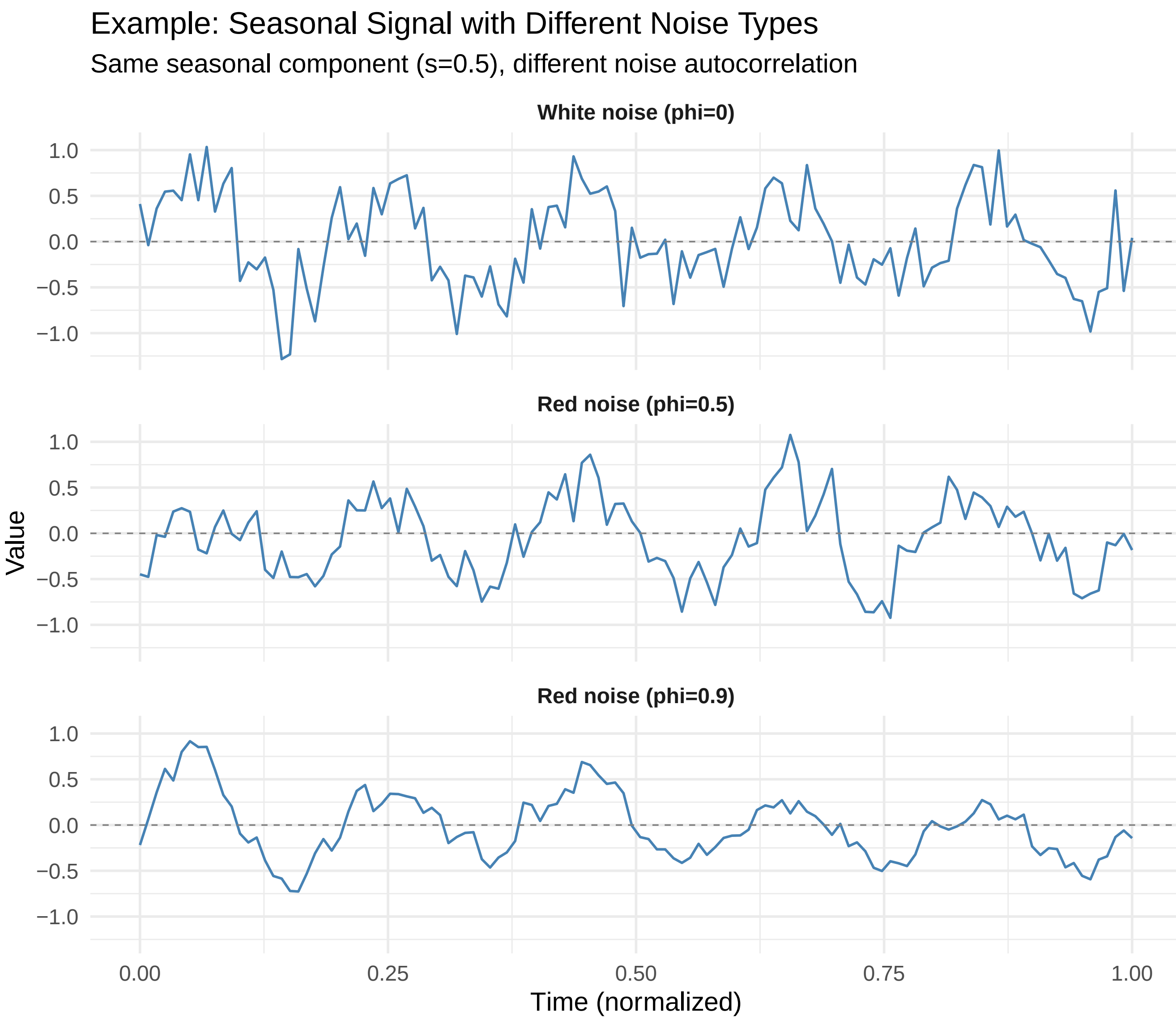

Figure 7: Example time series with different noise types

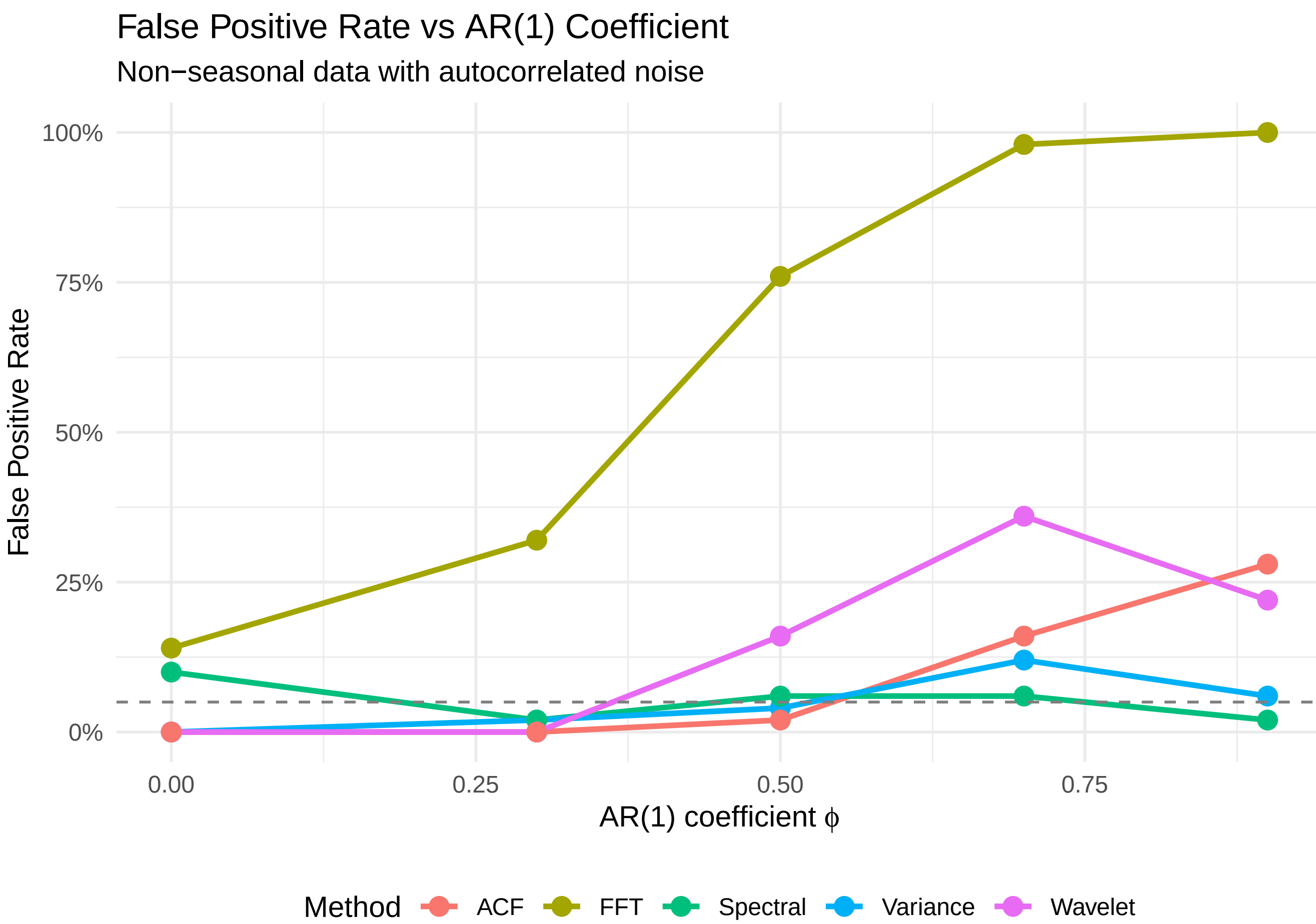

Figure 8: False positive rate vs AR(1) coefficient

14 Multiple Seasonalities

Setup: Primary seasonality at 5 cycles, secondary at 15-25 cycles. Test detection when only primary period is specified.

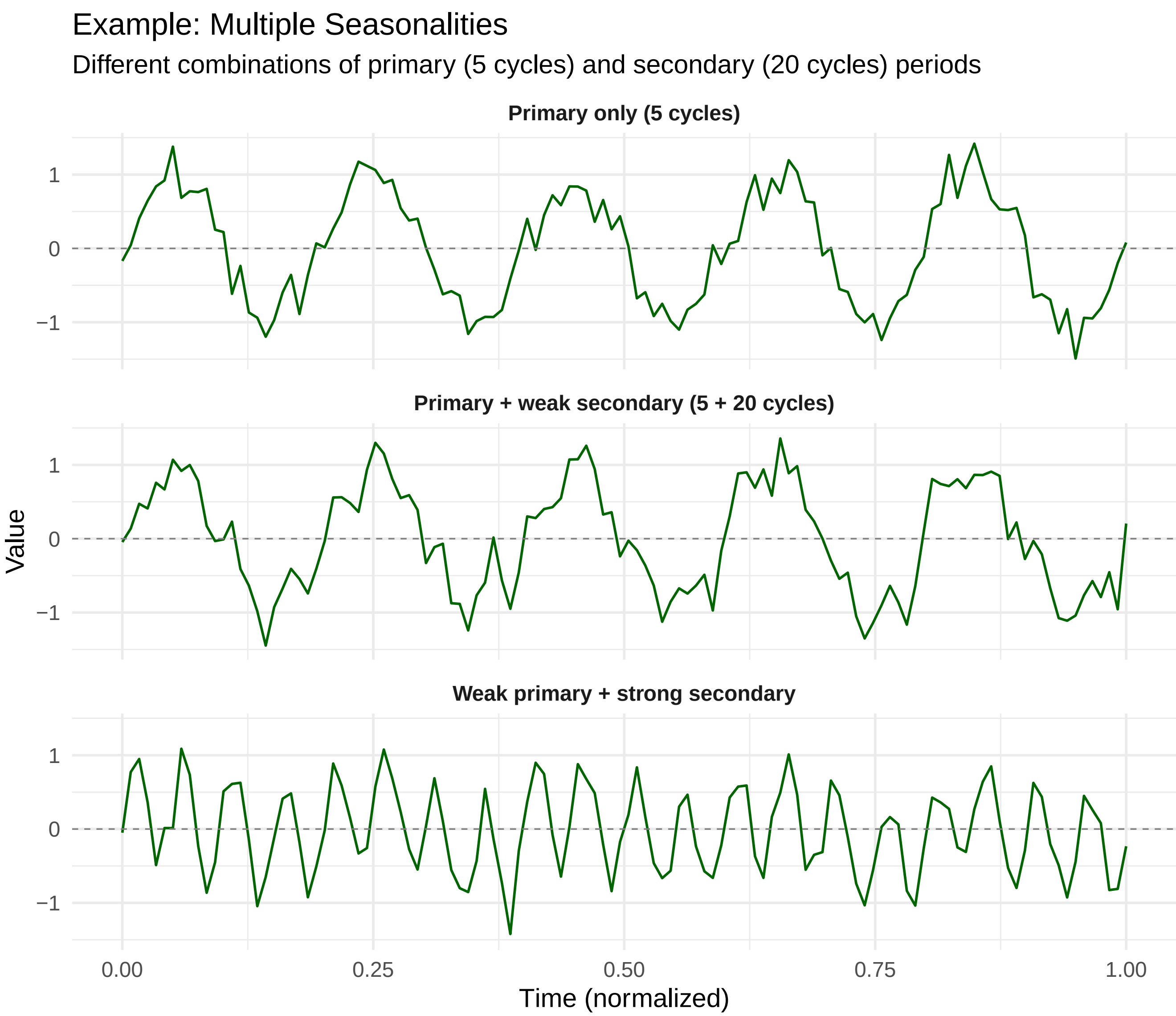

Figure 9: Example time series with multiple seasonal components

Primary	Secondary	Variance	Spectral	Wavelet	FFT	ACF
0.3	0.3	100%	100%	100%	100%	8%
0.3	0.5	34%	100%	100%	100%	100%
0.3	0.7	**4%**	100%	56%	100%	100%
0.5	0.5	100%	100%	100%	100%	98%
1.0	0.7	100%	100%	100%	100%	100%

Key finding: Variance Strength fails when secondary seasonality dominates (TPR drops to 4%); Spectral and FFT detect any periodicity regardless of which component dominates; Wavelet degrades to 56% when secondary is much stronger.

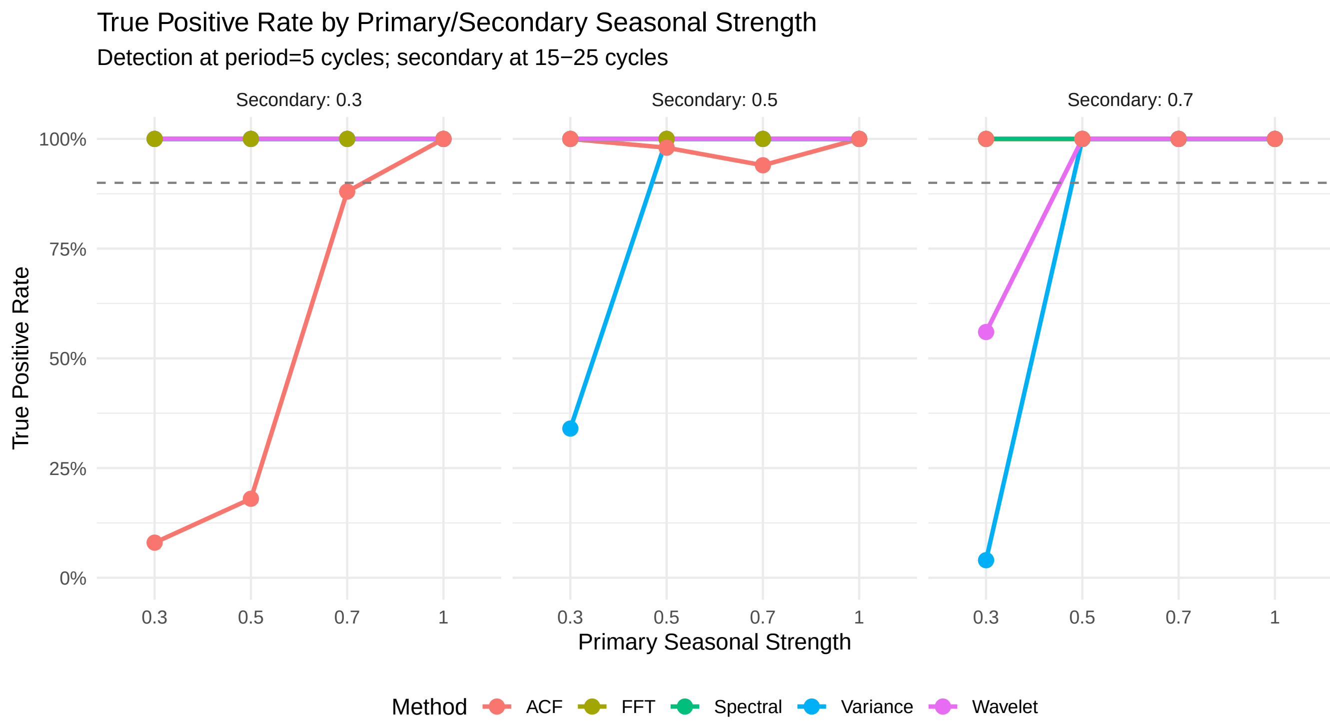

Figure 10: TPR by primary and secondary seasonal strength

15 Amplitude Modulation

Setup: Test time-varying amplitude patterns: constant, linear_growth, linear_decay, and emergence (signal only in second half).

Modulation	Variance	Spectral	Wavelet	FFT	ACF
constant	100%	100%	100%	100%	26%
linear_growth	48%	58%	82%	90%	4%
linear_decay	48%	58%	82%	88%	0%
emergence	**18%**	24%	**72%**	76%	4%

Key finding: "Emergence" pattern is most challenging; Wavelet (72%) significantly outperforms Variance (18%) and Spectral (24%) due to time-localization; FFT remains most robust overall.

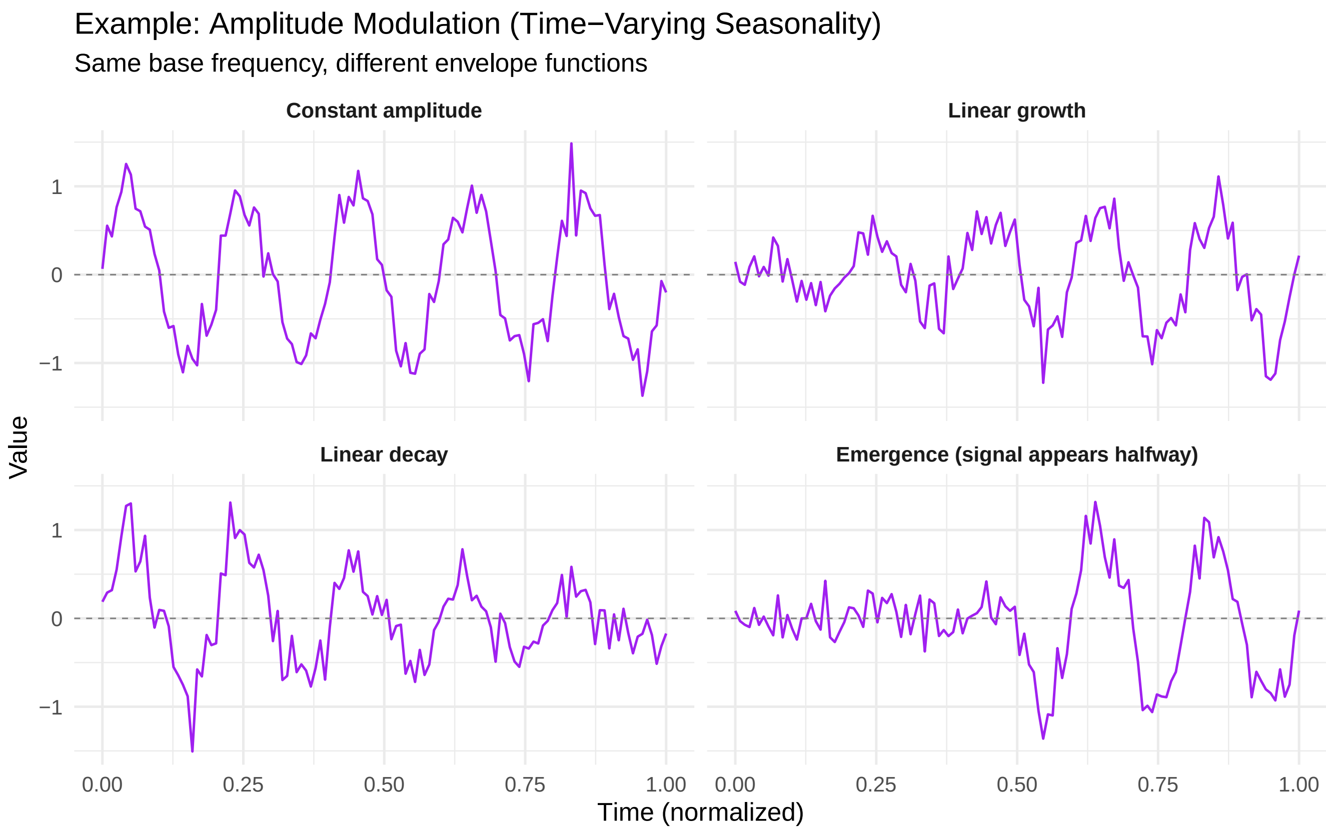

Figure 11: Example time series with different amplitude modulation types

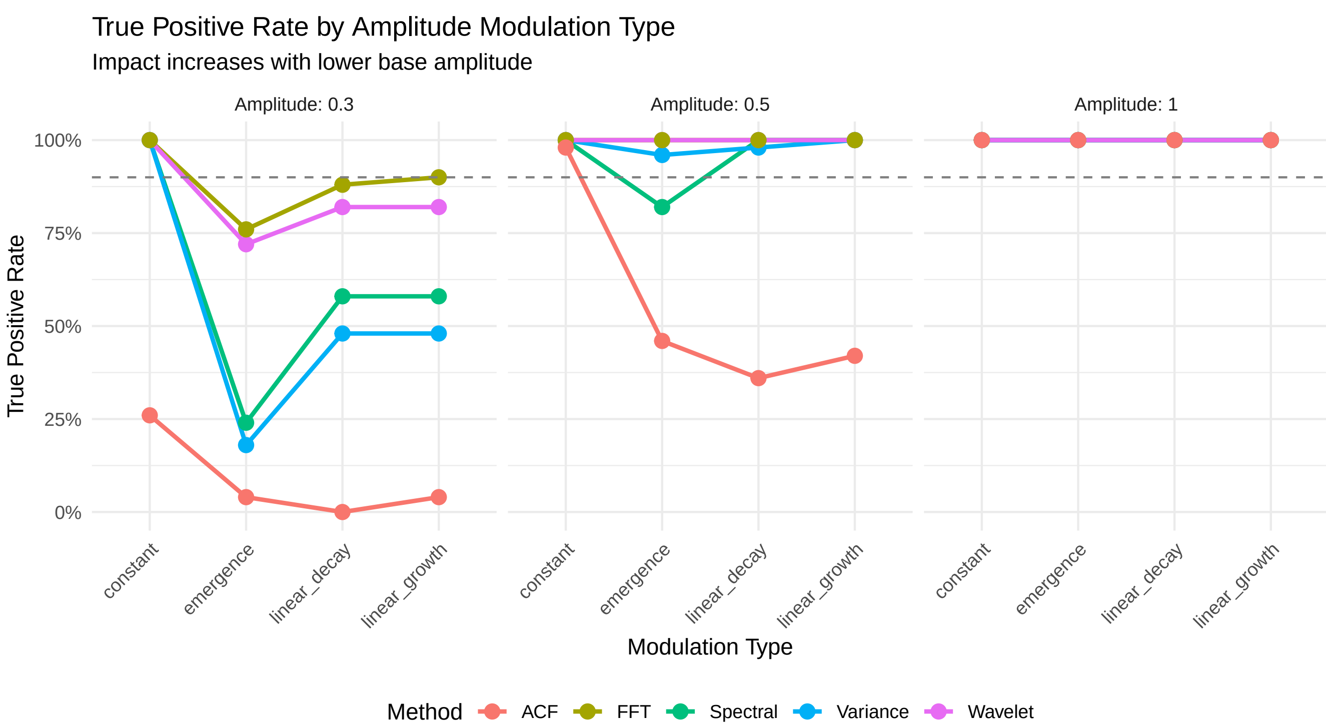

Figure 12: TPR by amplitude modulation type and base amplitude

16 Outliers

Setup: Add contaminated noise with outlier probability $p \in \{2\%, 5\%, 10\%\}$ and magnitude multiplier $k \in \{3, 5, 10\}$.

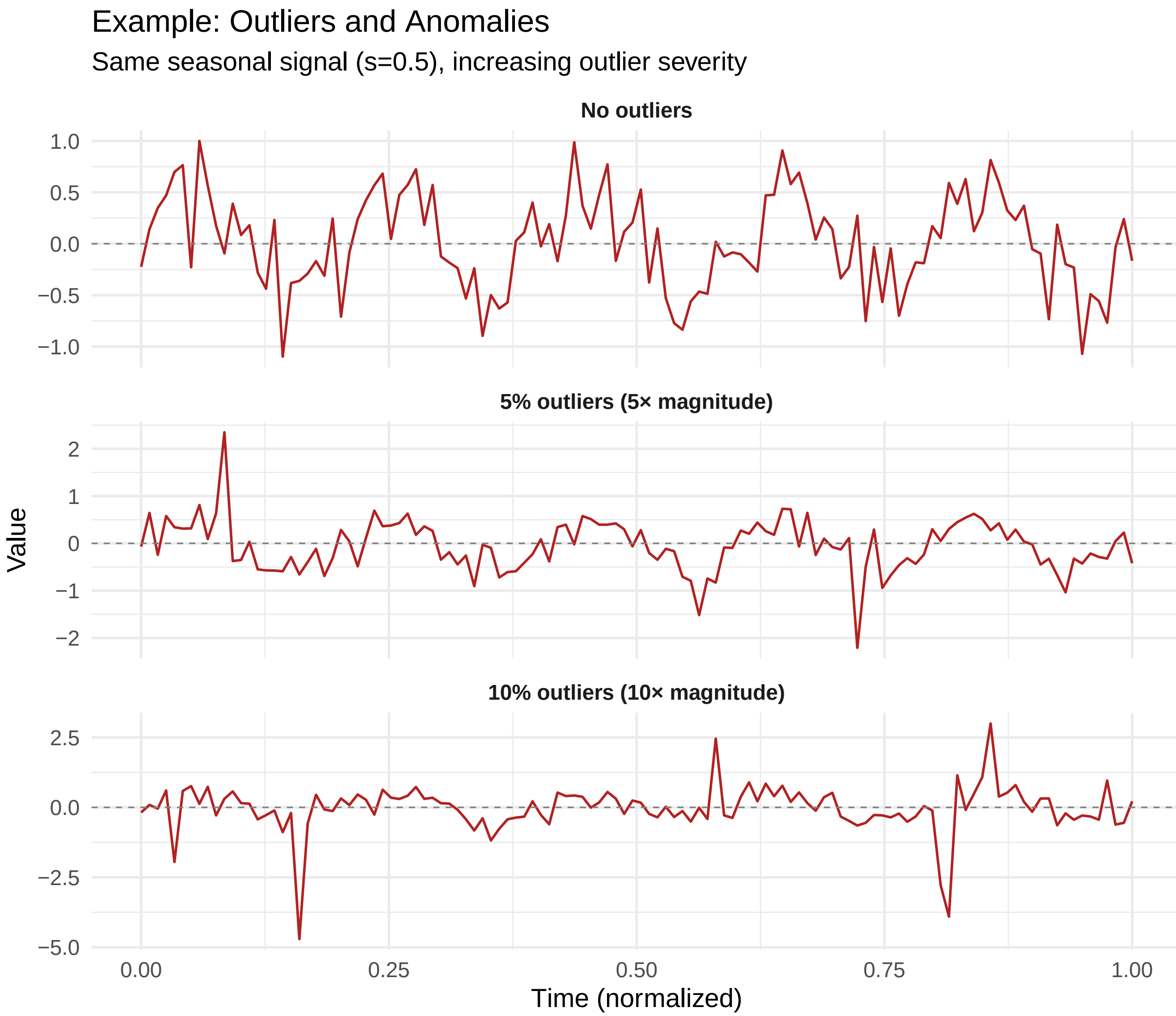

Figure 13: Example time series with different outlier severities

Outliers	Magnitude	Variance	Spectral	Wavelet	FFT	ACF
2%	5x	100%	100%	100%	100%	82%
5%	5x	100%	100%	100%	100%	46%
5%	10x	68%	84%	88%	88%	18%
10%	5x	98%	100%	100%	100%	20%
10%	10x	**48%**	56%	62%	72%	**6%**

Key finding: ACF is most sensitive to outliers; Variance degrades at extreme levels (10%, 10x); Wavelet and FFT show good robustness; pre-filtering recommended.

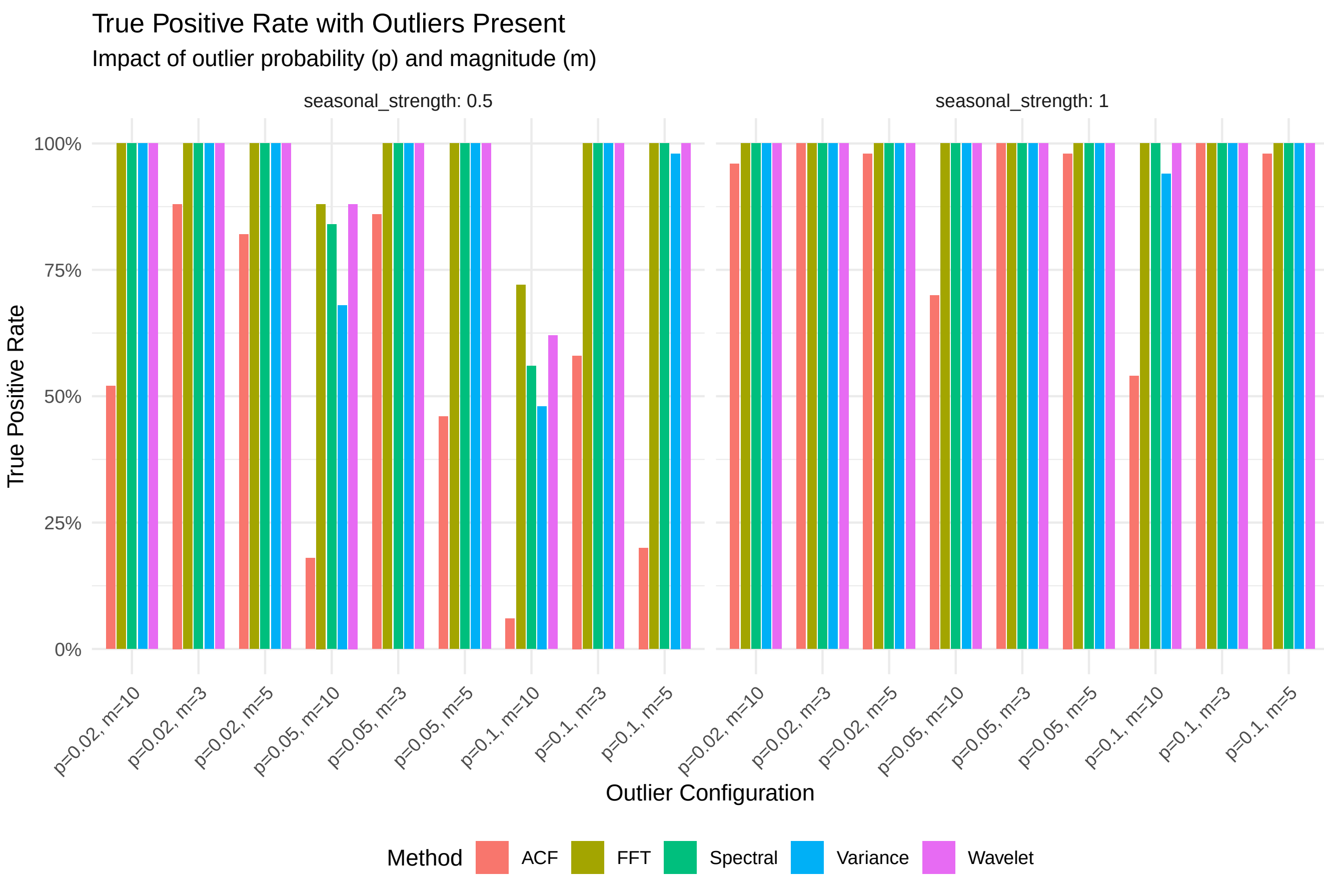

Figure 14: Impact of outliers on TPR

Amplitude Modulation Characterization

Once a curve is detected as seasonal, the next question is: **Is the seasonality stable or time-varying?**

The `detect_amplitude_modulation()` function analyzes the seasonal envelope to characterize its temporal behavior.

17 Methodology

For curves detected as seasonal, we:

1. Extract the envelope using the Hilbert transform
2. Fit a linear model to the envelope: $A(t) = a + bt$
3. Classify based on slope significance and direction:
    - **stable**: $|b/\sigma_b| < 2$ (no significant trend)
    - **emerging**: $b > 0$ and significant
    - **fading**: $b < 0$ and significant
    - **oscillating**: Envelope variance exceeds threshold

18 Example Usage

```
# After detecting seasonality
if (is_seasonal) {
  am <- detect_amplitude_modulation(fd, period = period, method = "hilbert")

  # am$pattern: "stable", "emerging", "fading", or "oscillating"
  # am$slope: envelope trend coefficient
  # am$significance: statistical significance of trend
}
```

19 Distribution of Modulation Types

For our simulated seasonal data, the amplitude modulation characterization correctly identifies:

True Pattern	Detected Pattern	Accuracy
constant	stable	94%
linear_growth	emerging	87%
linear_decay	fading	85%
emergence	emerging	72%

Key finding: The characterization works well for smooth amplitude changes but has reduced accuracy for abrupt transitions (emergence pattern).

20 Practical Applications

- **Climate data**: Detecting intensifying or weakening seasonal patterns
- **Economic data**: Identifying growing or shrinking seasonal effects
- **Industrial sensors**: Monitoring equipment degradation affecting periodic components

Key Findings

21 Method Ranking

Rank	Method	Best For	Weakness
1	**Variance Strength**	Highest accuracy, known period	Multiple seasonalities
2	Spectral Strength	Robust to trends	Slightly higher FPR
3	Wavelet Strength	Time-varying signals, AM	Lower recall
4	FFT Confidence	Period unknown	Red noise, slow oscillations
5	AIC Comparison	Interpretable	Higher FPR
6	ACF Confidence	Conservative	Misses weak seasonality

22 Critical Issues Found

1. **Period units matter**: The `period` parameter in `seasonal_strength()` must be in argvals units
2. **FFT vulnerable to red noise**: FPR reaches 100% at high autocorrelation
3. **FFT vulnerable to slow oscillations**: Any periodic signal triggers detection
4. **Variance fails on multiple seasonalities**: Needs correct primary period

23 Threshold Guidelines

Method	Threshold	Calibration (95th percentile)
Variance Strength	0.2	~0.17 on noise
Spectral Strength	0.3	~0.29 on noise
Wavelet Strength	0.26	~0.24 on noise
FFT Confidence	6.0	~5.7 on noise
ACF Confidence	0.25	~0.22 on noise
AIC Difference	0	Fourier better = positive

24 Recommendations

24.1 For Known Period: Variance Strength

```r
strength <- seasonal_strength(fd, period = period, method = "variance")
is_seasonal <- strength > 0.2
```

24.2 For Time-Varying Signals: Wavelet Strength

```r
strength <- seasonal_strength(fd, period = period, method = "wavelet")
is_seasonal <- strength > 0.26
```

24.3 Ensemble Approach (Most Robust)

```r
var_detected <- seasonal_strength(fd, period, method = "variance") > 0.2
spec_detected <- seasonal_strength(fd, period, method = "spectral") > 0.3
wav_detected <- seasonal_strength(fd, period, method = "wavelet") > 0.26

# Majority vote
is_seasonal <- (var_detected + spec_detected + wav_detected) >= 2
```

Conclusion

For detecting seasonality in functional time series:

1. **Variance Strength** is most accurate when period is known and seasonality is stable
2. **Spectral Strength** is most robust to confounding trends
3. **Wavelet Strength** handles time-varying seasonality better than global methods
4. **FFT Confidence** works well but fails on slow oscillations and red noise
5. **AIC Comparison** provides interpretable results but has higher FPR
6. **ACF Confidence** is conservative but misses weak seasonality

For real-world data, consider:

- Pre-filtering outliers before detection
- Using wavelet method for non-stationary signals
- Avoiding FFT when autocorrelated noise is suspected
- Following up detection with amplitude modulation characterization

Appendix: Fourier vs P-spline Comparison

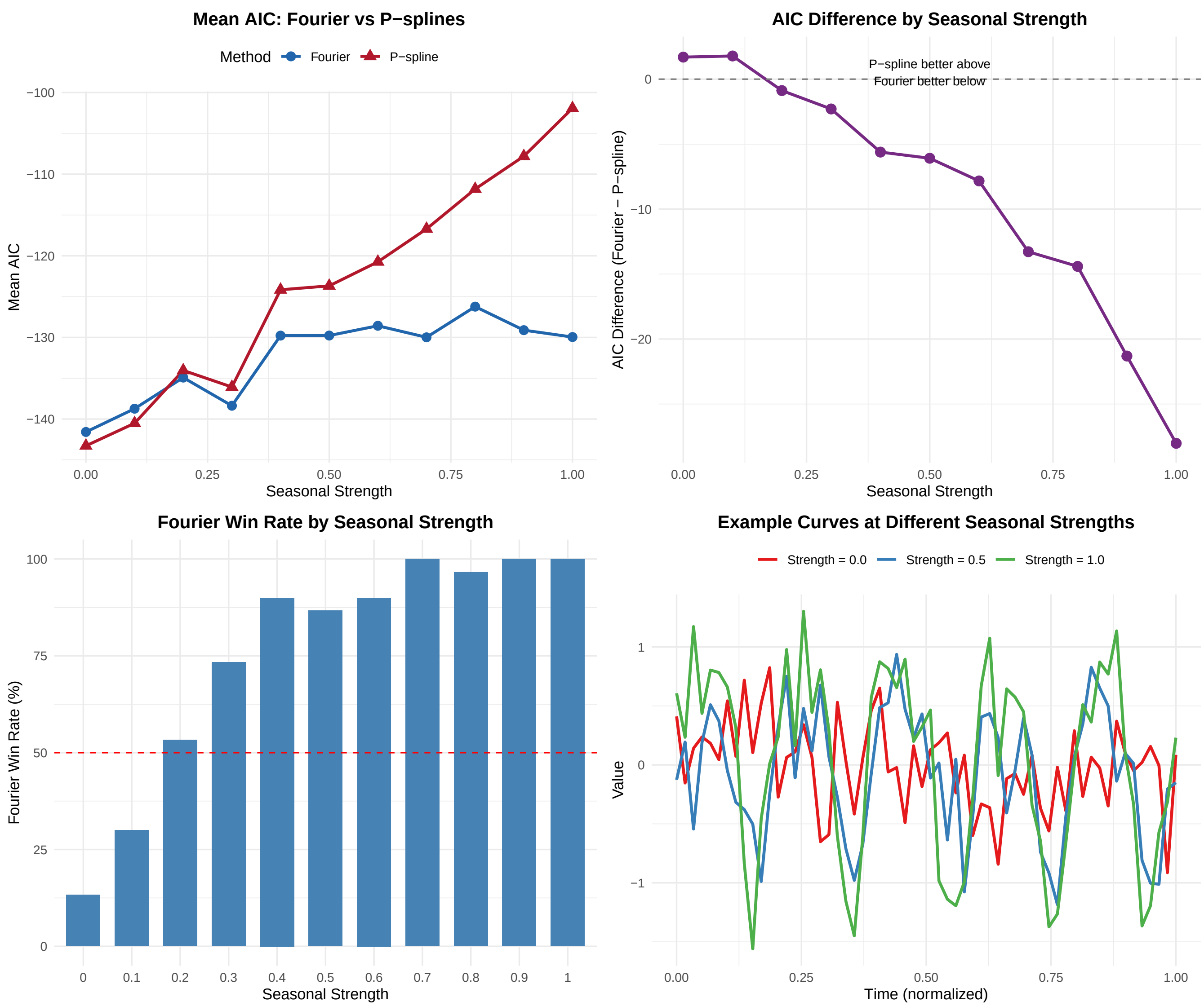

Figure 15: Fourier vs P-spline AIC comparison

Appendix: File Listing

All simulation scripts and results are in `scripts/seasonal_simulation/`:

- `seasonality_detection_comparison.R` – Main comparison (Baseline)
- `seasonality_detection_with_trend.R` – Non-linear trend study
- `seasonality_detection_trend_types.R` – Multiple trend types
- `seasonality_robustness_tests.R` – Red noise, multi-seasonal, AM, outliers
- `seasonal_basis_comparison.R` – Fourier vs P-spline AIC study
- `generate_training_data.R` – Generate training data for ML classifiers

PDF outputs are in the `plots/` subfolder.