Introduction

fdars Package

2025-12-29

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This report describes and compares five methods for detecting seasonality in functional time series data. We evaluate each method's performance across different scenarios including varying seasonal strengths, non-linear trends, and different trend types.

The goal is to answer: **Given a time series, how can we reliably determine if it contains a seasonal pattern?**

Detection Methods

1 AIC Comparison (Fourier vs P-spline)

Concept: If data is seasonal, a Fourier basis should fit better than P-splines because Fourier bases naturally capture periodic patterns.

Mathematical formulation:

For a curve $y(t)$, we fit two models:

1. **Fourier basis**: $\hat{y}(t) = \sum_{k=0}^{K} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$
2. **P-spline**: $\hat{y}(t) = \sum_{j=1}^{J} c_j B_j(t)$ with penalty $\lambda \int [\hat{y}''(t)]^2 dt$

We compute AIC for each:

$$\text{AIC} = n \log(\text{RSS}/n) + 2 \cdot \text{edf}$$

where RSS is the residual sum of squares and edf is the effective degrees of freedom.

Detection rule: Seasonality detected if $\text{AIC}_{\text{P-spline}} - \text{AIC}_{\text{Fourier}} > 0$

Interpretation: When Fourier has lower AIC, the periodic structure is significant enough to justify the global periodic assumption over the local flexibility of splines.

2 FFT Confidence

Concept: Use Fast Fourier Transform to detect dominant frequencies. Strong peaks in the periodogram indicate periodic components.

Mathematical formulation:

Given a time series $y_1, y_2, \ldots, y_n$, compute the discrete Fourier transform:

$$Y_k = \sum_{j=1}^{n} y_j e^{-2\pi i (j-1)(k-1)/n}$$

The periodogram (power spectrum) is:

$$P_k = |Y_k|^2$$

Detection score:

$$\text{Confidence} = \frac{\max_k P_k}{\text{mean}(P_k)}$$

Detection rule: Seasonality detected if Confidence > 6.0

Interpretation: A high ratio indicates one frequency dominates, suggesting periodicity rather than random noise.

3 ACF Confidence

Concept: Autocorrelation at the seasonal lag should be high for seasonal data.

Mathematical formulation:

The autocorrelation function at lag h is:

$$\rho_h = \frac{\sum_{t=1}^{n-h}(y_t - \bar{y})(y_{t+h} - \bar{y})}{\sum_{t=1}^{n}(y_t - \bar{y})^2}$$

For seasonal data with period p, we expect ρ_p to be significantly positive.

Detection rule: Seasonality detected if ACF confidence > 0.25

Interpretation: High autocorrelation at the seasonal lag indicates the pattern repeats.

4 Variance Strength

Concept: Decompose variance into seasonal and residual components. High seasonal variance ratio indicates seasonality.

Mathematical formulation:

Decompose the series: $y_t = T_t + S_t + R_t$ (trend + seasonal + residual)

The seasonal strength is:

$$\text{SS}_{\text{var}} = 1 - \frac{\text{Var}(R_t)}{\text{Var}(y_t - T_t)}$$

Detection rule: Seasonality detected if $\text{SS}_{\text{var}} > 0.2$

Interpretation: Values close to 1 mean the seasonal component dominates; values close to 0 mean residual noise dominates.

Important: The `period` parameter must be in the same units as `argvals`. For data normalized to [0,1] with 5 annual cycles, use `period = 0.2`.

5 Spectral Strength

Concept: Measure the proportion of spectral power at the seasonal frequency.

Mathematical formulation:

Using the periodogram P_k, identify the seasonal frequency $f_s = 1/\text{period}$.

$$\text{SS}_{\text{spectral}} = \frac{\sum_{k \in \mathcal{S}} P_k}{\sum_k P_k}$$

where $\mathcal{S}$ includes the seasonal frequency and its harmonics.

Detection rule: Seasonality detected if $\text{SS}_{\text{spectral}} > 0.3$

Interpretation: High values indicate spectral energy is concentrated at seasonal frequencies.

Simulation Studies

6 Simulation 1: Varying Seasonal Strength

6.1 Setup

This simulation tests how well each method detects seasonality at different signal strengths.

Parameters:

- 11 seasonal strength levels: 0.0, 0.1, . . . , 1.0
- 50 curves per strength level
- 5 years of monthly data (60 observations)
- Noise standard deviation: 0.3

Signal model:

$$y(t) = s \cdot [\sin(2\pi \cdot 5t) + 0.3\cos(4\pi \cdot 5t)] + \epsilon, \quad \epsilon \sim N(0, 0.3^2)$$

where s is the seasonal strength (0 = no seasonality, 1 = full seasonality).

Ground truth: A curve is classified as "truly seasonal" if $s \geq 0.2$.

6.2 Code

```r
library(fdars)
library(ggplot2)
library(tidyr)
library(dplyr)

set.seed(42)

# Configuration
n_strengths <- 11
n_curves_per_strength <- 50
```

```r
n_years <- 5
n_months <- n_years * 12
noise_sd <- 0.3

# Detection thresholds (calibrated to ~5% FPR on pure noise)
detection_thresholds <- list(
  aic_comparison = 0,
  fft_confidence = 6.0,
  acf_confidence = 0.25,
  strength_variance = 0.2,
  strength_spectral = 0.3
)

seasonal_strengths <- seq(0, 1, length.out = n_strengths)
t <- seq(0, 1, length.out = n_months)

# Generate seasonal curve
generate_seasonal_curve <- function(t, strength, noise_sd = 0.3) {
  n_cycles <- length(t) / 12
  seasonal <- strength * sin(2 * pi * n_cycles * t)
  seasonal <- seasonal + strength * 0.3 * cos(4 * pi * n_cycles * t)
  noise <- rnorm(length(t), sd = noise_sd)
  return(seasonal + noise)
}
```

6.3 Results

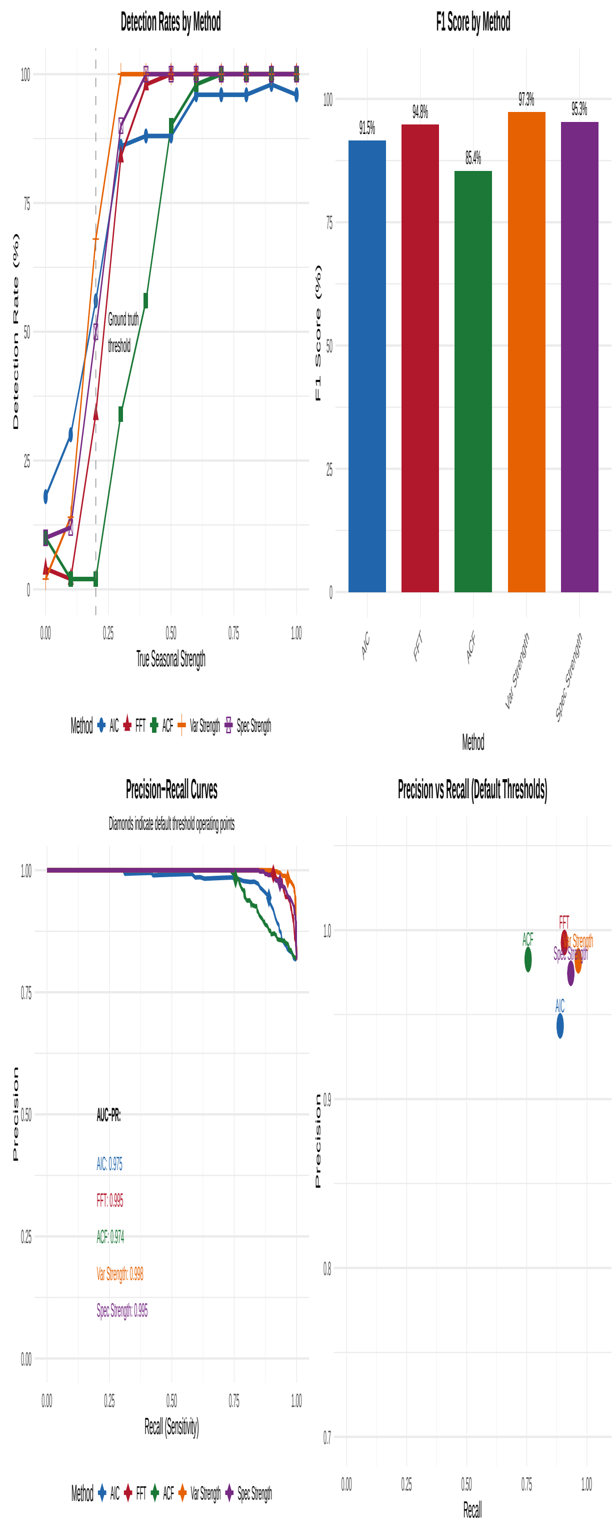

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Figure 1: Detection rates by seasonal strength

How to interpret:

- The x-axis shows the true seasonal strength (0 = pure noise, 1 = strong seasonality)
- The y-axis shows what percentage of curves each method classified as "seasonal"
- The vertical dashed line at 0.2 marks the ground truth threshold
- **Ideal behavior**: 0% detection below the threshold, 100% above

6.4 Classification Performance

Method	F1 Score	Precision	Recall	FPR	Specificity
Variance Strength	**97.3%**	98.2%	96.4%	2.0%	92.0%
Spectral Strength	95.3%	97.4%	93.3%	10.0%	89.0%
FFT Confidence	94.8%	99.3%	90.7%	4.0%	97.0%
AIC Comparison	91.5%	94.3%	88.9%	18.0%	76.0%
ACF Confidence	85.4%	98.3%	75.6%	10.0%	94.0%

How to interpret:

- **F1 Score**: Harmonic mean of precision and recall (higher is better)
- **Precision**: Of curves detected as seasonal, what % are truly seasonal?
- **Recall**: Of truly seasonal curves, what % did we detect?
- **FPR**: False Positive Rate - what % of non-seasonal curves were incorrectly flagged?

6.5 Precision-Recall Analysis

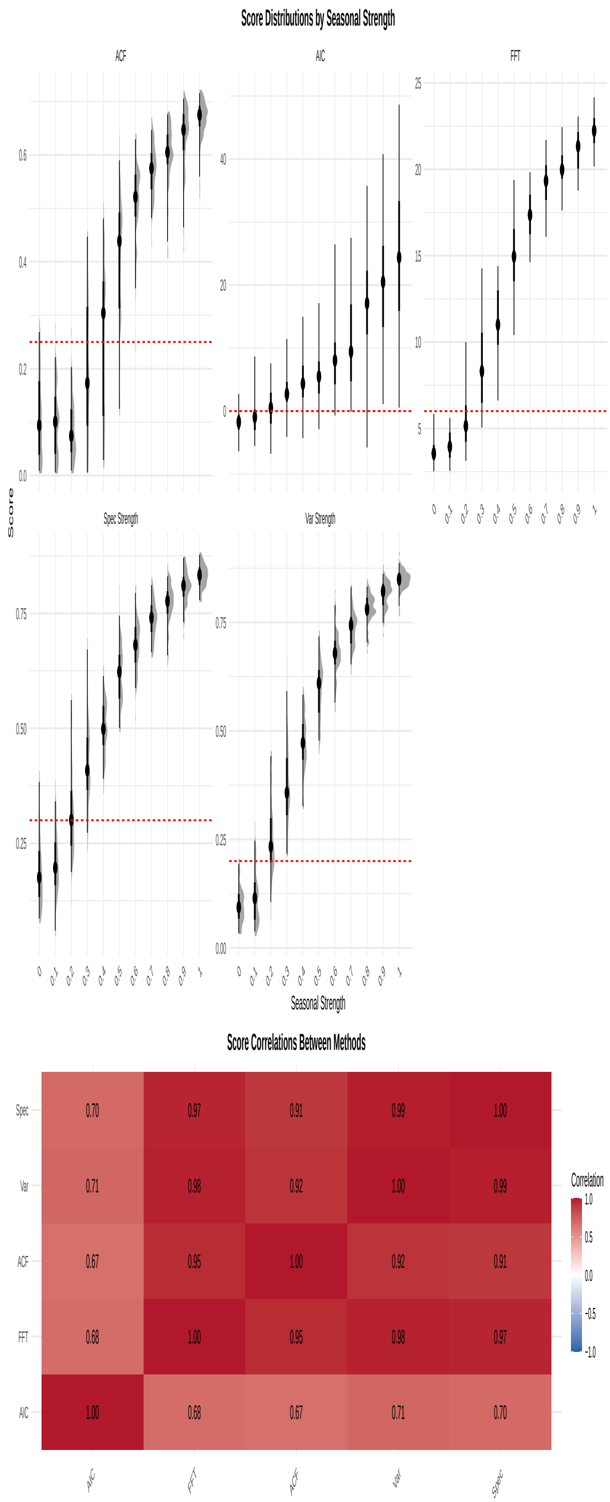

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Figure 2: Precision-Recall curves

How to interpret:

- Curves closer to the top-right corner are better
- The diamond markers show the operating point at the default threshold
- AUC-PR (Area Under the PR Curve) summarizes overall performance

7 Simulation 2: Non-linear Trend

7.1 Setup

This simulation tests robustness when non-linear trends are added to the seasonal signal.

Parameters:

- 6 seasonal strength levels × 6 trend strength levels
- 30 curves per combination
- Non-linear trend: quadratic + cubic + sigmoid components

Signal model:

$$y(t) = \text{Trend}(t, \tau) + \text{Seasonal}(t, s) + \epsilon$$

where τ is the trend strength and s is the seasonal strength.

Trend function:

$$\text{Trend}(t, \tau) = \tau \cdot [2(t - 0.5)^2 + 0.5(t - 0.3)^3 + 0.3 \cdot \sigma(10(t - 0.6)) - 0.5]$$

7.2 Code

```
# Non-linear trend function
generate_nonlinear_trend <- function(t, trend_strength) {
  quadratic <- 2 * (t - 0.5)^2
  cubic <- 0.5 * (t - 0.3)^3
  sigmoid <- 1 / (1 + exp(-10 * (t - 0.6)))
  trend <- trend_strength * (quadratic + cubic + 0.3 * sigmoid - 0.5)
  return(trend)
}

# Generate curve with trend + seasonal + noise
generate_curve <- function(t, seasonal_strength, trend_strength, noise_sd = 0.3) {
  trend <- generate_nonlinear_trend(t, trend_strength)
  n_cycles <- length(t) / 12
  seasonal <- seasonal_strength * sin(2 * pi * n_cycles * t)
  seasonal <- seasonal + seasonal_strength * 0.3 * cos(4 * pi * n_cycles * t)
  noise <- rnorm(length(t), sd = noise_sd)
  return(trend + seasonal + noise)
}
```

7.3 Results

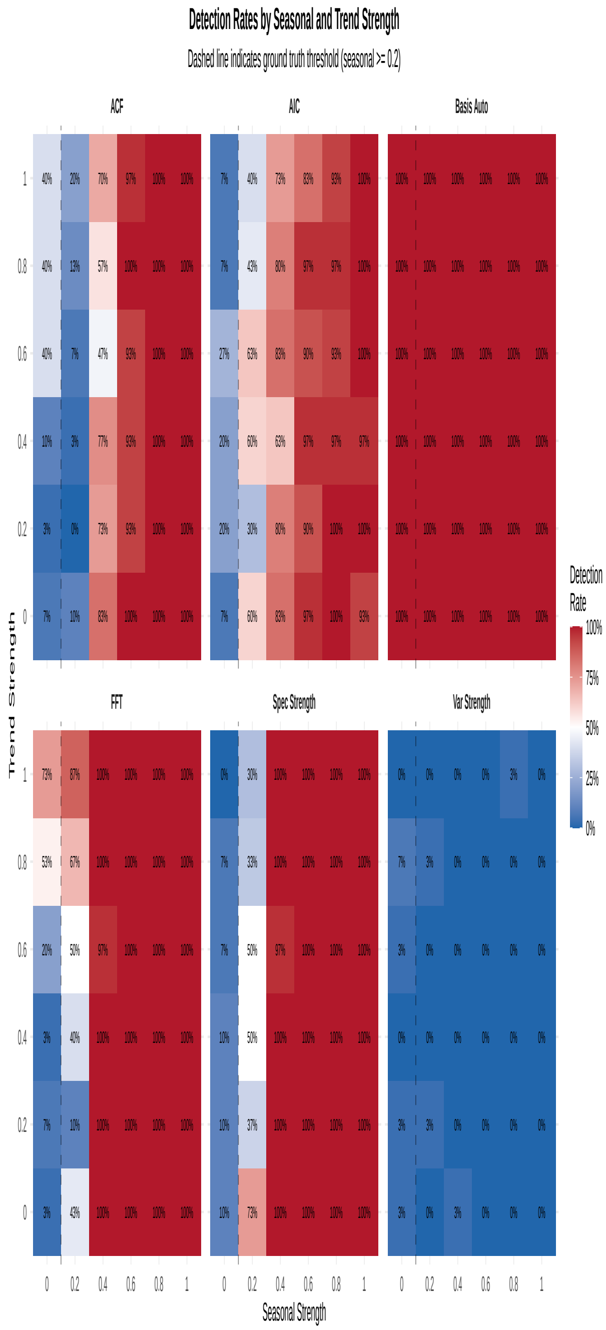

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Figure 3: Detection rates heatmap by seasonal and trend strength

How to interpret:

- Each cell shows the detection rate for a combination of seasonal strength (x) and trend strength (y)
- Blue = low detection rate, Red = high detection rate
- The dashed line separates non-seasonal (left) from seasonal (right) ground truth

7.4 F1 Score vs Trend Strength

Method	No Trend	Max Trend	F1 Drop
Spectral	96.3%	92.5%	3.9%
FFT	93.7%	91.8%	2.0%
AIC	92.2%	87.0%	5.7%
ACF	87.4%	83.5%	4.5%

How to interpret:

- **F1 Drop**: How much performance degrades when strong trends are present
- Lower drop = more robust to trends

7.5 False Positive Rate by Trend Strength

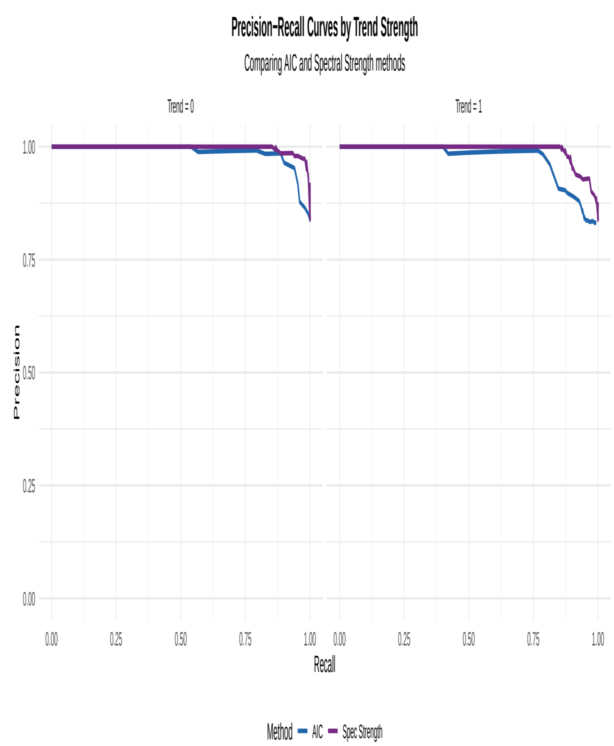

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Figure 4: FPR when no seasonality is present, across trend strengths

Key finding: FFT's FPR increases dramatically with trend strength because non-linear trends can create spurious peaks in the periodogram.

8 Simulation 3: Multiple Trend Types

8.1 Setup

This simulation tests which types of trends cause the most problems for each detection method.

Trend types tested:

1. **None**: Flat baseline
2. **Linear**: $f(t) = t - 0.5$
3. **Quadratic**: $f(t) = (t - 0.5)^2 - 0.25$
4. **Cubic**: $f(t) = 2(t - 0.5)^3$
5. **Exponential**: $f(t) = e^{2t}/e^2 - 0.5$
6. **Logarithmic**: $f(t) = \log(t + 0.1)$ (normalized)
7. **Sigmoid**: $f(t) = 1/(1 + e^{-10(t-0.5)}) - 0.5$
8. **Slow sine**: $f(t) = \sin(2\pi t)$ — one cycle over the entire series

8.2 Code

```
trend_functions <- list(
  none = function(t, strength) rep(0, length(t)),
  linear = function(t, strength) strength * (t - 0.5),
  quadratic = function(t, strength) strength * ((t - 0.5)^2 - 0.25),
  cubic = function(t, strength) strength * 2 * (t - 0.5)^3,
  exponential = function(t, strength) strength * (exp(2 * t) / exp(2) - 0.5),
  logarithmic = function(t, strength) {
    strength * (log(t + 0.1) - log(0.1)) / (log(1.1) - log(0.1)) - 0.5 * strength
  },
  sigmoid = function(t, strength) strength * (1 / (1 + exp(-10 * (t - 0.5))) - 0.5),
  slow_sine = function(t, strength) strength * sin(2 * pi * t)
)
```

8.3 Results

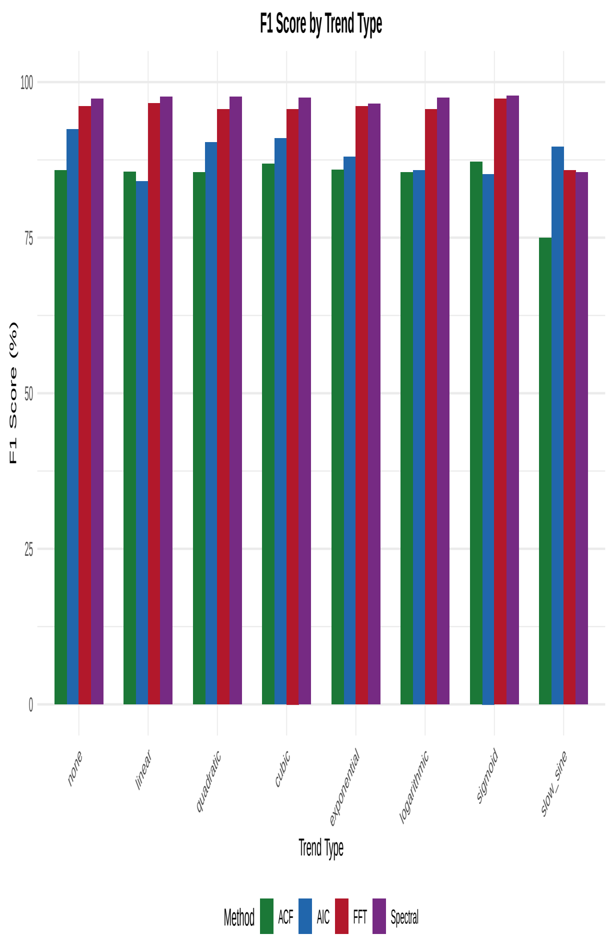

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Figure 5: F1 scores by trend type

8.4 FPR by Trend Type

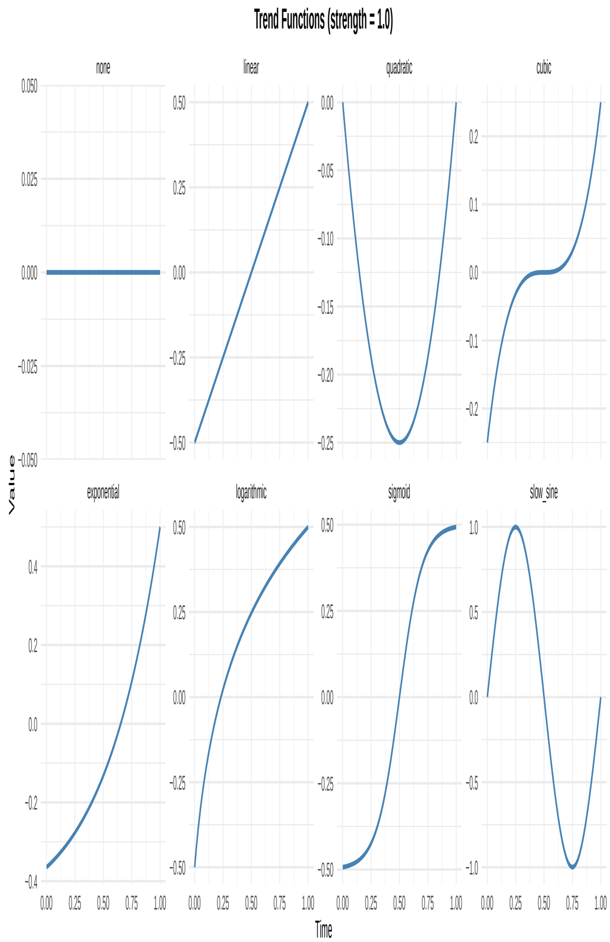

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Figure 6: Example trend functions

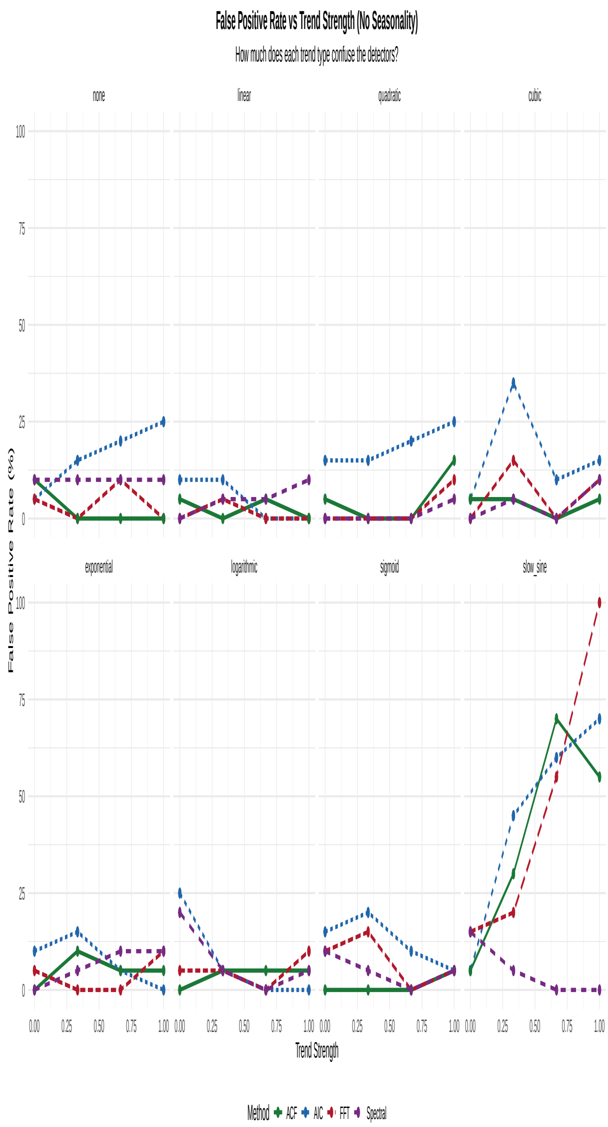

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Figure 7: FPR across trend types and strengths

How to interpret:

- Each panel shows one trend type
- Lines show how FPR changes as trend strength increases
- **Slow sine is catastrophic for FFT**: 100% FPR because FFT detects the slow oscillation as "seasonality"

8.5 Most Problematic Trend Types

Trend Type	FFT FPR	Spectral FPR	Issue
slow_sine	**100%**	0%	FFT detects non-seasonal oscillation
quadratic	10%	5%	Minor
sigmoid	5%	5%	Minor
linear	0%	10%	Handled well

Key Findings

9 Method Ranking

1. **Variance Strength** (F1=97.3%, FPR=2%): Best overall when period is known
2. **Spectral Strength** (F1=95.3%, FPR=10%): Most robust to different trend types
3. **FFT Confidence** (F1=94.8%, FPR=4%): Good but vulnerable to slow oscillations
4. **AIC Comparison** (F1=91.5%, FPR=18%): Interpretable but higher FPR
5. **ACF Confidence** (F1=85.4%, FPR=10%): Conservative, misses weak seasonality

10 Critical Issues Found

1. **Period units matter**: The `period` parameter in `seasonal_strength()` must be in argvals units, not raw time units (e.g., 0.2 not 12)
2. **FFT is vulnerable to slow oscillations**: Any periodic signal (even non-seasonal) triggers detection

Recommendations

11 For Unknown Datasets

Primary recommendation: Variance Strength

```
# Calculate period in argvals units
# If argvals is in [0,1] and you expect 5 annual cycles:
period_in_argvals_units <- 1 / 5  # = 0.2

strength <- seasonal_strength(fd,
                              period = period_in_argvals_units,
                              method = "variance",
```

```
                              detrend = "linear")
is_seasonal <- strength > 0.2
```

For robustness to unknown trends: Spectral Strength

```
strength <- seasonal_strength(fd,
                              period = period_in_argvals_units,
                              method = "spectral",
                              detrend = "linear")
is_seasonal <- strength > 0.3
```

Ensemble approach (most robust):

```
var_detected <- seasonal_strength(fd, period, method = "variance") > 0.2
spec_detected <- seasonal_strength(fd, period, method = "spectral") > 0.3
fft_detected <- estimate_period(fd, method = "fft")$confidence > 6.0

# Majority vote
is_seasonal <- (var_detected + spec_detected + fft_detected) >= 2
```

12 Threshold Guidelines

Method	Threshold	Calibration
Variance Strength	0.2	95th percentile of noise ~0.17
Spectral Strength	0.3	95th percentile of noise ~0.29
FFT Confidence	6.0	95th percentile of noise ~5.7
ACF Confidence	0.25	95th percentile of noise ~0.22
AIC Difference	0	Fourier better → positive difference

Calibration methodology: All thresholds were calibrated using pure noise data (seasonal strength = 0, no trend) by taking the 95th percentile of each method's score distribution. This ensures approximately 5% false positive rate on clean data. Note that FPR may increase when confounding trends are present (see Simulation 2 and 3 results).

Conclusion

For detecting seasonality in functional time series:

1. **Variance Strength** is the most accurate method when the seasonal period is known
2. **Spectral Strength** is most robust to confounding trends and unknown oscillations
3. **FFT Confidence** works well but is vulnerable to slow non-seasonal oscillations
4. **AIC Comparison** provides an interpretable alternative but has higher false positive rates
5. **ACF Confidence** is conservative (low FPR) but misses weak seasonality

The key insight is that simple variance-based decomposition outperforms more complex spectral methods when properly configured with the correct period parameter.

Appendix: Fourier vs P-spline Comparison

The AIC comparison method is based on the observation that Fourier bases naturally capture periodic patterns better than P-splines for seasonal data.

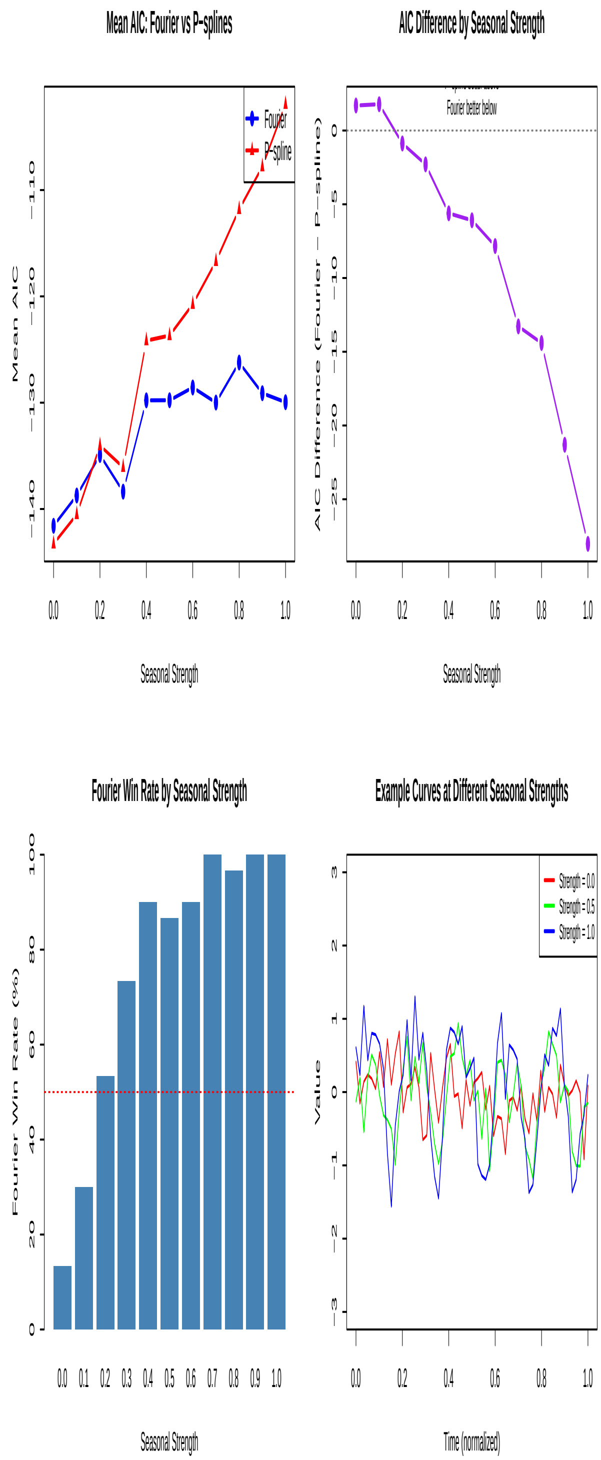

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Figure 8: Fourier vs P-spline AIC comparison

How to interpret:

- Top left: Mean AIC for both methods across seasonal strengths
- Top right: AIC difference (Fourier - P-spline) showing crossover point
- Bottom left: Fourier win rate at each strength level
- Bottom right: Example curves at different seasonal strengths

Appendix: File Listing

All simulation scripts and results are in `scripts/seasonal_simulation/`:

- `seasonality_detection_comparison.R` — Main comparison (Simulation 1)
- `seasonality_detection_with_trend.R` — Non-linear trend study (Simulation 2)
- `seasonality_detection_trend_types.R` — Multiple trend types (Simulation 3)
- `seasonal_basis_comparison.R` — Fourier vs P-spline AIC study

PDF outputs are in the `plots/` subfolder.