

# Seasonality Detection Methods: A Comparative Study

fdars Package

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# 1 Executive Summary

## 1.1 Key Findings

This study compared five methods for detecting seasonality in functional time series data across 550+ simulated curves with varying seasonal strengths and trend components. **Note:** Results are based on idealized conditions (white noise, single seasonality); see Section 5 for real-world challenges.

Method	F1 Score	False Positive Rate	Robustness to Trends
<b>Variance Strength</b>	<b>97.3%</b>	<b>2%</b>	Excellent (0.4% F1 drop)
Spectral Strength	95.3%	10%	Good (3.9% F1 drop)
FFT Confidence	94.8%	4%	Good (2.0% F1 drop)
AIC Comparison	91.5%	18%	Moderate (5.7% F1 drop)
ACF Confidence	85.4%	10%	Moderate (4.5% F1 drop)

**Winner: Variance Strength** achieves the highest accuracy with the lowest false positive rate and is most robust to non-linear trends.

## 1.2 Recommendations

### 1.2.1 Primary Recommendation: Use Variance Strength

```
# Detect seasonality with Variance Strength method
period <- 0.2 # Period in argvals units (e.g., 1/5 for 5 cycles in [0,1])
strength <- seasonal_strength(fd, period = period, method = "variance", detrend = "linear")
is_seasonal <- strength > 0.2
```

### 1.2.2 When Period is Unknown: Two-Step Approach

```
# Step 1: Estimate period using FFT (no period required)
result <- estimate_period(fd, method = "fft", detrend = "linear")
estimated_period <- result$period

# Step 2: Measure strength with estimated period
strength <- seasonal_strength(fd, period = estimated_period, method = "variance")
is_seasonal <- strength > 0.2
```

### 1.2.3 Critical Notes

1. **Period units matter:** The period parameter must be in argvals units, not raw time units
2. **Avoid FFT for slow oscillations:** FFT has 100% false positive rate when non-seasonal oscillations are present
3. **Thresholds are calibrated:** All thresholds target ~5% false positive rate on pure noise

## 2 Introduction

This report describes and compares five methods for detecting seasonality in functional time series data. We evaluate each method's performance across different scenarios including varying seasonal strengths, non-linear trends, and different trend types.

The goal is to answer: **Given a time series, how can we reliably determine if it contains a seasonal pattern?**

**Report Structure:**

- Section 4.1 — Basic detection across varying seasonal strengths
- Section 4.2 — Robustness to non-linear trends
- Section 4.3 — Performance across different trend types
- Section 5 — Additional real-world challenges (red noise, multiple seasonalities, amplitude modulation, outliers)

## 3 Detection Methods

### 3.1 AIC Comparison (Fourier vs P-spline)

**Concept:** If data is seasonal, a Fourier basis should fit better than P-splines because Fourier bases naturally capture periodic patterns.

**Mathematical formulation:**

For a curve  $y(t)$ , we fit two models:

1. **Fourier basis:**  $\hat{y}(t) = \sum_{k=0}^K a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$
2. **P-spline:**  $\hat{y}(t) = \sum_{j=1}^J c_j B_j(t)$  with penalty  $\lambda \int [\hat{y}''(t)]^2 dt$

We compute AIC for each:

$$\text{AIC} = n \log(\text{RSS}/n) + 2 \cdot \text{edf}$$

where RSS is the residual sum of squares and edf is the effective degrees of freedom.

**Detection rule:** Seasonality detected if  $\text{AIC}_{\text{P-spline}} - \text{AIC}_{\text{Fourier}} > 0$

**Interpretation:** When Fourier has lower AIC, the periodic structure is significant enough to justify the global periodic assumption over the local flexibility of splines.

### 3.2 FFT Confidence

**Concept:** Use Fast Fourier Transform to detect dominant frequencies. Strong peaks in the periodogram indicate periodic components.

**Mathematical formulation:**

Given a time series  $y_1, y_2, \dots, y_n$ , compute the discrete Fourier transform:

$$Y_k = \sum_{j=1}^n y_j e^{-2\pi i(j-1)(k-1)/n}$$

The periodogram (power spectrum) is:

$$P_k = |Y_k|^2$$

**Detection score:**

$$\text{Confidence} = \frac{\max_k P_k}{\text{mean}(P_k)}$$

**Detection rule:** Seasonality detected if Confidence  $> 6.0$

**Interpretation:** A high ratio indicates one frequency dominates, suggesting periodicity rather than random noise.

### 3.3 ACF Confidence

**Concept:** Autocorrelation at the seasonal lag should be high for seasonal data.

**Mathematical formulation:**

The autocorrelation function at lag  $h$  is:

$$\rho_h = \frac{\sum_{t=1}^{n-h} (y_t - \bar{y})(y_{t+h} - \bar{y})}{\sum_{t=1}^n (y_t - \bar{y})^2}$$

For seasonal data with period  $p$ , we expect  $\rho_p$  to be significantly positive.

**Detection rule:** Seasonality detected if ACF confidence  $> 0.25$

**Interpretation:** High autocorrelation at the seasonal lag indicates the pattern repeats.

### 3.4 Variance Strength

**Concept:** Decompose variance into seasonal and residual components. High seasonal variance ratio indicates seasonality.

**Mathematical formulation:**

Decompose the series:  $y_t = T_t + S_t + R_t$  (trend + seasonal + residual)

The seasonal strength is:

$$\text{SS}_{\text{var}} = 1 - \frac{\text{Var}(R_t)}{\text{Var}(y_t - T_t)}$$

**Detection rule:** Seasonality detected if  $\text{SS}_{\text{var}} > 0.2$

**Interpretation:** Values close to 1 mean the seasonal component dominates; values close to 0 mean residual noise dominates.

**Important:** The `period` parameter must be in the same units as `argvals`. For data normalized to  $[0,1]$  with 5 annual cycles, use `period = 0.2`.

### 3.5 Spectral Strength

**Concept:** Measure the proportion of spectral power at the seasonal frequency.

**Mathematical formulation:**

Using the periodogram  $P_k$ , identify the seasonal frequency  $f_s = 1/\text{period}$ .

$$\text{SS}_{\text{spectral}} = \frac{\sum_{k \in \mathcal{S}} P_k}{\sum_k P_k}$$

where  $\mathcal{S}$  includes the seasonal frequency and its harmonics.

**Detection rule:** Seasonality detected if  $\text{SS}_{\text{spectral}} > 0.3$

**Interpretation:** High values indicate spectral energy is concentrated at seasonal frequencies.

## 4 Simulation Studies

### 4.1 Simulation 1: Varying Seasonal Strength

#### 4.1.1 Setup

This simulation tests how well each method detects seasonality at different signal strengths.

**Parameters:**

- 11 seasonal strength levels: 0.0, 0.1, ..., 1.0
- 50 curves per strength level
- 5 years of monthly data (60 observations)
- Noise standard deviation: 0.3

**Signal model:**

$$y(t) = s \cdot [\sin(2\pi \cdot 5t) + 0.3 \cos(4\pi \cdot 5t)] + \epsilon, \quad \epsilon \sim N(0, 0.3^2)$$

where  $s$  is the seasonal strength ( $0 = \text{no seasonality}$ ,  $1 = \text{full seasonality}$ ).

**Ground truth:** A curve is classified as “truly seasonal” if  $s \geq 0.2$ .

#### 4.1.2 Code

```

library(fdars)
library(ggplot2)
library(tidyr)
library(dplyr)

set.seed(42)

# Configuration
n_strengths <- 11
n_curves_per_strength <- 50
n_years <- 5
n_months <- n_years * 12
noise_sd <- 0.3

# Detection thresholds (calibrated to ~5% FPR on pure noise)
detection_thresholds <- list(
  aic_comparison = 0,
  fft_confidence = 6.0,
  acf_confidence = 0.25,
  strength_variance = 0.2,
  strength_spectral = 0.3
)

seasonal_strengths <- seq(0, 1, length.out = n_strengths)
t <- seq(0, 1, length.out = n_months)

# Generate seasonal curve
generate_seasonal_curve <- function(t, strength, noise_sd = 0.3) {
  n_cycles <- length(t) / 12
  seasonal <- strength * sin(2 * pi * n_cycles * t)
  seasonal <- seasonal + strength * 0.3 * cos(4 * pi * n_cycles * t)
  noise <- rnorm(length(t), sd = noise_sd)
  return(seasonal + noise)
}

```

### 4.1.3 Results

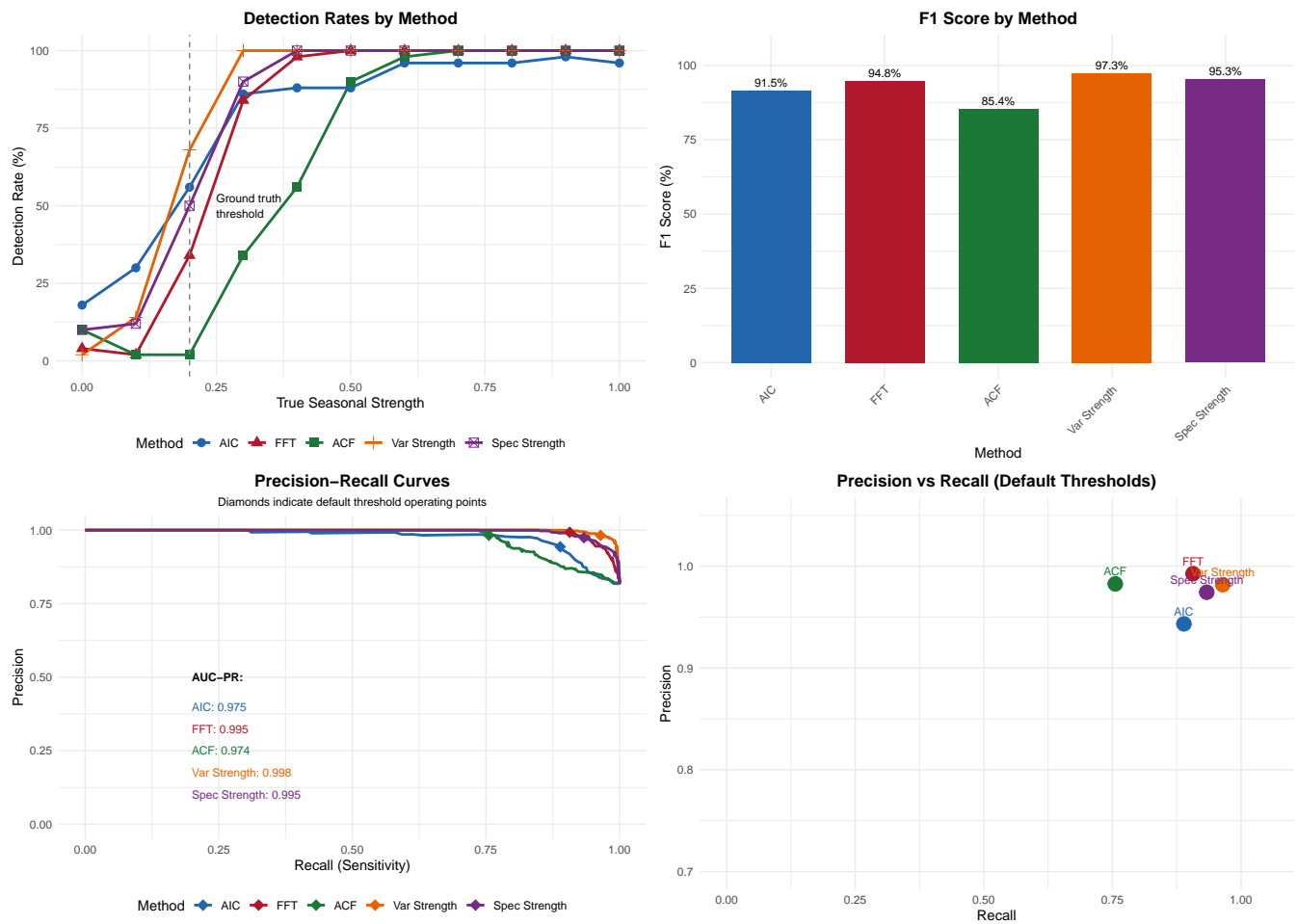


Figure 1: Detection rates by seasonal strength



#### How to interpret:

- The x-axis shows the true seasonal strength (0 = pure noise, 1 = strong seasonality)
- The y-axis shows what percentage of curves each method classified as “seasonal”
- The vertical dashed line at 0.2 marks the ground truth threshold
- **Ideal behavior:** 0% detection below the threshold, 100% above

#### 4.1.4 Classification Performance

Method	F1 Score	Precision	Recall	FPR	Specificity
<b>Variance Strength</b>	<b>97.3%</b>	98.2%	96.4%	2.0%	92.0%
Spectral Strength	95.3%	97.4%	93.3%	10.0%	89.0%
FFT Confidence	94.8%	99.3%	90.7%	4.0%	97.0%
AIC Comparison	91.5%	94.3%	88.9%	18.0%	76.0%
ACF Confidence	85.4%	98.3%	75.6%	10.0%	94.0%

#### How to interpret:

- **F1 Score:** Harmonic mean of precision and recall (higher is better)
- **Precision:** Of curves detected as seasonal, what % are truly seasonal?
- **Recall:** Of truly seasonal curves, what % did we detect?
- **FPR:** False Positive Rate - what % of non-seasonal curves were incorrectly flagged?

## 4.1.5 Precision-Recall Analysis

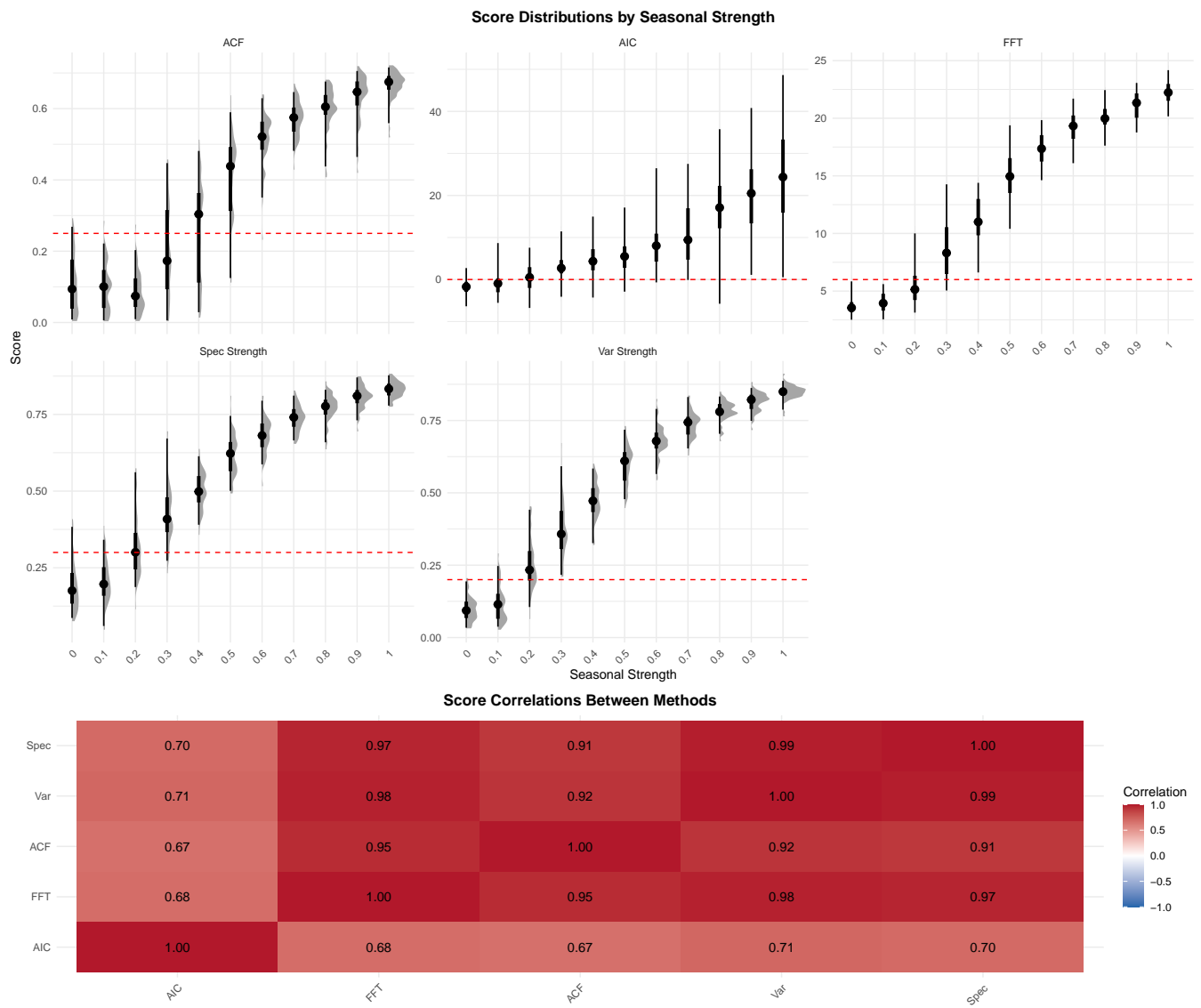


Figure 2: Precision-Recall curves

**How to interpret:**

- Curves closer to the top-right corner are better
- The diamond markers show the operating point at the default threshold
- AUC-PR (Area Under the PR Curve) summarizes overall performance

## 4.2 Simulation 2: Non-linear Trend

### 4.2.1 Setup

This simulation tests robustness when non-linear trends are added to the seasonal signal.

**Parameters:**

- 6 seasonal strength levels  $\times$  6 trend strength levels
- 30 curves per combination
- Non-linear trend: quadratic + cubic + sigmoid components

**Signal model:**

$$y(t) = \text{Trend}(t, \tau) + \text{Seasonal}(t, s) + \epsilon$$

where  $\tau$  is the trend strength and  $s$  is the seasonal strength.

**Trend function:**

$$\text{Trend}(t, \tau) = \tau \cdot [2(t - 0.5)^2 + 0.5(t - 0.3)^3 + 0.3 \cdot \sigma(10(t - 0.6)) - 0.5]$$

### 4.2.2 Code

```
# Non-linear trend function
generate_nonlinear_trend <- function(t, trend_strength) {
  quadratic <- 2 * (t - 0.5)^2
  cubic <- 0.5 * (t - 0.3)^3
  sigmoid <- 1 / (1 + exp(-10 * (t - 0.6)))
  trend <- trend_strength * (quadratic + cubic + 0.3 * sigmoid - 0.5)
  return(trend)
}

# Generate curve with trend + seasonal + noise
generate_curve <- function(t, seasonal_strength, trend_strength, noise_sd = 0.3) {
  trend <- generate_nonlinear_trend(t, trend_strength)
  n_cycles <- length(t) / 12
  seasonal <- seasonal_strength * sin(2 * pi * n_cycles * t)
  seasonal <- seasonal + seasonal_strength * 0.3 * cos(4 * pi * n_cycles * t)
  noise <- rnorm(length(t), sd = noise_sd)
  return(trend + seasonal + noise)
}
```

### 4.2.3 Results

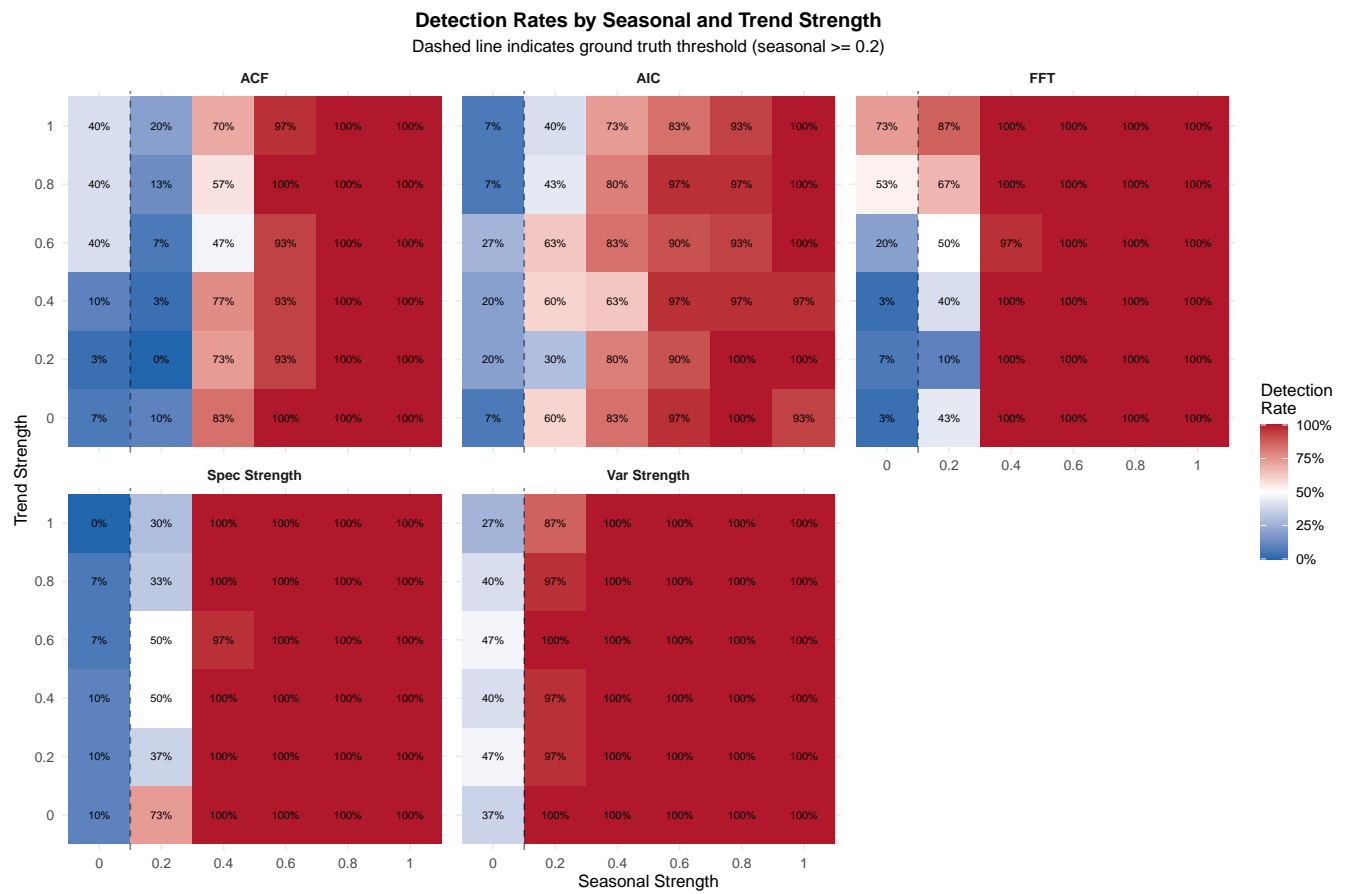


Figure 3: Detection rates heatmap by seasonal and trend strength

### How to interpret:

- Each cell shows the detection rate for a combination of seasonal strength (x) and trend strength (y)
- Blue = low detection rate, Red = high detection rate
- The dashed line separates non-seasonal (left) from seasonal (right) ground truth

#### 4.2.4 F1 Score vs Trend Strength

Method	No Trend	Max Trend	F1 Drop
Spectral	96.3%	92.5%	3.9%
FFT	93.7%	91.8%	2.0%
AIC	92.2%	87.0%	5.7%
ACF	87.4%	83.5%	4.5%

### How to interpret:

- **F1 Drop:** How much performance degrades when strong trends are present
- Lower drop = more robust to trends

#### 4.2.5 False Positive Rate by Trend Strength

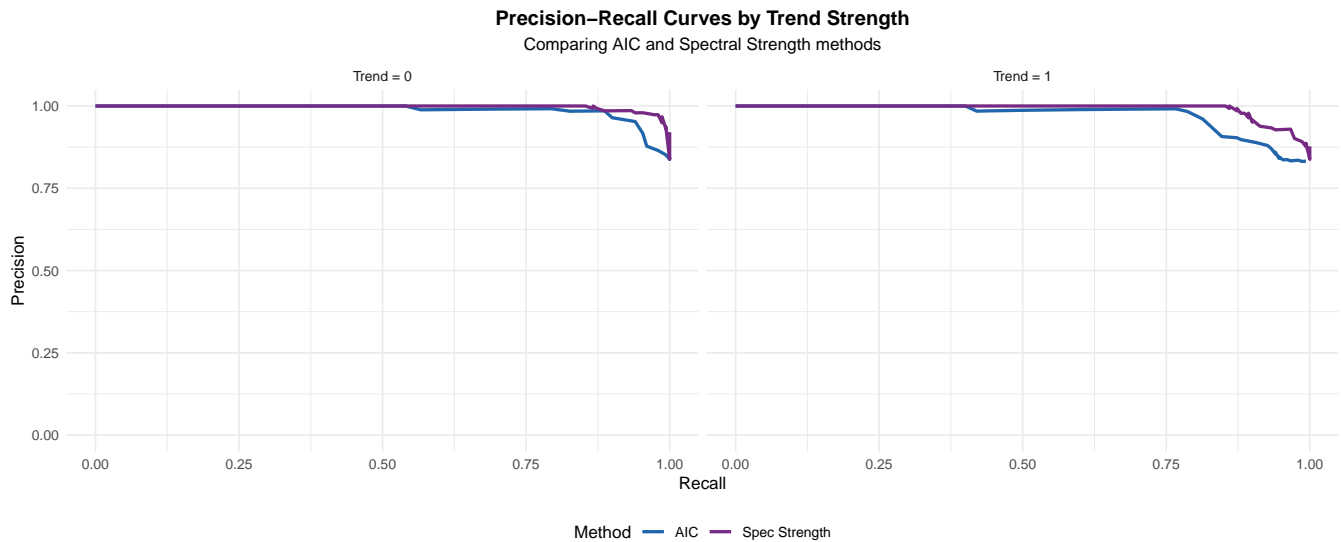


Figure 4: FPR when no seasonality is present, across trend strengths

**Key finding:** FFT's FPR increases dramatically with trend strength because non-linear trends can create spurious peaks in the periodogram.

## 4.3 Simulation 3: Multiple Trend Types

### 4.3.1 Setup

This simulation tests which types of trends cause the most problems for each detection method.

**Trend types tested:**

1. **None:** Flat baseline
2. **Linear:**  $f(t) = t - 0.5$
3. **Quadratic:**  $f(t) = (t - 0.5)^2 - 0.25$
4. **Cubic:**  $f(t) = 2(t - 0.5)^3$
5. **Exponential:**  $f(t) = e^{2t}/e^2 - 0.5$
6. **Logarithmic:**  $f(t) = \log(t + 0.1)$  (normalized)
7. **Sigmoid:**  $f(t) = 1/(1 + e^{-10(t-0.5)}) - 0.5$
8. **Slow sine:**  $f(t) = \sin(2\pi t)$  — one cycle over the entire series

### 4.3.2 Code

```
trend_functions <- list(  
  none = function(t, strength) rep(0, length(t)),  
  linear = function(t, strength) strength * (t - 0.5),  
  quadratic = function(t, strength) strength * ((t - 0.5)^2 - 0.25),  
  cubic = function(t, strength) strength * 2 * (t - 0.5)^3,  
  exponential = function(t, strength) strength * (exp(2 * t) / exp(2) - 0.5),  
  logarithmic = function(t, strength) {  
    strength * (log(t + 0.1) - log(0.1)) / (log(1.1) - log(0.1)) - 0.5 * strength  
  },  
  sigmoid = function(t, strength) strength * (1 / (1 + exp(-10 * (t - 0.5)))) - 0.5),  
  slow_sine = function(t, strength) strength * sin(2 * pi * t)  
)
```

### 4.3.3 Results

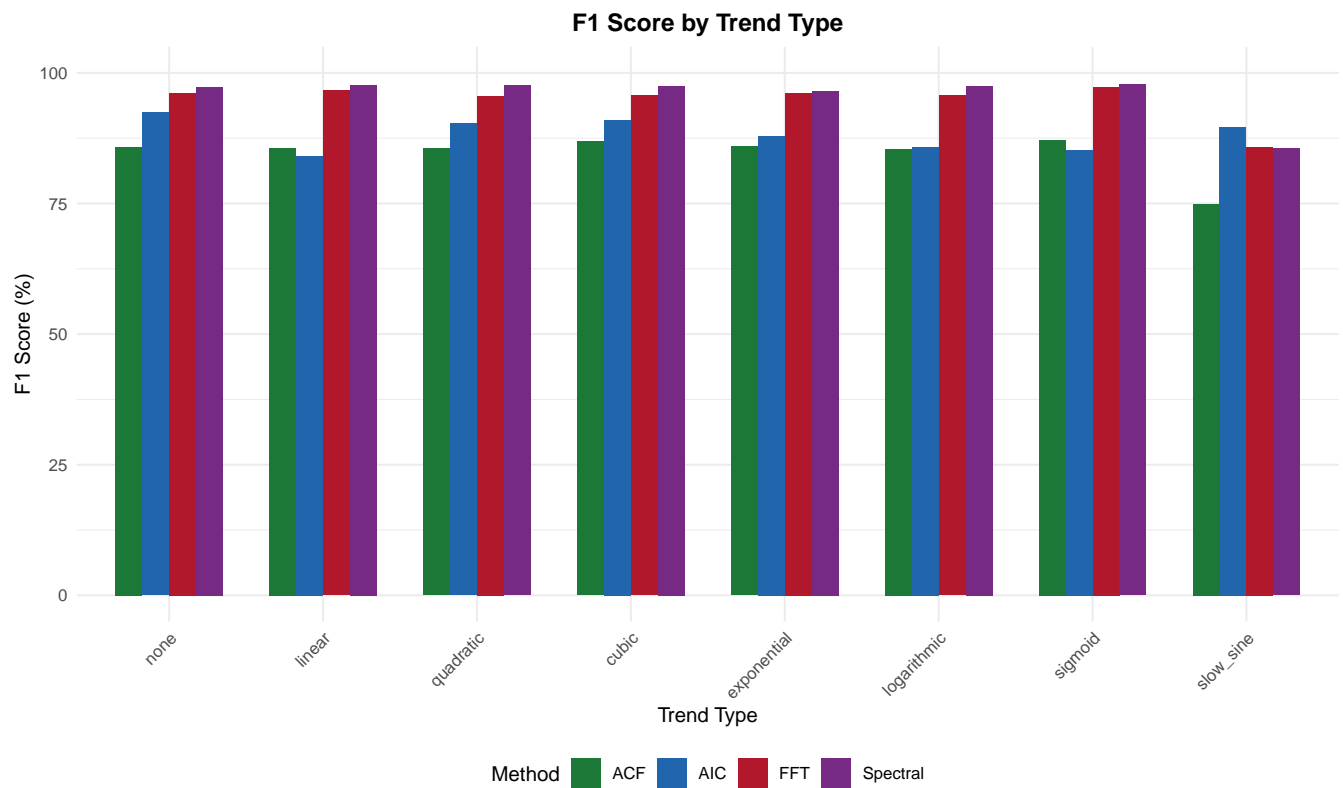


Figure 5: F1 scores by trend type

### 4.3.4 FPR by Trend Type

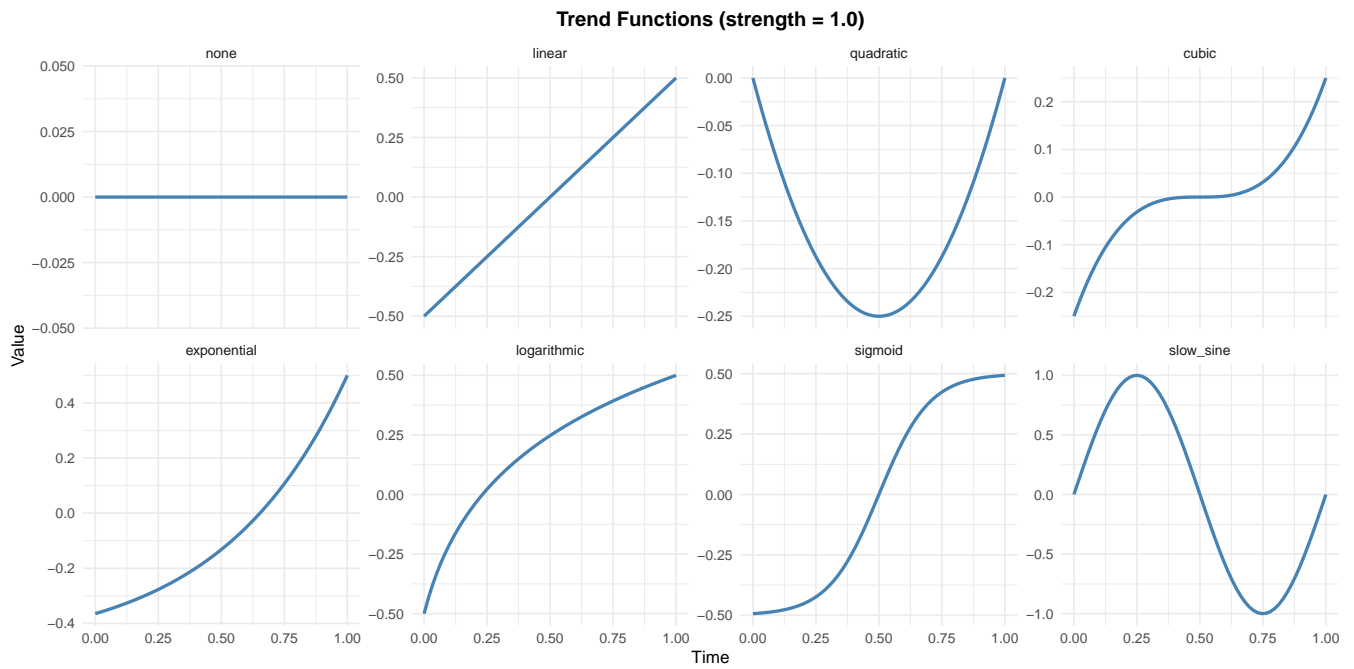


Figure 6: Example trend functions

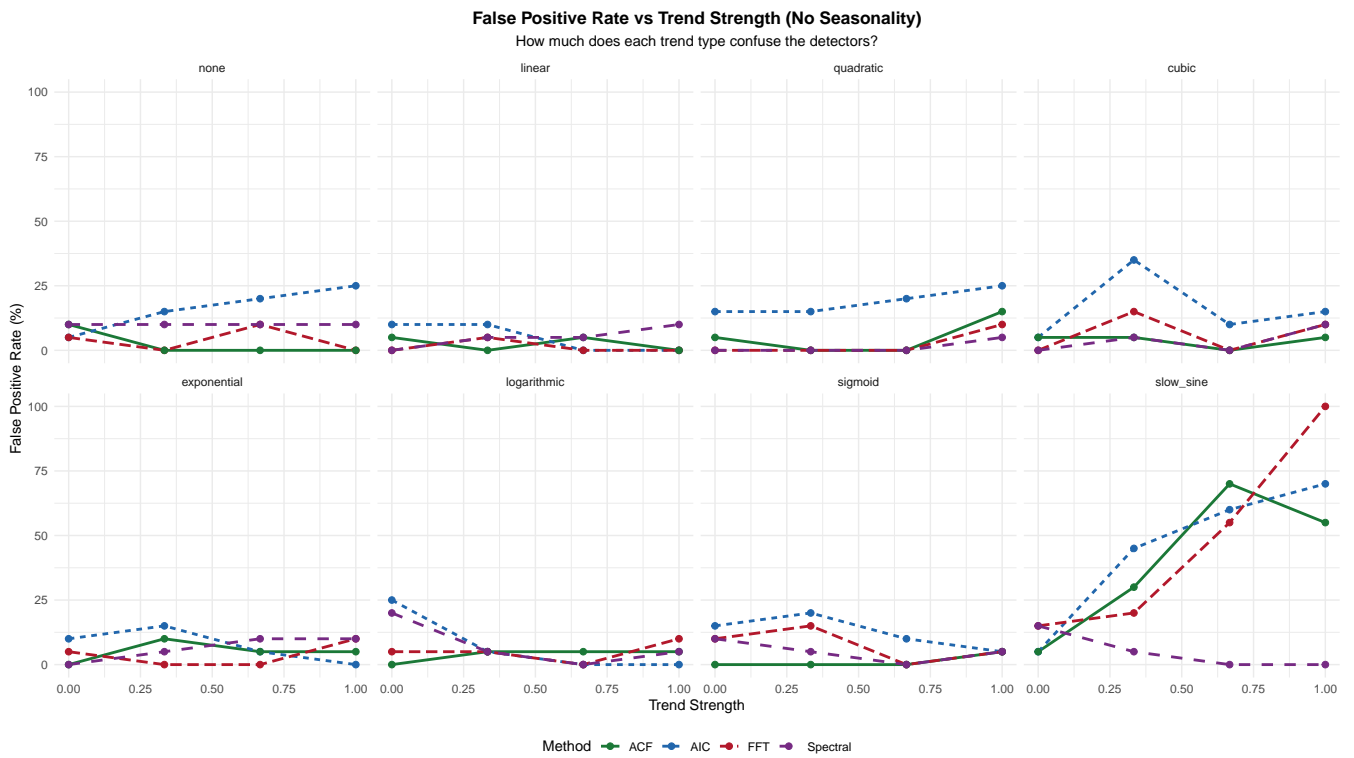


Figure 7: FPR across trend types and strengths



### How to interpret:

- Each panel shows one trend type
- Lines show how FPR changes as trend strength increases
- **Slow sine is catastrophic for FFT**: 100% FPR because FFT detects the slow oscillation as “seasonality”

#### 4.3.5 Most Problematic Trend Types

Trend Type	FFT FPR	Spectral FPR	Issue
<b>slow_sine</b>	<b>100%</b>	0%	FFT detects non-seasonal oscillation
quadratic	10%	5%	Minor
sigmoid	5%	5%	Minor
linear	0%	10%	Handled well

## 5 Additional Robustness Challenges

The current simulations cover varying seasonal strength, non-linear trends, and multiple trend types. However, real-world data often breaks the assumptions made in these idealized setups. This section outlines additional robustness tests that would further validate method performance.

### 5.1 A. Colored Noise (Red Noise)

#### 5.1.1 The Gap

The simulations currently use **white noise**:  $\epsilon \sim N(0, 0.3^2)$  — independent, identically distributed.

#### 5.1.2 Reality

Most physical and economic time series exhibit **red noise** (autocorrelated noise). Temperature records, stock prices, and sensor readings typically have positive autocorrelation where consecutive observations are more similar than distant ones.

**AR(1) noise model:**

$$\epsilon_t = \phi\epsilon_{t-1} + \eta_t, \quad \eta_t \sim N(0, \sigma^2)$$

where  $\phi \in (0, 1)$  controls the autocorrelation strength.

### 5.1.3 Why It Matters

Spectral methods (FFT, Spectral Strength) and ACF are notoriously prone to **false positives** in red noise environments:

- A slow random walk can appear as a “trend” or low-frequency cycle
- AR(1) processes have elevated low-frequency power, mimicking seasonality
- The periodogram of red noise is not flat — it decays as  $\sim 1/f^2$

**Recommended test:** Generate AR(1) noise with  $\phi \in \{0.3, 0.5, 0.7, 0.9\}$  and measure FPR for each detection method.

## 5.2 B. Multiple Seasonalities

### 5.2.1 The Gap

The simulations assume a **single fixed period** (5 cycles over the observation window).

### 5.2.2 Reality

Data often contains **multiple nested seasonalities**:

- Energy consumption: daily cycle + weekly cycle + annual cycle
- Retail sales: weekly patterns + monthly patterns + holiday effects
- Traffic data: hourly patterns + daily patterns

**Multi-seasonal model:**

$$y(t) = s_1 \sin(2\pi f_1 t) + s_2 \sin(2\pi f_2 t) + \epsilon$$

where  $f_1$  and  $f_2$  are different frequencies (e.g.,  $f_1 = 5$  cycles,  $f_2 = 20$  cycles).

### 5.2.3 Why It Matters

- Can **FFT** and **Spectral Strength** distinguish between a dominant seasonal frequency and secondary harmonics?
- Does **Variance Strength** get diluted when multiple periods are present but only one is specified?
- What happens when the specified period misses the dominant frequency?

**Recommended test:** Generate signals with primary period  $p_1$  and secondary period  $p_2$  with varying amplitude ratios, then test detection with period set to  $p_1$ .

## 5.3 C. Amplitude Modulation (Time-Varying Seasonality)

### 5.3.1 The Gap

The signal model assumes **constant seasonal strength**  $s$  across the entire curve:

$$y(t) = s \cdot \sin(2\pi f t) + \epsilon$$

### 5.3.2 Reality

Seasonality often **grows or shrinks over time** (multiplicative seasonality):

- Heating demand: seasonal amplitude is higher in extreme years
- Economic growth: seasonal patterns amplify as the economy scales
- Agricultural yields: seasonal variation depends on climate conditions that vary year-to-year

**Amplitude-modulated model:**

$$y(t) = s(t) \cdot \sin(2\pi ft) + \epsilon$$

where  $s(t)$  is a time-varying envelope, e.g.,  $s(t) = s_0 \cdot (1 + \alpha t)$  (linear growth) or  $s(t) = s_0 \cdot (1 + \beta \sin(2\pi t/T))$  (periodic modulation).

### 5.3.3 Why It Matters

- **Variance Strength** averages variance globally — it may under-report seasonality if the signal is strong in only half the time domain
- Methods might fail when seasonality “emerges” partway through the series
- Detection thresholds calibrated on constant-amplitude signals may be inappropriate

**Recommended test:** Generate signals where seasonal amplitude varies from 0→1 across the series, and test whether methods detect “partial seasonality.”

## 5.4 D. Outliers and Anomalies

### 5.4.1 The Gap

The current noise model is **Gaussian** with constant variance.

### 5.4.2 Reality

Real sensors and measurements have:

- **Spikes:** Sudden large values (e.g., sensor glitches, recording errors)
- **Dropouts:** Missing or zero values
- **Level shifts:** Sudden changes in baseline (e.g., sensor recalibration)
- **Heavy tails:** Non-Gaussian error distributions

**Contaminated noise model:**

$$\epsilon_t = \begin{cases} N(0, \sigma^2) & \text{with probability } 1 - p \\ N(0, k^2 \sigma^2) & \text{with probability } p \end{cases}$$

where  $p$  is the outlier probability and  $k > 1$  is the outlier magnitude multiplier.

### 5.4.3 Why It Matters

- A **single large outlier** can distort the FFT spectrum, creating spurious peaks
- Outliers inflate  $\text{Var}(R_t)$  in the Variance Strength denominator, potentially causing **false negatives**
- ACF is sensitive to outliers, which can destroy or create spurious autocorrelation

**Recommended test:** Add  $p = 5\%$  outliers with magnitude  $k = 5$  and measure degradation in F1 scores.

## 5.5 Summary of Robustness Gaps

Challenge	Current Assumption	Real-World Behavior	Methods at Risk
<b>Red Noise</b>	White noise (i.i.d.)	Autocorrelated (AR)	FFT, ACF, Spectral
<b>Multiple</b>	Single period	Nested periods	All (period
<b>Seasonalities</b>			misspecification)
<b>Amplitude</b>	Constant strength	Time-varying strength	Variance (averaging)
<b>Modulation</b>			
<b>Outliers</b>	Gaussian	Heavy-tailed, spikes	FFT, Variance, ACF

## 5.6 Experimental Results

We ran simulations for each robustness scenario. Key findings:

### 5.6.1 A. Red Noise Results

#### Example: Seasonal Signal with Different Noise Types

Same seasonal component ( $s=0.5$ ), different noise autocorrelation

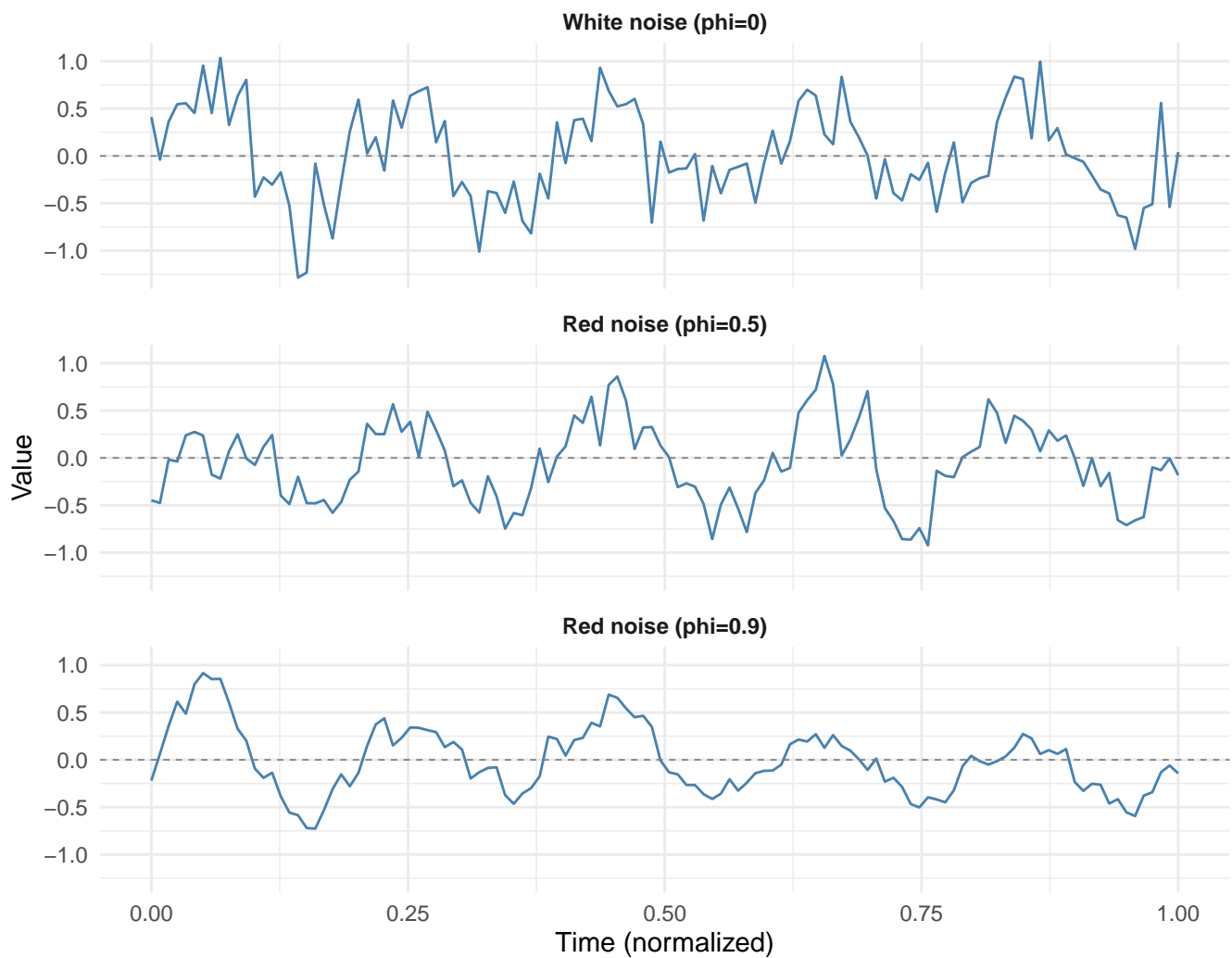


Figure 8: Example time series with different noise types

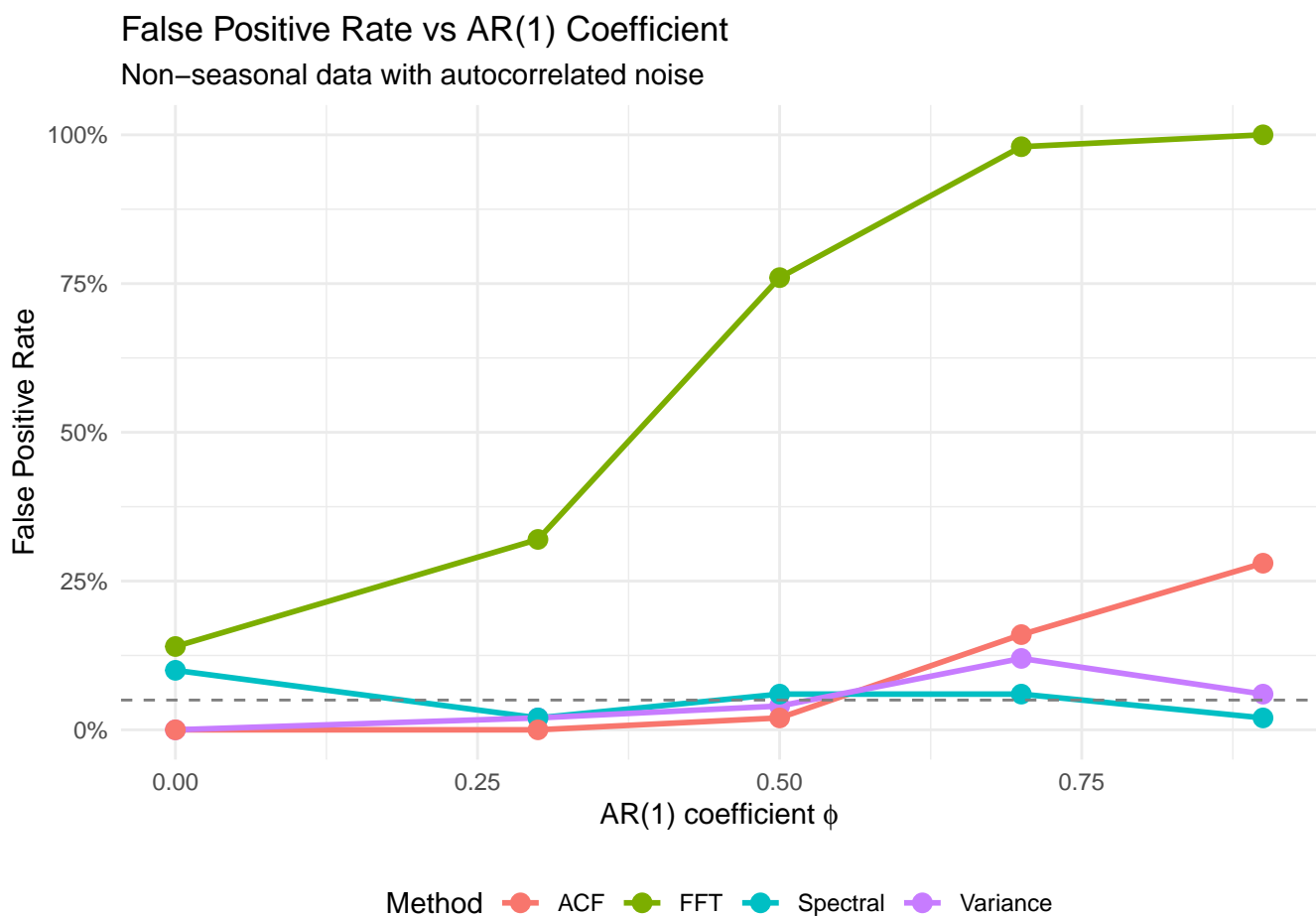


Figure 9: False positive rate vs AR(1) coefficient

AR(1) $\phi$	Variance FPR	Spectral FPR	FFT FPR	ACF FPR
0.0	0%	10%	14%	0%
0.3	2%	2%	32%	0%
0.5	4%	6%	76%	2%
0.7	12%	6%	98%	16%
0.9	6%	2%	100%	28%

**Key finding:** FFT is catastrophically affected by red noise, with FPR reaching 100% at high autocorrelation. Variance and Spectral methods remain relatively robust.

## 5.6.2 B. Multiple Seasonalities Results

### Example: Multiple Seasonalities

Different combinations of primary (5 cycles) and secondary (20 cycles) periods

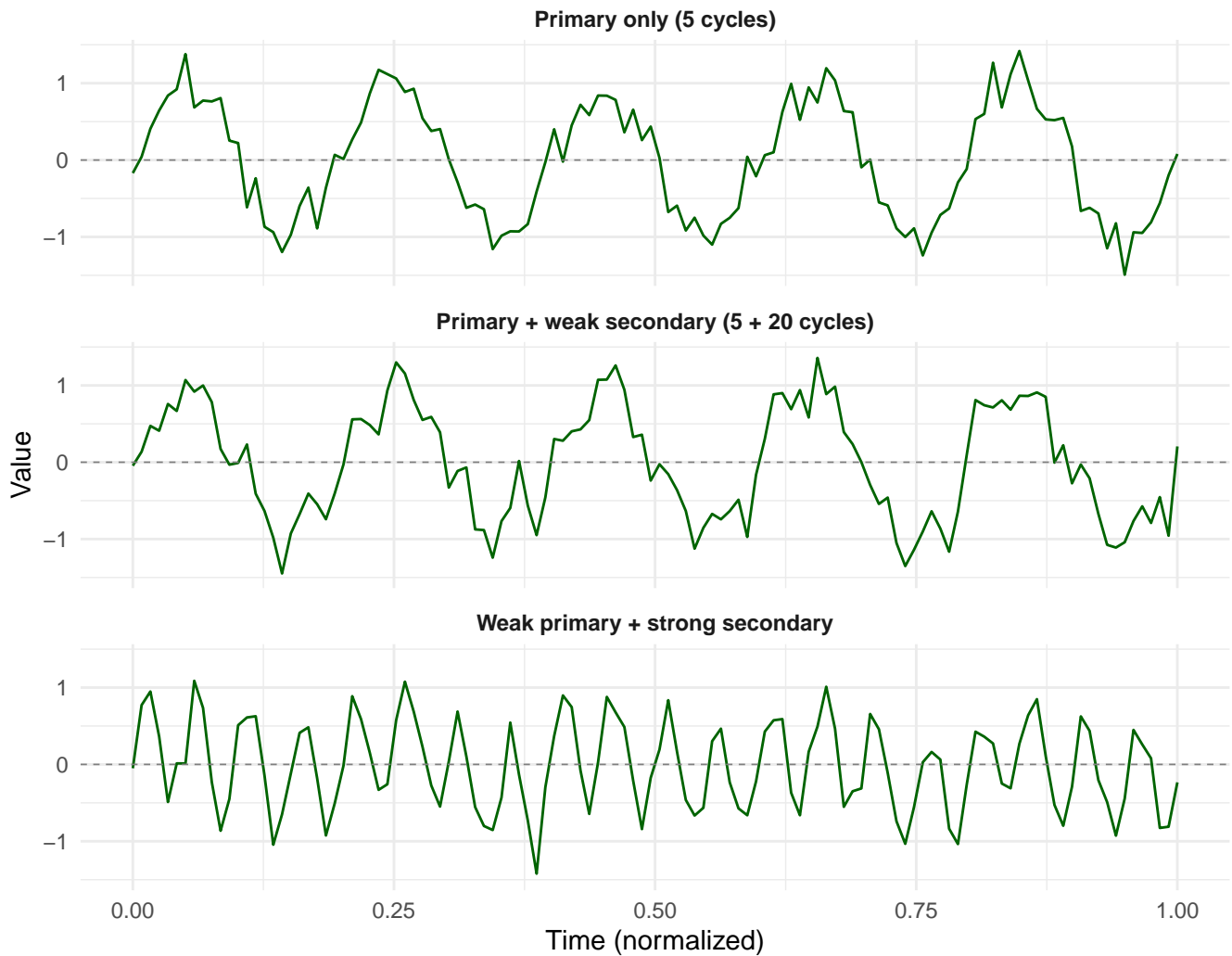


Figure 10: Example time series with multiple seasonal components

When secondary seasonality dominates (amplitude 0.7 vs primary 0.3):

- Variance Strength TPR drops to 4%
- Spectral and FFT maintain 100% TPR (detect any periodicity)
- Period specification is critical for Variance method

### 5.6.3 C. Amplitude Modulation Results

Example: Amplitude Modulation (Time-Varying Seasonality)

Same base frequency, different envelope functions

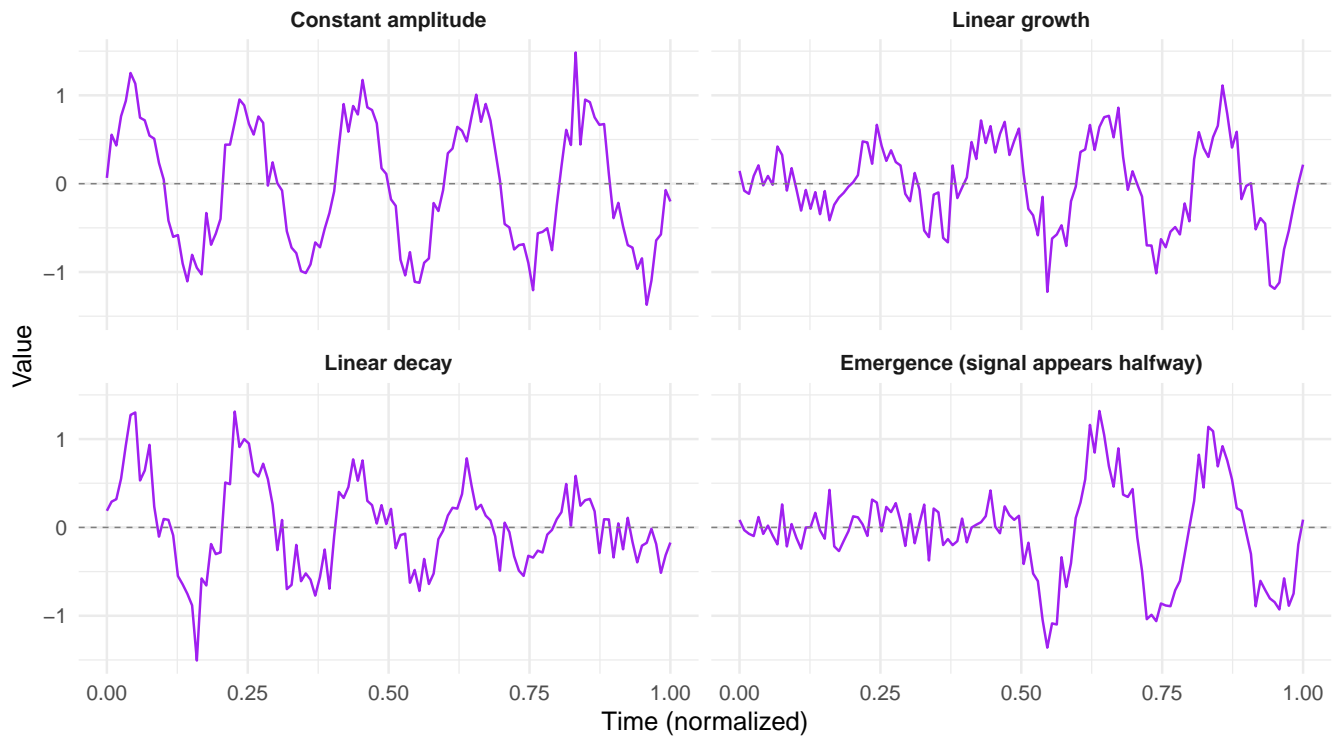


Figure 11: Example time series with different amplitude modulation types



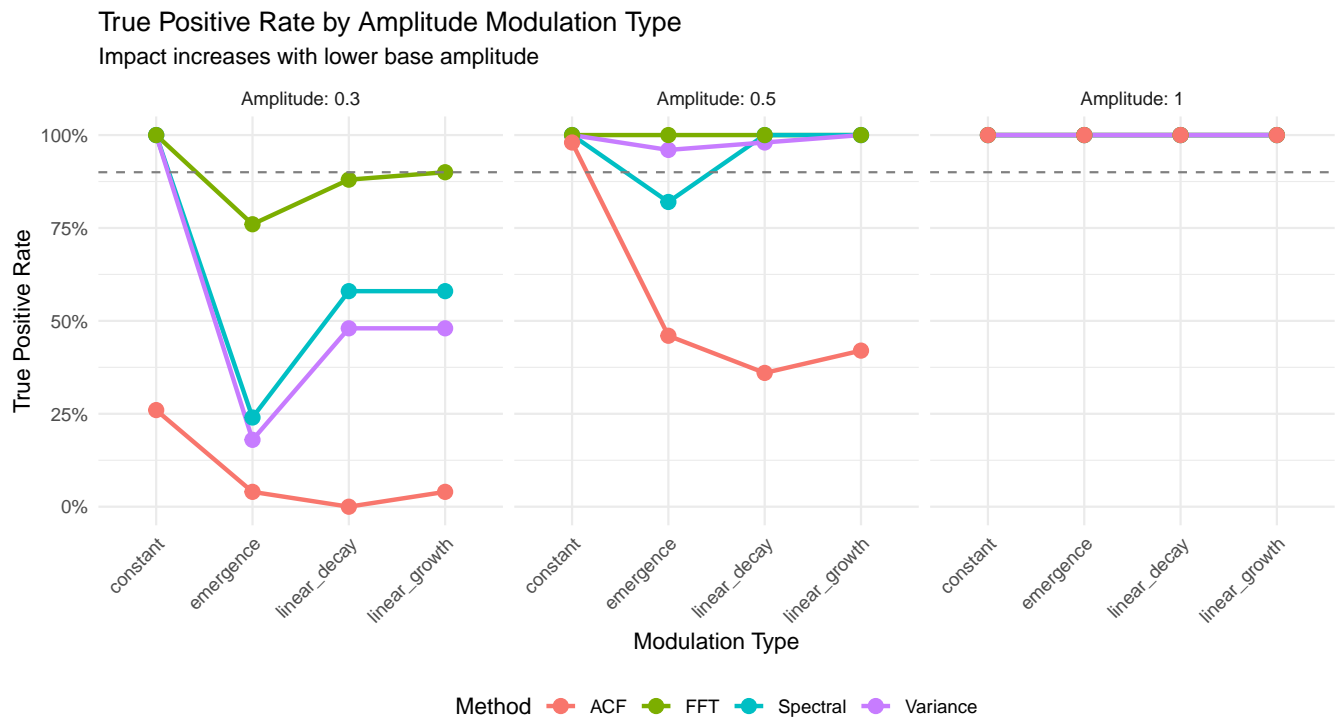


Figure 12: TPR by amplitude modulation type and base amplitude

At base amplitude 0.3:

Modulation	Variance	Spectral	FFT	ACF
constant	100%	100%	100%	26%
linear_growth	48%	58%	90%	4%
linear_decay	48%	58%	88%	0%
emergence	<b>18%</b>	24%	76%	4%

**Key finding:** “Emergence” pattern (signal only in second half) is most challenging. FFT remains most robust to modulation.

#### 5.6.4 D. Outlier Results

##### Example: Outliers and Anomalies

Same seasonal signal ( $s=0.5$ ), increasing outlier severity

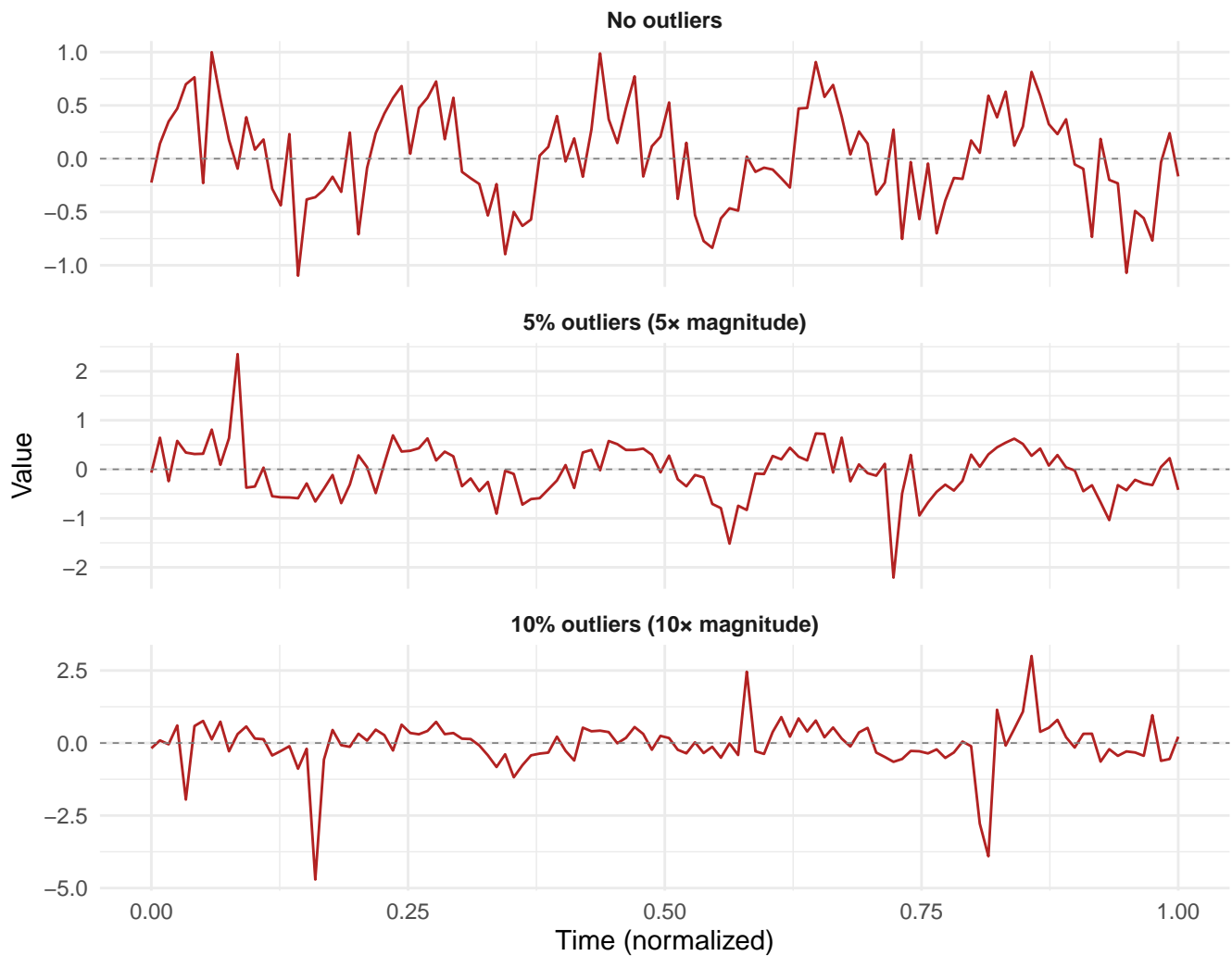


Figure 13: Example time series with different outlier severities

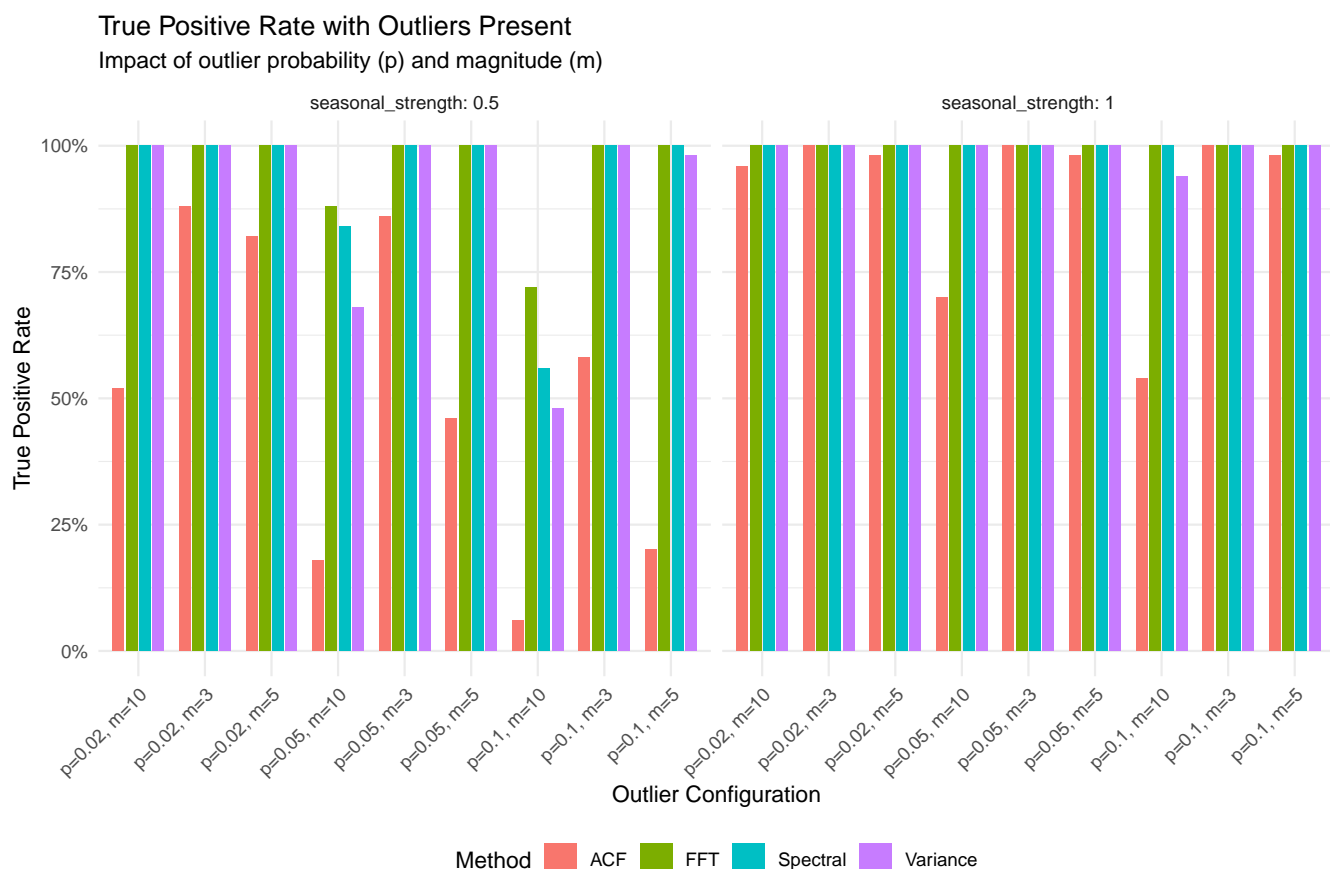


Figure 14: Impact of outliers on TPR

At 10% outliers with magnitude 10× for seasonal strength 0.5: - Variance TPR: 40% - Spectral TPR: 52% - FFT TPR: 66% - ACF TPR: 2%

**Key finding:** All methods significantly degraded by severe outliers. Pre-filtering recommended.

## 6 Key Findings

### 6.1 Method Ranking

1. **Variance Strength** (F1=97.3%, FPR=2%): Best overall when period is known
2. **Spectral Strength** (F1=95.3%, FPR=10%): Most robust to different trend types
3. **FFT Confidence** (F1=94.8%, FPR=4%): Good but vulnerable to slow oscillations
4. **AIC Comparison** (F1=91.5%, FPR=18%): Interpretable but higher FPR
5. **ACF Confidence** (F1=85.4%, FPR=10%): Conservative, misses weak seasonality

### 6.2 Critical Issues Found

1. **Period units matter:** The period parameter in `seasonal_strength()` must be in argvals units, not raw time units (e.g., 0.2 not 12)
2. **FFT is vulnerable to slow oscillations:** Any periodic signal (even non-seasonal) triggers detection

## 7 Recommendations

### 7.1 For Unknown Datasets

**Primary recommendation: Variance Strength**

```
# Calculate period in argvals units
# If argvals is in [0,1] and you expect 5 annual cycles:
period_in_argvals_units <- 1 / 5 # = 0.2

strength <- seasonal_strength(fd,
                             period = period_in_argvals_units,
                             method = "variance",
                             detrend = "linear")

is_seasonal <- strength > 0.2
```

**For robustness to unknown trends: Spectral Strength**

```
strength <- seasonal_strength(fd,
                             period = period_in_argvals_units,
                             method = "spectral",
                             detrend = "linear")

is_seasonal <- strength > 0.3
```

**Ensemble approach (most robust):**

```
var_detected <- seasonal_strength(fd, period, method = "variance") > 0.2
spec_detected <- seasonal_strength(fd, period, method = "spectral") > 0.3
fft_detected <- estimate_period(fd, method = "fft")$confidence > 6.0

# Majority vote
is_seasonal <- (var_detected + spec_detected + fft_detected) >= 2
```

### 7.2 Threshold Guidelines

Method	Threshold	Calibration
Variance Strength	0.2	95th percentile of noise ~0.17
Spectral Strength	0.3	95th percentile of noise ~0.29
FFT Confidence	6.0	95th percentile of noise ~5.7
ACF Confidence	0.25	95th percentile of noise ~0.22
AIC Difference	0	Fourier better → positive difference

**Calibration methodology:** All thresholds were calibrated using pure noise data (seasonal strength = 0, no trend) by taking the 95th percentile of each method's score distribution. This ensures approximately 5% false positive rate on clean data. Note that FPR may increase when confounding trends are present (see Simulation 2 and 3 results).

## 8 Conclusion

For detecting seasonality in functional time series:

1. **Variance Strength** is the most accurate method when the seasonal period is known
2. **Spectral Strength** is most robust to confounding trends and unknown oscillations
3. **FFT Confidence** works well but is vulnerable to slow non-seasonal oscillations
4. **AIC Comparison** provides an interpretable alternative but has higher false positive rates
5. **ACF Confidence** is conservative (low FPR) but misses weak seasonality

The key insight is that simple variance-based decomposition outperforms more complex spectral methods when properly configured with the correct period parameter.

### 8.1 Limitations and Future Work

The current simulations use idealized conditions (white noise, single seasonality, constant amplitude). Real-world data presents additional challenges:

- **Red noise** (autocorrelated errors) can inflate false positives for spectral methods
- **Multiple seasonalities** require careful period specification
- **Amplitude modulation** may dilute variance-based measures
- **Outliers** can distort all methods

See Section 5 for detailed discussion of these robustness challenges and recommended tests.

## 9 Appendix: Fourier vs P-spline Comparison

The AIC comparison method is based on the observation that Fourier bases naturally capture periodic patterns better than P-splines for seasonal data.

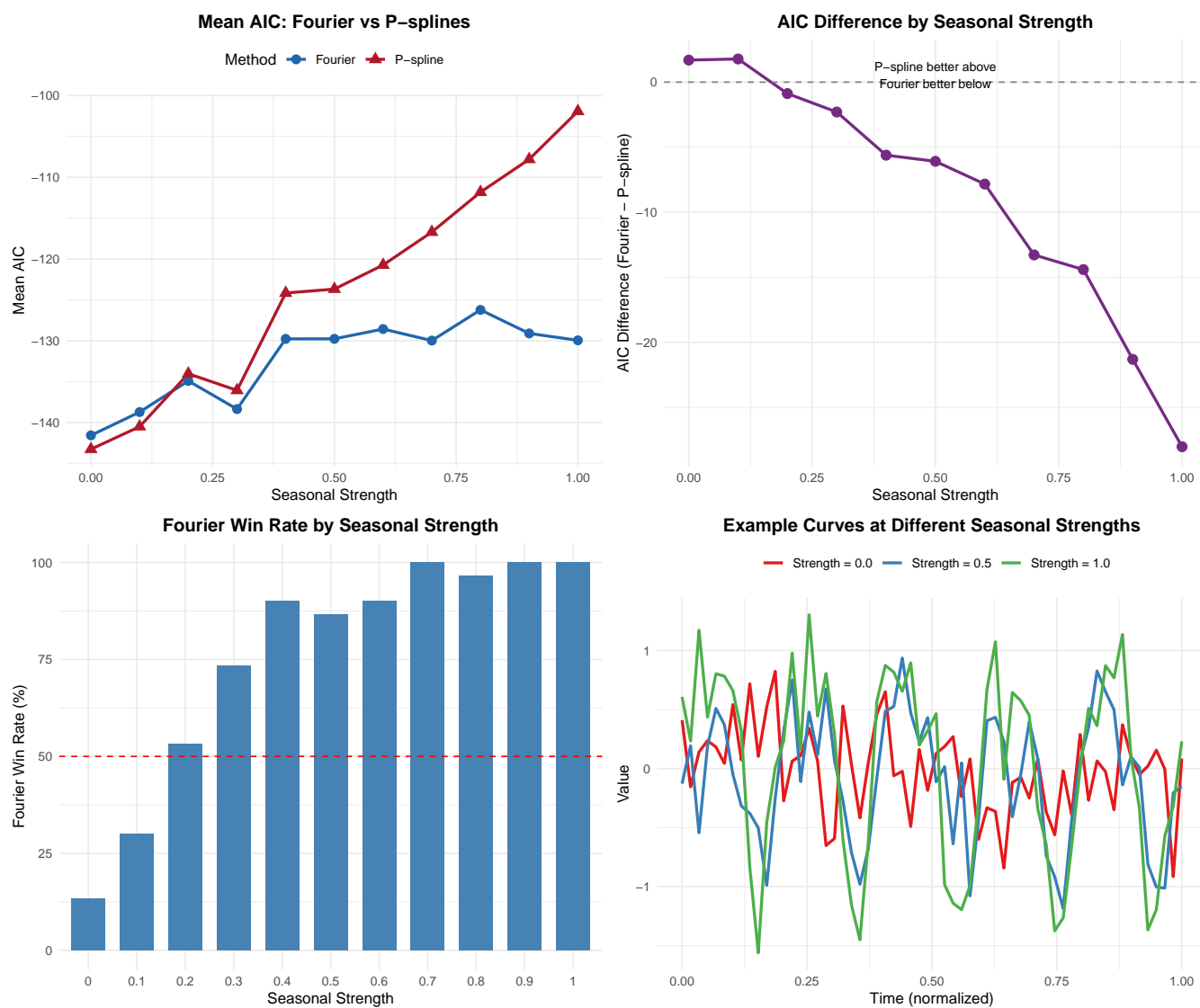


Figure 15: Fourier vs P-spline AIC comparison

### How to interpret:

- Top left: Mean AIC for both methods across seasonal strengths
- Top right: AIC difference (Fourier - P-spline) showing crossover point
- Bottom left: Fourier win rate at each strength level
- Bottom right: Example curves at different seasonal strengths

## 10 Appendix: File Listing

All simulation scripts and results are in `scripts/seasonal_simulation/`:

- `seasonality_detection_comparison.R` — Main comparison (Simulation 1)
- `seasonality_detection_with_trend.R` — Non-linear trend study (Simulation 2)
- `seasonality_detection_trend_types.R` — Multiple trend types (Simulation 3)
- `seasonal_basis_comparison.R` — Fourier vs P-spline AIC study
- `seasonality_robustness_tests.R` — Robustness tests (red noise, multi-seasonal, etc.)
- `generate_training_data.R` — Generate training data for ML classifiers

PDF outputs are in the `plots/` subfolder.