HYPERGRAPHS WITH ARBITRARILY SMALL CODEGREE TURÁN DENSITY

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Abstract. Let $k \geqslant 3$. Given a k-uniform hypergraph H, the minimum codegree $\delta(H)$ is the largest $d \in \mathbb{N}$ such that every $(k-1)$-set of $V(H)$ is contained in at least d edges. Given a k-uniform hypergraph F, the codegree Turán density $\gamma(F)$ of F is the smallest $\gamma \in [0,1]$ such that every k-uniform hypergraph on n vertices with $\delta(H) \geqslant (\gamma + o(1))n$ contains a copy of F. Similarly as other variants of the hypergraph Turán problem, determining the codegree Turán density of a hypergraph is in general notoriously difficult and only few results are known.

In this work, we show that for every $\varepsilon > 0$, there is a k-uniform hypergraph F with $0 < \gamma(F) < \varepsilon$. This is in contrast to the classical Turán density, which cannot take any value in the interval $(0, k!/k^k)$ due to a fundamental result by Erdős.

§1. Introduction

A k-uniform hypergraph (or k-graph) H consists of a vertex set $V(H)$ together with a set of edges $E(H) \subseteq V(H)^{(k)} = \{S \subseteq V(H) : |S| = k\}$. Given a k-graph F and $n \in \mathbb{N}$, the Turán number of n and F, $\mathrm{ex}(n, F)$, is the maximum number of edges an n-vertex k-graph can have without containing a copy of F. Since the main interest lies in the asymptotics, the *Turán density* $\pi(F)$ of a k-graph F is defined as

$$\pi(F) = \lim_{n \longrightarrow \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}} .$$

Determining the value of $\pi(F)$ for k-graphs (with $k \geqslant 3$) is one of the central open problems in combinatorics. In particular, the problem of determining the Turán density of the complete 3-graph on four vertices, i.e., $\pi(K_4^{(3)})$, was asked by Turán in 1941 [15] and Erdős [5] offered 1000\$ for its resolution. Despite receiving a lot of attention (see for instance the survey by Keevash [8]), this problem, and even the seemingly simpler problem of determining $\pi(K_4^{(3)-})$, where $K_4^{(3)-}$ is the $K_4^{(3)}$ minus one edge, remain open.

Several variations of this type of problem have been considered, see for instance [2, 6, 12] and the references therein. The variant that we are concerned with here asks how large the *minimum codegree* of an F-free k-graph can be. Given a k-graph $H = (V, E)$ and $S \subseteq V$, the degree $d(S)$ of S (in H) is the number of edges containing S, i.e., $d(S) = |\{e \in E : S \subseteq e\}|$. The *minimum codegree* of H is defined as $\delta(H) = \min_{x \in V^{(k-1)}} d(x)$.

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Given a k-graph F and $n \in \mathbb{N}$, Mubayi and Zhao [9] introduced the *codegree Turán number* $\mathrm{ex}_{\mathrm{co}}(n, F)$ of n and F as the maximum d such that there is an F-free k-graph H on n vertices with $\delta(H) \geqslant d$. Moreover, they defined the *codegree Turán density* $\gamma(F)$ of F as

$$\gamma(F) = \lim_{n \longrightarrow \infty} \frac{ex_{\mathrm{co}}(n, F)}{n}$$

and proved that this limit always exists. It is not hard to see that $\gamma(F) \leqslant \pi(F)$. The codegree Turán density of a family $\mathcal{F}$ of k-graphs is defined analogously.

Similarly as for the Turán density, determining the exact codegree Turán density of a given hypergraph can be very difficult and so it is only known for very few hypergraphs (see the table in [2]).

In this work, we show that there are k-graphs with arbitrarily small but strictly positive codegree Turán densities.

Theorem 1.1. *For every $\xi > 0$ and $k \geqslant 3$, there is a k-graph F with $0 < \gamma(F) < \xi$.*

Note that this is in stark contrast to the Turán density and the uniform Turán density, another variant of the Turán density that was introduced by Erdős and Sós [6]. Regarding the former, a classical result by Erdős [4] states that for no k-graph the Turán density is in the interval $(0, k!/k^k)$. Regarding the latter Reiher, Rödl, and Schacht [13] proved that for no 3-graph the uniform Turán density is in $(0, 1/27)$. Mubayi and Zhao [9] defined

$$\Gamma^{(k)} := \{\gamma(F) \colon F \text{ is a } k\text{-graph}\} \subseteq [0, 1]$$

and

$$\widetilde{\Gamma}^{(k)} := \{\gamma(\mathcal{F}) \colon \mathcal{F} \text{ is a family of } k\text{-graphs}\} \subseteq [0, 1]\,.$$

We remark that $\Gamma^{(k)} \subseteq \widetilde{\Gamma}^{(k)}$ and that similar sets have been studied for the classical Turán density (see, for instance, [1, 7, 11, 14]). Mubayi and Zhao [9] showed that $\widetilde{\Gamma}^{(k)}$ is dense in $[0, 1]$ and asked if this is also true for $\Gamma^{(k)}$. Their proof for $\widetilde{\Gamma}^{(k)}$ is based on showing that zero is an accumulation point of $\widetilde{\Gamma}^{(k)}$. Theorem 1.1 implies the same for $\Gamma^{(k)}$.

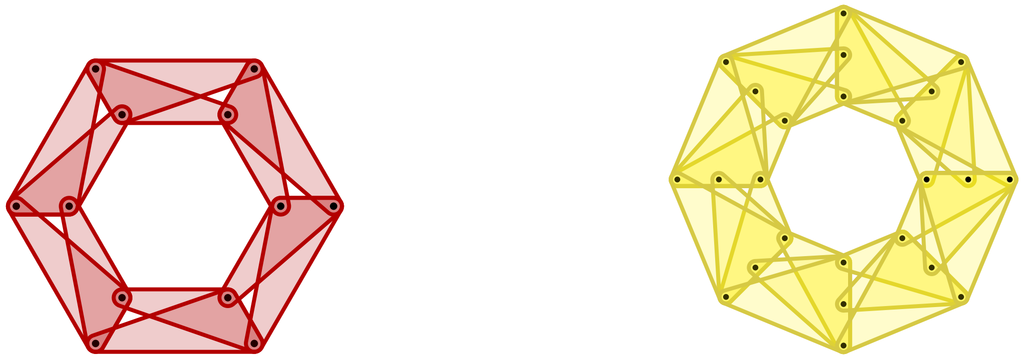
Corollary 1.2. *Zero is an accumulation point of $\Gamma^{(k)}$.*

Given a k-graph $H = (V, E)$ and a subset of vertices $A = \{v_1, \ldots, v_s\} \subseteq V$, we omit parentheses and commas and simply write $A = v_1 \cdots v_s$. For the proof of Theorem 1.1, we consider the following hypergraphs.

Definition 1.3. For integers $\ell \geqslant k \geqslant 2$, we define the *$k$-uniform zycle of length ℓ* as the k-graph $Z_\ell^{(k)}$ given by

$$\begin{aligned} V(Z_\ell^{(k)}) =& \{v_i^j \colon i \in [\ell], j \in [k-1]\}, \text{ and} \\ E(Z_\ell^{(k)}) =& \{v_i^1 v_i^2 \cdots v_i^{k-1} v_{i+1}^j \colon i \in [\ell], j \in [k-1]\}\,, \end{aligned}$$

where the sum of indices is taken modulo ℓ.



(A) Copy of $Z_6^{(3)}$ (B) Copy of $Z_8^{(4)}$

Observe that $Z_\ell^{(k)}$ has $(k-1)\ell$ vertices and $(k-1)\ell$ edges. Moreover, $Z_\ell^{(2)} = C_\ell$. When $k \in \mathbb{N}$ is clear from the context, we omit it in the notation.

The following bounds on the codegree Turán density of zycles imply Theorem 1.1.

Theorem 1.4. *Let $k \geqslant 3$. For every $d \in (0,1]$, there is an $\ell \in \mathbb{N}$ such that*

$$\frac{1}{2(k-1)^\ell} \leqslant \gamma(Z_\ell) \leqslant d\,.$$

In fact we show that $\gamma(Z_\ell) > 0$ for every $\ell \geqslant 3$ (see Lemma 2.6).

Finally, we prove that any proper subgraph of $Z_\ell^{(3)}$ has codegree Turán density zero. Let $Z_\ell^{(3)-}$ be the 3-graph obtained from $Z_\ell^{(3)}$ by deleting one edge.

Theorem 1.5. *Let $\ell \geqslant 3$. Then $\gamma(Z_\ell^{(3)-}) = 0$.*

To prove Theorem 1.5, we generalise a method developed by the authors together with Sales in [10].

§2. Proof of Theorem 1.4

Given a k-graph $H = (V, E)$, we define the neighbourhood of $x \in V^{(k-1)}$ as

$$N(x) = \{v \in V : x \cup \{v\} \in E\}\,.$$

Given a $(k-1)$-subset of vertices $e \in V^{(k-1)}$, we define the *back neighbourhood of* e and the *back degree of* e, respectively, by

$$\overleftarrow{N}(e) = \{f \in V^{(k-1)} \colon f \cup \{v\} \in E \text{ for every } v \in e\} \quad \text{and} \quad \overleftarrow{d}(e) = \big|\overleftarrow{N}(e)\big|\,.$$

Moreover, given a k-graph H and two disjoint $(k-1)$-sets of vertices $e, f \in V(H)^{(k-1)}$, we write $e \rhd f$ to mean $e \in \overleftarrow{N}(f)$. Thus, it is easy to see that Z_ℓ can be viewed as a sequence of $(k-1)$-sets of vertices $e_1, \dots, e_\ell$ such that $e_i \rhd e_{i+1}$ for every $i \in [\ell]$ (where the sum is taken modulo ℓ).

We split the proof in the lower and upper bound.

2.1. **Upper bound.** Here we prove the following lemma that yields the upper bound in Theorem 1.4.

Lemma 2.1. *Let $k \geqslant 3$. For every $d \in (0,1]$, there is a positive integer $\ell \in \mathbb{N}$ such that*

$$\gamma(Z_\ell) \leqslant d\,.$$

We will make use of the following lemma due to Mubayi and Zhao [9].

Lemma 2.2. *Fix $k \geqslant 2$. Given $\varepsilon, \alpha > 0$ with $\alpha + \varepsilon < 1$, there exists an $m_0 \in \mathbb{N}$ such that the following holds for every n-vertex k-graph H with $\delta(H) \geqslant (\alpha+\varepsilon)n$. For every integer m with $m_0 \leqslant m \leqslant n$, the number of m-sets $S \subseteq V(H)$ satisfying $\delta(H[S]) \geqslant (\alpha + \varepsilon/2)m$ is at least $\frac{1}{2}\binom{n}{m}$.*

For positive integers f, c and a k-graph F on f vertices, denote the *c-blow-up of F* by $F(c)$. This is the f-partite k-graph $F(c) = (V,E)$ with $V = V_1 \dot\cup \dots \dot\cup V_f$, $|V_i| = c$ for $1 \leqslant i \leqslant f$, and $E = \{v_{i_1} \cdots v_{i_k} : v_{i_j} \in V_{i_j}$ for every $j \in [k]$ and $i_1, \dots, i_k \in E(F)\}$.

By cyclically going around the vertices, it is easy to check that the blow-up of a zycle of length r contains zycles whose length is a multiple of r.

Fact 2.3. *For $k, r \geqslant 3$ and $c \in \mathbb{N}$, we have $Z_{cr} \subseteq Z_r(c)$.*

The following supersaturation result follows from a standard application of Lemma 2.2 combined with a classical result by Erdős [4].

Proposition 2.4. *Let $t, k, c \in \mathbb{N}$ with $k \geqslant 2$ and let $\mathcal{F} = \{F_1, \dots F_t\}$ be a finite family of k-graphs with $|V(F_i)| = f_i$ for all $i \in [t]$. For every $\varepsilon > 0$, there exists a $\zeta > 0$ such that for sufficiently large $n \in \mathbb{N}$, the following holds. Every n-vertex k-graph H with $\delta(H) \geqslant (\gamma(\mathcal{F}) + \varepsilon)n$ contains $\zeta\binom{n}{f_i}$ copies of F_i for some $i \in [t]$. Consequently, H contains a copy of $F_i(c)$.*

Proof. Given t, k, c and $\varepsilon > 0$, let $m_0 \in \mathbb{N}$ be given by Lemma 2.2, and let $C \in \mathbb{N}$ with $C^{-1} \ll c^{-1}$. Let $m \in \mathbb{N}$ with $m^{-1} \ll \varepsilon, m_0^{-1}, C^{-1}, f_i^{-1}, k^{-1}, t^{-1}$, and set

$$\zeta = \frac{1}{2t\binom{m}{\max_i f_i}}\,.$$

Now let $n \in \mathbb{N}$ be sufficiently large, i.e., $n^{-1} \ll \zeta$. Let H be given as in the statement of the lemma. Due to Lemma 2.2, at least $\frac{1}{2}\binom{n}{m}$ induced m-vertex subhypergraphs of H have minimum codegree at least $(\gamma(\mathcal{F}) + \varepsilon/2)m$. Since m is sufficiently large, each of those subgraphs will contain a copy of a hypergraph in $\mathcal{F}$. Therefore, there exists an $i \in [t]$ such that there are at least $\frac{1}{2t}\binom{n}{m}$ induced m-vertex subgraphs of H containing a copy of F_i.

Set $F = F_i$ and $f = f_i$, and define an auxiliary f-uniform hypergraph G_F by $V(G_F) = V(H)$ and $E(G_F) = \{S \in V(H)^{(f)} \colon F \subseteq H[S]\}$. By the counting above, we have

$$|E(G_F)| \geqslant \frac{1}{2t}\frac{\binom{n}{m}}{\binom{n-f}{m-f}} = \frac{1}{2t\binom{m}{f}}\binom{n}{f} \geqslant \zeta\binom{n}{f}\,.$$

A result by Erdős [4] implies that G_F contains a copy of $K_f^{(f)}(C)$. Each edge of $K_f^{(f)}(C)$ corresponds to (at least) one embedding of F into H, in one of the at most $f!$ possible ways that F could be embedded into the f vertex classes of $K_f^{(f)}(C)$ (viewed as vertex sets of H). Thus, when colouring the edges of $K_f^{(f)}(C)$ accordingly, Ramsey's theorem entails that there is a $K_f^{(f)}(c) \subseteq K_f^{(f)}(C)$ for which all embeddings of F follow the same permutation. This yields a copy $F(c)$ in H. □

No we are ready to prove Lemma 2.1.

Proof of Lemma 2.1. Given $k \geqslant 3$ and $d \in (0,1)$ (since for $d = 1$ the statement is clear), take $t = \lceil d^{-2(k-1)} \rceil + 1$ and $\ell = (2t)!$. We first prove the following claim.

Claim 1. $\gamma(Z_2, Z_4, \dots, Z_{2t}) \leqslant d\,.$

Proof of the claim:

Let $\varepsilon \ll 1/k, 1/t, 1-d$ and pick $n \in \mathbb{N}$ with $n^{-1} \ll \varepsilon$. Let $H = (V,E)$ be a k-graph on n vertices with $\delta(H) \geqslant (d+\varepsilon)n$. We shall prove that $Z_{2r} \subseteq H$ for some $r \in \{1, \dots, t\}$. To this end, we find a sequence of $(k-1)$-sets of vertices $e_1, \dots, e_{2r} \in V^{(k-1)}$ with $e_i \rhd e_{i+1}$ for every $i \in [2r]$ (where the sum is modulo $2r$). First, we show that there is a sequence of pairwise disjoint $(k-1)$-sets of vertices $e_1, e_3, \dots, e_{2t-1} \in V^{(k-1)}$ such that

$$\left|N(e_{2i-1})^{(k-1)} \cap \overleftarrow{N}(e_{2i+1})\right| > \frac{1}{t-1}\binom{n}{k-1} + t(k-1)n^{k-2}\,, \tag{2.1}$$

for every $i \in [t-1]$.

Pick e_1 arbitrarily. We choose $e_3, \dots, e_{2t-1}$ iteratively as follows. Suppose that for $j \in [t-1]$, we have already found a sequence $e_1, \dots, e_{2j-1}$ satisfying (2.1) for every $i \leqslant j$. Let $U_j = \bigcup_{i\in[j]} e_{2i-1}$ and note that $|U_j| \leqslant (k-1)t \leqslant \frac{\varepsilon n}{2}$. The following identity holds by a double counting argument, and the inequality follows from the minimum codegree condition

$$\sum_{e\in(V\smallsetminus U)^{(k-1)}} |N(e_{2j-1})^{(k-1)} \cap \overleftarrow{N}(e)| = \sum_{e\in N(e_{2j-1})^{(k-1)}} \binom{|N(e)\smallsetminus U|}{k-1} \geqslant \binom{(d+\frac{\varepsilon}{2})n}{k-1}^2 .$$

Therefore, by averaging there is an $e_{2j+1} \in (V \smallsetminus U_j)^{(k-1)}$ such that

$$\begin{aligned}\left|N(e_{2j-1})^{(k-1)} \cap \overleftarrow{N}(e_{2j+1})\right| &\geqslant \frac{\binom{(d+\frac{\varepsilon}{2})n}{k-1}^2}{\binom{n}{k-1}} \geqslant \left(d+\frac{\varepsilon}{4}\right)^{2(k-1)}\binom{n}{k-1} \\ &\geqslant d^{2(k-1)}\binom{n}{k-1} + t(k-1)n^{k-2} \\ &\geqslant \frac{1}{t-1}\binom{n}{k-1} + t(k-1)n^{k-2}\,.\end{aligned}$$

Hence, after t steps we found $e_1, e_3, \dots, e_{2t-1} \in V^{(k-1)}$ satisfying (2.1) for every $i \in [t-1]$.

Note that the number of $(k-1)$-sets containing at least one vertex in $\bigcup_{i\in[t]} e_{2i-1}$ is at most $t(k-1)n^{k-2}$. Thus, because of (2.1), the pigeonhole principle implies that there are indices $i, j \in [t-1]$ with $i < j$ and $e_{2i} \in \bigcap_{s \in \{i,j\}} \left(N(e_{2s-1})^{(k-1)} \cap \overleftarrow{N}(e_{2s+1}) \right)$ such that e_{2i} is disjoint from each of $e_1, e_3, \ldots, e_{2t-1}$. In particular, we have

$$e_{2i} \rhd e_{2i+1} \qquad \text{and} \qquad e_{2j-1} \rhd e_{2i}\,. \tag{2.2}$$

Next we choose the other $(k-1)$-sets with even indices in the sequence forming Z_{2r}. We shall choose $j-i-1$ pairwise disjoint $(k-1)$-sets $e_{2i+2}, \ldots, e_{2j-2} \in V^{(k-1)}$ such that $e_{2m} \in N(e_{2m-1}) \cap \overleftarrow{N}(e_{2m+1})$ for every $i < m < j$ (note that if $j = i+1$, we are done). In other words, for $i < m < j$, we need

$$e_{2m-1} \rhd e_{2m} \rhd e_{2m+1}\,. \tag{2.3}$$

Moreover, the e_{2m} have to be disjoint from the already chosen sets in the sequence. Each set $e \in V(H)^{(k-1)}$ can intersect at most $(k-1)n^{k-2}$ other elements of $V(H)^{(k-1)}$. Thus, we can greedily pick disjoint the even sets $e_{2m} \in N(e_{2m-1})^{(k-1)} \cap \overleftarrow{N}(e_{2m+1})$ one by one for each $i < m < j$. Indeed, for every $m \leqslant j-i-1$, the number of $(k-1)$-sets in $N(e_{2m-1})^{(k-1)} \cap \overleftarrow{N}(e_{2m+1})$ which do not intersect any previously chosen $(k-1)$-set in the sequence is at least

$$|N(e_{2m-1})^{(k-1)} \cap \overleftarrow{N}(e_{2m+1})| - 2t(k-1)n^{k-2} \overset{(2.1)}{\geqslant} \frac{1}{t-1}\binom{n}{k-1} - t(k-1)n^{k-2} \;>\; 0\,.$$

This means that we can always pick an $e_{2m} \in N(e_{2m-1})^{(k-1)} \cap \overleftarrow{N}(e_{2m+1})$ that is disjoint from all previously chosen sets.

Putting (2.2) and (2.3) together yields that the $(k-1)$-sets $e_{2i}, e_{2i+1}, \ldots, e_{2j-1}$, form a zycle of length $2(j-i) \leqslant 2t$. This concludes the proof of the claim. ■

Let $0 < \varepsilon \ll 1/\ell$ and $m \geqslant \ell/2$. Let $n \in \mathbb{N}$ with $n^{-1} \ll \varepsilon$ and let H be a k-graph with $\delta(H) \geqslant (d+\varepsilon)n$. We shall prove that $Z_\ell \subseteq H$. Notice that Proposition 2.4 and Claim 1 imply that H contains a copy of $Z_{2r}(m)$ with $r \in \{1, \ldots, t\}$. Applying Fact 2.3 with $c = \frac{\ell}{2r} \leqslant m$, we obtain a copy of Z_ℓ in H as desired. □

2.2. **Lower bound.** The following construction will provide an example of a Z_ℓ-free hypergraph with large minimum codegree.

Definition 2.5. Let $n, p, k \in \mathbb{N}$ be such that p is a prime, $k \geqslant 2$ and $p \mid n$. We define the n-vertex k-graph $\mathbb{F}_p^{(k)}(n)$ as follows. The vertex set consists of p disjoint sets of size $\frac{n}{p}$ each, i.e., $V(\mathbb{F}_p^{(k)}(n)) = V_0 \cup \ldots \cup V_{p-1}$ with $|V_i| = \frac{n}{p}$ for all $i \in [p]$. Given a vertex $v \in V(\mathbb{F}_p^{(k)}(n))$ we write $\mathfrak{f}(v) = i$ if and only if $v \in V_i$ for $i \in \{0, 1, \ldots, p-1\}$. We define the edge set of $\mathbb{F}_p^{(k)}(n)$ by

$$v_1 \cdots v_k \in E(\mathbb{F}_p^{(k)}(n)) \Leftrightarrow \begin{cases} \mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_k) \equiv 0 \bmod p \text{ and } \mathfrak{f}(v_i) \neq 0 \text{ for some } i \in [k], \text{ or} \\ \mathfrak{f}(v_{\sigma(1)}) = \cdots = \mathfrak{f}(v_{\sigma(k-1)}) = 0 \text{ and } \mathfrak{f}(v_{\sigma(k)}) = 1 \text{ for some } \sigma \in S_k\,. \end{cases}$$

When k is obvious from the context, we omit it from the notation and we always consider the indices of the clusters modulo p.

Lemma 2.6. *Let $k \geqslant 3$. For every $\ell \geqslant 2$, we have $\frac{1}{2(k-1)^\ell} \leqslant \gamma(Z_\ell^{(k)})$.*

Proof. Given $k \geqslant 3$ and $\ell \geqslant 2$, let $n, p \in \mathbb{N}$ be such that $p \mid n$, p is a prime larger than k, and $n^{-1} \ll p^{-1} < \frac{1}{(k-1)^\ell+1}$. Observe that by the Bertrand–Chebyshev theorem we might take $p \leqslant 2(k-1)^\ell$. We shall prove that

$$\delta(\mathbb{F}_p(n)) = \tfrac{n}{p} \geqslant \frac{n}{2(k-1)^\ell} \qquad \text{and} \qquad Z_\ell \nsubseteq \mathbb{F}_p(n)\,. \tag{2.4}$$

To check the codegree condition in (2.4), take a $(k-1)$-set of vertices $v_1, \dots, v_{k-1}$. If there is an $i \in [k-1]$ such that $\mathfrak{f}(v_i) \neq 0$, then let j be the only solution in $\{0, 1, \dots, p-1\}$ to the equation

$$\mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_{k-1}) + x \equiv 0 \pmod{p}\,.$$

Then, $N(v_1 \cdots v_{k-1}) \supseteq V_j$ and therefore $d(v_1 \cdots v_{k-1}) \geqslant \frac{n}{p}$. If $f(v_i) = 0$ for all $i \in [k-1]$, then $N(v_1 \cdots v_{k-1}) = V_1$ and we obtain $d(v_1 \cdots v_{k-1}) = \frac{n}{p}$.

To check the second part of (2.4), assume that there are $r \geqslant 2$ and sets $e_1, \dots, e_r \in V(\mathbb{F}_p(n))^{(k-1)}$ forming a copy of Z_r, i.e., we have $e_i \rhd e_{i+1}$ for all i. Here, and for the rest of the proof, we take the sum of indices of the e_i's to be modulo r. We shall prove that

$$r > \ell\,. \tag{2.5}$$

The following claim states that there is an i_0 for which e_{i_0} is completely contained in one of the clusters of $\mathbb{F}_p(n)$. Moreover, that cluster is not V_0.

Claim 2. *There is an $i_0 \in [r]$ and a $j \in [p-1]$ such that $e_{i_0} \subseteq V_j$.*

Proof of the claim: Fix any $i \in [r]$, let $e_i = v_1 \cdots v_{k-1}$, and pick $v_k \in e_{i+1}$ arbitrarily. We consider four cases.

Case (1): $|e_i \cap V_0| = k-1$.

By Definition 2.5 and since $v_1 \cdots v_k \in E(\mathbb{F}_p(n))$, we have $v_k \in V_1$. Since we picked $v_k \in e_{i+1}$ arbitrarily, we have that $e_{i+1} \subseteq V_1$ and finish the proof of this case by taking $i_0 = i+1$.

Case (2): $|e_i \cap V_0| < k-2$.

Let $j \equiv -(\mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_{k-1})) \bmod p$. By Definition 2.5 and since $v_1 \cdots v_k \in E(\mathbb{F}_p(n))$, we have

$$0 \equiv \mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_k) \equiv \mathfrak{f}(v_k) - j\,.$$

This means that $v_k \in V_j$ and since we picked $v_k \in e_{i+1}$ arbitrarily, similarly as above we get $e_{i+1} \subseteq V_j$. If $j \not\equiv 0$, we finish by taking $i_0 = i+1$. If $j \equiv 0$, the claim follows from Case *(1)* for e_{i+1} instead of e_i.

Case (3): $|e_i \cap V_0| = k-2$ and $|e_i \cap V_1| = 0$.

This case follows from similar arguments as the previous one.

Case (4): $|e_i \cap V_0| = k-2$ and $|e_i \cap V_1| = 1$.

By Definition 2.5, we either have $v_k \in V_0$ or $v_k \in V_{p-1}$. Thus, since we picked $v_k \in e_{i+1}$ arbitrarily, we certainly have $e_{i+1} \subseteq V_0 \cup V_{p-1}$. Hence, $|e_{i+1} \cap V_1| = 0$ and so the proof follows from Cases (1) - (3) for e_{i+1} instead of e_i. ■

We now show that for every $i \in [r]$,

$$\text{if } e_i \subseteq V_j \text{ with } j \not\equiv 0 \bmod p, \text{ then } e_{i+1} \subseteq V_{(1-k)j}\,. \tag{2.6}$$

Indeed, let $e_i = v_1 \cdots v_{k-1} \subseteq V_j$ and pick $v_k \in e_{i+1}$ arbitrarily. Since $\mathfrak{f}(v_i) \equiv j \bmod p$ for $i \in [k-1]$, we have

$$\mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_{k-1}) \equiv (k-1)j \pmod{p}\,.$$

Therefore, since $e_i \rhd e_{i+1}$ implies $v_1 \cdots v_k \in E(\mathbb{F}_p(n))$ and because $\mathfrak{f}(v_i) \equiv j \not\equiv 0 \bmod p$ for $i \in [k-1]$, we have

$$0 \equiv \mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_k) \equiv (k-1)j + \mathfrak{f}(v_k) \pmod{p}\,.$$

Hence $\mathfrak{f}(v_k) \equiv (1-k)j$, meaning that $v_k \in V_{(1-k)j}$. Since we picked $v_k \in e_{i+1}$ arbitrarily, we have $e_{i+1} \subseteq V_{(1-k)j}$ proving (2.6).

Finally, we are ready to show (2.5). Let i_0 and j be given by Claim 2. As p is a prime, $\mathbb{F}_p$ is a field. Together with $j \not\equiv 0$, this entails that $(1-k)^s j \not\equiv 0 \pmod{p}$ for all $s \in [r]$. Thus, r applications of (2.6) imply that

$$e_{i_0+r} \subseteq V_m \text{ with } m \equiv (1-k)^r j \pmod{p}\,.$$

Since $e_{i_0+r} = e_{i_0} \in V_j$, we have $(1-k)^r j \equiv j \pmod{p}$, and as $j \not\equiv 0$, we have $(1-k)^r \equiv 1$. Recalling that we chose p such that $p > (k-1)^\ell + 1$, (2.5) follows. □

§3. Proof of Theorem 1.5

3.1. Method. As mentioned in the introduction, to prove Theorem 1.5 we apply the method developed by the authors together with Sales in [10].

Definition 3.1. Given a k-graph $H = (V, E)$, a *picture* is a tuple $(v, m, \mathcal{L}, \mathcal{B})$, where

- *(i)* $v \in V$,
- *(ii)* $m \in \mathbb{N}$,
- *(iii)* $\mathcal{L}$ is a collection of m-tuples $\mathcal{L} \subseteq (V \smallsetminus \{v\})^m$, and
- *(iv)* $\mathcal{B} \subseteq [m]^{(k-1)}$ is a fixed family of $(k-1)$-subsets of $V(H)$,

such that for every $(x_1, \ldots, x_m) \in \mathcal{L}$ and every $i_1 \cdots i_{k-1} \in \mathcal{B}$, the k-sets $vx_{i_1} \cdots x_{i_{k-1}}$ are edges of H. That is to say, $x_{i_1} \cdots x_{i_{k-1}}$ is an edge in the link of H at v.

We use pictures to find a copy of a k-graph F on H. Roughly speaking, we say that a picture is *nice* if it 'encodes' a set of edges that would yield a copy of F, but whose existence we cannot (yet) guarantee when considering the link of H at v.

Definition 3.2. Given k-graphs F and $H = (V, E)$, and vertex set $S \subseteq V$, we say that a picture $(v, m, \mathcal{L}, \mathcal{B})$ is *S-nice for F*, if for every $w \in S$ and every $(x_1, \ldots, x_m) \in \mathcal{L}$, the hypergraph with vertex set V and edge set

$$E \cup \{wx_{i_1} \cdots x_{i_{k-1}} : i_1 \cdots i_{k-1} \in \mathcal{B}\}$$

contains a copy of F.

If F is clear from the context, we speak simply of S-nice pictures. The following lemma describes how the existence of S-nice pictures implies that H contains a copy of F.

Lemma 3.3. *Let F be a k-graph. Given $\xi, \zeta > 0$ and $c, m \in \mathbb{N}$, let $n \in \mathbb{N}$ such that $n^{-1} \ll \xi, \zeta, |V(F)|^{-1}, c^{-1}, m^{-1}$, and let H be an n-vertex k-graph.*

Suppose that there are $m \in \mathbb{N}$ and $\mathcal{B} \subseteq [m]^{(k-1)}$ such that for every $S \subseteq V(H)$ with $|S| \geqslant c$, there is an S'-nice picture $(v, m, \mathcal{L}, \mathcal{B})$, with $v \in S$, $S' \subseteq S$, $|S'| \geqslant \xi|S|$, and $|\mathcal{L}| \geqslant \zeta n^m$. Then H contains a copy of F.

Proof. Let $t = \lceil \zeta^{-1} \rceil + 1$. By iteratively applying the conditions of the lemma, we find a nested sequence of subsets $V(H) = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_t$ such that for $i \in [t]$, there are S_i-nice pictures $(v_i, m, \mathcal{L}_i, \mathcal{B})$ satisfying $v_i \in S_{i-1}$, $|S_i| \geqslant \xi^i n > c$, and $|\mathcal{L}_i| \geqslant \zeta n^m$.

Since $t \geqslant \zeta^{-1} + 1$, by the pigeonhole principle, there are two indices $0 < i < j \leqslant t$ such that $\mathcal{L}_i \cap \mathcal{L}_j \neq \varnothing$. Let $(x_1, \ldots, x_m) \in \mathcal{L}_i \cap \mathcal{L}_j$. Then because $(v_i, m, \mathcal{L}_i, \mathcal{B})$ is an S_i-nice picture and $v_j \in S_{j-1} \subseteq S_i$, Definition 3.2 guarantees that

$$E(H) \cup \{v_j x_{i_1} \cdots x_{i_{k-1}} : i_1 \cdots i_{k-1} \in \mathcal{B}\}$$

contains a copy of F. Since $(v_j, m, \mathcal{L}_j, \mathcal{B})$ is a picture, Definition 3.1 yields $v_j x_{i_1} \cdots x_{i_{k-1}} \in E(H)$ for all $i_1 \cdots i_{k-1} \in \mathcal{B}$. Thus, we conclude that this copy of F is in fact in H. $\square$

Now we apply Lemma 3.3 to prove Theorem 1.5.

3.2. **Proof of Theorem 1.5.** Let $\ell \geqslant 3$ be an integer and let $\varepsilon > 0$. Let $\xi, \zeta > 0$, and let $n, c \in \mathbb{N}$ such that $n^{-1} \ll c^{-1} \ll \zeta, \xi \ll \varepsilon$. Let H be a 3-graph with $\delta(H) \geqslant \varepsilon n$. We aim to show that $Z_\ell^- \subseteq H$. Set $m = 2$ and $\mathcal{B} = \big\{\{1,2\}\big\}$, then due to Lemma 3.3, we only need to prove that for every $S \subseteq V(H)$ of size at least c, there is an S'-nice picture $(v, 2, \mathcal{L}, \big\{\{1,2\}\big\})$ with $v \in S$, $S' \subseteq S$, $|S'| \geqslant \xi|S|$, and $|\mathcal{L}| \geqslant \zeta n^2$.

Given $S \subseteq V(H)$ with $|S| \geqslant c$, take any vertex $v \in S$ and let $V = V(H) \smallsetminus \{v\}$. Observe that using the minimum codegree condition and the above hierarchy, we have

$$\sum_{bb' \in V^{(2)}} |N_{L_v}(b) \cap N_{L_v}(b') \cap S| = \sum_{u \in S \smallsetminus \{v\}} \binom{d_{L_v}(u)}{2} \geqslant \binom{\varepsilon n}{2}(|S| - 1) \geqslant \xi \binom{n}{2} |S|, \qquad (3.1)$$

where L_v denotes the link of H at v. Thus, by averaging there is a pair $b_1, b_2 \in V$ such that $|N_{L_v}(b_1) \cap N_{L_v}(b_2) \cap S| \geqslant \xi|S|$. We pick $S' \subseteq N_{L_v}(b_1) \cap N_{L_v}(b_2) \cap S$ with $|S'| = \lceil \xi|S| \rceil$.

Since $\delta(H) - 2\ell - |S'| \geqslant \varepsilon n/2 \geqslant 2$ we can greedily pick pairwise disjoint pairs of vertices $e_1, \ldots, e_{\ell-2} \in (V(H) \smallsetminus S')^{(2)}$ such that

$$b_1 b_2 = e_1 \rhd e_2 \rhd \cdots \rhd e_{\ell-2}\,. \qquad (3.2)$$

Now let $R = \bigcup_{i\in[\ell-2]} e_i$ and take

$$\mathcal{L} = \{(x_1, x_2) \in V^2 \colon x_1 \in N_H(e_{\ell-2}) \smallsetminus R \text{ and } x_2 \in N_H(x_1 v) \smallsetminus R\}\,.$$

Note that $|\mathcal{L}| \geqslant (\delta(H)/2)^2 \geqslant \varepsilon^2 n^2/4 \geqslant \zeta n^2$. Further, since $x_1x_2 \in E(L_v)$ for every $(x_1, x_2) \in \mathcal{L}$, $(v, m, \mathcal{L}, \mathcal{B})$ is a picture in H. Moreover, observe that it is S'-nice. Indeed, we only we need to check that for any $u \in S'$ and $(x_1, x_2) \in \mathcal{L}$, the hypergraph with edges $E(H) \cup \{ux_1x_2\}$ contains a copy of Z_ℓ^-. For this, note that in $E(H) \cup \{ux_1x_2\}$ we have $x_1x_2 \rhd uv$. Further, $u \in S'$ and the choice of b_1 and b_2 imply $uv \rhd b_1b_2$. Together with (3.2), this gives $x_1x_2 \rhd uv \rhd e_1 \rhd \cdots \rhd e_{\ell-2}$, and using the fact that $x_1 \in N(e_{\ell-2})$, we obtain a copy of Z_ℓ^- (where the missing edge is $x_2e_{\ell-2}$).

§4. Concluding Remarks

Following a very similar proof as that for Theorem 1.5, we can show a general upper bound for $\gamma(Z_\ell^{(3)})$ for every $\ell \geqslant 3$.

Proposition 4.1. *For $\ell \geqslant 3$, $\gamma(Z_\ell^{(3)}) \leqslant 1/2$.*

Proof. Given $\ell \geqslant 3$ and $\varepsilon > 0$, let $\xi, \zeta > 0$ and $n, c \in \mathbb{N}$ such that $n^{-1} \ll c^{-1} \ll \zeta, \xi \ll \varepsilon$. Let H be a 3-graph with $\delta(H) \geqslant \big(\frac{1}{2} + \varepsilon\big)n$. We aim to show that $Z_\ell \subseteq H$. As in the proof of Theorem 1.5, we pick $m = 2$ and $\mathcal{B} = \big\{\{1, 2\}\big\}$ and due to Lemma 3.3, we only need to prove that for every $S \subseteq V(H)$ of size at least c, there is an S'-nice picture $(v, 2, \mathcal{L}, \big\{\{1, 2\}\big\})$ with $v \in S$, $S' \subseteq S$, $|S'| \geqslant \xi|S|$, and $|\mathcal{L}| \geqslant \zeta n^2$.

For the first part of the proof we proceed as in the proof of Theorem 1.5 and we only use $\delta(H) \geqslant \varepsilon n$. In particular, we obtain two vertices $b_1, b_2 \in V(H) \smallsetminus \{v\} =: V$ and a set $S' \subseteq N_{L_v}(b_1) \cap N_{L_v}(b_1) \cap S$ with $|S'| = \lceil \xi|S|\rceil$. Moreover, we again greedily pick pairwise disjoint pairs of vertices $e_1, \dots, e_{\ell-2} \in (V \smallsetminus S')^{(2)}$ satisfying (3.2). The set $\mathcal{L}$ is chosen differently. Set $R = \bigcup_{i\in[\ell-2]} e_i$ and

$$\mathcal{L} = \{(x_1, x_2) \in V^2 \colon x_1, x_2 \in N(e_{\ell-2}) \smallsetminus R \text{ and } x_1x_2 \in E(L_v)\}\,. \tag{4.1}$$

Observe that given $x_1 \in N(e_{\ell-2}) \smallsetminus R$, any vertex $x_2 \in (N(xv) \cap N(e_{\ell-2})) \smallsetminus R$, gives rise to $(x_1, x_2) \in \mathcal{L}$. Furthermore, since $\delta(H) \geqslant (1/2 + \varepsilon)n$,

$$|(N(xv) \cap N(e_{\ell-2})) \smallsetminus R| \geqslant \varepsilon n - 2\ell \geqslant \frac{\varepsilon}{2} n\,,$$

and similarly we have $N(e_{\ell-2}) \smallsetminus R \geqslant n/2$. Therefore, we obtain $|\mathcal{L}| \geqslant \varepsilon n^2/4$, and since $x_1x_2 \in E(L_v)$ for all $(x_1, x_2) \in \mathcal{L}$, $(v, m, \mathcal{L}, \mathcal{B})$ is a picture in H.

To see that the tuple $(v, m, \mathcal{L}, \mathcal{B})$ is indeed an S'-nice picture, we shall prove that for every $u \in S'$ and $(x_1, x_2) \in \mathcal{L}$, the hypergraph with (vertex set $V(H)$ and) edges $E(H) \cup \{ux_1x_2\}$ contains a copy of Z_ℓ. Indeed, the definition of $\mathcal{L}$ implies $x_1x_2v \in E(H)$ and therefore $x_1x_2 \rhd uv$ in $E(H) \cup \{ux_1x_2\}$. Also due to the definition of $\mathcal{L}$, we have $x, y \in N(e_{\ell-2})$ and thus, $e_{\ell-2} \rhd xy$. Moreover, $u \in S'$ and the choice of b_1 and b_2 entails $uv \rhd b_1b_2 = e_1$. Combining this with (3.2), we obtain $uv \rhd e_1 \rhd \dots e_{\ell-2} \rhd x_1x_2 \rhd uv$, that is a copy of Z_ℓ, in $E(H) \cup \{ux_1x_2\}$. $\square$

It would be interesting to know whether Proposition 4.1 is sharp for some $\ell \geqslant 3$. The following construction gives a lower bound of $1/3$ for the codegree Turán density of any zycle of length not divisible by 3. Let $n \in \mathbb{N}$ be divisible by 3 and let $H = (V, E)$, where $V = V_1 \cup V_2 \cup V_3$ with $|V_i| = n/3$ and $E = \{uvw \in V^{(3)} \colon u, v \in V_i \text{ and } w \in V_{i+1}\}$, where the sum is taken modulo 3. It is not hard to check that $\delta(H) \geqslant n/3$ and that $Z_\ell \nsubseteq H$ for every ℓ not divisible by 3.

Observe that $Z_2^{(3)} = K_4^{(3)}$. For this 3-graph, a well-known conjecture by Czygrinow and Nagle [3] states that $\gamma(Z_2^{(3)}) = \gamma(K_4^{(3)}) = 1/2$. Regarding the next case, $Z_3^{(3)}$, note that its codegree Turán density is not bounded by the previous construction. The following 3-graph entails $\gamma(Z_3^{(3)}) \geqslant 1/4$, and in fact it provides the same lower bound for every $Z_\ell^{(3)}$ with ℓ not divisible by 4. Let $n \in \mathbb{N}$ divisible by 4 and let $H = (V, E)$, where $V = V_1 \cup V_2 \cup V_3 \cup V_4$ with $|V_i| = n/4$. Define the edges of H as

$$E = \{xyz \colon x, y \in V_i \text{ and } z \in V_{i+1}\} \cup \{xyz \colon x \in V_1, y \in V_2, z \in V_3 \cup V_4\},$$

where the sum of indices is taken modulo 4. Clearly, $\delta(H) \geqslant n/4$. To see that $Z_\ell \nsubseteq H$ for ℓ not divisible by 4, it can be checked that all zycles are of the form $e_1 \rhd \cdots \rhd e_r$ such that $e_i \subseteq V_{j_i}$ for some $j_i \in [4]$. Together with Proposition 4.1, this yields

$$\frac{1}{4} \leqslant \gamma(Z_3^{(3)}) \leqslant \frac{1}{2}.$$

Problem 4.2. *Determine the value of $\gamma(Z_3^{(3)})$.*

On a different note, recall that Theorem 1.5 states that $Z_\ell^{(3)}$ is (inclusion) minimal with respect to the property of having strictly positive codegree Turán density. It would be interesting to know if this also holds for larger uniformities.

Question 4.3. *For $k > 3$ and sufficiently large ℓ, what are the minimal subgraphs $F \subseteq Z_\ell^{(k)}$ with $\gamma(F) > 0$?*

Let $e_1, \ldots, e_\ell$ be pairwise disjoint $(k-1)$-sets of vertices. Consider the k-graph whose edges are given by $e_1 \rhd \cdots \rhd e_\ell$ plus one additional edge of the form $e_\ell \cup \{v\}$ with $v \in e_1$. Following the same arguments as in the proof of Theorem 1.5, we obtain that this k-graph has codegree Turán density zero for every $\ell \geqslant 3$.

In order to prove that the lower bound of Lemma 2.6 in Subsection 2.2, we introduce the k-graphs $\mathbb{F}_p^{(k)}(n)$ that have large minimum codegree and are $Z_\ell^{(k)}$-free for small ℓ. It would be interesting to study the codegree Turán density of $\mathbb{F}_p^{(k)}(n)$ itself. Observe however, that for $n \geqslant pk$ we have $K_{k+1}^{(k)-} \subseteq \mathbb{F}_p^{(k)}(n)$, which suggests that this problem might be very difficult for general n.

It is perhaps more natural to study the codegree Turán density of the following k-graph. For $p > k$, let $\widetilde{\mathbb{F}}_p^{(k)}$ be the k-graph on $p(k-1)$ vertices with $V(\widetilde{\mathbb{F}}_p^{(k)}) = V_1 \cup \ldots \cup V_p$ where $|V_i| = k-1$ for every $i \in [p]$ and whose edges are given by

$$v_1 \cdots v_k \in E(\widetilde{\mathbb{F}}_p^{(k)}) \iff \mathfrak{f}(v_1) + \cdots + \mathfrak{f}(v_k) \equiv 0 \bmod p,$$

where the function $\mathfrak{f}\colon V(\widetilde{\mathbb{F}}_p^{(k)}) \longrightarrow [p]$ is analogous as in Definition 2.5.

Problem 4.4. *For $k \geqslant 3$, determine the codegree Turán density of $\widetilde{\mathbb{F}}_p^{(k)}$.*

Consider the indices of the clusters $V_1, \dots, V_p$ of $\widetilde{\mathbb{F}}_p^{(k)}$ to be modulo p. Observe for $j \in [p]$, we have $V_j \cup \{v\} \in E(\widetilde{\mathbb{F}}_p^{(k)})$ for every $v \in V_{(1-k)j}$. It follows that

$$V_1 \rhd V_{1-k} \rhd \cdots \rhd V_{(1-k)^{p-2}} \rhd V_{(1-k)^{p-1}} = V_1\,,$$

where the last identity is given by Fermat's little theorem. Hence, there is an $\ell \leqslant p-1$ such that $Z_\ell \subseteq \widetilde{\mathbb{F}}_p^{(k)}$ and therefore, Lemma 2.6 yields $\gamma(\widetilde{\mathbb{F}}_p^{(k)}) \geqslant \frac{1}{2(k-1)^p} > 0$.

Question 4.5. *For $k \geqslant 3$, is it true that $\lim\limits_{p \longrightarrow \infty} \gamma(\widetilde{\mathbb{F}}_p^{(k)}) = 0$?*

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