Extremal Set Theory

Lecture notes

Preamble and notation

In this part of the course we will study extremal-type questions for subsets of a given finite set.

We will use the same standard notation introduced in the previous parts of the the course. Namely, given $n \in \mathbb{N}$, we denote the set of the first n natural numbers $\{1, 2, \ldots, n\}$ by $[n]$. Further, for $a < b \leqslant n$ the 'interval'

$$\{x\colon a \leqslant x \leqslant b\} = \{a, a+1, \ldots, b\} \subseteq [n]$$

will be denoted by $[a, b]$. For a set $X \subseteq [n]$, the set of all subsetes of X is denoted by 2^X and for $k \leqslant n$ we write $\binom{X}{k}$ to refer to the set of all subsets of X of size k.

Sometimes we will bring notation from hypergraph theory. In particular, given a family of sets $\mathcal{F} \subseteq 2^X$, we sometimes call the elements of X *vertices* and we refer to a set $F \in \mathcal{F}$ as an *edge of* $\mathcal{F}$. Moreover, for simplicity, we sometimes we omit parenthesis and commas coming from set theoretic notation. For instance, instead of $\{a, b, c\}$ we will simply write abc.

The following will be a very useful formula for the binomial coefficient

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

In Section 1 we introduce the classical Erdős-Ko-Rado Theorem for intersecting families. We will study generalisations and variations of this basic result. Then in Section 2 we will study an elegant proof technique introduced by Katona in the 70s. As we will see, this technique can be applied to prove several classical results in extremal set theory, as well as some not-so-classical results. We end these notes with Section 3, in which we study extremal results for the so-called *shadow* of a family of subsets.

§1. Intersecting problems

Definition 1.1. We say that a family of subsets $\mathcal{F}$ is *intersecting* if for every two sets $A, B \in \mathcal{F}$ we have $A \cap B \neq \varnothing$.

Similarly, we say that two families $\mathcal{F}$ and $\mathcal{H}$ are *cross-intersecting* if for every two sets $A \in \mathcal{F}$ and $B \in \mathcal{H}$ we have $A \cap B \neq \varnothing$.

The following are examples of intersecting families.

Example 1.2. Let $\mathcal{B} \in 2^{[n]}$ be the family of sets of size at least $\lfloor \frac{n}{2} \rfloor + 1$. This is

$$\mathcal{B} := \left\{X \in 2^{[n]}\colon |X| \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1\right\} \subseteq 2^{[n]}.$$

Observe that for every two sets $X, Y \in \mathcal{B}$ we have $|X| + |Y| \geqslant n + 1$, and therefore, $X \cap Y \neq \varnothing$.

Example 1.3. Let $F(1) \in 2^{[n]}$ be the family of all sets containing the element 1, i.e. $\mathcal{F} := \{F \in 2^{[n]} \colon 1 \in F\}$. Obviously, $1 \in F_1 \cap F_2$ for every $F_1, F_2 \in \mathcal{F}$, and hence $F(1)$ is intersecting.

Example 1.4. Let n be even and let $\mathcal{A} = \{X \in 2^{[n]} \colon |X| = n/2 \text{ and } 1 \in X\}$. Let $\mathcal{B}$ be the family from Example 1.2 and observe that

$$\mathcal{F} := \mathcal{A} \cup \mathcal{B}$$

is intersecting.

The following is a very natural question, and is one of the starting points of Extremal Set Theory.

Question 1.5. Given $n \in \mathbb{N}$, what is the largest possible intersecting family on n vertices?

The answer is easy to obtain.

Theorem 1.6. *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family. Then, $|\mathcal{F}| \leqslant 2^{n-1}$.*

Proof. Let $\overline{\mathcal{F}} = \{F^c \colon F \in \mathcal{F}\}$. It is easy to see that $|\mathcal{F}| = |\overline{\mathcal{F}}|$. Moreover, since $\mathcal{F}$ is intersecting, for each set $A \in 2^{[n]}$ either A or A^c is in $\mathcal{F}$, but not both. In other words,

$$\mathcal{F} \cap \overline{\mathcal{F}} = \varnothing\,.$$

Hence $2|\mathcal{F}| = |\mathcal{F}| + |\overline{\mathcal{F}}| = |\mathcal{F} \cup \overline{\mathcal{F}}| \leqslant 2^n$. □

Remark 1.7. Observe that due to the family $\mathcal{F}(1)$ from Example 1.3, Theorem 1.6 is best possible.

We can go a bit further and deduce the following surprisingly powerful result.

Theorem 1.6 (reprise)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family. Then, there is an intersecting family $\mathcal{F}' \supseteq \mathcal{F}$ with $|\mathcal{F}'| \leqslant 2^{n-1}$.*

The same problem, becomes substantially more difficult when we ask for the maximum possible size of an intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ for a given $k \geqslant 2$. This problem was solved by Erdős, Ko, and Rado in 1938[1] and is one of the most influential results in extremal set theory.

Theorem 1.8. *(Erdős-Ko-Rado Theorem [2]) Let $n \geqslant 2k$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then, $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.*

In order to prove Theorem 1.8, Erdős, Ko, and Rado introduced a set operation that we will define in the next subsection.

For $2k \geqslant n+1$ the problem is trivial, since every two k-sets intersect. In other words, the family $\mathcal{F} = \binom{[n]}{k}$ is itself intersecting and obviously of maximal size.

If $\mathcal{F}(1)$ is the family given in Example 1.3, then the family $\mathcal{F}_k(1) := \mathcal{F}(1) \cap \binom{[n]}{k}$ shows that Theorem 1.8 is best possible.

[1]For many geopolitical reasons (among other issues), it was only publish in 1961

1.1. **Shifting for Erdős-Ko-Rado.** The following operation is the basis for the proof of Theorem 1.8 (and for many other proofs!).

Definition 1.9. Given a family of sets $\mathcal{F} \subseteq 2^{[n]}$ and two numbers $i, j \in [n]$ we define the (i, j)-*shift* of a set $F \in 2^{[n]}$ by

$$\sigma^{\mathcal{F}}_{i,j}(F) := \begin{cases} (F \smallsetminus j) \cup i & \text{if } i \notin F, j \in F, \text{ and } (F \smallsetminus j) \cup i \notin \mathcal{F} \\ F & \text{otherwise}. \end{cases}$$

When $\mathcal{F}$ is clear form the context, we will omit it from the notation. Moreover, we define the (i, j)-*shift* of $\mathcal{F}$ by

$$\sigma_{i,j}(\mathcal{F}) = \{\sigma^{\mathcal{F}}_{i,j}(F) \colon F \in \mathcal{F}\}.$$

Roughly speaking, given $\mathcal{F} \subseteq 2^{[n]}$, the operation $\sigma_{i,j}(\cdot)$ replaces the vertex j with i, whenever it is possible (i.e. when j is present and i is not) and whenever it creates a *new* set in the family $\mathcal{F}$. For example, for the family $\mathcal{F} = \{123, 234, 345, 456, 156\} \subseteq \binom{[6]}{3}$ we have

$$\begin{aligned} \sigma_{2,5}(\mathcal{F}) &= \{\sigma_{2,5}(123), \sigma_{2,5}(234), \sigma_{2,5}(345), \sigma_{2,5}(456), \sigma_{2,5}(156)\} \\ &= \{\quad 123, \quad 234, \quad 345, \quad 246, \quad 126 \quad\}. \end{aligned}$$

Observe that $\mathcal{F}, \sigma_{2,5}(\mathcal{F}) \subseteq \binom{[6]}{3}$ and that $|\mathcal{F}| = |\sigma_{2,5}(\mathcal{F})|$. Furthermore, both $\mathcal{F}$ and $\sigma_{25}(\mathcal{F})$ are intersecting. In fact, these properties are always preserved.

Proposition 1.10. *Let $\mathcal{F} \subseteq 2^{[n]}$ and $i, j \in [n]$, then*

(*a*) $|\mathcal{F}| = |\sigma_{i,j}(\mathcal{F})|$,

(*b*) *for every $F \in \mathcal{F}$, we have $|F| = |\sigma_{i,j}(F)|$, and*

(*c*) *if $\mathcal{F}$ is intersecting, then $\sigma_{i,j}(\mathcal{F})$ is intersecting as well.*

Proof. Since (*a*) and (*b*) are easy to prove we focus only on (*c*).

Suppose $\mathcal{F}$ is intersecting. Given $A, B \in \mathcal{F}$ we shall prove

$$\sigma_{i,j}(A) \cap \sigma_{i,j}(B) \neq \varnothing.$$

First, if $\sigma_{i,j}(A) = A$ and $\sigma_{i,j}(B) = B$, then $\sigma_{i,j}(A) \cap \sigma_{i,j}(B) = A \cap B \neq \varnothing$, since $\mathcal{F}$ is intersecting. Second, if $\sigma_{i,j}(A) \neq A$ and $\sigma_{i,j}(B) \neq B$, then $\sigma_{i,j}(A) = (A \smallsetminus j) \cup i$ and $\sigma_{i,j}(B) = (B \smallsetminus j) \cup i$. Hence, $i \in \sigma_{i,j}(A) \cap \sigma_{i,j}(B) \neq \varnothing$. Thus, without loss of generality, we may assume $\sigma_{i,j}(A) = A$ and $\sigma_{i,j}(B) = (B \smallsetminus j) \cup i$.

If $i \in A$, then $i \in \sigma_{i,j}(A) \cap \sigma_{i,j}(B) \neq \varnothing$, and we would be done. Thus, we may assume the opposite and conclude

$$i \notin A \qquad \text{and} \qquad i \notin B. \tag{1.1}$$

Note that $\sigma_{i,j}(A) \cap \sigma_{i,j}(B) = A \cap \big((B \smallsetminus j) \cup i\big) \supseteq A \cap B \smallsetminus j$. Hence, if $A \cap B \neq j$, then $\sigma_{i,j}(A) \cap \sigma_{i,j}(B) \neq \varnothing$ and we would be done. Hence, we may assume

$$A \cap B = j, \tag{1.2}$$

and in particular, $j \in A$. That, together with (1.1) and the fact that $\sigma_{i,j}(A) = A$ yield $(A \smallsetminus j) \cup i \in \mathcal{F}$. But then, the fact that

$$\big((A \smallsetminus j) \cup i\big) \cap B \;\subseteq\; A \cap B \smallsetminus j \overset{(1.2)}{=} \varnothing \,.$$

is a contradiction, since $\mathcal{F}$ is intersecting. □

The following definition is crucial for the proof of Theorem 1.8.

Definition 1.11. A family $\mathcal{F} \subseteq 2^{[n]}$ is called *shifted* if for every $1 \leqslant i < j \leqslant n$ we have $\sigma_{i,j}(\mathcal{F}) = \mathcal{F}$.

Observe that for a family to be shifted it is require that $\sigma_{i,j}(\mathcal{F}) = \mathcal{F}$ only for $i < j$ (not for *every* pair i, j).

It is not hard to see that after finitely many iterative applications of the shifting we will end up with a shifted family. In particular, we have the following lemma.

Lemma 1.12. *Given an intersecting family $\mathcal{F} \subseteq 2^{[n]}$ there is a family $\mathcal{F}^\star \subseteq 2^{[n]}$ such that*

(i) *$\mathcal{F}^\star$ is shifted,*
(ii) *$\mathcal{F}^\star$ is intersecting,*
(iii) *$|\mathcal{F}| = |\mathcal{F}^\star|$, and*
(iv) *for $k \in [n]$, we have $|\mathcal{F} \cap \binom{[n]}{k}| = |\mathcal{F}^\star \cap \binom{[n]}{k}|$; in particular, if $\mathcal{F} \subseteq \binom{[n]}{k}$ then $\mathcal{F}^\star \subseteq \binom{[n]}{k}$.*

Proof. We first prove that after finitely many shifting operations applied consecutively to $\mathcal{F}$ we obtained a shifted family (i.e. satisfying (i)).

For a set $A \in 2^{[n]}$ define the parameter $\varphi(A) := \sum_{a \in A} a$ and for $\mathcal{F}$ define

$$\varphi(\mathcal{F}) := \sum_{A \in \mathcal{F}} \varphi(A) = \sum_{A \in \mathcal{F}} \sum_{a \in A} a \,.$$

Observe that $\varphi(\mathcal{F}) = 0$ if and only if $\mathcal{F} = \varnothing$ or $\mathcal{F} = \{\varnothing\}$. Thus, since $\mathcal{F}$ is intersecting we have $\varphi(\mathcal{F}) > 0$.

If $\mathcal{F}$ is already shifted then there is nothing to prove. Suppose $\mathcal{F}$ is not shifted, meaning there are $1 \leqslant i < j \leqslant n$ such that $\sigma_{i,j}(\mathcal{F}) \neq \mathcal{F}$. For at least one set $A \in \mathcal{F}$ we have $\sigma_{i,j}(A) = (A \smallsetminus j) \cup i$. Moreover, $\varphi(\sigma_{i,j}(A)) = \varphi(A) - j + i < \varphi(A)$. Summing over the whole family $\mathcal{F}$, we obtain

$$0 < \varphi(\sigma_{i,j}(\mathcal{F})) < \varphi(\mathcal{F}) \,,$$

where the first inequality follows from the fact that $\sigma_{i,j}(\mathcal{F})$ is non-empty.

By the reasoning above, for a non shifted family, the value of $\varphi(\cdot)$ reduces by at least one after applying the shifting operation. Hence, after at most $\varphi(\mathcal{F})$ iterative applications of shifting we must obtain a shifted family $\mathcal{F}^\star$, i.e. satisfying (i).

Finally, observe that due to Proposition 1.10 shifting preserves the size of the sets $F \in \mathcal{F}$, the size of $\mathcal{F}$ itself, and the property of being intersecting, therefore (ii)-(iv) are preserved under the any number of consecutive shifting operations. □

Now we prove Theorem 1.8.

Proof of Theorem 1.8. We proceed by induction on $k+n$. If $k=1$ the statement is obvious for every $n \geqslant 2$. For $n = 2k$ and $k \geqslant 1$, observe that

$$\binom{n}{k} = 2\binom{n-1}{k-1},$$

and therefore we may proceed as in the proof of Theorem 1.6. That is, as $\mathcal{F}$ is intersecting, we have $\mathcal{F} \cap \overline{\mathcal{F}} = \varnothing$. Thus, $2|\mathcal{F}| = |\mathcal{F}| + |\overline{\mathcal{F}}| \leqslant |\mathcal{F} \cup \overline{\mathcal{F}}| \leqslant \binom{n}{k} = 2\binom{n-1}{k-1}$ as desired.

Now, fix $k > 1$ and $n > 2k$ and suppose by induction that the statement of the theorem holds for every pair n', k' with $n' + k' < n + k$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family and apply Lemma 1.12 to obtain an shifted family $\mathcal{F}^\star$ satisfying *(ii)*-*(iv)*. In particular, we have $|\mathcal{F}| = |\mathcal{F}^\star|$ and $\mathcal{F}^\star \subseteq \binom{[n]}{k}$.

Consider the families

$$\mathcal{F}^\star(n) = \left\{ F \setminus n \in \binom{[n-1]}{k-1} : F \in \mathcal{F}^\star \text{ and } n \in F \right\} \quad \text{and}$$

$$\mathcal{F}^\star(\overline{n}) = \left\{ F \in \binom{[n-1]}{k} : F \in \mathcal{F}^\star \text{ and } n \notin F \right\},$$

and observe that $|\mathcal{F}^\star| = |\mathcal{F}^\star(n)| + |\mathcal{F}^\star(\overline{n})|$.

Note that $\mathcal{F}^\star(\overline{n}) \subseteq \mathcal{F}^\star$, and therefore, $\mathcal{F}^\star(\overline{n})$ is intersecting. We shall prove that $\mathcal{F}^\star(n)$ is intersecting as well. Indeed, let $A, B \in \mathcal{F}^\star(n)$ and suppose for a contradiction that $A \cap B = \varnothing$. As $|A| + |B| = 2k - 2 < n - 2$ there must be an $i \in [n-1]$ such that $i \notin A \cup B$. Since $A \cup n \in \mathcal{F}^\star$ and $\mathcal{F}^\star$ is shifted, we have $A \cup i \in \mathcal{F}$. However, this yields

$$(A \cup i) \cap (B \cup n) = \varnothing,$$

which is a contradiction as $A \cup i, B \cup n \in \mathcal{F}^\star$ and $\mathcal{F}^\star$ is intersecting. We conclude that $\mathcal{F}^\star(n)$ is intersecting.

Applying the induction hypothesis to both $\mathcal{F}^\star(n) \subseteq \binom{[n-1]}{k}$ and $\mathcal{F}^\star(\overline{n}) \subseteq \binom{[n-1]}{k-1}$ we obtain

$$|\mathcal{F}^\star| = |\mathcal{F}^\star(n)| + |\mathcal{F}^\star(\overline{n})| \leqslant \binom{(n-1)-1}{k-1} + \binom{(n-1)-1}{(k-1)-1} = \binom{n-1}{k-1},$$

as desired. $\square$

1.2. **Larger intersections.**

Definition 1.13. Given $t \geqslant 1$, we say that a family of subsets $\mathcal{F} \subseteq 2^{[n]}$ is t-intersecting if for every two sets $A, B \in \mathcal{F}$ we have

$$|A \cap B| \geqslant t.$$

Obviously $\mathcal{F}$ is 1-intersecting if and only if it is intersecting. The following lemma can be proven along the lines of Proposition 1.10.

Lemma 1.14. *Given $i, j \in [n]$, if $\mathcal{F}$ is t-intersecting then $\sigma_{i,j}(\mathcal{F})$ is also t-intersecting.* $\square$

The original paper by Erdős, Ko, and Rado [2] contained the following result is proved as well.

Theorem 1.15. *Let $n, k, t \in \mathbb{N}$ with $\binom{k}{t}^3 \leqslant \frac{n-t}{k-t}$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a t-intersecting family. Then*

$$|\mathcal{F}| \leqslant \binom{n-t}{k-t}.$$

Proof. Let $\mathcal{F}$ be a t-intersecting family of maximal size. Because of Lemma(s) 1.14 (and 1.12) we may assume that $\mathcal{F}$ is shifted.

First we prove that there are two sets $A, B \in \mathcal{F}$ with $|A \cap B| = t$. Indeed, suppose that for every two sets in $\mathcal{F}$ the intersection is of size at least $t+1$. Since $\mathcal{F}$ is shifted, we have that $[k] \in \mathcal{F}$. Observe that for every set $F \in \mathcal{F}$ we have $|[2, k+1] \cap F| \geqslant |[1, k] \cap F| - 1 \geqslant (t+1) - 1 = t$ (recall that $[a, b]$ denotes the set of all integers between a and b). Therefore, since $\mathcal{F}$ is maximal, the set $[2, k+1] \in \mathcal{F}$. In general, given an interval $[a, a+k-1] \in \mathcal{F}$ for every $F \in \mathcal{F}$ we have

$$|F \cap [a+1, a+k]| \geqslant |F \cap [a, b]| - 1 \geqslant (t+1) - 1 = t\,,$$

and therefore $[a+1, b+1] \in \mathcal{F}$ by maximally. Hence, $[k], [k+1, 2k] \in \mathcal{F}$, however, this is a contradiction, as $[k] \cap [k+1, 2k] = \varnothing$.

Now we take $A, B \in \mathcal{F}$ with $|A \cap B| = t$. If for every set $C \in \mathcal{F}$ we have $A \cap B \subseteq C$, then $\mathcal{F} \subseteq \{F \in \binom{[n]}{k} \colon A \cap B \subseteq F\}$ which is of size at most $\binom{n-t}{k-t}$ and we would be done. Thus, we may assume there is a set $C \subseteq F$ for which $A \cap B \nsubseteq C$. In particular, since $\mathcal{F}$ is t-intersecting we have

$$|A \cap B| = t, \qquad |A \cap C|, |B \cap C| \geqslant t, \qquad \text{and} \qquad |A \cap B \cap C| < t. \tag{1.3}$$

Roughly speaking, we want to bound the number of sets in $\mathcal{F}$ by counting the numbers of sets in $\mathcal{F}$ that intersect in A, B, and C exactly in three given sets $X \in \binom{A}{t}$, $Y \in \binom{B}{t}$, and $Z \in \binom{C}{t}$ respectively. Then, we sum over all possible sets X, Y, Z.

For every set $D \in \mathcal{F} \smallsetminus \{A, B, C\}$ define $D_A := D \cap A$, $D_B := D \cap B$, and $D_C := D \cap C$ and note that, since $\mathcal{F}$ is t-intersecting, we have $|D_A|, |D_B|, |D_C| \geqslant t$. Moreover, note that

$$|D_A \cup D_B \cup D_C| > t\,, \tag{1.4}$$

otherwise, if $D_A \cup D_B \cup D_C = t$, then we would have $D_A = D_B = D_C \subseteq A \cap B \cap C$ which contradicts (1.3).

Now, for any three sets $X \in \binom{A}{t}$, $Y \in \binom{B}{t}$, and $Z \in \binom{C}{t}$ define the function

$$\mathfrak{f}(X, Y, Z) := |\{D \in \mathcal{F} \colon D_A \supseteq X, D_B \supseteq Y, \text{ and } D_C \supseteq Z\}|\,,$$

and set $\ell := |X \cup Y \cup Z|$. Because of (1.4) we may assume $\ell \geqslant t+1$. Hence, we have

$$\mathfrak{f}(X,Y,Z) \leqslant \binom{n-\ell}{k-\ell} = \binom{n-\ell}{n-k} \leqslant \binom{n-(t+1)}{n-k} = \binom{n-t-1}{k-t-1}.$$

Finally, we finish the proof by observing that

$$\begin{aligned}|\mathcal{F}| &\leqslant \sum_{X\in\binom{A}{t}} \sum_{Y\in\binom{B}{t}} \sum_{Z\in\binom{C}{t}} \mathfrak{f}(X,Y,Z) \\ &\leqslant \binom{k}{t}^3 \binom{n-t-1}{k-t-1} \\ &\leqslant \frac{n-t}{k-t}\binom{n-t-1}{k-t-1} = \binom{n-t}{k-t},\end{aligned}$$

where the last inequality follows from the bound $\binom{k}{t}^3 \leqslant \frac{n-t}{k-t}$ in the statement of the theorem. □

Understanding the behaviour of maximal t-intersecting families when the inequality $\binom{k}{t}^3 \geqslant \frac{n-t}{k-t}$ does not hold turned out to be a very hard problem. It was only solved by Aswelde and Kachatrian [1] in 1999, more than 60 years after Erdős, Ko, and Rado's result proven.

1.3. **Non-trivial Families.**

Definition 1.16. A family $\mathcal{F} \subseteq 2^{[n]}$ is called *trivial* if $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$.

In other words, $\mathcal{F}$ is trivial if there is an element $x \in [n]$ which is contained in every edge. Erdős-Ko-Rado Theorem states that trivial families attained the maximum possible size of a k-uniform intersecting family. A natural question is to study what is the maximum possible size when we restrict ourselves to non-trivial families.

The following family is intersecting and non-trivial.

Example 1.17. Take $H \subseteq \binom{[n]}{k}$ defined as

$$\mathcal{H} = \{[2,k+1]\} \cup \left\{F \in \binom{[n]}{k} : 1 \in F \text{ and } F \cap [2,k+1] \neq \varnothing\right\}$$

Hilton and Milner [6] proved that Example 1.17 is indeed extremal.

Theorem 1.18. *(Hilton-Milner Theorem [6]) Let $k \geqslant 2$ and $n \geqslant 2k$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family, then*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$$

The original proof of Theorem 1.18 was hard and long. The following proof introduced by Frankl [4] is more fundamental (but still not so short). To make a clearer presentation, we will divide the proof in two lemmas. The first one reduces the problem to the case in which $\mathcal{F}$ is shifted.

Lemma 1.19. *Let $k \geqslant 2$ and $n \geqslant 2k$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a maximal non-trivial intersecting family, then there is a shifted non-trivial intersecting family $\mathcal{F}^\star$ with $|\mathcal{F}^\star| = |\mathcal{F}|$.*

Proof of Lemma 1.19. Let $\mathcal{F}$ be a maximal non-trivial intersecting family. Due to Proposition 1.10 the shifting operation does not change the size of the family nor the fact that it is intersecting. Thus, if after applying shifting for every $i < j \leqslant n$ the resulting family is still non-trivial, then we are done.

Hence we may assume there are $i < j \leqslant n$ such that $\mathcal{F}$ is non-trivial but $\sigma_{i,j}(\mathcal{F})$ is trivial. Without loss of generality, we may rename the vertices and assume that $i = 1$ and $j = 2$. Since $\sigma_{1,2}(\mathcal{F})$ is trivial but $\mathcal{F}$ is not, it is not hard to see that for every set $A \in \sigma_{1,2}(\mathcal{F})$ we have $1 \in A$. Moreover, for every $A \in \mathcal{F}$ we have $A \cap \{1, 2\} \neq \varnothing$. Thus, since $\mathcal{F}$ is maximal, it holds that

$$\mathcal{H}_{12} = \left\{ B \in \binom{[n]}{k} : 1, 2 \in B \right\} \subseteq \mathcal{F} \,. \tag{1.5}$$

Observe that for every $3 \leqslant i < j \leqslant n$ we have $\sigma_{i,j}(\mathcal{H}_{12}) = \mathcal{H}_{12}$. Therefore, for every $3 \leqslant i < j \leqslant n$ the family $\sigma_{i,j}(\mathcal{F})$ is non-trivial. Indeed, the only way in which $\sigma_{i,j}(\mathcal{F})$ could be trivial wold be if $i \in \bigcap_{A \in \sigma_{i,j}(\mathcal{F})} A$. However, that cannot happen since $\mathcal{H}_{12} \subseteq \sigma_{i,j}(\mathcal{F})$ and there is a set $B \notin H_{12}$ with $i \neq B$. Hence, we may apply shifting to $\mathcal{F}$ for all $3 \leqslant i < j \leqslant n$ and the family will still be non-trivial (intersecting, and of the same size). Denote the resulting family by $\mathcal{F}'$.

Using the shifting and the fact that $\mathcal{F}'$ is non-trivial, it is easy to see that both $[k+1] \smallsetminus 1, [k+1] \smallsetminus 2 \in \mathcal{F}'$. In fact, as $\mathcal{H}_{12} \subseteq \mathcal{F}'$, we have $\binom{[k+1]}{k} \subseteq \mathcal{F}'$.

Finally, for every $1 \leqslant i \leqslant j \leqslant n$ we have that $\sigma_{i,j}\left(\binom{[k+1]}{k}\right) = \binom{[k+1]}{k}$ and hence $\binom{[k+1]}{k} \subseteq \sigma_{i,j}(\mathcal{F}')$. In particular, as $\binom{[k+1]}{k}$ is non-trivial, $\sigma_{i,j}(\mathcal{F}')$ is non-trivial.

We obtain the desire $\mathcal{F}^\star$ after applying shifting for every $1 \leqslant i < j \leqslant n$. $\square$

In the second lemma deal with the case in which $\mathcal{F}$ is shifted.

Lemma 1.20. *Let $k \geqslant 2$ and $n \geqslant 2k$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a shifted non-trivial intersecting family, then*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \,.$$

The proof of Lemma 1.20 is based on a series of operations performed to the sets in $\mathcal{F}$.

Proof of Lemma 1.20. We first shall prove that for every set $A \in \mathcal{F}$, there is an integer $i \in [n]$ with

$$|A \cap [2i]| \geqslant i\,. \tag{1.6}$$

Indeed, let $A = \{a_1, \dots, a_k\}$ be an increasing ordering of the elements of A, that is $1 \leqslant a_1 < a_2 < \dots < a_k \leqslant n$. For a contradiction, assume that for all $j \in [k]$ we have $a_j > 2j$. Since the $\mathcal{F}$ is shifted, the set of even numbers $F = \{2, 4, \dots, 2k\} \in \mathcal{F}$. Indeed for every $j \in [k]$ note that $2j \in (A \smallsetminus a_j) \cup 2j \in \sigma_{2j,a_j}(\mathcal{F})$. Thus, F results from a at most k shifting operations applied to A. Moreover, after k shifting operations applied to F, we obtain the set of odd numbers $F' = \{1, 3, \dots, 2k-1\} \in \mathcal{F}$. However, $F \cap F' = \varnothing$, contradicting the fact that $\mathcal{F}$ is intersecting.

Now, we define ℓ_A as the maximum possible integer satisfying (1.6). That is, for each set $A \in \mathcal{F}$, define

$$\ell_A := \max\{i \in [n] \colon |A \cap [2i]| \geqslant i\}\,.$$

Observe that for $A = \{a_1, \dots, a_k\}$ with $1 \leqslant a_1 < a_2 < \dots < a_k \leqslant n$ we have

$$a_i > 2i \qquad \text{for every } i > \ell_A\,. \tag{1.7}$$

Therefore, as above, after $k - \ell_A$ shifting operations, we obtain

$$(A \cap [2\ell_A]) \cup \{2\ell_A + 2, 2\ell_A + 4, \dots, 2k\} \in \mathcal{F}\,. \tag{1.8}$$

Now, for each $A \in \mathcal{F}$ define the set

$$A^{\Delta} := (A^c \cap [2\ell_A]) \cup (A \cap [2\ell_A + 1, n]) = A \Delta [2\ell_A]\,,$$

where $S \Delta T := (S \smallsetminus T) \cup (T \smallsetminus S)$ is the *symmetric difference* between two sets. The following claim gives some of the mains properties of the $(\cdot)^{\Delta}$ operation.

Claim 1. *We have*

(i) $|A \cap [2\ell_A]| = |A^{\Delta} \cap [2\ell_A]| = \ell_A$,

(ii) for $\ell > \ell_A$, *we have* $|A \cap [2\ell]| = |A^{\Delta} \cap [2\ell]| < \ell$, *and*

(iii) if $1 \notin A$, *then* $1 \in A^{\Delta}$ *and* $\ell_A \geqslant 2$.

Proof of the claim: For *(i)*, observe that $|A \cap [2\ell_A]| \geqslant \ell_A$ simply by definition of ℓ_A. If $|A \cap [2\ell_A]| \geqslant \ell_A + 1$, then $|A \cap [2(\ell_A + 1)]| \geqslant \ell_A + 1$, contradicting he maximality of ℓ_A. Then, $|A \cap [2\ell_A]| = \ell_A$, and $|A^{\Delta} \cap [2\ell_A] = \ell_A$ follows directly from the definition of A^{Δ}.

For *(ii)*, let $\ell > \ell_A$ and note that for every $a \in [2\ell_A + 1, \ell]$ we have $a \in A$ if and only if $a \in A^{\Delta}$. Thus,

$$\begin{aligned} |A \cap [2\ell]| &= |A \cap [2\ell_A]| + |A \cap [2\ell_A + 1, \ell]| \\ &= |A^{\Delta} \cap [2\ell_A]| + |A^{\Delta} \cap [2\ell_A + 1, \ell]| \\ &= |A^{\Delta} \cap [2\ell]|\,, \end{aligned}$$

where we used *(i)*. For the inequality in *(ii)*, note that $|A \cap [2\ell]| \geqslant \ell$ yields a contradiction with the maximality of ℓ_A.

Finally, for *(iii)*, note that if $\ell_A = 1$ then $2 \in A$. Therefore, (1.8) implies the set of even numbers $F = \{2, 4, \dots, 2k\} \in \mathcal{F}$. After shifting operations on F, we get that the set of odd numbers $F' = \{1, 3, \dots, 2k-1\} \in \mathcal{F}$. This contradicts the fact that $\mathcal{F}$ is intersecting. Hence $\ell_A \geqslant 2$, and therefore, $1 \in A^{\Delta}$. ■

Set $\mathcal{F}(\bar{1}) = \{A \in \mathcal{F} \colon 1 \notin A\}$ and define the following ma $\varphi \colon \mathcal{F}(\bar{1}) \longrightarrow \binom{[2,n]}{k-1}$ by

$$\varphi(A) := A^{\Delta} \smallsetminus 1 .$$

The following claim summarises the main properties of $\varphi(\cdot)$.

Claim 2. *We have*

(a) φ is injective,
(b) $\varphi(\mathcal{F}(\bar{1})) \cap \mathcal{F}(1) = \varnothing$, and
(c) for $A \in \mathcal{F}(\bar{1}) \smallsetminus \{[2, k+1]\}$ we have $\varphi(A) \cap [2, k+1] \neq \varnothing$,

where $\mathcal{F}(1) = \left\{A \in \binom{[n] \smallsetminus 1}{k-1} \colon A \cup 1 \in \mathcal{F}\right\}$.

Proof of the claim: For *(a)*, take two different sets $A, B \in \mathcal{F}(\bar{1})$, and observe that if $\ell_A < \ell_B$ then, due to *(ii)*, we have

$$|A \Delta [2\ell_B]| < \ell_B = |B \Delta [2\ell_B]| ,$$

meaning that $\varphi(A) \neq \varphi(B)$. If $\ell_A = \ell_B = \ell$, then $A \neq B$ implies

$$\varphi(A) = A^{\Delta} \smallsetminus 1 = A\Delta[2\ell] \smallsetminus 1 \neq B\Delta[2\ell] \smallsetminus 1 = B^{\Delta} \smallsetminus 1 = \varphi(B) .$$

For *(b)*, suppose there are $A, B \in \mathcal{F}$ with $\varphi(A) = B \smallsetminus 1$ and set $\ell := \ell_A$. Because of (1.8), we have

$$F_1 = (A \cap [2\ell]) \cup \{2\ell_A + 2, 2\ell_A + 4, \dots, 2k\} \in \mathcal{F} .$$

Moreover, since $B \cap [2\ell + 1, n] = A \cap [2\ell + 1, n]$ and (1.7) we have

$$F_2 = (B \cap [2\ell]) \cup \{2\ell_A + 1, 2\ell_A + 3, \dots, 2k - 1\} \in \mathcal{F} ,$$

as well. However, $B \cap [2\ell] = A^c \cap [2\ell]$ yields that $F_1 \cap F_2 = \varnothing$, contradicting the fact that $\mathcal{F}$ is intersecting.

For *(c)* observe that, due to *(iii)* and *(i)*, if $2\ell_A \leqslant k + 1$, then

$$|A^{\Delta} \cap [k+1]| \geqslant |A^{\Delta} \cap [2\ell_A]| = \ell_A \geqslant 2 .$$

Thus $|\varphi(A) \cap [2, k+1]| \geqslant |A^{\Delta} \cap [2\ell_A]| - 1 \geqslant 1$, meaning $\varphi(A) \cap [2, k+1] \neq \varnothing$. When $2\ell_A \geqslant k + 1$ observe that if $\varphi(A) \cap [2, k+1] = \varnothing$ then

$$\varphi(A) \cap [2, k+1] = (A^{\Delta} \smallsetminus 1) \cap [2, k+1] = A^c \cap [2, k+1] = \varnothing ,$$

meaning that $A \subseteq [2, k+1]$. As $|A| = k$, we have $A = [2, k+1]$. ■

Now we use the properties of $\varphi(\cdot)$ to finish the proof. Since $\mathcal{F}$ is shifted and non-trivial we have

$$A_\star := [2, k+1] \in \mathcal{F}\,.$$

As $\mathcal{F}$ is intersecting, for every $A \in \mathcal{F}(1)$ we have $A \cap A_\star \neq \varnothing$. Further, because of (*c*) for every $A \in \mathcal{F}(\bar{1}) \smallsetminus \{A_\star\}$ we have $\varphi(A) \cap A_\star \neq \varnothing$. Therefore,

$$\mathcal{F}(1) \cup \varphi\big(\mathcal{F}(\bar{1}) \smallsetminus \{A_\star\}\big) \subseteq \left\{A \in \binom{[2,n]}{k-1} : A \cap A_\star \neq \varnothing\right\}\,.$$

Hence,

$$\begin{aligned}
|\mathcal{F}| - 1 &= |\mathcal{F}(1)| + |\mathcal{F}(\bar{1})| - 1\\
&= |\mathcal{F}(1)| + |\varphi(\mathcal{F}(\bar{1}))| - 1\\
&= \big|\mathcal{F} \cup \varphi\big(\mathcal{F}(\bar{1}) \smallsetminus \{A_\star\}\big)\big|\\
&\leqslant \left|\left\{A \in \binom{[2,n]}{k-1} : A \cap A_\star \neq \varnothing\right\}\right| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1},
\end{aligned}$$

where the second identity follows from (*a*) and the third one from (*a*) and (*b*). $\square$

It is it clear how Theorem 1.18 follows from Lemmas 1.19 and 1.20.

§2. Proofs via Katona's circle

In this section we introduce the so-called circle method, which was introduced by Katona in the 70s [8, 9]. It was famously applied to give an ~~amazing~~ insightful new proof of Erdős-Ko-Rado [8]. Later, this method helped to prove many other results (for a survey, see [5]).

2.1. Basic definitions.

We need to introduce the notion of *circles* and *arcs*. Before giving a formal definition, let us consider the following rough description. Loosely speaking, a circle is a circular arrangement of the elements in $[n]$ that describes which pairs of elements are 'consecutive'. More precisely, for a permutation $(a_1, a_2, \ldots, a_n)$ of the elements of $[n]$, all pairs (a_i, a_{i+1}) for every $i \in [n-1]$ and (a_n, a_1) are considered to be consecutive elements in the circle induced by the permutation. Consequently, we say that two permutations $(a_1, a_2, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ induce the same circle if their consecutive elements are the same. For example $(a_1, a_2, \ldots, a_n)$ and $(a_2, a_3, \ldots, a_n, a_1)$ induce the same circle. Given a fixed circle $(a_1, \ldots, a_n)$ and $i < k \leqslant n$, a set of consecutive elements of the form $\{a_i, a_{i+1}, \ldots, a_{i+k}\}$ is called *arc*, where we take the sum of indices modulo n.

The informal definitions above are enough to understand the Katona's circle method. However, to avoid possible confusion, let us introduce the formal definition.

Definition 2.1 (circles and arcs)**.** Given $n \in \mathbb{N}$ and two permutations $\pi_1 = (a_1, \ldots, a_n) \in S_n$ and $\pi_2 = (b_1, \ldots, b_n) \in S_n$, we say they are *circularly equivalent* denoted by $\pi_1 \sim \pi_2$ if there is a $k \in [n]$ such that $a_i = b_{i+k}$ for every $i \in [n]$, where the sum is taken modulo n. It is easy to see that $\sim$ is an equivalent relation.

The equivalence classes given by the relation of circular equivalence are called *circles*, and the set of circles is denoted by $\mathcal{C}_n = S_n/\sim$. We will slightly abuse notation and will not distinguish between the circle $[(a_1, \ldots, a_n)]$ and the permutation $(a_1, \ldots, a_n)$. In other words, we will consider a circle to be any of its representatives.

Finally, given a circle $\pi = (a_1, \ldots, a_n) \in \mathcal{C}_n$ and $k \in [n-1]$ we define the set of *arcs in* π *of length* k by

$$\mathcal{A}(\pi, k) := \{a_i, \ldots, a_{i+k-1} \in 2^{[n]} : i \in [n]\},$$

where the sum is taken modulo n. Moreover, the sets of all arcs will be denoted by $\mathcal{A}(\pi) = \bigcup_{k \in [n-1]} \mathcal{A}(\pi, k)$. Given an arc $a_i, \ldots, a_{i+k-1} \in \mathcal{A}(\pi, k)$ we say that a_i is its *first* element and that a_{i+k-1} is its *last* element.

Note that 'arcs of length zero' and 'arcs of length n' are excluded from these definitions.

Given $n \in \mathbb{N}$, $k \in [n-1]$, and $\pi \in \mathcal{C}_n$ the following identities are easy to deduce

$$|\mathcal{C}_n| = (n-1)!, \qquad |\mathcal{A}(\pi, k)| = n, \qquad \text{and} \qquad |\mathcal{A}(\pi)| = n(n-1)\,.$$

To show the method in action, let us show the following elegant proof of Erdős-Ko-Rado Theorem given by Katona [8].

2.2. **Erdős-Ko-Rado Theorem (reprise).**

Proof of Theorem 1.8. Let $n \geqslant 2k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Given a circle $\pi \in \mathcal{C}_n$, define

$$\mathcal{F}(\pi) := \mathcal{F} \cap \mathcal{A}(\pi)\,,$$

and note that $\mathcal{F}(\pi)$ is intersecting.

Claim 3 (Erdős-Ko-Rado in the circle)**.** *For every* $\pi \in \mathcal{C}_n$ *we have* $|\mathcal{F}(\pi)| \leqslant k$*.*

Proof of the claim: Pick any arc $A = a_1, \ldots, a_k \in \mathcal{F}(\pi)$, where a_1 and a_k are the first and last element respectively. As $\mathcal{F}(\pi)$ is intersecting for every $B \in \mathcal{F}(\pi) \setminus \{A\}$ either the first or last element of B lies in A. Since A is the arc of length k whose first element is a_1, the first element of B cannot be a_1. Similarly a_k cannot by the last element of B.

Let L_i be the arc of length k ending in a_i and F_i the arc of length k starting in a_{i+1}. Hence

$$\mathcal{F}(\pi) \subseteq \{L_i : i \in [k-1]\} \cup \{F_i : i \in [k-1]\}\,.$$

Since $n \geqslant 2k$, we have $L_i \cap F_i = \varnothing$, and therefore, for every $i \in [k-1]$ at most one of the two sets L_i or F_i is in $\mathcal{F}(\pi)$. Adding the initial set A we obtain $|\mathcal{F}(\pi)| \leqslant (k-1) + 1 = k$. ∎

Now, for every set $A \in \mathcal{F}$, there are $k!(n-k)!$ circles $\pi \in \mathcal{C}_n$ such that $A \in \mathcal{F}(\pi)$. That entails

$$\sum_{\pi \in \mathcal{C}_n} |\mathcal{F}(\pi)| = k!(n-k)!|\mathcal{F}|\,.$$

Applying Claim 3 and recalling that $|\mathcal{C}_n| = (n-1)!$ we have

$$|\mathcal{F}| = \frac{1}{k!(n-k)!} \sum_{\pi \in \mathcal{C}_n} |\mathcal{F}(\pi)| \leqslant \frac{k(n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}\,.$$

□

2.3. **Sperner Theorem via Katona's circle.** A family $\mathcal{F} \subseteq 2^{[n]}$ is called an *antichain* if there are no two sets $A, B \in \mathcal{F}$ with $A \subseteq B$. Observe that if all sets in $\mathcal{F}$ are of the same size, then $\mathcal{F}$ is an antichain.

Sperner's Theorem states that the maximum possible size of antichain is attained by a family that consists in all sets of certain size. In other words, the extremal families are of the form $\binom{[n]}{k}$, and it is easy to see that the maximum of the binomial coeficient is attained when $k = \lfloor n/2 \rfloor$.

Theorem 2.2 (Sperner's Theorem)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. Then*

$$|\mathcal{F}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}\,.$$

The first proof was given by Sperner in 1928 [13]. The following proof was presented by Lubell in a *one-page* paper [12].

Proof of Theorem 2.2. Set $m = \lfloor \frac{n}{2} \rfloor$. Note that the function $f(k) = \binom{n}{k}$ attain its maximum for $k = m$. That is $\binom{n}{k} \leqslant \binom{n}{m}$, for every $k \in [n]$. Diving by $n!$ we obtain

$$\frac{1}{k!(n-k)!} \leqslant \frac{1}{m!(n-m)!}\,. \tag{2.1}$$

We now use the Katona's method. Given $\pi \in \mathcal{C}_n$ let $\mathcal{F}(\pi) = \mathcal{A}(\pi) \cap \mathcal{F}$, and note that $\mathcal{F}(\pi)$ is an antichain.

Claim 4 (Circular Sperner)**.** *For every $\pi \in \mathcal{C}_n$ we have $|\mathcal{F}(\pi)| \leqslant n$.*

Proof of the claim: If there are two sets $A, B \in \mathcal{F}$ with the same last element then we have $A \subseteq B$ or $B \subseteq A$. Therefore, as $\mathcal{F}(\pi)$ is an antichain, for every $i \in [n]$ there is at most one set $\mathcal{F}(\pi)$ ending in i. This yields, $|\mathcal{F}(\pi)| \leqslant n$. ■

As in the proof of Theorem 1.8 observe that for $A \in \mathcal{F}$, there are $|A|!(n-|A|)!$ circles $\pi \in \mathcal{C}_n$ such that $A \in \mathcal{F}(\pi)$. Hence, we have

$$\begin{aligned} |\mathcal{F}| &= \sum_{\pi \in \mathcal{C}_n} \sum_{A \in \mathcal{F}(\pi)} \frac{1}{|A|!(n-|A|)!} \\ &\overset{(2.1)}{\leqslant} \sum_{\pi \in \mathcal{C}_n} \sum_{A \in \mathcal{F}(\pi)} \frac{1}{m!(n-m)!} \\ &\leqslant \sum_{\pi \in \mathcal{C}_n} \frac{n}{m!(n-m)!} = \frac{n(n-1)!}{m!(n-m)!} = \binom{n}{m}, \end{aligned}$$

where the last inequality follows from Claim 4 $\square$

§3. Kruskal-Katona Theorem: Minimal shadows

3.1. Preliminaries.

Definition 3.1. Given a family of sets $\mathcal{F} \subseteq \binom{[n]}{k}$, the *shadow of* $\mathcal{F}$ is defined by

$$\partial \mathcal{F} = \left\{ X \in \binom{[n]}{k-1} : \text{ there is an } x \in [n] \text{ with } X \cup x \in \mathcal{F} \right\}.$$

In other words, the shadow of $\mathcal{F}$ is the family of all $k-1$ subsets of sets in $\mathcal{F}$. The main question behind Kruskal-Katona Theorem (Theorem 3.9) is the following.

Question 3.2. Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size m, how small $\partial \mathcal{F}$ can be?

Both Katona [7] and Kruskal [10] independently found a surprisingly detailed answer to this question. The proof we present here was introduced by Frankl [3].

Before, going any further let us prove the following two preliminary lemmas. The first one state that we can essentially restrict our attention to shifted families.

Lemma 3.3. *Given* $\mathcal{F} \subseteq \binom{[n]}{k}$*, we have that*

$$\partial(\sigma_{i,j}(\mathcal{F})) \subseteq \sigma_{i,j}(\partial \mathcal{F}).$$

Proof. Let $X \in \partial(\sigma_{i,j}(\mathcal{F}))$, and therefore, there are $x \in [n]$ and $A \in \mathcal{F}$ such that $X \cup x = \sigma_{i,j}(A)$. We shall prove that $X \in \sigma_{i,j}(\partial F)$. Indeed, this follows from a straightforward case analysis.

Suppose first that $\sigma_{i,j}(A) = A$. Then $X \cup x = A$, and hence $X \in \partial \mathcal{F}$. Note that if $\sigma_{i,j}(X) = X$ we are done, as that implies that $X \in \sigma_{i,j}(\partial \mathcal{F})$. If $x = i$, then $(X \setminus j) \cup i = (X \setminus j) \cup x = A \setminus j$, implying that $(X \setminus j) \cup i \in \partial F$. Therefore, $\sigma_{i,j}(X) = X$ (as $(X \setminus j) \cup i$ is already in $\partial \mathcal{F}$). If $x \neq i$, we have $j \in A$ and $i \notin A$, thus, since $\sigma_{i,j}(A) = A$, we have $(A \setminus j) \cup i \in \mathcal{F}$. That yields $(X \setminus j) \cup i = ((A \setminus j) \cup i) \setminus x \in \partial \mathcal{F}$, again meaning that $\sigma_{i,j}(X) = X$ as desired.

Now suppose $\sigma_{i,j}(A) = (A \setminus j) \cup i$. If $x = i$, then $\sigma_{i,j}(A) \setminus i = A \setminus j = X$ which yields $X \in \partial \mathcal{F}$. Since $j \notin X$, we have $\sigma_{i,j}(X) = X$, implying that $X \in \sigma_{i,j}(\partial \mathcal{F})$.

If $x \neq i$, then $A \smallsetminus x = (X \smallsetminus i) \cup j)$ meaning that $(X \smallsetminus i) \cup j \in \partial\mathcal{F}$. If we suppose further that $\sigma_{i,j}((X \smallsetminus i) \cup j) \neq (X \smallsetminus i) \cup j$, then we have

$$\sigma_{i,j}((X \smallsetminus i) \cup j) = \Big(((X \smallsetminus i) \cup j) \smallsetminus j\Big) \cup i = X\,, \tag{3.1}$$

meaning that $X \in \sigma_{i,j}(\partial\mathcal{F})$. Then, we may assume that $\sigma_{i,j}((X \smallsetminus i) \cup j) = (X \smallsetminus i) \cup j$. Considering the identity in (3.1) we have that X is 'blocked' in $\partial\mathcal{F}$. In other words, $X \in \partial\mathcal{F}$. Therefore, since $\sigma_{i,j}(X) = X$, we have $X \in \sigma_{i,j}(\partial\mathcal{F})$. □

Proposition 3.3 means that applying the shifting operation will only decrease the size of the shadow. Hence, the minimal is attained by a shifted family.

The following lemma gives useful properties for the shadows of shifted families.

Lemma 3.4. *Let $\mathcal{F}\binom{[n]}{k}$ be a shifted family. Then, we have that*

(1) $\partial\big(\mathcal{F}(\bar{1})\big) \subseteq \mathcal{F}(1)$ and

(2) $|\partial\mathcal{F}| = |\mathcal{F}(1)| + |\partial(\mathcal{F}(1))|\,,$

where $\mathcal{F}(1) = \Big\{A \in \binom{[n]\smallsetminus 1}{k-1} \colon A \cup 1 \in \mathcal{F}\Big\}$ and $\mathcal{F}(\bar{1}) = \Big\{A \in \binom{[n]\smallsetminus 1}{k} \colon 1 \notin A \in \mathcal{F}\Big\}$.

Proof. For (1) let $X \in \partial\big(\mathcal{F}(\bar{1})\big)$, then there is an $x \in [n] \smallsetminus 1$ such that $x \cup X \in \mathcal{F}(\bar{1})$. As $\mathcal{F}$ is shifted, we have $\sigma_{1,x}(X \cup x) = X$, meaning that $X \cup 1 \in \mathcal{F}$. Thus, $X \in \mathcal{F}(1)$.

To prove (2), let

$$\mathcal{H}_1 = \{X \in \partial\mathcal{F} \colon 1 \in X\} \qquad \text{and} \qquad \mathcal{H}_{\bar{1}} = \{X \in \partial F \colon 1 \notin X\}\,.$$

Note that $\partial\mathcal{F} = \mathcal{H}_1 \dot\cup \mathcal{H}_{\bar{1}}$. We shall prove that $|\mathcal{F}(1)| = |\mathcal{H}_{\bar{1}}|$ and $|\partial(\mathcal{F}(1))| = |\mathcal{H}_1|$.

Take an arbitrary $X \in \mathcal{H}_{\bar{1}}$. Then there is an $x \in [n]$ with $X \cup x \in \mathcal{F}$ and since $\mathcal{F}$ is shifted, we have $X \cup 1 \in \mathcal{F}$ as well (note that we might have $x = 1$). Hence $X \in \mathcal{F}(1)$. Reciprocally, if we take an arbitrary set $X \in \mathcal{F}(1)$, then $X \cup 1 \in \mathcal{F}$, and then $X \in \partial\mathcal{F}$, with $1 \notin X$. In other words, $X \in \mathcal{H}_{\bar{1}}$. Consequently, we have $\mathcal{H}_{\bar{1}} = \mathcal{F}(1)$.

To show that $|\partial(\mathcal{F}(1))| = |\mathcal{H}_1|$ define $\mathcal{H}_1' = \Big\{Y \in \binom{[n]}{k-2} \colon Y \cup 1 \in \mathcal{H}_1\Big\}$ and note

$$|\mathcal{H}_1'| = |\mathcal{H}_1| \tag{3.2}$$

Directly form the definition, for any set $Y \in \mathcal{H}_1'$, there is a $y \in [n] \smallsetminus 1$ such that $Y \cup y \cup 1 \in \mathcal{F}$. Similarly, for every $Y \in \partial(\mathcal{F}(1))$, there is an $y \in [n] \smallsetminus 1$ with $Y \cup y \in \mathcal{F}(1)$, meaning that $Y \cup y \cup 1 \in \mathcal{F}$. In other words,

$$\begin{aligned} Y \in \mathcal{H}_1' &\Leftrightarrow Y \cup 1 \in \mathcal{H}_1 \\ &\Leftrightarrow \exists y \in [n] \smallsetminus 1 \text{ such that } Y \cup 1 \cup y \in \mathcal{F} \\ &\Leftrightarrow \exists y \in [n] \smallsetminus 1 \text{ such that } Y \cup y \in \mathcal{F}(1) \\ &\Leftrightarrow Y \in \partial\big(\mathcal{F}(1)\big)\,. \end{aligned}$$

That clearly yields $\mathcal{H}_1' = \partial\big(\mathcal{F}(1)\big)$. Applying (3.2) we obtain $|\mathcal{H}_1| = |\partial\big(\mathcal{F}(1)\big)|$. □

3.2. **Vanilla version.** Before stating the full version, we present the following simplified version, originally introduced by Lovász [11].

Theorem 3.5 (Kruskal-Katona - Vanilla version [11])**.** *Let $k \leqslant m$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| \geqslant \binom{m}{k}$, then*

$$|\partial \mathcal{F}| \geqslant \binom{m}{k-1}.$$

Proof. Because of Proposition 3.3 we may assume $\mathcal{F}$ is shifted. We proceed by induction on $m+k$. When $k=1$ or $m=k$ the statement in trivial and give us the base case.

First, we shall prove

$$|\mathcal{F}(1)| \geqslant \binom{m-1}{k-1}. \tag{3.3}$$

Suppose that (3.3) does not hold, i.e. $|\mathcal{F}(1)| < \binom{m-1}{k-1}$. Since $|\mathcal{F}| = |\mathcal{F}(\bar{1})| + |\mathcal{F}(1)|$, we have

$$|\mathcal{F}(\bar{1})| = |\mathcal{F}| - |\mathcal{F}(1)| > \binom{m}{k} - \binom{m-1}{k-1} = \binom{m-1}{k}.$$

Then, we can use the induction hypothesis for $\mathcal{F}(\bar{1})$, obtaining $|\partial\big(\mathcal{F}(\bar{1})\big)| \geqslant \binom{m-1}{k-1}$. However, this is a contradiction, since *(1)* in Lemma 3.4 yields $|\partial\big(\mathcal{F}(\bar{1})\big)| \leqslant |\mathcal{F}(1)| < \binom{m-1}{k-1}$ due to our supposition. Therefore, (3.3) holds.

Now, due to (3.3) we may apply the induction hypothesis now to $\mathcal{F}(1)$ and deduce that $|\partial\big(\mathcal{F}(1)\big)| \geqslant \binom{m-1}{k-2}$. Therefore, due to *(2)* from Lemma 3.4 we have

$$|\partial \mathcal{F}| = |\mathcal{F}(1)| + |\partial\big(\mathcal{F}(1)\big)| \geqslant \binom{m-1}{k-1} + \binom{m-1}{k-2} = \binom{m}{k-1}$$

□

3.3. **Kruskal-Katona Theorem.** Observe that we can apply Theorem 3.5 only when the size of the family $|\mathcal{F}|$ is of the form $\binom{m}{k}$. Furthermore, for every $x \in \mathbb{R}$ define

$$\binom{x}{k} := \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}.$$

Then the proof of Theorem 3.5 still holds answering Question 3.2 whenever $|\mathcal{F}|$ is of the form $\binom{x}{k}$ for any real $x \in \mathbb{R}$.

To state the full version of the theorem, we need the following definition.

Definition 3.6. The *colexicographical order,* $\lessdot$ is a total order $\binom{\mathbb{N}}{k}$, the set of k-sets of natural numbers, defined by

$$A \lessdot B \iff \max A \setminus B \leqslant \max B \setminus A\,.$$

Intuitively, we have that sets containing large elements will appear later in the colexicographical order. For example for $k=3$, the first 3-sets in the colexicographical order are

$$123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136 \tag{3.4}$$

Observe that no element containing the number 6 appear until all elements from $\binom{[5]}{3}$ have already appeared.

Definition 3.7. Given $m \geqslant k$ we define $\mathscr{L}^{(k)}(m)$ to be the first m sets of size k in the colexicographical order.

Thus, (3.4) corresponds to the set $\mathscr{L}^{(3)}(12)$. The following fact is easy to see.

Fact 3.8. *Let $k \leqslant m \leqslant n$ and $A \in \binom{[n]}{k}$ be a set such that $m \in A$. Then for every $B \in \binom{[m-1]}{k}$ we have $B \prec A$.* $\square$

In other words, in the colexicographical order, a set containing $m \in \mathbb{N}$ is larger than all elements in $\binom{[m-1]}{k}$. That is to say, if $m \in A$, then $A \notin \mathscr{L}^{(k)}(\binom{m-1}{k})$. Moreover, we have

$$\mathscr{L}^{(k)}\left(\binom{n}{k}\right) = \binom{[n]}{k}.$$

Kruskal-Katona Theorem states that the families that minimise the shadow are precisely the initial segments of the colexicographical order.

Theorem 3.9 (Kruskal-Katona)**.** *For every family $\mathcal{F} \subseteq \binom{[n]}{k}$ such that $|\mathcal{F}| = m$ we have*

$$|\partial\mathcal{F}| \geqslant |\partial\mathscr{L}^{(k)}(m)|.$$

Theorem 3.9 is stronger than the vanilla version given in Theorem 3.5. However, the main ideas for the proof of the former, are already present in the proof of the later.

The main difference between Theorems 3.5 and 3.9 is that the size of $\mathcal{F}$ in Theorem 3.9 is not necessarily of the form $\binom{m}{k}$. To bring the two theorems together, assume $m \in \mathbb{N}$ can be 'represented' as a sum of binomial coefficients:

$$m = \sum_{j=\ell}^{k} \binom{a_j}{j}, \tag{3.5}$$

for integers $a_k > a_{k-1} > \cdots > a_\ell \geqslant \ell \geqslant 1$. Indeed, it is not hard to prove that every natural number $k \leqslant m \in \mathbb{N}$ has a representation like (3.5). It is called the *cascade representation of m.*

The following lemma brings together the cascade representation and the initial segment of the colexicographical order.

Lemma 3.10. *Let $k \leqslant m$ with $m = \sum_{j\in[\ell,k]} \binom{a_j}{j}$ for integers $a_k > a_{k-1} > \cdots > a_\ell \geqslant \ell \geqslant 1$. Then,*

$$|\partial\mathscr{L}^{(k)}(m)| = \sum_{j=\ell}^{k} \binom{a_j}{j-1}.$$

Proof. We proceed by induction on k. Note that for $k = 1$ the result follows easily for every $m \geqslant 1$.

We need the following claim about binomial coefficients.

Claim 5. *Given $a \geqslant k \geqslant 2$*

$$\sum_{j=1}^{k-1}\binom{a-j}{k-j} < \binom{a}{k-1}.$$

Before proving Claim 5, let us first see how it is helpful for the proof of the lemma. Observe that since $a_k > a_{k-1} > \cdots > a_\ell \geqslant \ell \geqslant 1$ it holds

$$m = \sum_{j\in[\ell,k]}\binom{a_j}{j} \leqslant \sum_{j=0}^{k-1}\binom{a_k-i}{k-i} < \binom{a_k}{k} + \binom{a_k}{k-1} = \binom{a_k+1}{k},$$

where the last inequality follows from Claim 5. Then we have

$$\binom{a_k}{k} \leqslant m < \binom{a_k+1}{k}. \tag{3.6}$$

Let $m_2 = m - \binom{a_k}{k} = \sum_{j=\ell}^{k-1}\binom{a_j}{j}$ and observe that its cascade representation is the same as in m except for the largest term. From Definition 3.6, Fact 3.8, and (3.6) it is easy to see that

$$\mathscr{L}^{(k)}(m) = \binom{[a_k]}{k} \cup \left\{A \cup a_{k+1} \colon A \in \mathscr{L}^{(k-1)}(m_2)\right\}.$$

Note that for every set $A \in \partial\mathscr{L}^{(k-1)}(m_2)$ it trivially holds that $a_{k+1} \cup A \notin \binom{[a_k]}{k-1}$. Therefore, we deduce

$$|\partial\mathscr{L}^{(k)}(m)| = \binom{a_k}{k-1} + \left|\partial\mathscr{L}^{(k-1)}(m_2)\right| = \binom{a_k}{k-1} + \sum_{j=\ell}^{k-1}\binom{a_j}{j-1} = \sum_{j=\ell}^{k}\binom{a_j}{j-1},$$

where we applied the induction hypothesis for $\mathscr{L}^{(k-1)}(m_2)$.

We now show the proof of Claim 5.

Proof of the claim: We proceed by induction on $a+k$. For $k=2$ the statement is trivial for every $a \geqslant 2$ and given $k \geqslant 2$, the statement is trivial for $a = k$. Let $a \geqslant k \geqslant 3$ and apply the induction hypothesis (twice) to see that

$$\begin{aligned}\sum_{j=1}^{k-1}\binom{a-j}{k-j} &= \sum_{i=1}^{k-1}\binom{a-j-1}{k-j} + \binom{a-j-1}{k-j-1} \\ &< \binom{a-1}{k-1} + \binom{a-1}{k-2} = \binom{a}{k-1}.\end{aligned}$$

■

□

Finally we are ready for the proof of Theorem 3.9. We follow the same structure as the proof of Theorem 3.5.

Proof of Theorem 3.9. Because of Proposition 3.3 we may assume $\mathcal{F}$ is shifted. We proceed by induction on $m+k$. When $k=1$ or $m=1$ the statement in trivial and give us the base case.

Let the cascade representation of m be given by

$$m = \sum_{j=\ell}^{k} \binom{a_j}{j},$$

for integers $a_k > a_{k-1} > \cdots > a_\ell \geqslant \ell \geqslant 1$.

We first prove the following claim

Claim 6. $|\mathcal{F}(1)| \geqslant \displaystyle\sum_{j=\ell}^{k} \binom{a_j - 1}{j-1}$.

Proof of the claim: Let $m_2 = \sum_{j=\ell}^{k} \binom{a_j-1}{j-1}$ and suppose that the claim does not hold, i.e. $|\mathcal{F}(1)| < m_2$. Since $|\mathcal{F}| = |\mathcal{F}(\bar{1})| + |\mathcal{F}(1)|$, we have

$$|\mathcal{F}(\bar{1})| = |\mathcal{F}| - |\mathcal{F}(1)| > m - m_2 = \sum_{j=\ell}^{k} \binom{a_j}{j} - \binom{a_j-1}{j-1} = \sum_{j=\ell}^{k} \binom{a_j-1}{j} \tag{3.7}$$

Suppose that

$$|\partial\big(\mathcal{F}(\bar{1})\big)| \geqslant m_2\,, \tag{3.8}$$

then *(1)* in Lemma 3.4 would yield $m_2 \leqslant |\partial\big(\mathcal{F}(\bar{1})\big)| \leqslant |\mathcal{F}(1)|$ which contradicts our supposition that $|\mathcal{F}(1)| < m_2$.

We finish the proof of the claim by showing (3.8). We consider two cases.

First Case: $a_\ell > \ell$**.** Due to (3.7) we may apply the induction hypothesis to $\mathcal{F}(\bar{1})$ to obtain $|\partial\big(\mathcal{F}(\bar{1})\big)| \geqslant |\mathscr{L}^k(m_2)|$. In this case we have $a_k - 1 > a_{k-1} - 1 > \cdots > a_\ell - 1 \geqslant \ell$ meaning that $\sum_{j\in[\ell,k]} \binom{a_j-1}{j}$ is a cascade representation. Hence Lemma 3.10 yields

$$|\partial\big(\mathcal{F}(\bar{1})\big)| \geqslant |\mathscr{L}^k(m_2)| = \sum_{j=\ell}^{k} \binom{a_j-1}{j-1} = m_2\,,$$

as desired.

Second Case: $a_\ell = \ell$**.** As before, the inequality (3.7) allow us to apply the induction hypothesis to $\mathcal{F}(\bar{1})$ to obtain $|\partial\big(\mathcal{F}(\bar{1})\big)| \geqslant |\mathscr{L}^k(m_2)|$. However, $\sum_{j\in[\ell,k]} \binom{a_j-1}{j}$ is *not* a cascade representation.

Let t to be the maximum $j \in [\ell,k]$ such that $a_j = j$, and note that since $a_t > a_{t-1} > \cdots > a_\ell = \ell \geqslant 1$ we have $a_j = j$ for every $j \in [\ell,t]$. Since $\binom{a_j-1}{j} = \binom{j-1}{j} = 0$ for every $j \in [\ell,t]$, inequality (3.7) yields

$$|\mathcal{F}(\bar{1})| \geqslant m_2 + 1 = \sum_{j=t+1}^{k} \binom{a_j-1}{j} + 1 = \sum_{j=t+1}^{k} \binom{a_j-1}{j} + \binom{t}{t},$$

which is a cascade representation. Therefore, applying the induction hypothesis and Lemma 3.10 as before, we obtain

$$\begin{aligned}|\partial(\mathcal{F}(\bar{1}))| &\geqslant |\mathscr{L}^k(m_2)| \\ &= \sum_{j=t+1}^{k}\binom{a_j-1}{j-1}+\binom{t}{t-1} \\ &= \sum_{j=t+1}^{k}\binom{a_j-1}{j-1}+\sum_{j=1}^{t}\binom{j-1}{j-1} \geqslant \sum_{j=\ell}^{k}\binom{a_j-1}{j-1} = m_2\,,\end{aligned}$$

where we use the fact that $a_j = j$ for $j \in [\ell, t]$.

Hence, in both cases, inequality (3.8) holds. ■

Using Claim 6 we may apply the induction hypothesis now to $\mathcal{F}(1)$ and deduce

$$|\partial\big(\mathcal{F}(1)\big)| \geqslant |\partial\mathscr{L}^{(k-1)}(m_2)|\,.$$

Observe that if $\ell > 1$, then $m_2 = \sum_{j=\ell}^{k}\binom{a_j-1}{j-1} = \sum_{j=\ell-1}^{k-1}\binom{a_{j+1}-1}{j}$ is a cascade representation for $k-1$. Thus we may apply Lemma 3.10 and obtain

$$|\partial\big(\mathcal{F}(1)\big)| \geqslant |\partial\mathscr{L}^{(k-1)}(m_2)| = \sum_{j=\ell}^{k}\binom{a_j-1}{j-2}\,. \tag{3.9}$$

If $\ell = 1$, then we have $m_2 - 1 = \sum_{j=\ell+1}^{k}\binom{a_j-1}{j-1} = \sum_{j=\ell}^{k-1}\binom{a_{j+1}-1}{j}$ is a cascade representation for $k-1$. Thus, as before, we may deduce

$$|\partial\big(\mathcal{F}(1)\big)| \geqslant |\partial\mathscr{L}^{(k-1)}(m_2)| \geqslant |\partial\mathscr{L}^{(k-1)}(m_2-1)| = \sum_{j=\ell+1}^{k}\binom{a_j-1}{j-2}$$

Since $\binom{a_\ell-1}{\ell-2} = \binom{a_\ell-1}{-1} = 0$ inequality (3.9) also holds for this case.

Finally, due to the formula given by *(2)* in Lemma 3.4, we conclude

$$\begin{aligned}|\partial\mathcal{F}| &= |\mathcal{F}(1)| + |\partial\big(\mathcal{F}(1)\big)| \\ &\geqslant \sum_{j=\ell}^{k}\binom{a_j-1}{j-1}+\binom{a_j-1}{j-2} \\ &= \sum_{j=\ell}^{k}\binom{a_j}{j-1} = |\partial\mathscr{L}^{(k)}(m)|\,,\end{aligned}$$

where the first inequality follows from Claim 6 and (3.9) and the last identity follows from Lemma 3.10. □

It is possible extend Kruskal-Katona theorem in the following way. Given $t < k \leqslant n$ and a family $\mathcal{F} \subseteq \binom{n}{k}$, define the *t-shadow* of $\mathcal{F}$ by

$$\partial^t\mathcal{F} = \left\{X \in \binom{n}{t}\colon \text{ there is a } Y \in \binom{n}{k-t} \text{ such that } X \cup Y \in \mathcal{F}\right\}.$$

The following is a generalisations of Theorem 3.9.

Theorem 3.11 (Kruskal-Katona). *Let $t < k \leqslant n$. For every family $\mathcal{F} \subseteq \binom{[n]}{k}$ such that $|\mathcal{F}| = m$ we have*

$$|\partial^t \mathcal{F}| \geqslant |\partial^t \mathscr{L}^{(k)}(m)|\,.$$

$\square$

3.4. **Lexicographical ordering.**

Definition 3.12. The *lexicographical order,* $\prec$ is a total order $\binom{\mathbb{N}}{k}$, the set of k-sets of natural numbers, defined by

$$A \prec B \iff \min A \smallsetminus B < \min B \smallsetminus A\,.$$

Let the *reversed colexicographical ordering* $\lessdot_r$ be the colexicographical ordering after the reversal of the interval $[n]$. In other words, $\lessdot_r$ is defined by

$$A \lessdot_r B \iff \min A \smallsetminus B > \min B \smallsetminus A\,, \tag{3.10}$$

and note that (3.10) is exactly the inverse of the lexicographical ordering.

Definition 3.13. Given $n, m \geqslant k$ we define $L^{(k)}(m)$ to be the first m sets of size k on the interval $[n]$ in the lexicographical order.

The following fact is easy to prove.

Fact 3.14. *Let $n \geqslant m \geqslant k$.*

$$L^{(k)}(m) = \overline{\mathscr{L}_r^{(n-k)}(m)}\,,$$

where $\mathscr{L}_r^{(n-k)}(m)$ denotes the first m sets of size k on the interval $[n]$ in reversed colexicographical ordering.

Proof. Let $A, B \in \binom{[n]}{k}$ and observe that

$$A \prec B \Leftrightarrow \min A \smallsetminus B < \min B \smallsetminus A \Leftrightarrow \min B^c \smallsetminus A^c < \min A^c \smallsetminus B^c \Leftrightarrow A^c \lessdot_r B^c\,,$$

Which yields the desired result. $\square$

Now we are ready for the the following is a simple application of Theorem 3.11.

Theorem 3.15. *Given $k \leqslant n$, let $\mathcal{F} \subseteq \binom{[n]}{k}$. If $\mathcal{F}$ is intersecting, then $L^{(k)}(|\mathcal{F}|)$ is intersecting.*

Proof. We start the proof with the following claim.

Claim 7. *$\mathcal{F}$ is intersecting if and only if $\mathcal{F} \cap \partial^k \overline{\mathcal{F}} = \varnothing$.*

Proof of the claim:

$\Rightarrow$) Suppose that $\mathcal{F}$ is intersecting and that there is a set $A \in \mathcal{F} \cap \partial^k \overline{\mathcal{F}}$. That means that there is a set $B \supseteq A$ such that $B \in \overline{\mathcal{F}}$. In particular, $B^c \in \mathcal{F}$ which is a contradiction, since $A \cap B^c = \varnothing$.

$\Leftarrow$) Suppose that $\mathcal{F} \cap \partial^k \overline{\mathcal{F}} = \varnothing$ and that $\mathcal{F}$ is not intersecting. In particular there are two sets $A, B \in \mathcal{F}$ such that $A \cap B = \varnothing$. In other words, $A \subseteq B^c$. This is a contradiction as it implies that $A \in \partial^k \overline{\mathcal{F}} \cap \mathcal{F}$.

■

Fix $m = |\mathcal{F}|$ and $m' = |\partial^k \mathscr{L}^{(n-k)}(m)|$ and observe that due to Fact 3.14.

$$L^{(k)}(m) \cap \partial^k \overline{L^{(k)}(m)} = L^{(k)}(m) \cap \partial^k \mathscr{L}_r^{(n-k)}(m) = L^{(k)}(m) \cap \mathscr{L}_r^{(k)}(m') \quad (3.11)$$

where the last identity follows from the fact that the k-shadow of the first $(n-k)$-sets in the reversed colexicographical ordering are also the first k-sets of the reversed colexigraphical ordering.

Since the the reversed colexicographical ordering is exactly the inverse of the lexicographical ordering, the set $\mathscr{L}_r^{(k)}(m')$ corresponds exactly with the m' *last* k-sets in $\binom{[n]}{k}$ in the lexicographical ordering. Therefore, due to Claim 7 if $L^{(k)}(m)$ is not intersecting, we have

$$L^{(k)}(m) \cap \partial^k \overline{L^{(k)}(m)} \neq \varnothing \iff m + m' > \binom{n}{k}.$$

Finally, due to Theorem 3.11, this implies

$$|\mathcal{F}| + |\partial^k \overline{\mathcal{F}}| \geqslant |\mathcal{F}| + |\partial^k \mathscr{L}^{(n-k)}(m)| = m + m' > \binom{n}{k}.$$

However this yields $\mathcal{F} \cap \partial^k \overline{\mathcal{F}} \neq \varnothing$ contradicting the fact that $\mathcal{F}$ is intersecting.

□

References

[1] R. Ahlswede and L. H. Khachatrian, "A pushing-pulling method: New proofs of intersection theorems," *Combinatorica*, vol. 19, pp. 1–15, 1999 (↑ 7).

[2] P. Erdős, C. Ko, and R. Rado, "Intersection theorems for systems of finite sets," *Quart. J. Math. Oxford Ser. (2)*, vol. 12, pp. 313–320, 1961, DOI: 10.1093/qmath/12.1.313 (↑ 2, 6).

[3] P. Frankl, "A new short proof for the Kruskal-Katona theorem," *Discrete Math.*, vol. 48, no. 2-3, pp. 327–329, 1984, DOI: 10.1016/0012-365X(84)90193-6 (↑ 14).

[4] P. Frankl, "A simple proof of the hilton–milner theorem," *Moscow Journal of Combinatorics and Number Theory*, vol. 8, no. 2, pp. 97–101, 2019 (↑ 8).

[5] P. Frankl, "Old and new applications of katona's circle," *European Journal of Combinatorics*, vol. 95, p. 103 339, 2021 (↑ 11).

[6] A. J. Hilton and E. C. Milner, "Some intersection theorems for systems of finite sets," *The Quarterly Journal of Mathematics*, vol. 18, no. 1, pp. 369–384, 1967 (↑ 7).

[7] G. Katona, "A theorem of finite sets," *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, New York-London, 1968, pp. 187–207 (↑ 14).

[8] G. O. Katona, "A simple proof of the erdös-chao ko-rado theorem," *Journal of Combinatorial Theory, Series B*, vol. 13, no. 2, pp. 183–184, 1972 (↑ 11, 12).

[9] G. O. Katona, "Extremal problems for hypergraphs," *Combinatorics: Proceedings of the NATO Advanced Study Institute held at Nijenrode Castle, Breukelen, The Netherlands 8–20 July 1974*, Springer, 1975, pp. 215–244 (↑ 11).

[10] J. B. Kruskal, "The number of simplices in a complex," *Mathematical optimization techniques*, vol. 10, pp. 251–278, 1963 (↑ 14).

[11] L. Lovász, "Combinatorial problems and exercises." American Mathematical Soc., 2007, vol. 361 (↑ 16).

[12] D. Lubell, "A short proof of Sperner's lemma," *J. Combinatorial Theory*, vol. 1, p. 299, 1966 (↑ 13).

[13] E. Sperner, "Ein Satz über Untermengen einer endlichen Menge," *Math. Z.*, vol. 27, no. 1, pp. 544–548, 1928, DOI: 10.1007/BF01171114 (↑ 13).