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Time scaling in PID controller tuning

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The paper examines the problem of tuning the parameters of a PID controller for the frequently assumed first-order plus dead time plant. The original contribution is to look at the problem from a time-scaling viewpoint so that the controller parameters can be given as a function of the time delay to time constant ratio. This enables the controller parameter results of different rules to be easily compared and reveals those that are not consistent under time scaling.

Key words: controller; FOPDT plant; performance index; PID; time scaling.

Nomenclature

ω	Frequency
ω_n	Normalized frequency
ω_o	Natural frequency
ζ	Damping ratio
s	Complex variable of the Laplace transform
s_n	Normalized complex variable of the Laplace transform
τ	Dead time
T	Time constant
K_o	Critical gain
ρ	Dead time to time constant ratio (τ/T)
ϕ	Phase margin
T_i	Integral time constant
T_d	Derivative time constant
μ	T_i/T_d

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Figures 1–4 appear in colour online: http://tim.sagepub.com

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1. Introduction

Amplitude and time scaling were commonly used when simulating control systems on analogue computers. The former was required because of the limited voltage range of the amplifier outputs and the latter for various aspects of convenience, eg, a slow-speed simulation if it was required to obtain hard copy using a pen recorder or a high-speed one if observations on an oscilloscope were required. It is also quite usual in textbooks to see Bode diagrams and step responses plotted for normalized first- and second-order transfer functions but beyond that, there is typically no further mention of normalized time scales. This is unfortunate, as an understanding of time scaling can often lead to a better understanding of various aspects of control theory, particularly classical design. This paper discusses one aspect of time scaling, namely what it reveals about tuning a PID controller for a first-order plus dead time plant (FOPDT), a transfer function that is commonly used as an approximate model for many process control plants. Also, this model fit was suggested by Ziegler and Nichols (Z-N) to a plant step response for application of their first tuning rule (Ziegler and Nichols, 1942). Since the Z-N work, many papers and even a significant part of one book (O'Dwyer, 2006) have been written on suggested rules for PID controller tuning of this transfer function. Nowhere, however, does it appear to be clearly pointed out that unless they satisfy the time-scale properties derived in this paper, which some do not, then the tuning rules will not be consistent in ensuring the same response property for different time-scaled plant transfer functions. The Z-N first tuning rule is shown to satisfy the time-scale property and interestingly, when approached from this viewpoint, is shown to be the easiest formulation that might give reasonable response results.

Relay autotuning may be regarded as a modern way of applying the Z-N loop cycling approach. It overcomes a practical difficulty of the Z-N method of trying to achieve something near to a loop oscillation in loops having long time constants, non-linear effects and noise. The Z-N rules based on the loop cycling method are examined, and it is shown that these also satisfy the time-scaling property for the resulting system. In the next section, time scaling and some tuning rules based on the FOPDT plant are discussed and compared. The following section discusses relay autotuning and presents a frequency response interpretation of the Z-N cycling rules. Some conclusions are then given in Section 4.

2. Time scaling

The general second-order transfer function with no zeros is usually written as $\bar{\omega}_o^2/(s^2+2\zeta s\omega_o+\omega_o^2)$ and it is normalized by replacing s/ω_o by s_n to give $1/(s_n^2+2\zeta s_n+1)$. This means that any information regarding the first transfer function can be found from the normalized transfer function with the frequency ω of the first given by $\omega = \omega_n \omega_o$. In dealing with the FOPDT transfer function

$$G_p(s) = \frac{K_p e^{-s\tau}}{1 + sT} \tag{1}$$

one can time scale by replacing sT by s_n to give the normalized plant transfer function

$$G_{np}(s) = \frac{K_p e^{-\rho s_n}}{1 + s_n} \tag{2}$$

where $\rho = \tau/T$. If the plant in (1) is controlled by an ideal PID controller in the error channel with transfer function $K_c[1 + sT_d + (1/sT_i)]$, then this becomes $K_c[1 + s_n T'_d + (1/s_n T'_i)]$, in normalized form where $T'_d = T_d/T$ and $T'_i = T_i/T$, and the normalized open-loop transfer function can be written

$$G_{nol}(s) = \frac{Ke^{-s\rho}}{1 + s}[1 + sT_d + (1/sT_i)] \tag{3}$$

where $K = K_p K_c$ and the primes and subscript n on s have been omitted for convenience, but an n has been added to the subscript of the open-loop transfer function. Thus if one wishes to design the system to have a certain property, say, minimization of an integral performance criterion for the closed-loop step response, and a unique solution exists, then the parameters K, T_d and T_i will be functions of the parameter ρ. They may, therefore, be written $K = f_1(\rho)$, $T_i = f_2(\rho)$ and $T_d = f_3(\rho)$. Then for this same property to be satisfied for the unscaled system, which may be referred to as a consistent design procedure, the controller parameters must have the form $K_c = f_1(\rho)/K_p$, $T_i = Tf_2(\rho)$ and $T_d = Tf_3(\rho)$.

It is interesting to consider what might be simple functions of ρ to give reasonable system behaviour. Since experience suggests that as ρ increases the gain should be reduced and the time constants increased, then a suitable simple choice could be $f_1(\rho) = k_1/\rho, f_2(\rho) = k_2\rho$ and $f_3(\rho) = k_3\rho$. These are indeed the Z-N rules with $k_1 = 1.2$, $k_2 = 2$ and $k_3 = 0.5$. Other tuning rules, which satisfy the time-scaling property, are the Cohen and Coon (1953) (C-C), Zhuang and Atherton (1993) (Z-A) and Wang *et al*. (1995) (W-J-C) rules, where the functions are more complicated and are given for reference in Table 1.

The Z-A rules were given to minimize various integral performance criteria so that the coefficients depend on the criterion chosen.

A rule that does not satisfy the time-scaling property is that of Chien *et al*. (1952) for set point response, as the integral term is not expressible in the required form. Interestingly, however, the rules given for disturbance rejection do satisfy time scaling.

Two other plant transfer functions, which can also be normalized in terms of the parameter ρ are

$$G_1(s) = \frac{K_p e^{-s\tau}}{s(1 + sT)} \tag{4}$$

Table 1 ρ formulas

	$f_1(\rho)$	$f_2(\rho)$	$f_3(\rho)$
Z-N	$1.2/\rho$	2ρ	0.5ρ
C-C	$\dfrac{16+3\rho}{12\rho^2}$	$\dfrac{\rho(32+6\rho)}{13+8\rho}$	$\dfrac{4\rho}{11+2\rho}$
Z-A	$a_1\rho^{b_1}$	$\dfrac{1}{a_2+b_2\rho}$	$a_3\rho^{b_3}$
W-J-C	$\dfrac{(0.53+0.73\rho)(1+0.5\rho)}{\rho(1+\rho)}$	$1+0.5\rho$	$\dfrac{0.5\rho}{1+0.5\rho}$

and

$$G_2(s) = \frac{K_p e^{-s\tau}}{(1+sT)^n} \tag{5}$$

For time-scaling consistency, the required controller parameters must again be in the form $K_c = f_1(\rho)/K_p$, $T_i = Tf_2(\rho)$ and $T_d = Tf_3(\rho)$. Several rules for both PD and PID controllers for the plant $G_1(s)$ are given in O'Dwyer (2006). Some of the PD controllers suggested used a zero to cancel the plant pole so that the result is a proportional control of an integrating plant. Since the book does not present the controller formulas for the plants of Equations (1) and (4) in the simpler time-scaled form, checking their consistency requires some algebra. One rule given for the plant of Equation (4) (McMillan, 1984) has derivative and integral terms for the PID controller of the correct form but the controller gain given by $1.111/(\tau K_p\rho(1+[1/\rho]^{0.65})^2)$ is not, because of the presence of τ.

3. Relay autotuning

The relay autotuning approach to setting PID parameters has received significant attention in recent years, since it was suggested by Åström and Hagglund (1984). The approach provides an improved practical procedure for estimating the critical, or ultimate, gain and critical frequency parameters employed in the Z-N loop cycling rules. Its major advantage is that the loop limit cycle is established automatically for typical process plant transfer functions and its amplitude can be controlled by the relay level settings. Furthermore, by adjusting these settings, one can gain information about possible non-linearity in the plant. The disadvantage is that in using the describing function method to calculate these two parameters, the values obtained are approximate and are more in error the greater the distortion in the limit cycle.

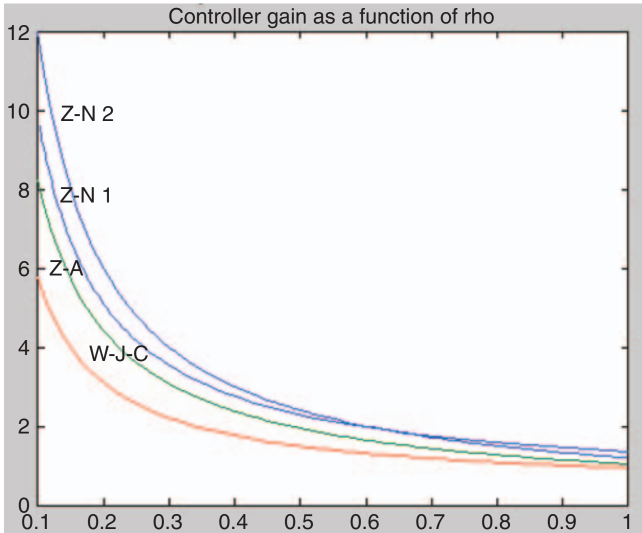

Figure 1 Graphs of controller gain, $f_1(\rho)$, versus ρ

If accurate estimates are required, however, and time for performing the experiment is not a problem, then this can be achieved by adding a known tuned filter to the loop.

Z-N suggested the following choices for the controller parameters: $K_c = 0.6K_o$, $T_i = 0.5T_o$ and $T_d = 0.125T_o$ for the loop cycling method. Here K_o is the critical gain, ie, $1/K_o$ is the gain when the phase shift of the plant is 180° and T_o is the period corresponding to the critical frequency ω_o of this point, ie, $T_o = 2\pi/\omega_o$. For the normalized plant, ω_o is given by the solution of $\tan^{-1}(\omega_o) + \rho\omega_o = \pi$ and K_o from $K_o = (1 + \omega_o^2)^{1/2}/K_p$. Thus, from the first equation, ω_o is a function of ρ and consequently therefore so are T_o and K_pK_c. Thus for the normalized plant K_pK_c, T_d and T_i are functions of ρ as required for consistency, and the controller parameters they give can be compared with other consistent rules.

The comparisons are given in Figures 1–3 for values of ρ between 0.1 and 1. The Z-A rules are plotted for minimization of the ISTE. The C-C result has been omitted from Figure 1, as the normalized controller gain it gives increases rapidly as ρ decreases. This is shown in Figure 4 for values of ρ between 0.3 and 1. The Z-N results are labelled 1 and 2, with the latter being those for the loop cycling method. From the graphs, several observations about the methods are possible. Both the Z-N methods give comparable parameters with the results for the controller gain being particularly close.

Furthermore, it can be seen that the Z-A and W-J-C results are quite similar for all three coefficients. The C-C and Z-N results differ significantly from these for the integral time constant, whilst the C-C results give an excessive gain for small ρ. Another significant point to note is that, unlike the Z-N rules, the Z-A and W-J-C ones do not use a constant value of μ, the ratio of $f_2(\rho)$ to $f_3(\rho)$, but one that increases in both

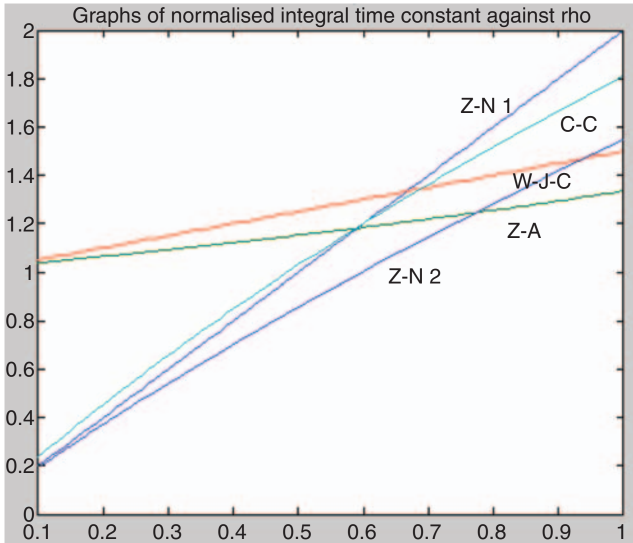

Figure 2 Graphs of integral time constant, $f_2(\rho)$, versus ρ

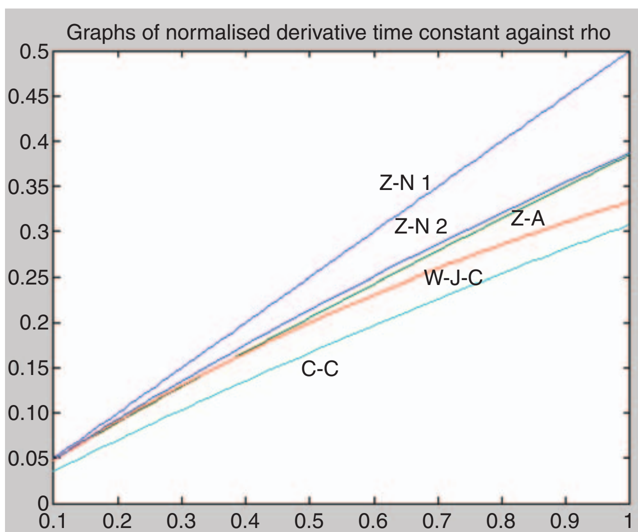

Figure 3 Graphs of derivative time constant, $f_3(\rho)$, versus ρ

cases significantly with decrease in ρ. The W-J-C method gives the largest change from 4.5 for $\rho = 1$ to just over 12 for $\rho = 0.2$. As might be expected, the closed-loop step responses for the Z-A and W-J-C methods typically show a relatively small overshoot for all values of ρ, whereas the Z-N methods show a higher overshoot, which increases as ρ decreases

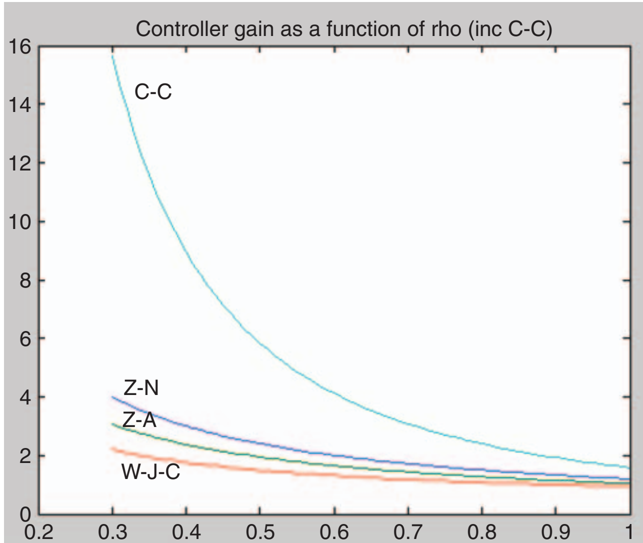

Figure 4 Graphs of $f_1(\rho)$ versus ρ (from 0.3)

A further aspect of the relay autotuning, ie, Z-N loop cycling, method of tuning, which is only occasionally mentioned in the literature (Åström and Hagglund, 1995), is that, in terms of the open-loop frequency domain properties, the suggested controller parameters move the critical frequency to the point with gain 0.66 and phase -155° on the resulting compensated open-loop frequency response locus $G_{nol}(j\omega)$. This is a simple way of tuning and if one wishes to choose this point to have a different gain, g, and phase, $-(180^\circ - \varphi)$, and with the T_i to T_d ratio 4 as used by Z-N, then this can be done using the following three equations to obtain the normalized parameters

$$\rho\omega_o + \tan^{-1}\omega_o = \pi \tag{4}$$

$$T_d = \frac{[1 + \tan(\phi/2)]}{2\omega_o[1 - \tan(\phi/2)]} \tag{5}$$

$$K = \frac{4g\omega_o T_d(1 + \omega_o^2)^{1/2}}{(1 + 4\omega_o^2 T_d^2)} \tag{6}$$

There seem to be no major reasons for choosing $T_i/T_d = 4$. Certainly, from a viewpoint of expediency, it makes the two zeros of the PID controller transfer function real and equal. This then simplifies a frequency domain design using a Bode diagram, which was frequently used decades ago, as the approximation is a V characteristic.

For the general choice of $T_i/T_d = \mu$, then the equations become

$$\rho\omega_o + \tan^{-1}\omega_o = \pi \quad (7)$$

$$T_d = \frac{\mu\tan\phi + [\mu^2\tan^2\phi + 4\mu]^{1/2}}{2\mu\omega_o} \quad (8)$$

$$K = \frac{\mu g\omega_o T_d(1+\omega_o^2)^{1/2}}{[\mu^2\omega_o^2 T_d^2 + (1-\mu\omega_o^2 T_d^2)^2]^{1/2}} \quad (9)$$

If one has an estimate of ρ for an FOPDT plant transfer function then, from the results presented earlier for the Z-A and W-J-C methods, a choice of μ greater than 4 would seem reasonable.

4. Conclusion

This paper, by considering time scaling in PID controller design for a plant modelled as an FOPDT transfer function, has brought out some points that are extremely important but have received insignificant consideration in the literature. The consideration of time scaling in design means that controllers for all FOPDT plants with the same dead time to time constant ratio ρ can be designed from one set of results to meet a defined specification. Two other plants for which this approach can be used have also been mentioned, one in particular $Ke^{-s\tau}/(1+sT)^n$, where n is an integer, is also often used as an approximation for a process transfer function.

In essence, time scaling allows a solution with one less parameter involved, eg, for a plant with an SOPDT transfer function that is two time constants and a dead time; results can be obtained in terms of two time ratios. Time scaling can be useful in other design approaches, for instance in the synthesis of closed-loop transfer functions with a particular property (Atherton, 2006), and for the design of other fixed structure controllers such as phase lag and lead transfer functions.

Furthermore, this paper has only addressed obtaining the parameters for an ideal PID controller in the error channel. In many industrial applications, the D term is fed from the output rather than the error so that similar formulas for the controller parameters in terms of ρ can be obtained for this case, which may be denoted as PI-D control. This was done for integral performance criteria in Zhuang and Atherton (1993) as well as parameters to minimize the disturbance response.

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