Pendulum Project Report 2: Period Versus Pendulum Length and Q Fac tor Versus Pendulum Length

Michelle Deng

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Introduction

The purpose of this project is to investigate four relationships about pendulum: period versus initial angle, amplitude decay for determining Q, period versus length, and the dependence of Q factor on length. The experiments confirmed theoretical relationships and provided quantitative insights into the damping process. The parameters for all fitted models were obtained using SciPy's curve fit regression.

The period (T) of the pendulum was expected to be a function of the length (L). The primary period relationship found was the simple formula, which is approximately:

$$T = 2\sqrt{L} \tag{1}$$

where T is the period and L is the length of pendulum.

The Quality factor (Q) was calculated by the relationship:

$$Q = \pi\tau/T \tag{2}$$

where τ is the damping time constant that describe the exponential decay of the amplitude, and T is the period.

The angular amplitude decay was modeled by:

$$\theta(t) = \theta_0 e^{-t/\tau} cos(2\pi\frac{t}{T} + \varphi_0) \tag{3}$$

Where $\theta(t)$ is the angle between the pendulum and the vertical axis at time t, and θ_0 is the initial amplitude, and φ_0 is the initial phase.

The data showed that the period (T) is largely independent of the initial angle (θ_0), especially for the angle in range of $(-0.685, 0.685)$ rad.

The fitting resulted in the model:

$$T = (0.015 \pm 0.002)\theta^2 + (-0.002 \pm 0.001)\theta + (1.482 \pm 0.002) \tag{4}$$

The experiment proved that the period (T) is proportional to the square root of the length (L) of pendulum, consistent with the theory. The power-law fit yielded the highly consistent result:

$$T = (2.0030 \pm 0.0005)L^{(0.5016 \pm 0.0003)} \tag{5}$$

The Q was found to display a non-linear dependence on length. The data was best fitted with a quadratic relationship:

$$Q = (500 \pm 1)L^2 - (428.3 \pm 0.9)L + (127.5 \pm 0.2) \tag{6}$$

This result indicates that the Q increases at an accelerating rate as the length gets longer, which means that the effect of frictional damping decreases for longer pendulums.

Experimental Setup

The pendulum was designed with two lightweight parallel strings of adjustable length so that the pendulum would not have conical circular motion. The strings were suspended from a desk lamp arm, and a locker as a dense mass at the bottom, which made the mass concentrate on the end of the pendulum.

A physical protractor was fixed at the end of the pendulum on the desk lamp arm to measure the initial angle for each trial.

Figure 1: Pendulum Setup

Methods and Procedure

- Period vs. Angle

The pendulum was released from an initial angle between -4π/9 and 4π/9 with an increment of π/18. Three trials were conducted for each initial angle. The total time from the pendulum begins to swing until it stopped was recorded for each trial, and the formula $T = \frac{t}{n}$ was used to calculate the average period for each trial, where t is the total time for swinging, n is the number of oscillations, and T is the period. The T used to find the best fitting line is the average of three periods from each trial.

To minimize the type A uncertainty, 3 trials were assigned for each initial angle, and the average period is used to find the relationship between period and initial angle. For type B uncertainties, a timer with a precision of 0.01 second and a protractor with a precision of 1°, which means that the uncertainty in angle was ±0.5°, corresponding to ±0.0087 rad, while the uncertainty for period was 0.01.

- Q Factor

A trial with an initial angle of 7π/18 was recorded into a video with a precision of 0.01 second. This relatively large initial angle was selected to analyze the amplitude decay more clearly, since the underdamped behavior of the pendulum becomes easier to track at higher amplitudes. The circular motion of the pendulum was analyzed. Tracker was used to record the string's angle relative to the vertical axis (in radians) for each frame, with a time interval $\Delta t = \frac{1}{30}s$. A timer with a precision of 0.01 second and a ruler with a precision of 0.001 m was used. The amplitude decay data were fitted in Python fitting function provided by the professor. The angular amplitude $\theta(t)$ was fitted in with an exponential decay model: $y = a\, e^{\frac{-t}{b}}$. The formula

$$\theta(t) = \theta_0 e^{-t/\tau} cos(2\pi \frac{t}{T} + \varphi_0) \qquad (3)$$

is given, so the output b corresponds to τ. The formula

$$Q = \pi \frac{\tau}{T} \qquad (2)$$

was used to find the Q factor, while T is the period and τ is the damping time constant.

For both time and angle measurements, type B uncertainties were considered. A digital timer with a resolution of 0.01 s was used, and the uncertainties in time were ±0.005 s based on the timer precision, and the uncertainty in angle was ±0.01 rad based on the data collected by Tracker.

- Period vs. Length

Three trials were performed with different lengths of pendulum, decreasing the uncertainty caused by contingency. The length of pendulum varied by changing the length of the rope to an increment of approximate 10 cm. For trials of different lengths, the initial angle was controlled between -0.685 to 0.685 rad using a physical protractor at the end of the pendulum, preventing the effect by the coefficient C in Equation (9) and made the period closer to theoretical value. From Equation (9)(10)(11)(12), it is shown that B is experimentally zero while C is nonzero, meaning that as the initial angle increases, the effect of C would increase, which leads to a bias when measuring the period of each pendulum length.

- Q Factor vs. Length

Method 1 was used to find the Q factor of the pendulum with different lengths, since method 1 has more specific uncertainties and finding the best fitted parameters is the more rigorous method to get τ, which would also make the Q factor more accurate.

Results and Data Analysis

- Period vs. Angle

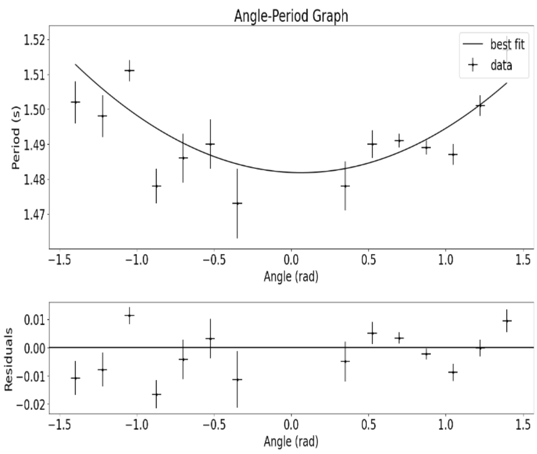

Figure 1: Angle-Period Graph: Measured period versus initial angle with one-sigma uncertainties. A quadratic curve is fitted to the data, and the calculation shown below indicates no systematic deviation from the model.

A. Observations

The peri of 1.48-1.51 s across all measured initial angles. The data points fitted in with a quadratic function curve with relatively small error bars. The curve is symmetrical about $\theta = 0$, and the minimum period appears at $\theta = 0$.

B. Analysis

The data of the period and the initial angle fitted into a quadratic model of the form:

$$T = a\theta^2 + b\theta + c \tag{7}$$

and parameters:

$$a = 0.015 \pm 0.002 \tag{8}$$

$$b = -0.002 \pm 0.001 \tag{9}$$

$$c = 1.482 \pm 0.002 \tag{10}$$

The fitted equation can also be expressed as the formular by the theoretical period T_0:

$$T = T_0(1 + B\theta_0 + C\theta_0^2) \tag{11}$$

With parameters:

$$T_0 = 1.482 \pm 0.002 \tag{12}$$

$$B = -0.001 \pm 0.002 \tag{13}$$

$$C = 0.010 \pm 0.002 \tag{14}$$

Coefficient B experimentally equals zero within 2σ, indicating that there is no asymmetry in the quadratic curve and in the pendulum. There is no measurable asymmetry between positive and negative initial angles. The coefficient C has a small magnitude; however, C is experimentally nonzero, meaning that the period would slightly increase as the initial angle increases, and the difference between practical period and theoretical period would narrow as the initial angle decreases to zero.

To determine the range of initial angle for which the effect by C is negligible, a 1% deviation criterion was applied. Using the formula:

$$\frac{\Delta T}{T_0} = |B||\theta| + |C|\theta^2 \tag{15}$$

where ΔT is the change of period. As $\frac{\Delta T}{T_0} = 0.01$, the maximum of B and C are taken as

$$B_{max} = |-0.001| + 2 * 0.002 = 0.005 \quad (16)$$
$$C_{max} = |0.010| + 2 * 0.002 = 0.014 \quad (17)$$

The roots of the quadratic function are

$$\theta_{max} \approx \pm 0.685\, rad \quad (18)$$

In conclusion, C experimentally equals to zero when the initial angle is in the range of (-0.685, 0.685).

$T_0 = 1.482 \pm 0.002\, s$ is fitted with the theoretically prediction using Equation (2), while theoretically $L = 0.549\, m$, which is slightly longer than the practical length of 0.500 m

C. Uncertainties

The uncertainties in the period measurements include both type A and type B uncertainties. Type B uncertainties arise from the measurement instruments, such as the time. Type A uncertainties originate from statistical variations among the different trials. These uncertainties were incorporated into the regression analysis, and they are reflected in the uncertainties of the fitted parameters.

- Q Factor

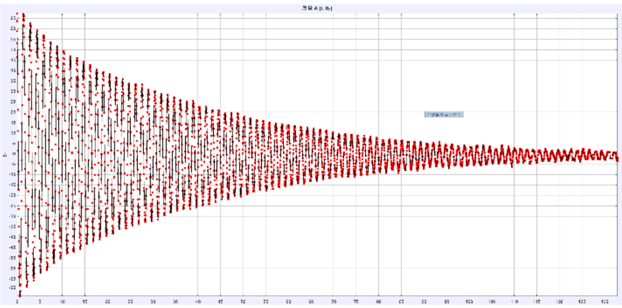

Figure 3: Time (seconds)-Angle (rad) graph of the pendulum motion, showing the exponential decay characteristics of a damped harmonic oscillation.

A. Observations

The angular amplitude gradually decreases over time, forming an exponential decay trend, as shown in Figures 3 and 4. The oscillation remains symmetric during the motion.

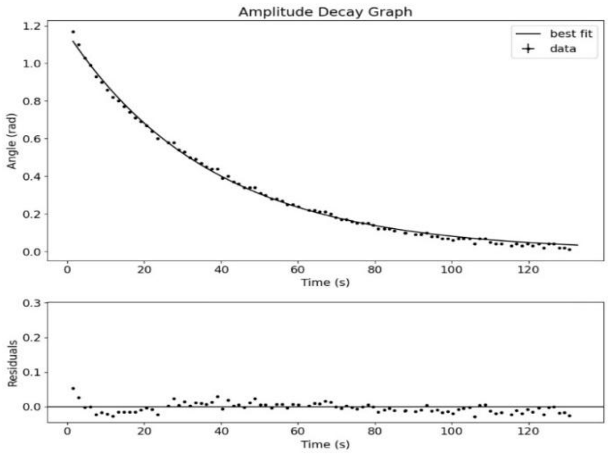

Figure 2: Amplitude decay of the pendulum obtained from peaks in figure 3, with an exponential decay fit curve.

B. Analysis

The data of peaks were fitted into an exponential model:

$$\theta(t) = a\, e^{-t/b} \quad (19)$$

with best fitted parameters:

$$a = 1.16191 \pm 0.00004 \quad (20)$$
$$b = 37.572 \pm 0.002 \quad (21)$$

Given the formula:

$$\theta(t) = a\, e^{-t/\tau} \quad (19)$$

The time constant is determined:

$$\tau = 37.572 \pm 0.002 \quad (22)$$

Using the fitted period I equation (10), the Q factor was calculated as:

$$Q_1 = \pi \frac{\tau}{T_0} = 79.6 \pm 0.1 \quad (23)$$

For verification, an alternative method of estimating Q factor was used by counting the total number of oscillations n until the amplitude decreased to:

$$e^{-\pi/3} \sim 35\% \quad (24)$$

The number of oscillations N at the time is 26.

$$Q_2 \sim kN = 3 * 26 = 78 \pm 3 \quad (25)$$

So, the difference between Q_1 and Q_2 is:

$$\Delta Q = Q_1 - Q_2 = 2 \pm 3 \quad (26)$$

Because ΔQ within 2σ, it is experimentally zero.

$$Q_1 = Q_2 \tag{27}$$

C. Uncertainties

For method 1, the uncertainty of Q_1 was calculated by propagating the relative uncertainties of τ and T_0. For method 2, the uncertainty of Q_2 mainly comes from counting the number of oscillations, giving that $u(Q_2) = 3$ since $u(N) = 1$.

- Period vs. Length

A. Observations and Analysis

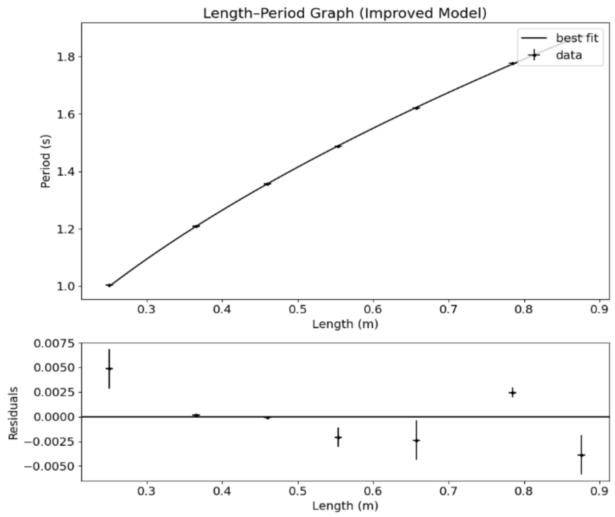

Figure 5: Measured period and pendulum length with a power-law fit.

The collected data were fitted to the power law function:

$$T = kL^n \tag{28}$$

With best fitted parameters:

$$k = 2.0030 \pm 0.0005 \tag{29}$$

$$n = 0.5016 \pm 0.0003 \tag{30}$$

The fitted function between length and period was close to:

$$T \simeq 2\sqrt{L} \tag{1}$$

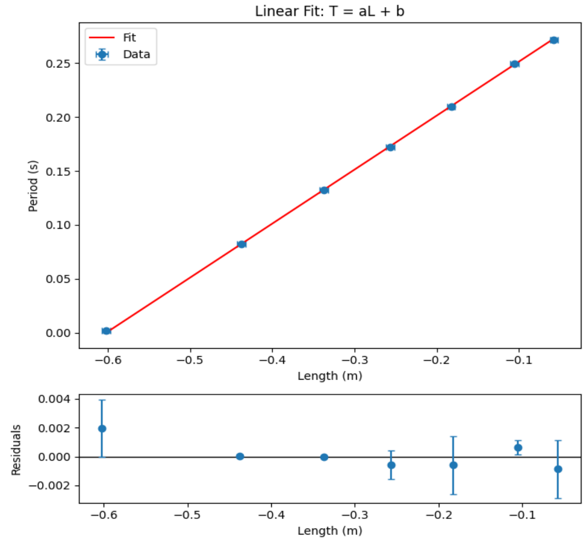

Figure 6: Linear fit of $\log(T)$ and $\log(L)$, with corresponding residuals shown below.

To verify the parameters, log(L) and log(T) were used to fit to another linear function:

$$\log(T) = a\log(L) + b \tag{31}$$

With best fitted parameters:

$$a = 0.5012 \pm 0.0009 \tag{32}$$

$$b = 0.3015 \pm 0.0003 \tag{33}$$

Comparing the two pairs of parameters, it is easy to find that:

$$n \sim a \tag{34}$$

$$\log(k) \sim b \tag{35}$$

The fitted model

$$T = (2.0030 \pm 0.0005)L^{(0.5016 \pm 0.0003)} \tag{4}$$

is in excellent agreement with equation (31). So, the equation (31) has been proved to be consistent. However, the rigorous version of equation (31) is:

$$T \simeq 2\pi\sqrt{L/g} \tag{36}$$

Which corrects the unit.

B. Uncertainty

The main sources of uncertainty in the period measurements were type B uncertainties which were caused by timer precision. The uncertainties lead by angle was decreased by using small initial angle. When using the log test for the Equation

(31), extremely small uncertainties, 0.0009 and 0.0003, were produced, which proved that the data strongly fit the power-law model.

- Q Factor vs. Length

A. Observations and Analysis

The relationship between Q factor and the length of the pendulum can be fitted to a quadratic model:

$$Q = aL^2 + bL + c \tag{37}$$

With best fitted parameters:

$$a = 585 \pm 1 \tag{38}$$

$$b = -428.3 \pm 0.9 \tag{39}$$

$$c = 127.5 \pm 0.2 \tag{40}$$

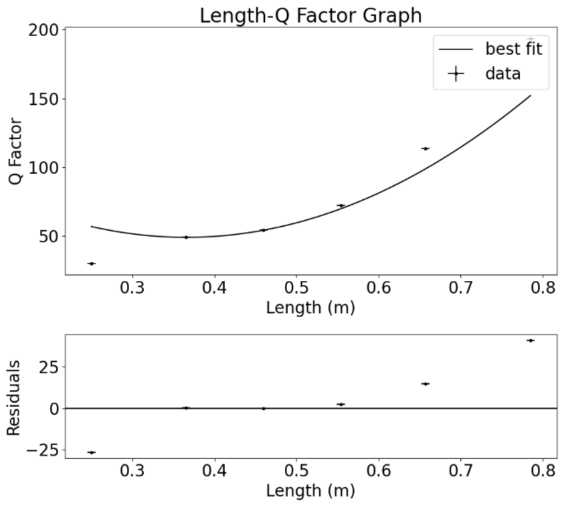

Figure 7: Q factor and pendulum length fitted with a quadratic curve.

As the length of pendulum increases, the Q factor of the pendulum also increases. The Q factor grows slowly with length, indicating that longer pendulum experience smaller relative energy losses per cycle. This behavior is likely due to air resistance and other frictional effects between the strings and the knot at the pivot. As the pendulum gets longer, the total stored mechanical energy increases, while the friction is less important compared to the total stored energy.

B. Uncertainty

In the Q factor analysis, uncertainties arise mainly because the fitted damping constant τ.

Conclusion

In this experiment, the relationship between pendulum length and period and the dependence of the Q factor on length were investigated. The period-length data could be best fitted in with a power-law model as equation (3), which is consistent with the theoretical equation (31).

The Q factor displayed a non-linear dependence on pendulum length with a trend of quadratic function. This behavior reflects that the frictional effect on the pendulum is reducing while the pendulum length increases, since the work done by the friction is less important compared to the total energy of the pendulum.

Uncertainties in the experiment were small and did not affect the main result.

Overall, the experiment confirms the theoretical relationship between the period and length and the dependence between length and Q factor, providing deeper insight into damping process of pendulums.