Sphere Fitting

Part I

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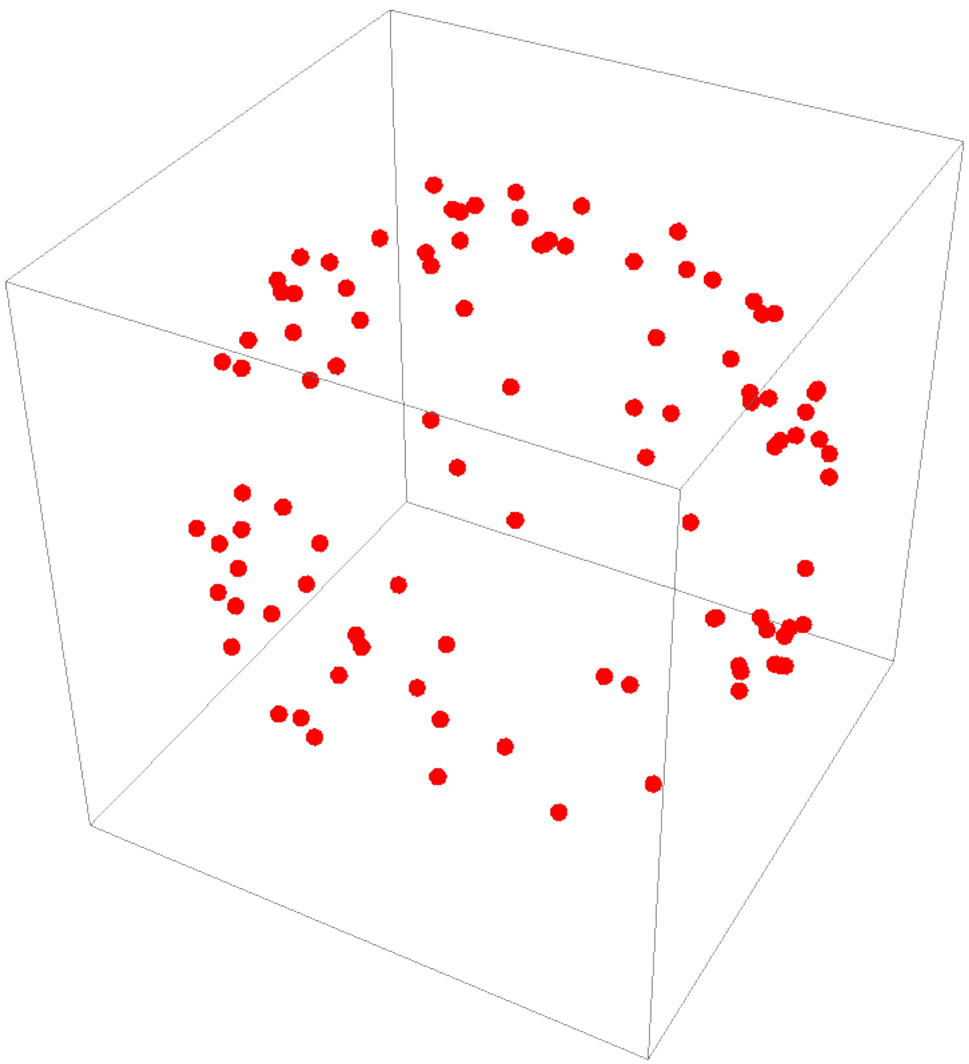
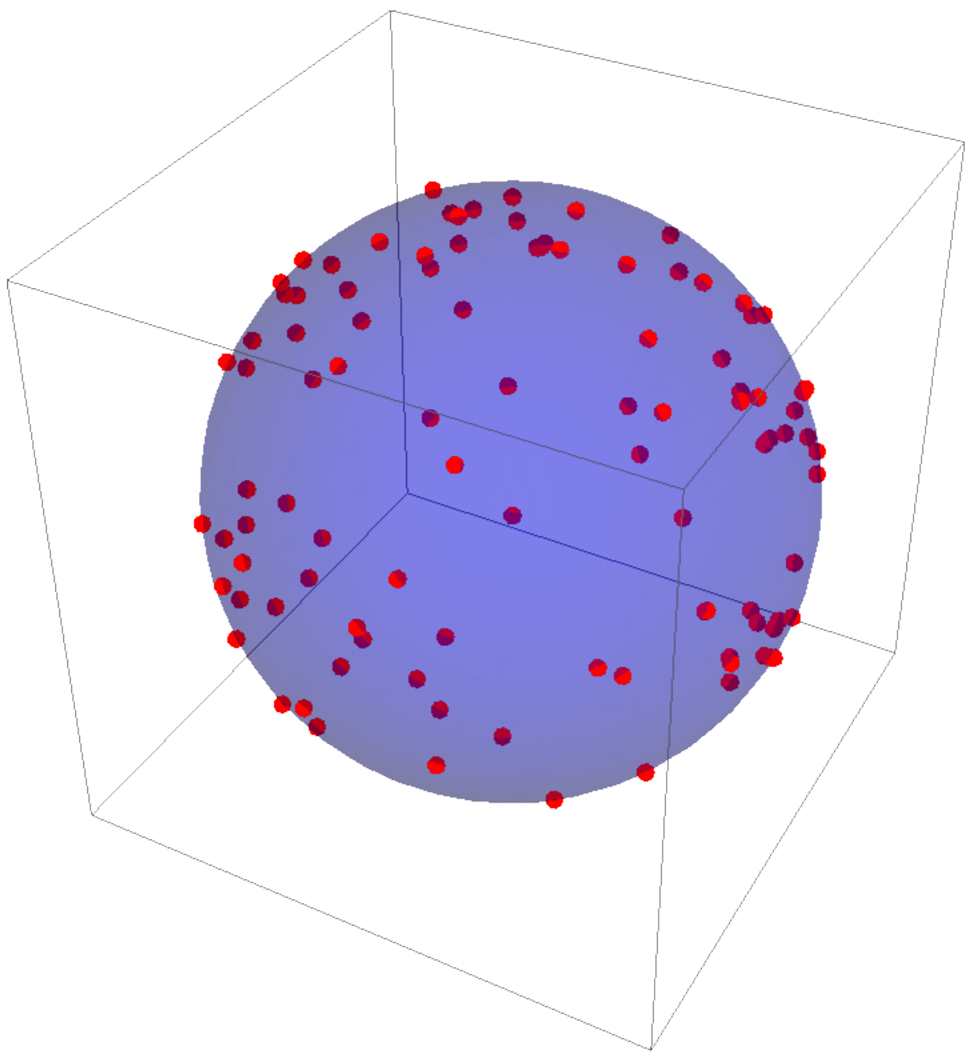
What's Sphere Fitting?





Why Sphere Fitting?

How do we decide what a best fit is?

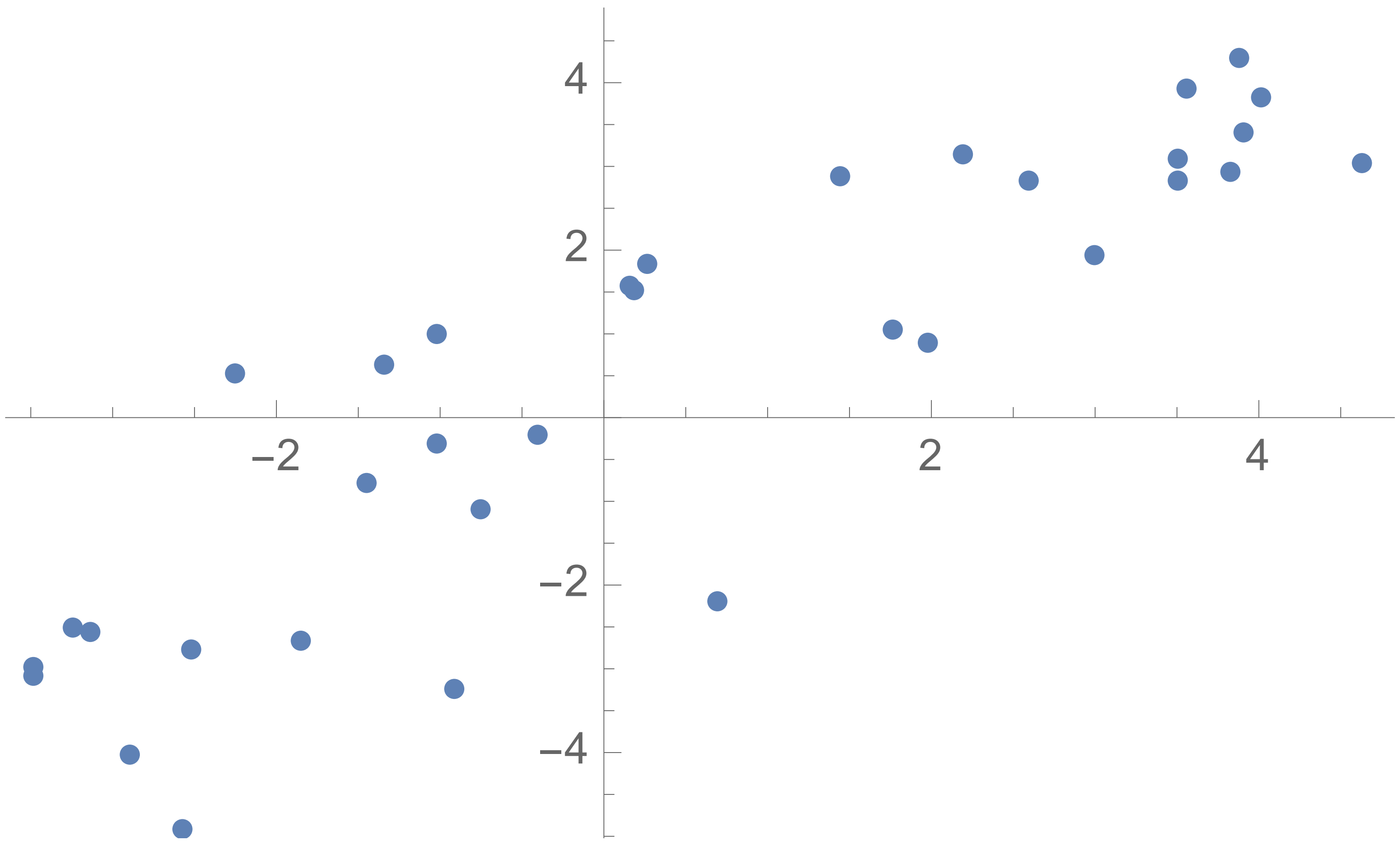
Digression:

- fit to a line using least-squares

Algebraic Approach

Data:

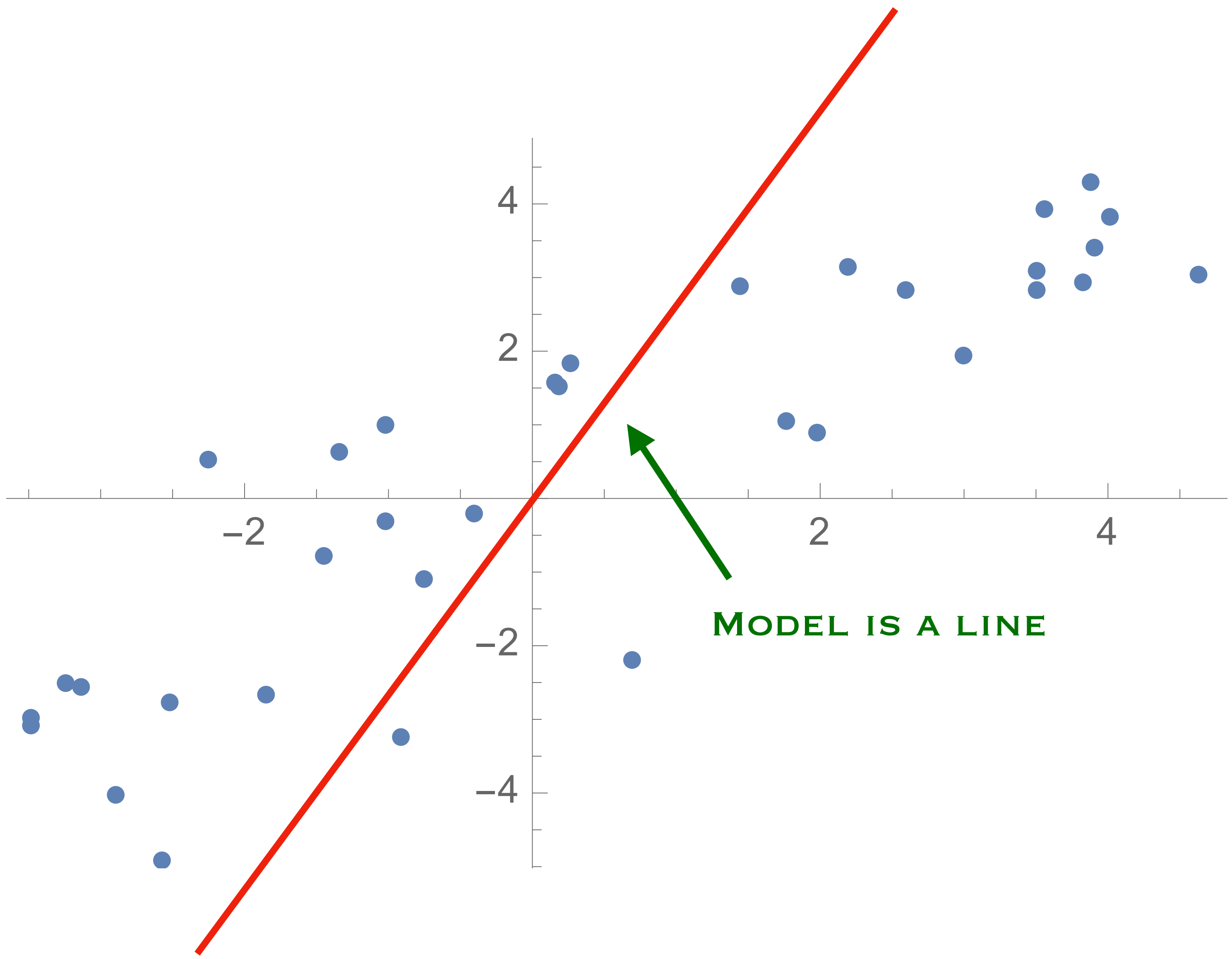
$\{(x_i, y_i)\}_{i=1}^{n}$



Algebraic Approach

Model:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = ax + b$$

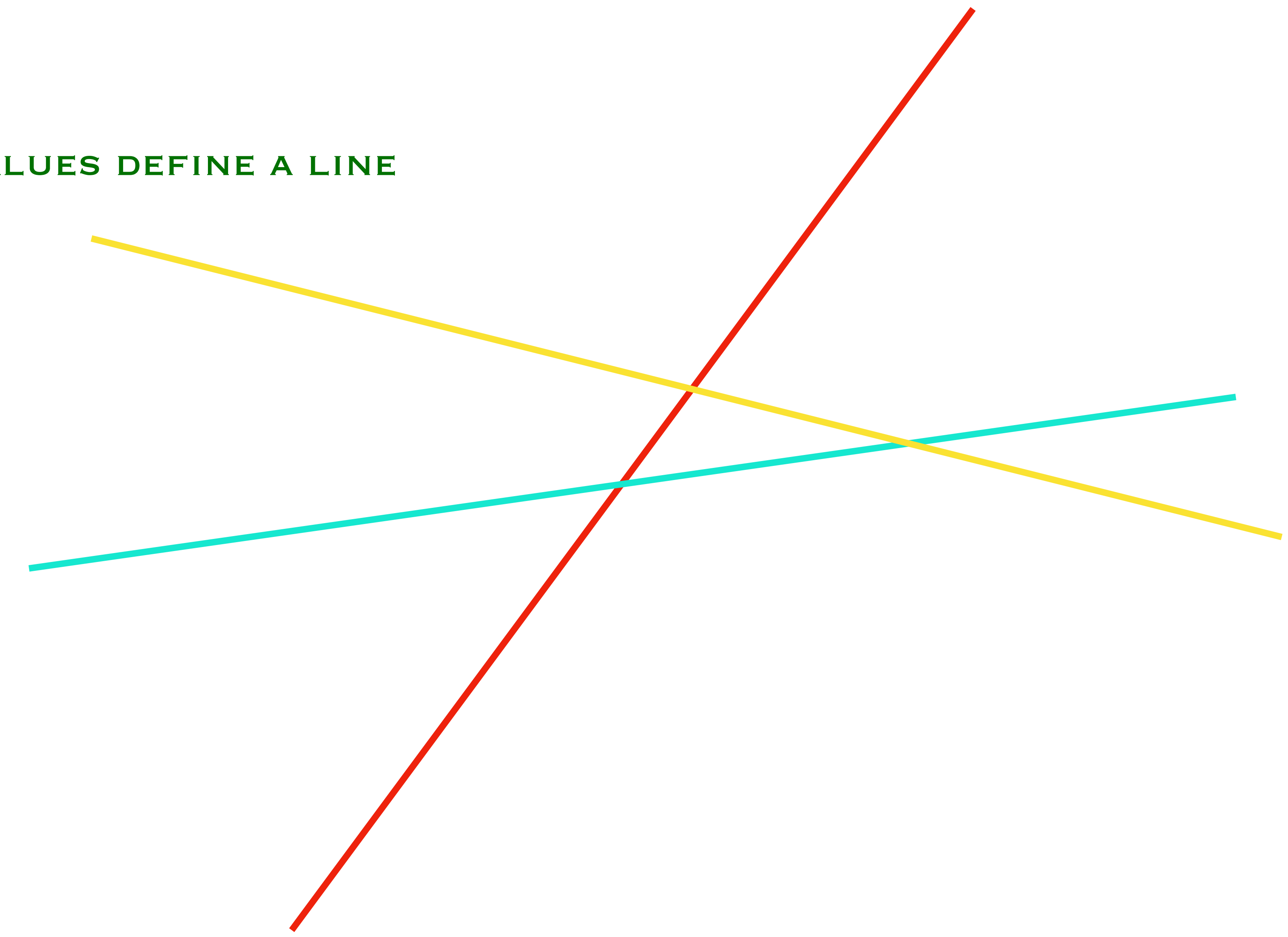


Algebraic Approach

Parameters:

a, b

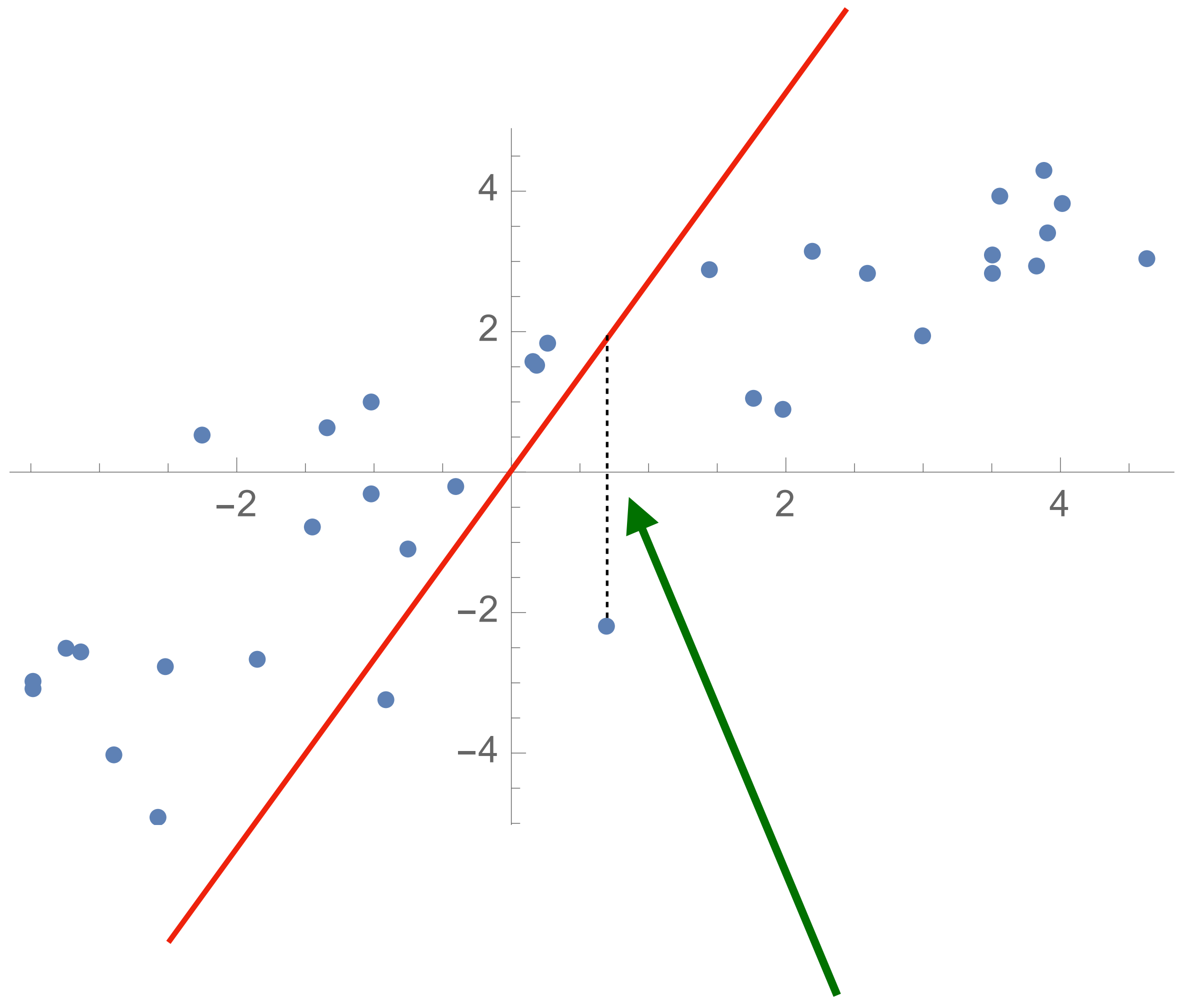
Each pair of values define a line



Algebraic Approach

Define a residual function for each data point:

$$g_i : \mathbb{R}^2 \to \mathbb{R} \quad \text{such that} \quad g_i(a,b) = f(x_i) - y_i = ax_i + b - y_i$$



RESIDUAL IS SIMPLY A "VERTICAL" SIGNED DISTANCE TO THE LINE

Algebraic Approach

Define a functional as the sum of squares of residuals:

$$G : \mathbb{R}^2 \to \mathbb{R} \quad \text{such that} \quad G(a,b) = \sum_{i=1}^{n} \left(g_i(a,b)\right)^2$$

It is a quadratic function for the case of a line

Algebraic Approach

Solution:

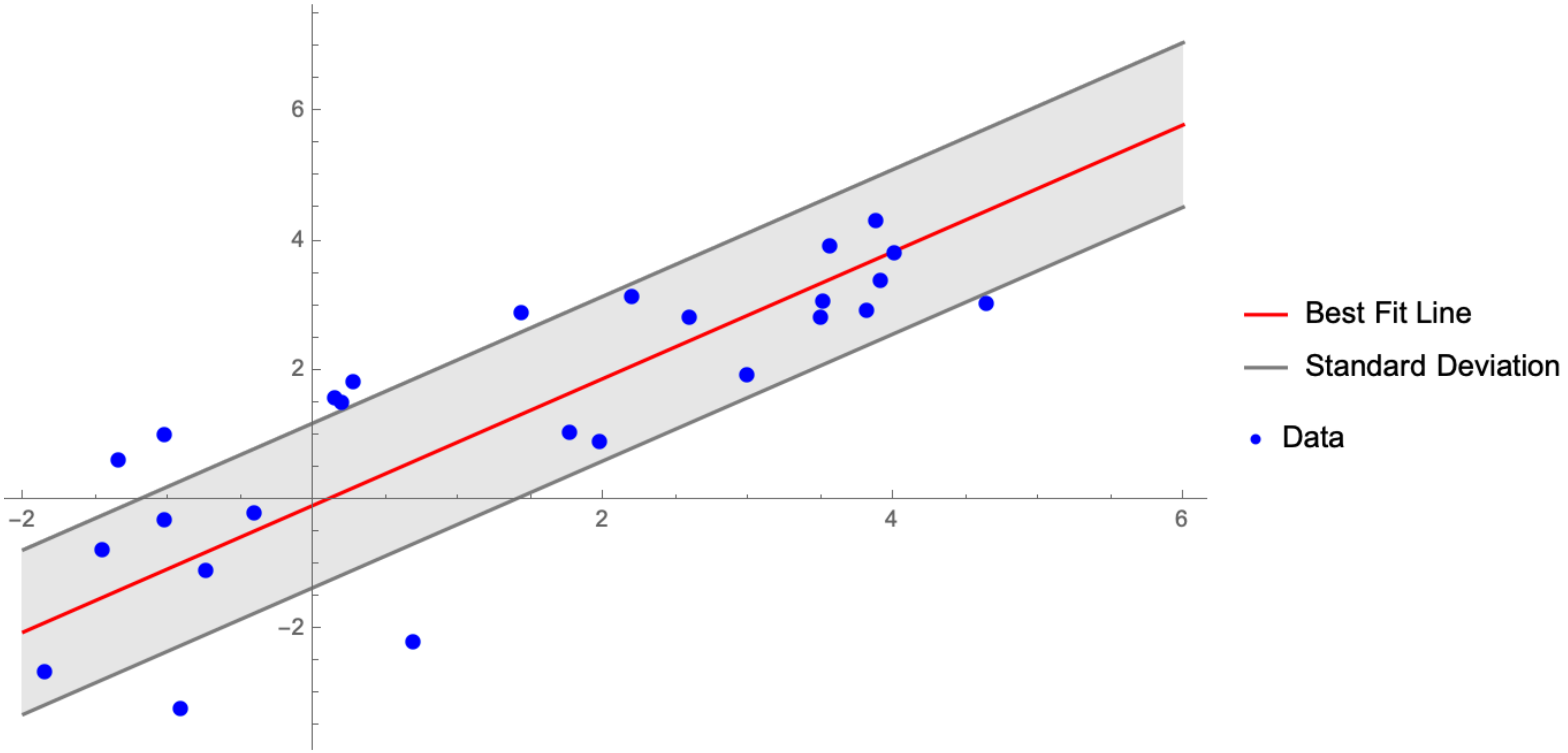
$$\operatorname{argmin}_{(a,b)\in\mathbb{R}^2}\{G(a,b)\}$$

MAY NOT BE UNIQUE!

Algebraic Approach

Solution:

$$\text{argmin}_{(a,b)\in\mathbb{R}^2}\{G(a,b)\}$$



Algebraic Approach

This is an instance of the classic linear least-squares problem:

$$\left\{\begin{array}{ccccccc} ax_1 & + & b & - & y_1 & = & 0 \\ ax_2 & + & b & - & y_2 & = & 0 \\ & & & & & \vdots & \\ ax_n & + & b & - & y_n & = & 0 \end{array}\right. \implies \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies X\boldsymbol{c} - \boldsymbol{y} = \mathbf{0}$$

We need to solve

$$\operatorname{argmin}_{\boldsymbol{c}\in\mathbb{R}^2} \{\|X\boldsymbol{c} - \boldsymbol{y}\|_2\}$$

but we actually solve

$$\operatorname{argmin}_{\boldsymbol{c}\in\mathbb{R}^2} \left\{\|X\boldsymbol{c} - \boldsymbol{y}\|_2^2\right\} \quad (\text{which is equivalent to } \operatorname{argmin}_{(a,b)\in\mathbb{R}^2} \{G(a,b)\})$$

A Simple Algebraic Approach

Consider the problem of fitting a circle to a set of planar points

Let us define the functional as

$$G(r, x_c, y_c) = \sum_{i=1}^{n} (\pi r^2 - \pi((x_i - x_c)^2 + (y_i - y_c)^2))^2$$

AREA OF A CIRCLE OF RADIUS R

Observe that the residual functions are:

$$g_i(r, x_c, y_c) = \pi r^2 - \pi((x_i - x_c)^2 + (y_i - y_c)^2)$$

A Simple Algebraic Approach

Let

$$J(r, x_c, y_c) = \frac{G(r, x_c, y_c)}{\pi^2} = \sum_{i=1}^{n} (r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2))^2$$

So,

$$\frac{\partial J}{\partial r}(r, x_c, y_c) = 4\left[\sum_{i=1}^{n}\left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2)\right)\right] r$$

$$\frac{\partial J}{\partial x_c}(r, x_c, y_c) = 4\left[\sum_{i=1}^{n}\left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2)\right)\right] (x_c - x_i)$$

$$\frac{\partial J}{\partial y_c}(r, x_c, y_c) = 4\left[\sum_{i=1}^{n}\left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2)\right)\right] (y_c - x_i)$$

A Simple Algebraic Approach

We want to find

$$(r, x_c, y_c) \in \mathbb{R}^3$$

such that

$$\frac{\partial J}{\partial r}(r, x_c, y_c) = \frac{\partial J}{\partial x_c}(r, x_c, y_c) = \frac{\partial J}{\partial y_c}(r, x_c, y_c) = 0$$

This can lead us to the classic linear least-squares problem, but...

A Simple Algebraic Approach

From

$$\frac{\partial J}{\partial r}(r, x_c, y_c) = 0$$

we get

$$nr^2 = \sum_{i=1}^{n} \Big((x_i - x_c)^2 + (y_i - y_c)^2 \Big)$$

A Simple Algebraic Approach

From

$$\frac{\partial J}{\partial x_c}(r, x_c, y_c) = 0$$

we get

$$\sum_{i=1}^{n} \left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2) \right) x_i = \sum_{i=1}^{n} \left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2) \right) x_c$$

A Simple Algebraic Approach

From

$$nr^2 = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right)$$

and

$$\sum_{i=1}^{n} \left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2) \right) x_i = \sum_{i=1}^{n} \left(r^2 - ((x_i - x_c)^2 + (y_i - y_c)^2) \right) x_c$$

we get

$$r^2 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right) x_i$$

Similarly,

$$r^2 \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right) y_i$$

A Simple Algebraic Approach

Using the notation

$$\textstyle\sum_x = \sum_{i=1}^n x_i \qquad \sum_y = \sum_{i=1}^n y_i \qquad \sum_{x^2} = \sum_{i=1}^n x_i^2 \qquad \sum_{y^2} = \sum_{i=1}^n y_i^2$$

$$\textstyle\sum_{xy} = \sum_{i=1}^n x_i y_i \qquad \sum_{x^3} = \sum_{i=1}^n x_i^3 \qquad \sum_{y^3} = \sum_{i=1}^n y_i^3 \qquad \sum_{x^2y} = \sum_{i=1}^n x_i^2 y_i$$

$$\textstyle\sum_{xy^2} = \sum_{i=1}^n x_i y_i^2$$

A Simple Algebraic Approach

equation

$$nr^2 = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right)$$

becomes

$$nr^2 = \sum_{x^2} - 2 \sum_x x_c + n x_c^2 + \sum_{y^2} - 2 \sum_y y_c + n y_c^2$$

A Simple Algebraic Approach

and equation

$$r^2 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right) x_i$$

becomes

$$r^2 \sum_x = \sum_{x^3} -2 \sum_{x^2} x_c + \sum_x x_c^2 + \sum_{xy^2} -2 \sum_{xy} y_c + \sum_x y_c^2$$

A Simple Algebraic Approach

and equation

$$r^2 \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right) y_i$$

becomes

$$r^2 \sum_y = \sum_{x^2 y} -2 \sum_{xy} x_c + \sum_y x_c^2 + \sum_{y^3} -2 \sum_{y^2} y_c + \sum_y y_c^2$$

A Simple Algebraic Approach

Multiply

$$nr^2 = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right)$$

by

$$\sum_x$$

and subtract

$$n \left(r^2 \sum_x \right) = n \left(\sum_{x^3} - 2 \sum_{x^2} x_c + \sum_x x_c^2 + \sum_{xy^2} - 2 \sum_{xy} y_c + \sum_x y_c^2 \right)$$

to give

$$\sum_{x^2} \sum_x - n \sum_{x^3} - 2x_c \left(\left(\sum_x \right)^2 - n \sum_{x^2} \right) + \sum_x \sum_{y^2} - n \sum_{xy^2} - 2y_c \left(\sum_x \sum_y - n \sum_{xy} \right) = 0$$

A Simple Algebraic Approach

Multiply

$$nr^2 = \sum_{i=1}^{n} \left((x_i - x_c)^2 + (y_i - y_c)^2 \right)$$

by

$$\sum_y$$

and subtract

$$n \left(r^2 \sum_y \right) = n \left(\sum_{x^2 y} - 2 \sum_{xy} x_c + \sum_y x_c^2 + \sum_{y^3} - 2 \sum_{y^2} y_c + \sum_y y_c^2 \right)$$

to give

$$\sum_{x^2} \sum_y - n \sum_{x^2 y} - 2x_c \left(\sum_x \sum_y - n \sum_{xy} \right) + \sum_y \sum_{y^2} - n \sum_{y^3} - 2y_c \left(\left(\sum_y \right)^2 - n \sum_{y^2} \right) = 0$$

A Simple Algebraic Approach

Now we solve a linear system of 2 equations and 2 unknowns:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= 2\left(\left(\sum_x\right)^2 - n\sum_{x^2}\right), & a_{12} &= 2\left(\sum_x \sum_y - n\sum_{xy}\right) \\
a_{21} &= 2\left(\sum_x \sum_y - n\sum_{xy}\right), & a_{22} &= 2\left(\left(\sum_y\right)^2 - n\sum_{y^2}\right) \\
b_1 &= \sum_{x^2}\sum_x - n\sum_{x^3} + \sum_x \sum_{y^2} - n\sum_{xy^2}, & b_2 &= \sum_{x^2}\sum_y - n\sum_{y^3} + \sum_y \sum_{y^2} - n\sum_{x^2 y}
\end{aligned}$$

A Simple Algebraic Approach

Once we obtain the coordinates of the center, the radius is given as:

$$r^2 = \frac{1}{n}\left(\sum_{x^2} 2\sum_x x_c + nx_c^2 + \sum_{y^2} -2\sum_y y_c + ny_c^2\right)$$

A Simple Algebraic Approach

This approach is credited to:

Samuel M. Thomas and Y. T. Chan
A simple approach for the estimation of circular arc center and radius, *Computer Vision, Graphics, and Image Processing* (CVGIP), 45(3), March, p. 362-370, 1989.

Its extension to the 3d case is trivial and can be found at:

Y. D. Sumith
Fast geometric fit algorithm for sphere using exact solution, ArXiv, 2015.

https://dblp.org/db/journals/corr/corr1506.html#YD15

Experiment

Considered a sphere of radius 5 centered at (0,0,0)

Generated 100 points uniformly distributed over the sphere

Added white noise to the points using standard deviation equal to

- 0.5000 (~10.00% of the radius)
- 0.0500 (~ 1.00% of the radius)
- 0.0050 (~ 0.10% of the radius)
- 0.0005 (~ 0.01% of the radius)

Ran algorithm for computing center and radius from the points

Results

StdDev	Radius	Center	SoSR
0.5000	5.10152	(+0.76577800, -0.01023420, -0.12833200)	2990.160000
0.0500	4.99944	(-0.01199920, +0.00508314, +0.01469530)	26.48840000
0.0050	5.00016	(-0.00018245, -0.00057873, -0.00375501)	0.254553000
0.0005	5.00006	(+0.00001887, +0.00012307, +0.00008819)	0.002555000

Implementation

C++ 17 and Eigen 3.3.9

Unit tests

Final Remarks

Algebraic vs Geometric Fitting (difference in the residuals):