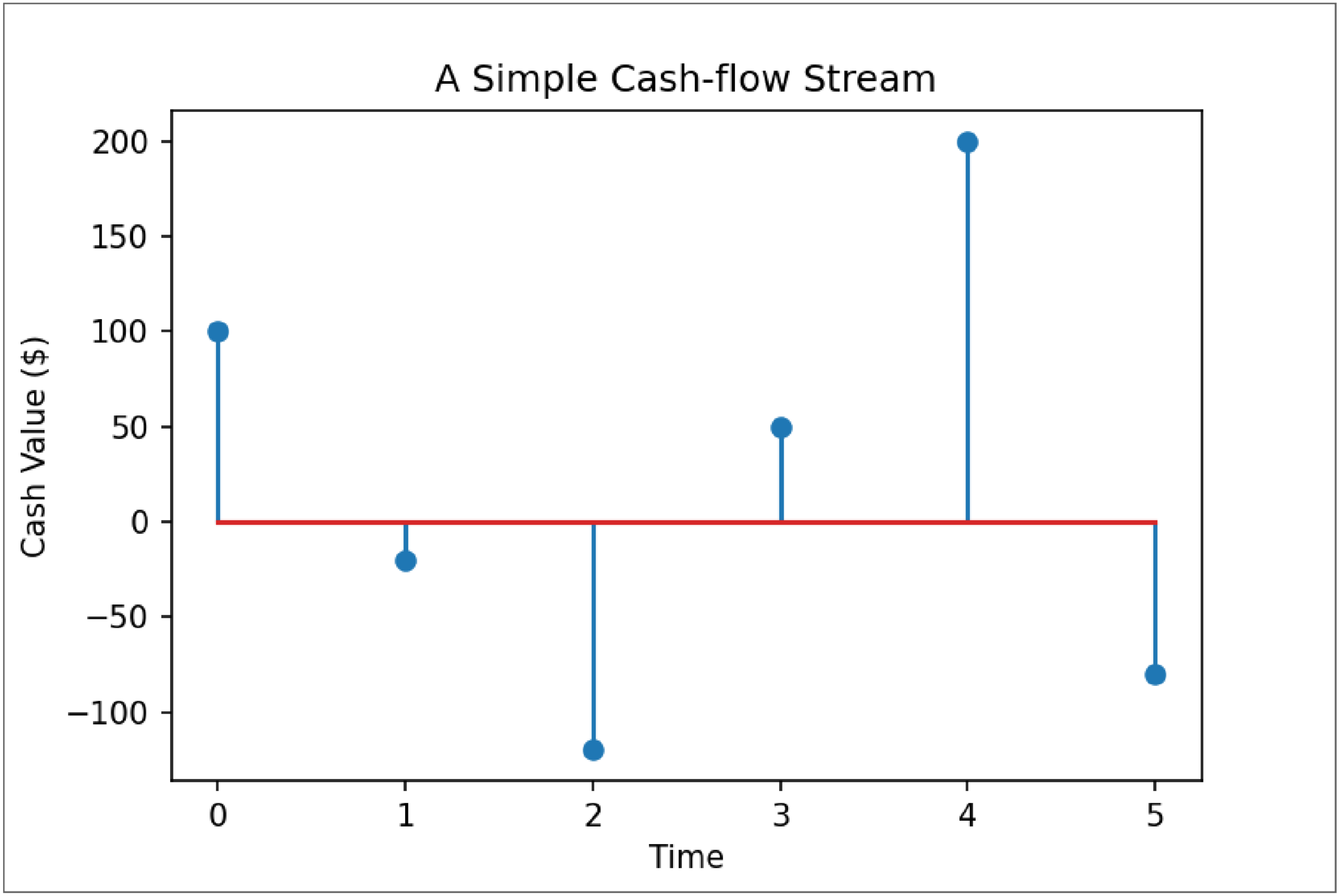
Chapter 1 - Introduction

Investment Science


A Simple Cash-flow Stream
Cash Value ($)
200
150
100
50
0
−50
−100
0
1
2
3
4
5
Time

- All investments are cash flow streams:
    - Consider things in terms of their *cash value*.
    - Cash is *fungible* (mutually interchangeable). This allows for direct comparison: **The Comparison Principle**.
    - This concept changed the world: barter economy to a money economy.
- What are the dangers?
    - Diablo 3: The real money auction house debacle (GDC 2019 - Cursed Problems in Game Design).
    - When everything is fungible, nothing is unique.
    - Obviously, not everything should have a cash value.

- Therefore, investment science is using *scientific tools* (mathematical analysis) to customize cash flow streams.
    - Many ways to summarize / describe a cash flow stream: NPV, IRR, duration…
- Investment science generally occurs within the confines of a market, which provides a structure to the science. Adds the applied part to the applied mathematics.

Basic Market Properties

- **Arbitrage**
  - **Example**: A simple borrow and lend.
  - Assuming "no arbitrage" provides a lot of structure to the market and prevents degenerate solutions.
- **Risk Aversion**
  - **Example**: Coin-flip vs certain cash.
  - People are risk averse. They need to be rewarded to take on risk.

Problems We Are Going to Solve

- The Pricing Problem:
  - What is the appropriate price of an investment (or project or product), given the rest of the market?
- The Hedging Problem:
  - How can we protect ourselves from the risks of a cash flow stream?
- The Portfolio Problem:
  - If we have a bunch of cash, how should we invest among the products in the market?
- Combination and Consumption Problems.

Chapter 2 - The Basic Theory of Interest

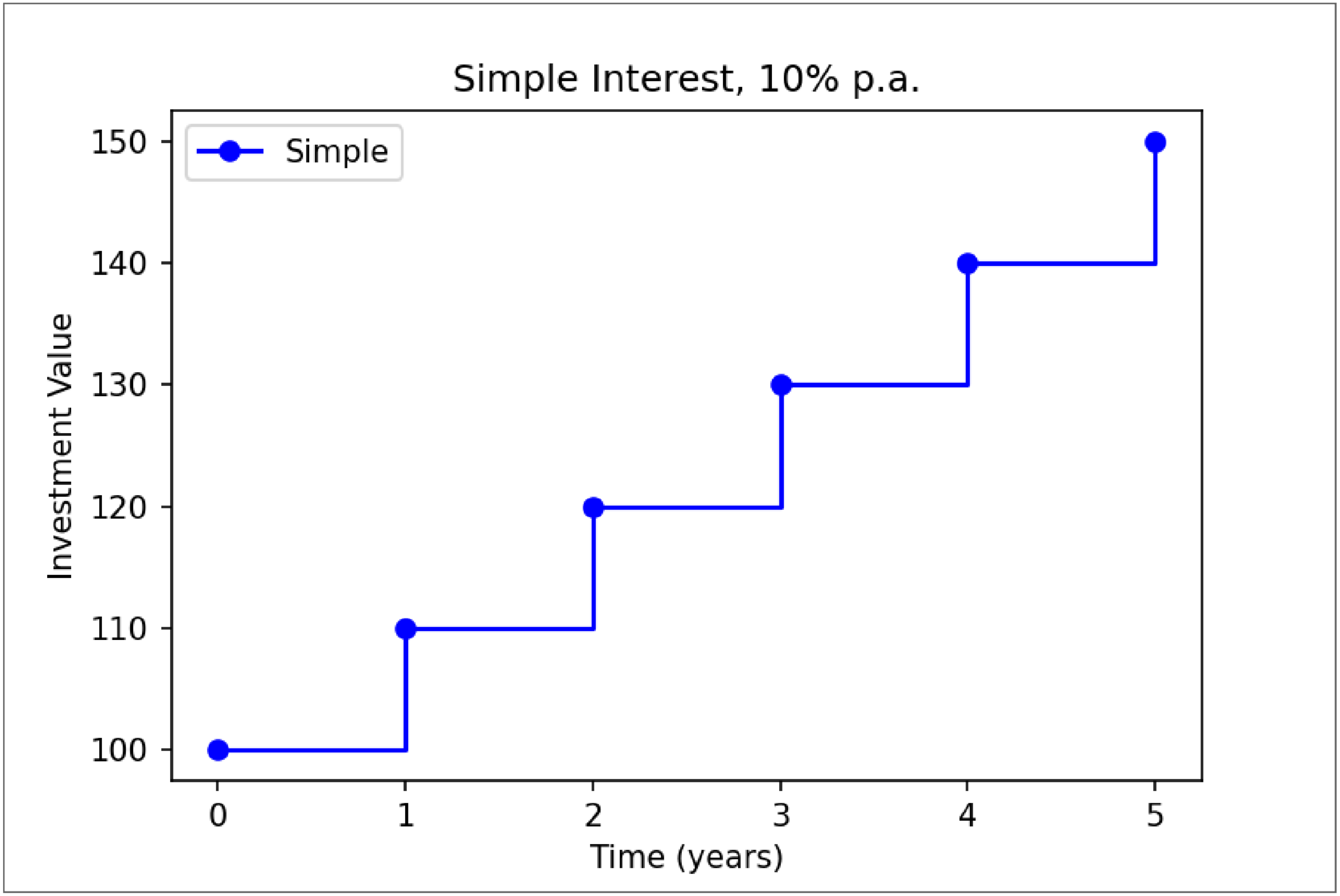
“Compound interest is the eighth wonder of the world. He who understands it, earns it … he who doesn't … pays it.” - Albert Einsten

Principal and Interest

Simple Interest:

$$V = A(1 + rn)$$

- V is the final value
- A is the amount invested
- r is the annual (yearly) simple interest rate
- n is the number of years



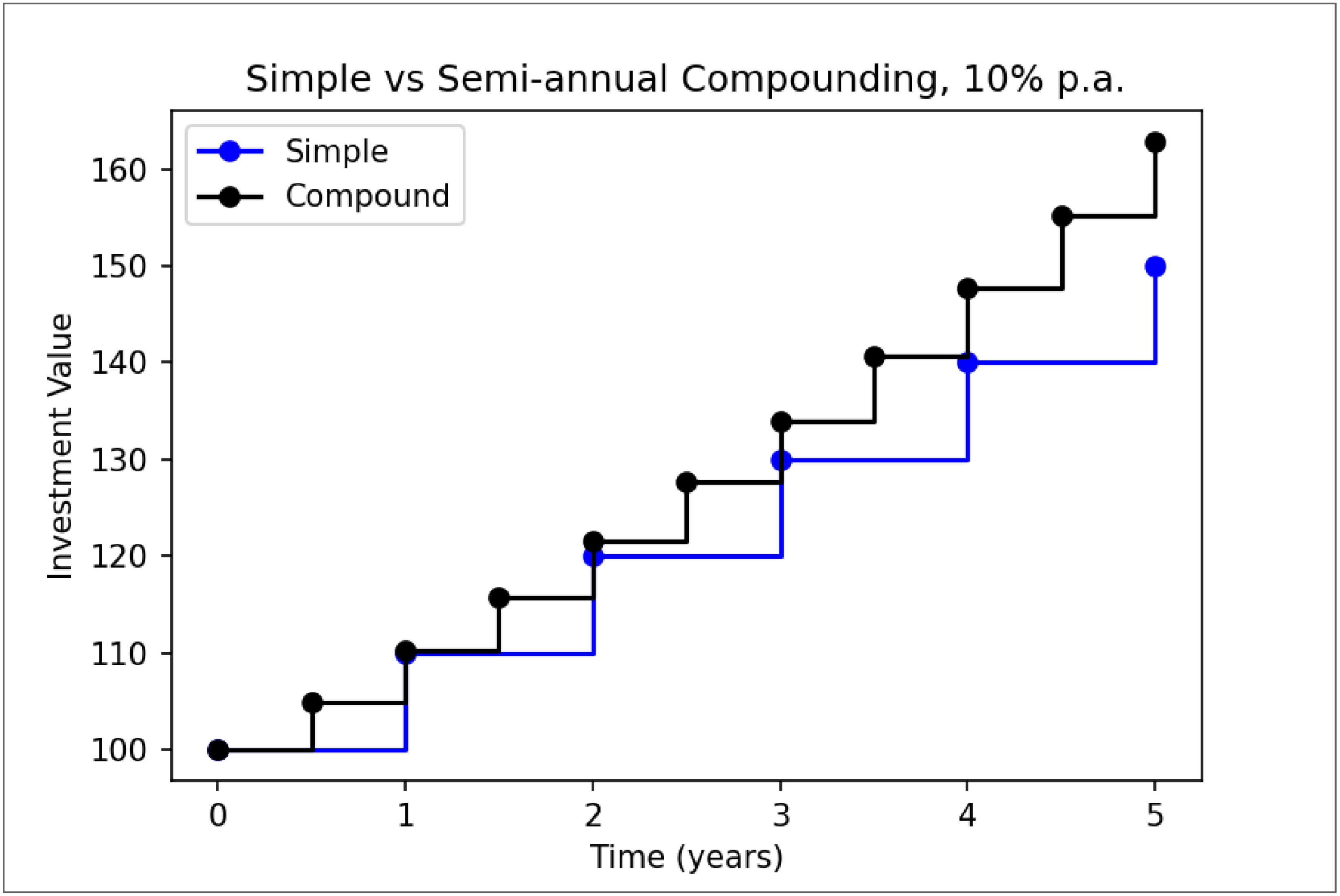
Simple Interest, 10% p.a.
Simple
150
140
130
120
110
100
Investment Value
0
1
2
3
4
5
Time (years)

Compound Interest:

$$V = A(1 + \frac{r}{m})^k$$

- V is the final value
- A is the initial amount invested
- r is the annual (yearly) interest rate
- m is the compounding period, i.e., "How often does the interest compound?"
- k is the number of compounding periods, i.e., "How many times has the interest compounded?"

Compound interest gives rise to *geometric growth* because of its power form (the exponent).


Simple vs Semi-annual Compounding, 10% p.a.
Simple
Compound
Investment Value
160
150
140
130
120
110
100
0
1
2
3
4
5
Time (years)

Nominal vs Effective Rate

- Although the compound interest is quoted as an annual rate, e.g., "8% per year compounded monthly", this is not the actual growth of the initial amount, A, over the course of a year.
- This quoted rate, is known as the *nominal rate*. Nominal means "in name only", which means "not really". This is "not really" the rate.
- The rate that A actually grows over a year, denoted r', is known as the *effective rate*.
- Thus,

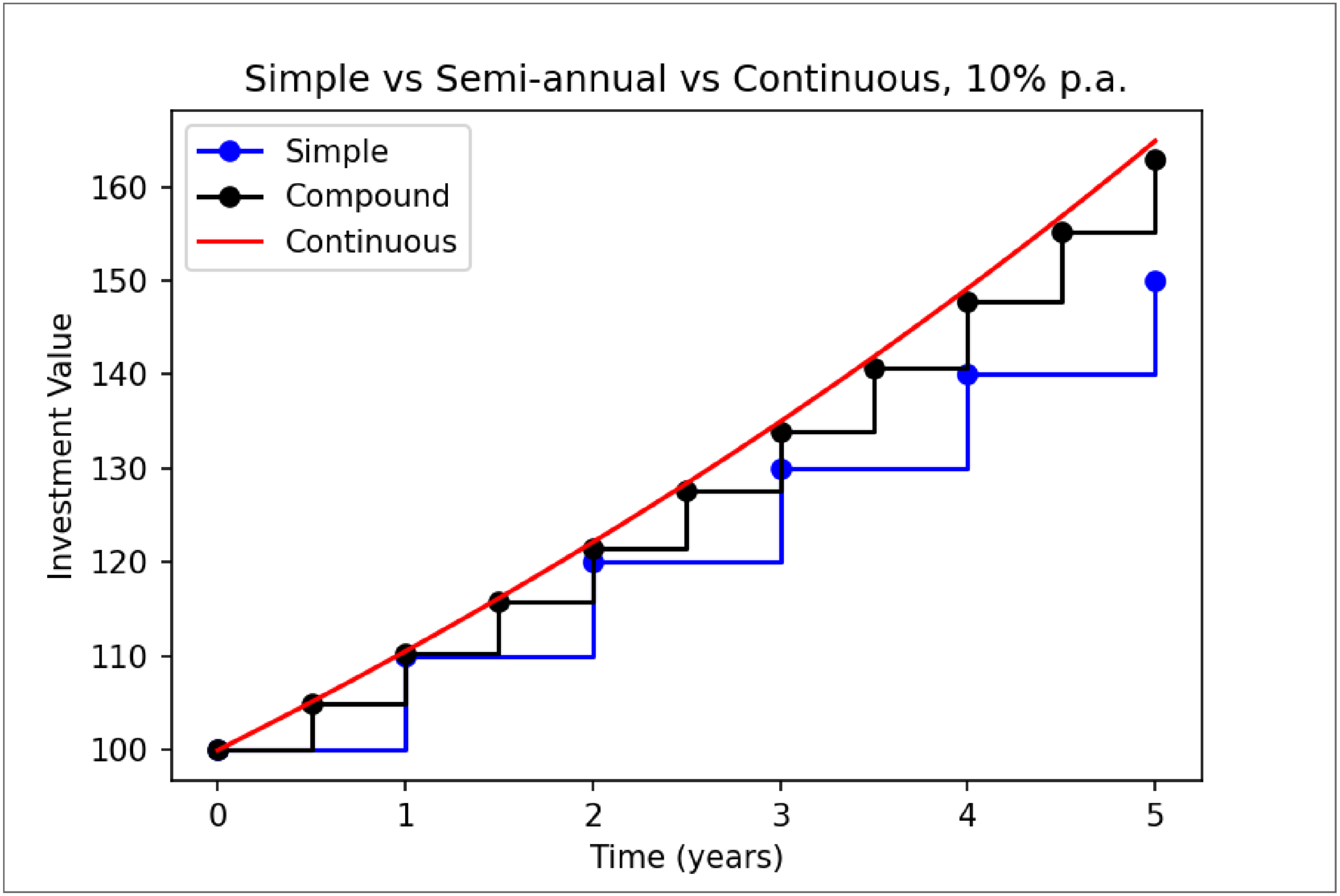
$$r' = (1 + \frac{r}{m})^{m}.$$

Continuous Compounding:

$$V = Ae^{rt}$$

- V is the final value
- A is the initial amount invested
- r is the annual (yearly) interest rate
- e is the natural number or Euler's number, the base of the natural logarithm
- t is the time invested (in years)

Continuous compounding gives rise to *exponential growth*.


Simple vs Semi-annual vs Continuous, 10% p.a.
Simple
Compound
Continuous
Investment Value
Time (years)
100
110
120
130
140
150
160
0
1
2
3
4
5

Debt and Money Markets

- Debt works the same way as investments, although usually different rates apply.
- Different rates apply to different circumstances and it can be very challenging to figure out which interest rate to use. We'll return to this problem in the future.

Present Value

- The value of a cash amount can be projected forward or backwards in time by using the appropriate interest rate.
- Bringing a value backwards in time is known as *discounting*.
- **Example:**
    - I owe Karl $\$100$ exactly one year from now. The prevailing simple interest rate is r. How much money do I need today to meet this obligation?
    - $x(1+r) = 100$
    - $x = 100\frac{1}{1+r}$
- $d := \frac{1}{1+r}$ is the *discount factor* (corresponding to the simple rate, r) that I use to obtain the *present value* of a cash flow 1 year from now.

- There are discount factors corresponding to each interest rate and compounding convention.

Present and Future Values of Streams

The Ideal Bank

- Has the same interest rate for deposits and loans, which applies equally to all amounts.
- No transaction fees or services costs.
- The *constant ideal bank* has a single, constant interest rate.

Future Value

Future Value of a Cash Flow Stream: Consider a nominal annual interest rate, r, compounding at m periods per year. Given an equally-spaced cash flow stream $(x_0, x_1, \ldots, x_n)$, which each cash flow occuring at the start of a period, the future value of this cash flow stream at time n (the start of period n) is:

$$FV = x_0(1 + \frac{r}{m})^n + x_1(1 + \frac{r}{m})^{n-1} + \ldots + x_n.$$

Present Value

Present Value of a Cash Flow Stream: Consider a nominal annual interest rate, r, compounding at m periods per year. Given an equally-spaced cash flow stream $(x_0, x_1, \ldots, x_n)$, which each cash flow occuring at the start of a period, the present value of this cash flow stream at time 0 is:

$$PV = x_0 + x_1 \frac{1}{1+\frac{r}{m}} + x_2 \frac{1}{(1+\frac{r}{m})^2} + \ldots + x_n \frac{1}{(1+\frac{r}{m})^n}.$$

Present Value and an Ideal Bank

- **Key Idea**: An ideal bank can always transform one cash flow stream into another, iff they have the same present value.