3 - PROOF BY INDUCTION - SOLUTIONS

3.1(5.1 of [IMR]) Prove by induction that for all natural numbers n, $n^3 - n$ divisible by 3.

Proof. Let $p(n)$ be the predicate, $n^3 - n$
We use induction on n
Base case: For $n = 1$, $n^3 - n = 1 - 1 = 0$, so p(1) holds.
Inductive step: Suppose now as inductive hypothesis that, $k^3 - k$ where $k \in \mathbb{N}, k \geq 1$ i.e p(k) holds.
Then,

$$\begin{aligned} p(k+1) &= (k+1)^3 - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ p(k+1) - p(k) &= k^3 + 3k^2 + 2k - (k^3 - k) \\ &= 3k^2 + 3k \\ &= 3(k^2 + 1) \\ p(k+1) &= p(k) + 3(k^2 + 1) \end{aligned}$$

As both $p(k)$ and $3k^2 + 3k$ are divisible by 3, then the sum, namely $p(k+1)$ must also be divisible by 3. This proves the inductive step.
Conclusion: Hence, by induction, $p(n)$ is true for all natural numbers n. $\square$

3.6 Let (u_n) be the sequence of numbers defined by

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1, \\ u_{k+1} &= u_{k-1} + u_k, \text{ for } k \geq 2 \end{aligned}$$

(These are Fibonacci numbers).
Prove by induction on n that

$$u_n^2 = u_{n-1}u_{n+1} + (-1)^{n-1}$$

for all natural numbers n such that $n \geq 2$

Proof. We use (strong) induction on n
Let p(n) be the statement: $u_n^2 = u_{n-1}u_{n+1} + (-1)^{n-1}$
Base case: $n = 1, (u_1)^2 = 1^2 = 1, u_1^2 = u_0u_2 + (-1)^0 = 0 + 1 = 1$ so p(1) holds
$n = 2, (u_2)^2 = 1^2 = 1, u_2^2 = u_1u_3 + (-1)^1 = 1 * 2 - 1 = 1$ so p(2) holds
Inductive Step: Suppose now as inductive hypothesis that $p(r)$ is true for all natural numbers $r \leq k$, where $k \geq 2$
Then, $(u_{k+1})^2 = (u_{k-1} + u_k)^2 = u_{k-1}^2 + 2u_{k-1}u_k + u_k^2 = (u_{k-2}u_k + (-1)^{k-2}) + 2u_{k-1}u_k + (u_{k-1}u_{k+1} + (-1)^{k-1})$
Then, $p(k+1) = u_{(k+1)-1}u_{(k+1)+1} + (-1)^{(k+1)-1} = u_ku_{k+2} + (-1)^k$
Conclusion: Hence, by induction, $p(n)$ is true for all natural numbers $n \geq 2$ $\square$