5 - SUBSPACES - SOLUTIONS

1 (2.8.5) Let

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$$

Determine if $\mathbf{w}$ is in subspace if $\mathbb{R}^3$ generated by $\mathbf{v_1}$ and $\mathbf{v_2}$.

Solution. *The system*

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} = \mathbf{w}$$

is found to be inconsistent. Therefore, $\mathbf{w}$ *is not in the subspace* $\mathbb{R}^3$. □

3 (2.8.11, .13) Give integers p and q such that Nul A is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$:

$$A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$$

Find a nonzero vector in Nul A and a nonzero vector in Col A

Solution. $p = 4$ *and* $q = 3$. *Nonzero vector in Nul A is the transposition of matrix:* $\begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix}$ *or any non trivial solution* $\mathbf{x}$ *of the equation* $\mathbf{Ax} = \mathbf{0}$. *For a nonzero vector in* Col A, *any column of* A *holds.* □

5 (2.8.23) Shown are a matrix A and echolon form for A:

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find a basis for Col A and a basis for Nul A.

Solution. *For Col A, we can take the two pivot columns of A:*

$$\left\{ \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \right\}$$

These pivoted columns are linearly independent. Therefore, they form a basis for Col A
To find a basis for Nul A *by solving the homogenous system of equations:*

$$\mathrm{A}\mathbf{X}{=}0$$

First, we should reduce A to reduced echolon form:

$$\begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_3 and x_4 are free variables. Let $x_3 = s$ and $x_4 = t$ and

$$x_1 = 4x_3 - 5x_4 = 4s - 5t$$
$$x_2 = -5x_3 + 6x_4 = -5s + 6t$$

Therefore, we can see that there exists and arbitrary solution which can be uniquley written as a linear combination of the free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4s - 5t \\ -5s + 6t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis of the solution space of $A\boldsymbol{x} = 0$, which is Nul A. □

13(*) Invert the matrices using row operations on the augmented matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution.

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

□

14(*) Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -1 & 4 & 2 \\ 6 & -1 & 7 & 5 \end{bmatrix}$$

Find a basis for Col A.

Solution. *Since*

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = 2 * \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

We get that

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \in \text{span}\{\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}\}$$

The vectors

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

are linearly independent. Thus, a basis for Col A *is:*

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} \right\}$$

$\square$