2 - NUMBER THEORY I - SOLUTIONS

2.3 Prove by contradiction that there do not exist integers m and n such that

$$24n + 3m^2 = 7$$

Proof. Suppose for contradiction that there does exist integers m and n such that $24n + 3m^2 = 7$. Then, since 24 and 3 share a factor of 3, $24n + 3m^2 = 7$ can be rewritten as $3(8n + m^2) = 7$, where $8n + m^2$ belongs to the integers. This implies that $3|7$ which is impossible. Therefore, there does not exist integers m and n such that $24n + 3m^2 = 7$. □

2.4 Let p be a natural number with $p > 1$. Assume the following about p:

If a and b are natural numbers with $p|ab$, then $p|a$ or $p|b$.

Prove that p must be prime

Proof. Suppose for contradiction, that p is non-prime, meaning p has factors excluding 1 and p. Therefore, there exists natural numbers a and b, not equal to 1 or p, such that, $a * b = p$ meaning $p|ab$. But, since a and b are less than p but greater than 0, then $p \nmid a$ or $p \nmid b$. As a result, the statement becomes false for non-primes as the hypothesis is true, but the conclusion is false.
Thus, p must be prime. □

2.7 (ii) Let $a, b, c \in \mathbb{Z}$. Prove the following using the contrapositive:

If $c \nmid ab$, then $c \nmid a$ and $c \nmid b$

This is the statement of the form $P \Rightarrow Q$ where:
  P is the statement $c \nmid ab$
  Q is the statement $(c \nmid a) \wedge (c \nmid b)$
To find the contrapositive:
  (not P) is the statement $c \mid ab$
  (not Q) is the statement $(c \mid a) \vee (c \mid b)$
and so the statement (not Q) $\Rightarrow$ (not P) reads

$$(c \mid a) \vee (c \mid b) \Rightarrow c \mid ab$$

where a, b and c are integers.

Proof. The contrapositive of the statement

$$c \nmid ab \Rightarrow (c \nmid a) \wedge (c \nmid b)$$

is the statement

$$(c \mid a) \vee (c \mid b) \Rightarrow c \mid ab$$

If we consider, $c \mid a$, then $a = pc$, where p is some integer, then $ab = (pc)b = (pb)c$ so $c \mid ab$. For $c \mid b$, then $b = qc$ where q is some integer, then $ab = (aq)c$ so $c \mid ab$, therefore the contrapositive holds true, as when the hypothesis is true, the conclusion is true. This is sufficient to prove the statement is true as the it can only be false when the hypothesis is true and the conclusion is false which is not the case. Thus, the statement is true. □