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Homework #1

Due 26/11/24

This homework consists of two parts, where the first (problems 1 to 4) is multiple-choice.

1. Let . $R = \begin{bmatrix} \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$Then:
   - ☐ R is a rotation matrix.
   - ☐ R^T is a rotation matrix
   - ☑ RR a rotation matrix.
   - ☐ $-R^T R^T$ is a rotation matrix.
2. Consider the following two elementary rotations:

   $R_Y(\alpha)$: rotation by an angle of α around the *Y* axis.

   $R_Z(\beta)$: rotation by an angle of β around the *Z* axis.

   Define a rotation matrix $R = R_Z(\alpha)R_Y(\beta)R_Z(\gamma)$ for some angles α,β and γ. Then:

   - ☑ Any rotation matrix can be written like this by proper choice of the angles.
   - ☐ For some μ and φ, *R* can also be written as $R = R_Y(\mu)R_Z(\varphi)$.
   - ☐ If α=-γ, then $R = R_Y(\beta)$.
   - ☐ If α=γ, then $R = R_Y(-\beta)$.
3. Consider the three eigenvectors x_i, *i*=1,2,3 of any rotation matrix. Then:
   - ☐ $x_i = Rx_i$ for all *i*.
   - ☐ Either $x_i = Rx_i$ or $x_i = -Rx_i$ for all *i*.
   - ☑ For at least one *i* . $x_i = Rx_i$
   - ☐ For at least one *i* . $x_i = -Rx_i$

4. Let *R* be a rotation matrix and *x* an arbitrary vector. Then:
   - ☐ $(Rx)^\wedge = x^\wedge$.
   - ☑ $(Rx)^\wedge = Rx^\wedge$.
   - ☐ $(Rx)^\wedge = x^\wedge R^T$.
   - ☐ $(Rx)^\wedge R = Rx^\wedge$.

1. Consider the following rotation matrix:

$$R = \begin{bmatrix} r_{11} & x_{12} & r_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

The parameters r_{ij} are known. Compute the unknow parameters x_{ij}. Is the solution unique? If not, how many solutions exist?

since R is a rotation matrix then: $R^T R = I \wedge RR^T = I$

$$R^T R = \begin{pmatrix} r_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ r_{13} & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} r_{11} & x_{12} & r_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} \cdot r_{11} + x_{21} \cdot x_{21} + x_{31} \cdot x_{31} & r_{11} \cdot x_{12} + x_{21} \cdot x_{22} + x_{31} \cdot x_{32} & r_{11} \cdot r_{13} + x_{21} \cdot 0 + x_{31} \cdot x_{33} \\ x_{12} \cdot r_{11} + x_{22} \cdot x_{21} + x_{32} \cdot x_{31} & x_{12} \cdot x_{12} + x_{22} \cdot x_{22} + x_{32} \cdot x_{32} & x_{12} \cdot r_{13} + x_{22} \cdot 0 + x_{32} \cdot x_{33} \\ r_{13} \cdot r_{11} + 0 \cdot x_{21} + x_{33} \cdot x_{31} & r_{13} \cdot x_{12} + 0 \cdot x_{22} + x_{33} \cdot x_{32} & r_{13} \cdot r_{13} + 0 \cdot 0 + x_{33} \cdot x_{33} \end{pmatrix}$$

we get the equations:

1. $r_{11}^2 + x_{21}^2 + x_{31}^2 = 1$
2. $x_{12}^2 + x_{22}^2 + x_{32}^2 = 1$
3. $r_{13}^2 + x_{33}^2 = 1$

4. $r_{11} x_{12} + x_{21} x_{22} + x_{31} x_{32} = 0$

6. $r_{11} r_{13} + x_{31} x_{33} = 0$

7. $r_{13} x_{12} + x_{33} x_{32} = 0$

in addition, since R is a rotation matrix the columns are orthonormal, therefore:

$r_{11} x_{12} + x_{21} x_{22} + x_{31} x_{32} = 0$

$r_{11} r_{13} + x_{31} x_{33} = 0$

$x_{12} r_{13} + x_{32} x_{33} = 0$

7. $r_{11} x_{21} + x_{12} x_{22} = 0$

8. $x_{11} x_{31} + x_{12} x_{32} + r_{13} x_{33} = 0$

9. $x_{21} x_{31} + x_{22} x_{32} = 0$

also, $\det(R) = 1$: $\det\begin{pmatrix} r_{11} & x_{12} & r_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} =$

$r_{13} \cdot \det\begin{pmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} + x_{33} \cdot \det\begin{pmatrix} r_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = r_{13} \cdot (x_{21}x_{32} - x_{22}x_{31}) + x_{33}(r_{11}x_{22} - x_{12}x_{21})$

since $RR^T = I$ as well :

$$R \cdot R^T = \begin{bmatrix} r_{11} & x_{12} & r_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \cdot \begin{bmatrix} r_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ r_{13} & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} r_{11}^2 + x_{12}^2 + r_{13}^2, & r_{11}x_{21} + x_{22} \cdot x_{12}, & r_{11}x_{31} + x_{12} \cdot x_{32} + r_{13}x_{33} \\ x_{21} r_{11} + x_{22} \cdot x_{12}, & x_{21}^2 + x_{22}^2, & x_{31} \cdot x_{21} + x_{32} \cdot x_{22} \\ x_{31} \cdot r_{11} + x_{32} \cdot x_{12} + x_{33} \cdot r_{13}, & x_{31} \cdot x_{21} + x_{32} \cdot x_{22}, & x_{31}^2 + x_{32}^2 + x_{33}^2 \end{bmatrix}$$

we get the equations:

(10) $r_{11}^2 + x_{12}^2 + r_{13}^2 = 1$ $\qquad r_{11}x_{21} + x_{12}x_{22} = 0$

(11) $x_{21}^2 + x_{22}^2 = 1$ $\qquad r_{11}x_{31} + x_{12}x_{32} + r_{13}x_{33} = 0$

(12) $x_{31}^2 + x_{32}^2 + x_{33}^2 = 1$ $\qquad x_{31}x_{21} + x_{32}x_{22} = 0$

$\rightarrow \quad x_{33} = \pm\sqrt{1 - r_{13}^2}$ $\qquad$ eq (3)

$\rightarrow \quad x_{31} = \dfrac{-r_{11} - r_{13}}{x_{33}} = \pm \dfrac{r_{11} + r_{13}}{\sqrt{1 - r_{13}^2}}$ $\qquad$ eq (6)

$\rightarrow \quad x_{32} = \pm\sqrt{1 - \dfrac{(r_{11} + r_{13})^2}{1 - r_{13}^2} - (1 - r_{13}^2)}$ $\qquad$ eq (12)

$\rightarrow \quad x_{12} = \pm\sqrt{1 - r_{11}^2 - r_{13}^2}$ $\qquad$ eq (10)

$\rightarrow \quad x_{22} = \pm\sqrt{1 - x_{12}^2 - x_{32}^2} = \pm\sqrt{1 - (1 - r_{11}^2 - r_{13}^2) - \left(1 - \dfrac{(r_{11} + r_{13})^2}{1 - r_{13}^2} - (1 - r_{13}^2)\right)}$ $\qquad$ eq (2)

$\rightarrow \quad x_{21} = \pm\sqrt{1 - r_{11}^2 - x_{31}^2} = \pm\sqrt{1 - r_{11}^2 - \left(1 - \dfrac{(r_{11} + r_{13})^2}{1 - r_{13}^2} - (1 - r_{13}^2)\right)}$ $\qquad$ eq (1)

$\Rightarrow$ for each x_{ij} we have 2 options, overall the solution is not unique, there are 2^6.

5. Consider the three elementary rotation matrices

$$C_Z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, C_Z(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $C(\psi,\theta,\phi) = C_X(\phi)C_Y(\theta)C_Z(\psi)$. Show that this is a rotation matrix. Solve the inverse problem: given a rotation matrix C, compute the angles ψ, θ, ϕ. Is this always possible?

יש טעות בשאלה: C_z כתובה פעמיים ומשתמשים ב C_x בהגדרה של C.
שאלנו את המרצה והוא אמר להחליף את $C_z(\phi)$ ב $C_x(\phi)$
שנמצאות בשקופיות ההרצאה:

$$C_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

$$C_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$C_z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C(\psi,\theta,\phi) = C_x(\phi)\, C_y(\theta)\, C_z(\psi)$$

C is a rotation matrix if:
1. $\det(C) = 1$
2. $C^T C = I$

1) $\det(C) = \det(C_x C_z C_z) = \det(C_x)\det(C_y)\det(C_z)$
$= 1$

כיוון ש C_x, C_y, C_z הן מטריצות
סיבוב.

2) $C^TC = (C_x C_y C_z)^T (C_x C_y C_z) = (C_z^T C_y^T C_x^T)(C_x C_y C_z)$

$= C_z^T C_y^T (C_x^T C_x) C_y C_z$

$= C_z^T C_y^T I C_y C_z$ ל C_x מטריצת סיבוב ולכן $C_x^T C_x = I$

$= C_z^T (C_y^T C_y) C_z$

$= C_z^T I C_z$ ל C_y מטריצת סיבוב ולכן $C_y^T C_y = I$

$= (C_z^T C_z)$

$= I$ ל C_z מטריצת סיבוב ולכן $C_z^T C_z = I$

$\Longrightarrow$ $C = C_x C_y C_z$ is a rotation matrix.

$$C(\psi,\theta,\phi) = \begin{pmatrix} \cos(\theta)\cos(\psi) , & \cos(\theta)\sin(\psi) , & -\sin(\theta) \\ \sin(\phi)\sin(\theta)\cos(\psi)-\cos(\phi)\sin(\psi) , & \sin(\phi)\sin(\theta)\sin(\psi)+\cos(\phi)\cos(\psi) , & \sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta)\cos(\psi)+\sin(\phi)\sin(\psi) , & \cos(\phi)\sin(\theta)\sin(\psi)-\sin(\phi)\cos(\psi) , & \cos(\phi)\cos(\theta) \end{pmatrix}$$

let A be a rotation matrix: $\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$

we'll check to see if $C(\psi,\theta,\phi) = A$ has a unique solution:

$$\left.\begin{aligned} a_1 &= \cos(\theta)\cos(\psi) \\ a_2 &= \cos(\theta)\sin(\psi) \end{aligned}\right\} \frac{a_2}{a_1} = \tan(\psi) \rightarrow \psi = \arctan\left(\frac{a_2}{a_1}\right)$$

$$a_3 = -\sin(\theta) \longrightarrow \begin{array}{l} \theta_1 = 2\pi - \arcsin(a_3) \\ \theta_2 = \pi + \arcsin(a_3) \end{array}$$

$$\left.\begin{array}{l} a_6 = \sin(\phi)\cos(\theta) \\ a_9 = \cos(\phi)\cos(\theta) \end{array}\right\} \frac{a_6}{a_8} = \tan(\phi) \longrightarrow \theta = \arctan\left(\frac{a_6}{a_8}\right)$$

we got 2 possible values for θ so the solution isn't unique unless $a_3 = 0 \longrightarrow \theta = \pi$.

6. Matlab exercise. Generate a rotation matrix *U* from three Euler angles. Compute the eigenvalues and eigenvectors. Verify that one of the eigenvalues is 1. Verify that the corresponding vector is invariant with respect to *U* (namely, if *x* is the vector, then *x*=*Ux*). Conclude it is an axis of rotation.

1. generating a random rotation matrix U:

```
alpha = randi([0, 360]); % Random rotation about X-axis
beta = randi([0, 360]);  % Random rotation about Y-axis
gamma = randi([0, 360]); % Random rotation about Z-axis
```

\# 3 random angles

```
alpha = deg2rad(alpha);
beta = deg2rad(beta);
gamma = deg2rad(gamma);
```

\# conversion to radians

```
Rx = [1, 0,          0;
      0, cos(alpha), -sin(alpha);
      0, sin(alpha), cos(alpha)];
Ry = [cos(beta),  0, sin(beta);
      0,          1, 0;
     -sin(beta),  0, cos(beta)];
Rz = [cos(gamma), -sin(gamma), 0;
      sin(gamma), cos(gamma),  0;
      0,          0,           1];
U = Rx * Ry * Rz;
```

\# $U = R_x(\alpha) R_y(\beta) R_z(\gamma)$

2. computing the eigenvectors and values:

```
[eigenvectors, eigenvalues] = eig(U);
```

3. finding the eigenvalue $\lambda = 1$:

```
[eigenval_close_to_1, idx] = min(abs(diag(eigenvalues) - 1));
invariant_vector = eigenvectors(:, idx);
```

4. proving invariance:

```
invariance_check = norm(U * invariant_vector - invariant_vector) < 1e-10;
```

\# $x = Ux$

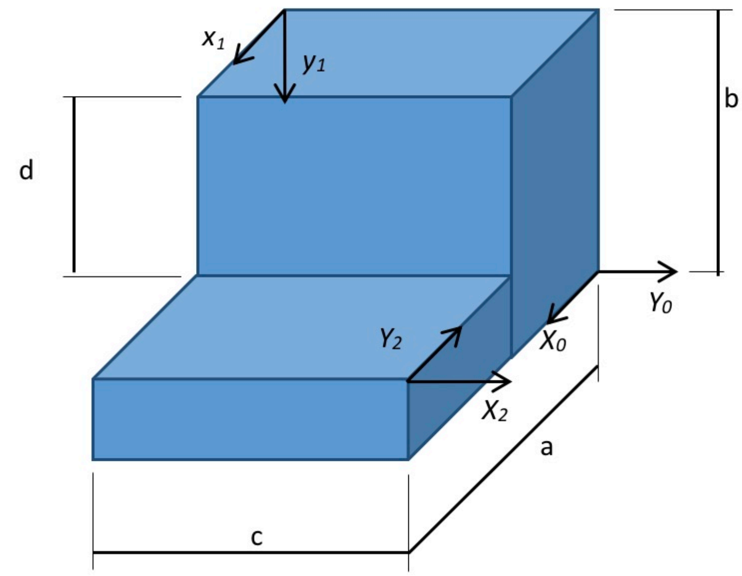
5. displaying the results:

```
disp('Random Euler Angles (Degrees):');
disp(['Alpha (X-axis): ', num2str(rad2deg(alpha)), '°']);
disp(['Beta (Y-axis): ', num2str(rad2deg(beta)), '°']);
disp(['Gamma (Z-axis): ', num2str(rad2deg(gamma)), '°']);
disp('Rotation Matrix U:');
disp(U);
disp('Eigenvalues:');
disp(diag(eigenvalues));
disp('Eigenvectors:');
disp(eigenvectors);
disp('Invariant Vector (Axis of Rotation):');
disp(invariant_vector);
disp('Invariance Check (U * x = x):');
disp(invariance_check);
```

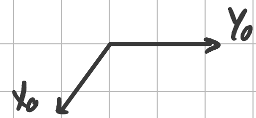
6. final results:

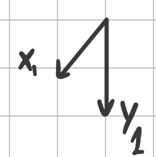
```
Random Euler Angles (Degrees):
Alpha (X-axis): 294°
Beta (Y-axis): 326°
Gamma (Z-axis): 45°
Rotation Matrix U:
    0.5862   -0.5862   -0.5592
    0.6488   -0.0736    0.7574
   -0.4851   -0.8068    0.3372
Eigenvalues:
   1.0000 + 0.0000i
  -0.0751 + 0.9972i
  -0.0751 - 0.9972i
Eigenvectors:
  -0.7843 + 0.0000i  -0.0206 + 0.4382i  -0.0206 - 0.4382i
  -0.0371 + 0.0000i   0.7066 + 0.0000i   0.7066 + 0.0000i
   0.6193 + 0.0000i   0.0163 + 0.5550i   0.0163 - 0.5550i
Invariant Vector (Axis of Rotation):
   -0.7843
   -0.0371
    0.6193
Invariance Check (U * x = x):
   1
```

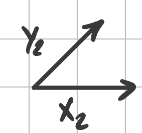
7. For the frames specified in the figure, compute ${}^1_0T$, ${}^2_1T$ and ${}^0_2T$.

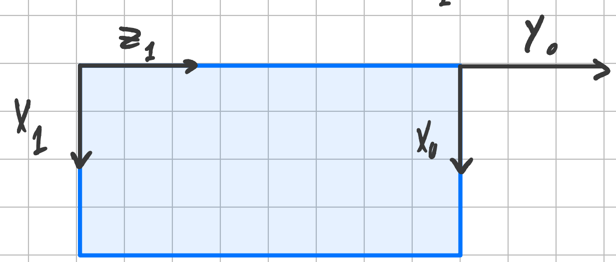


in the figure, we have 3 axis systems (?):

A_0: 

A_1: 

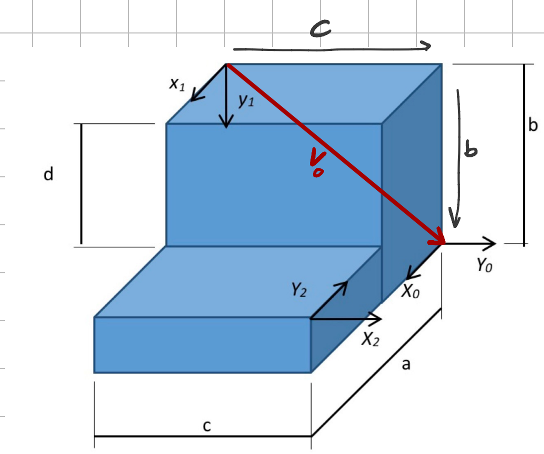
A_2: 



the rotation angle between A_0 and A_1 is $-90°$ around the x axis:

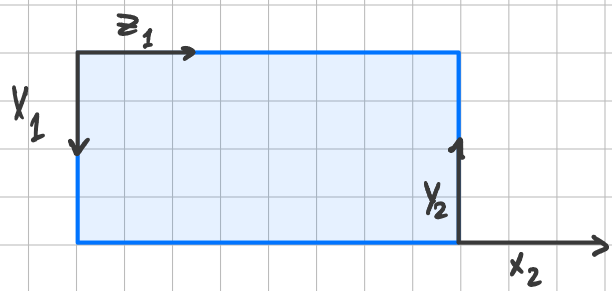
$$ {}^1_0R = {}^1_0R_x(-90)\,{}^1_0R_y(0)\,{}^1_0R_z(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-90) & \sin(-90) \\ 0 & -\sin(-90) & \cos(-90) \end{pmatrix} \begin{pmatrix} \cos 0 & 0 & -\sin 0 \\ 0 & 1 & 0 \\ \sin 0 & 0 & \cos 0 \end{pmatrix} \begin{pmatrix} \cos 0 & \sin 0 & 0 \\ -\sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} $$



$${}^{1}V_0 = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$$

$$\rightarrow T_0^1 = \begin{pmatrix} {}^1_0R & {}^1V_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

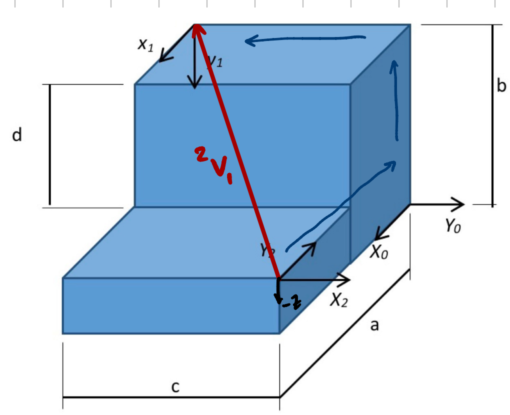


the rotation angles between A_1 and A_2 is $90°$ around the z axis then $-90°$ around the y axis:

$${}^2_1R = {}^2_1R_x(0)\,{}^2_1R_y(-90)\,{}^2_1R_z(90) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(0) & \sin(0) \\ 0 & -\sin(0) & \cos(0) \end{pmatrix} \begin{pmatrix} \cos(-90) & 0 & -\sin(-90) \\ 0 & 1 & 0 \\ \sin(-90) & 0 & \cos(-90) \end{pmatrix} \begin{pmatrix} \cos(90) & \sin(90) & 0 \\ -\sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$



$${}^2V_1 = a\hat{j} + d\hat{z} - c\hat{x} = \begin{pmatrix} -c \\ a \\ d \end{pmatrix}$$

$$\rightarrow T_1^2 = \begin{pmatrix} {}^2_1R & {}^2V_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -c \\ -1 & 0 & 0 & a \\ 0 & -1 & 0 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow T_0^2 = T_0^1 \cdot T_1^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & -c \\ -1 & 0 & 0 & a \\ 0 & -1 & 0 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^0 = \left(T_0^2\right)^{-1} = \begin{pmatrix} 0 & 0 & -1 & c+a \\ 0 & 1 & 0 & d-b \\ 1 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

שאלה 8

נחשב את T_0^1.. אנחנו יכולים להסיק ש-ס"י y מקובעים עם המרכז של כדור הארץ

$$R_0^1 = {}_zR_0^1(\Omega \Delta t)$$

וסיבוב מתרחש בזווית $\Omega \Delta t$ בכיוון החיובי של ציר z

$$T_0^1 = \begin{bmatrix} \cos(\Omega\Delta t) & \sin(\Omega\Delta t) & 0 & 0 \\ -\sin(\Omega\Delta t) & \cos(\Omega\Delta t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

עכשיו נחשב את T_1^2, במידה ואנחנו מסתובבים עם סיבוב כדור הארץ אז אנחנו מקבלים סיבוב בציר ה-Z ב-λ כלומר נקבל $R_1^2(\lambda)$. ואז נסובב ציר z בזווית: $-90-\theta \Leftarrow R^2_{1,y}(-90-\theta)$

$$R_1^2 = R^2_{1,y}(-90-\theta) \cdot R^2_{1,z}(\lambda)$$

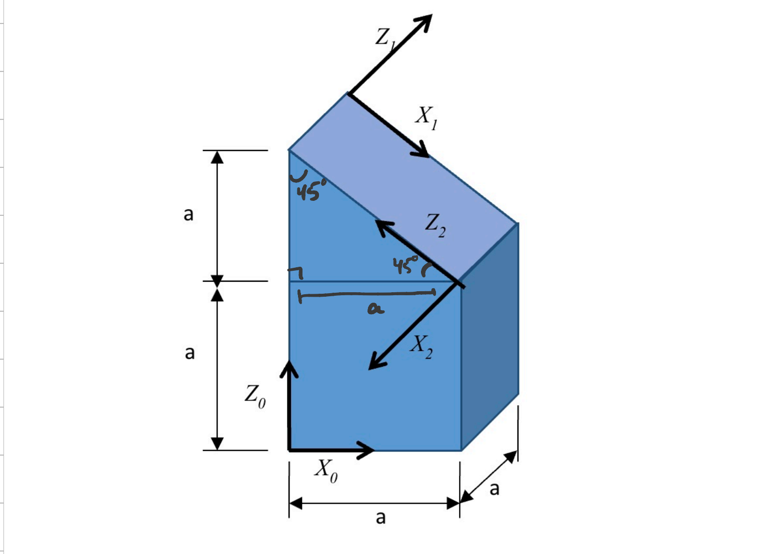
$$R_1^2 = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{bmatrix} * \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1^2 = \begin{bmatrix} -\sin\theta\cos\lambda & -\sin\theta\sin\lambda & \cos\theta \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\theta\cos\lambda & -\cos\theta\sin\lambda & -\sin\theta \end{bmatrix} \qquad P^2_{1-org} = \begin{bmatrix} 0 \\ 0 \\ r_{earth} \end{bmatrix}$$

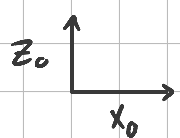
כלומר נקבל :

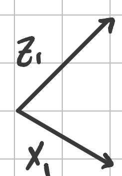
$$T_1^2 = \begin{bmatrix} -\sin\theta\cos\lambda & -\sin\theta\sin\lambda & \cos\theta & 0 \\ -\sin\lambda & \cos\lambda & 0 & 0 \\ -\cos\theta\cos\lambda & -\cos\theta\sin\lambda & -\sin\theta & r_{earth} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

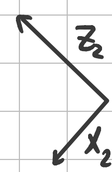
9. For the frames specified in the figure, compute ${}^1_0T$, ${}^2_1T$ and ${}^0_2T$.

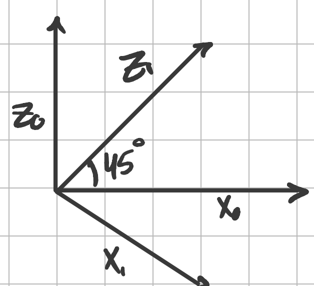


in the figure, we have 3 axis systems:

A_0: 

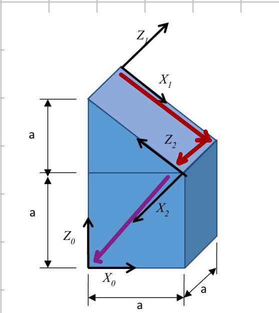
A_1: 

A_2: 



the rotation angle between A_0 and A_1 is $45°$ around the y axis:

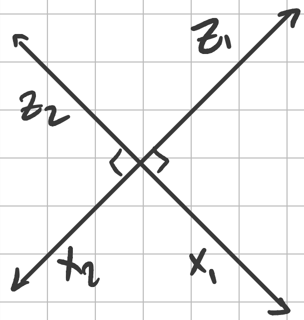
$$ {}^1_0R = {}^1_0R_x(0)\,{}^1_0R_y(45)\,{}^1_0R_z(0) = \begin{pmatrix} \cos 45 & 0 & -\sin 45 \\ 0 & 1 & 0 \\ \sin 45 & 0 & \cos 45 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} $$



הקו הסגול מקביל
לציר $\hat{z}_1$.

$$ {}^1V_0 = \sqrt{2}a\,\hat{x} - a\,\hat{y} - \sqrt{2}a\,\hat{z} $$

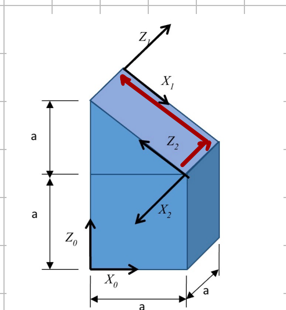
$$ T_0^1 = \begin{pmatrix} {}^1_0R & {}^1V_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & \sqrt{2}a \\ 0 & 1 & 0 & -a \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & -\sqrt{2}a \\ 0 & 0 & 0 & 1 \end{pmatrix} $$



the rotation angle between A_1 and A_2 is 180° around the z axis then 90° around the y axis:

$$ {}^2_1R = {}^2_1R_x(0)\,{}^2_1R_y(90)\,{}^2_1R_z(180) = \begin{pmatrix} \cos(90) & 0 & -\sin(90) \\ 0 & 1 & 0 \\ \sin(90) & 0 & \cos(90) \end{pmatrix} \begin{pmatrix} \cos(180) & \sin(180) & 0 \\ -\sin(180) & \cos(180) & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} $$



$$ {}^2V_1 = -a\,\hat{j} + \sqrt{2}\,a\,\hat{z} $$

$$ T_1^2 = \begin{pmatrix} {}^2_1R & {}^2V_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -a \\ -1 & 0 & 0 & \sqrt{2}a \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ T_0^2 = T_0^1 \cdot T_1^2 = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & \sqrt{2}a \\ 0 & 1 & 0 & -a \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & -\sqrt{2}a \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -a \\ -1 & 0 & 0 & \sqrt{2}a \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & (\sqrt{2}-1)a \\ 0 & 1 & 0 & -2a \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 & (1-\sqrt{2})a \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ T_2^0 = (T_0^2)^{-1} = \begin{pmatrix} 2-1/\sqrt{2} & 0 & -\sqrt{2}/2 & -\sqrt{2}(\sqrt{2}a-a) \\ 0 & 1 & 0 & 2a \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

10. Consider the pan-tilt camera illustrated in the figure:

Compute the line-of-sight as a function of the pan (azimuth) and tilt (elevation) angles. Show in the figure which measurements should be taken from the unit.

define the line-of-sight to be the X axis, the camera can move as follows:

1. Pan: when the camera moves from left to right and vice versa, the rotation axis is Z.
2. Tilt: when the camera moves up and down the rotation axis is Y.

at any given point, the rotation can be described as: $R_{10} = R_Y(\theta)\, R_Z(\psi)$

$$= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix}$$

we defined X to be the line of sight axis, therfore:

$$\text{line-of-sight} = R_{10} \cdot \hat{X} = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{line-of-sight} = \begin{pmatrix} \cos\theta\cos\psi \\ -\sin\psi \\ \sin\theta\cos\psi \end{pmatrix}$$

11. Repeat for the tilt-pan camera in the figure. For which combination of angles you get the same line-of-sight for the two configurations? For which the same rotation matrix?

following the exact steps from the previous question, defining the X axis as the line of sight:

at any given point, the rotation can be described as: $R_{11} = R_z(\beta)\, R_y(\alpha)$

$$\rightarrow \text{line-of-sight} = R \cdot \hat{X} = \begin{pmatrix} \cos\beta\cos\alpha & \sin\beta & -\cos\beta\sin\alpha \\ -\sin\beta\cos\alpha & \cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\alpha \\ -\sin\beta\cos\alpha \\ \sin\alpha \end{pmatrix}$$

to find for which angles we get the same line of sight:

$$\begin{pmatrix} \cos\theta\cos\psi \\ -\sin\psi \\ \sin\theta\cos\psi \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\alpha \\ -\sin\beta\cos\alpha \\ \sin\alpha \end{pmatrix}$$

① $\cos\theta\cos\psi = \cos\beta\cos\alpha$

② $-\sin\psi = -\sin\beta\cos\alpha$

③ $\sin\theta\cos\psi = \sin\alpha$

$$\rightarrow \frac{eq\ 3}{eq\ 1} : \frac{\sin\theta\cancel{\cos\psi}}{\cos\theta\cancel{\cos\psi}} = \frac{\sin\alpha}{\cos\beta\cos\alpha} \rightarrow \tan\theta = \cos\beta\tan\alpha$$

$$\theta = \arctan(\cos\beta\tan\alpha)$$

$$\frac{eq\ 2}{eq\ 1} : \frac{-\sin\psi}{\cos\psi\cos\theta} = \frac{-\sin\beta\cos\alpha}{\cos\alpha\cos\beta} \rightarrow \frac{\tan\psi}{\cos\theta} = \tan\beta$$

$$\psi = \arctan(\tan\beta\cos(\arctan(\cos\beta\tan\alpha)))$$

so for any $\psi, \theta, \alpha, \beta$ that satisfy these equations we get the same line-of-sight configuration.

also:

$$R_{10} = R_{11}$$

$$\begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\alpha & \sin\beta & -\cos\beta\sin\alpha \\ -\sin\beta\cos\alpha & \cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$\rightarrow \quad \begin{aligned} \cos\psi &= \cos\beta & &\rightarrow & \psi &= \pm\beta \\ \cos\theta &= \cos\alpha & &\rightarrow & \theta &= \pm\alpha \\ 0 &= \sin\alpha\sin\beta & &\rightarrow & \alpha &= k\cdot\pi \quad \text{or} \quad \beta = k\cdot\pi \qquad k\in\mathbb{N} \\ \sin\theta\sin\psi &= 0 & &\rightarrow & \theta &= k\cdot\pi \quad \text{or} \quad \psi = k\cdot\pi \qquad k\in\mathbb{N} \end{aligned}$$

נניח בה"כ כי $\theta = \alpha = k\cdot\pi$:

$$\rightarrow \quad \cos\theta = \cos\alpha = (-1)^k \quad \text{and} \quad \sin\theta = \sin\alpha = 0$$

in order to satisfy $\cos\theta\sin\psi = \sin\beta$ we require that:

① $\psi = \beta \Rightarrow \cos\theta = 1$, $\theta = 2\pi k$

② $\psi = -\beta \rightarrow \cos\theta = -1$, $\theta = \pi + 2\pi k$

$$\Rightarrow \quad R_{10} = \begin{pmatrix} (-1)^k\cos\psi & (-1)^k\sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & (-1)^k \end{pmatrix}$$

$$R_{11} = \begin{pmatrix} (-1)^k\cos\beta & \sin\beta & 0 \\ (-1)^k\sin\beta & \cos\beta & 0 \\ 0 & 0 & (-1)^k \end{pmatrix}$$

for k that is odd we get $(-1)^k = -1$, and for k that is even we get $(-1)^k = 1$, basically it doesn't matter.

overall: for $\psi = \beta$, $\theta = \alpha$ or $\psi = -\beta$, $\theta = \alpha$ we get the same rotation matrix.