Introduction to Robotics

Tala Saba - 324030220
Adan Sarhan - 324849256

Homework #4

Due 2/1/2025

This homework consists of two parts, where the first (problems 1 to 3) is multiple-choice.

1. Let ${}^{A}\Omega_{A,B}$ denote the angular velocity of frame *A* with respect to *B* in the *A* frame, and ${}^{A}\Omega_{B.A}$ the angular velocity of frame *B* with respect to *A* in the *A* frame. Then:
   - ☐ ${}^{A}\Omega_{B.A} = 0.$
   - ☐ ${}^{A}\Omega_{B.A} = {}^{B}_{A}R\,{}^{A}\Omega_{A,B}.$
   - ☑ ${}^{A}\Omega_{B.A} = -\,{}^{A}\Omega_{A,B}.$
   - ☐ ${}^{A}\Omega_{B.A} = {}^{B}_{A}R\,{}^{A}\Omega_{A,B}\,{}^{A}_{B}R.$
2. Which of the following is always correct?
   - ☐ ${}^{A}\left({}^{B}V_{A-ORG}\right) = {}^{A}\left({}^{A}V_{B-ORG}\right).$
   - ☐ ${}^{A}\left({}^{B}V_{A-ORG}\right) = {}^{A}\left({}^{B}V_{B-ORG}\right)$
   - ☑ ${}^{A}\left({}^{B}V_{A-ORG}\right) = -\,{}^{A}\left({}^{A}V_{B-ORG}\right)$
3. Let ${}^{C}_{A}R = {}^{C}_{B}R\,{}^{B}_{A}R$. Then:
   - ☐ ${}^{A}\Omega_{A,C} = {}^{A}\Omega_{A,B} + {}^{A}_{B}R\,{}^{B}\Omega_{A,C}.$
   - ☐ ${}^{A}\Omega_{A,C} = {}^{A}\Omega_{A,B} + {}^{B}\Omega_{B,C}.$
   - ☐ ${}^{A}\Omega_{A,C} = {}^{A}\Omega_{A,B} \times {}^{B}\Omega_{B,C}$
   - ☑ ${}^{A}\Omega_{A,C} = {}^{A}\Omega_{A,B} + {}^{A}_{B}R\,{}^{B}\Omega_{B,C}.$

4. Given the rotation matrix:

$$ {}^{B}_{A}R(t) = R_X(\phi(t))R_Y(\theta(t))R_Z(\psi(t)) $$

Compute ${}^{B}\Omega_{A,B}$ as a function of the derivatives of the Euler Angles.

Given that frame A is a constant base frame and frame B rotates around it:

$$ {}^{B}\Omega_{A,B} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_{\text{in frame } B} = w_1 \cdot \hat{x}_B + w_2 \cdot \hat{y}_B + w_3 \cdot \hat{z}_B = \dot{\phi} \cdot \hat{x}_B + \dot{\theta} \cdot \hat{y}_B + \dot{\psi} \cdot \hat{z}_B $$

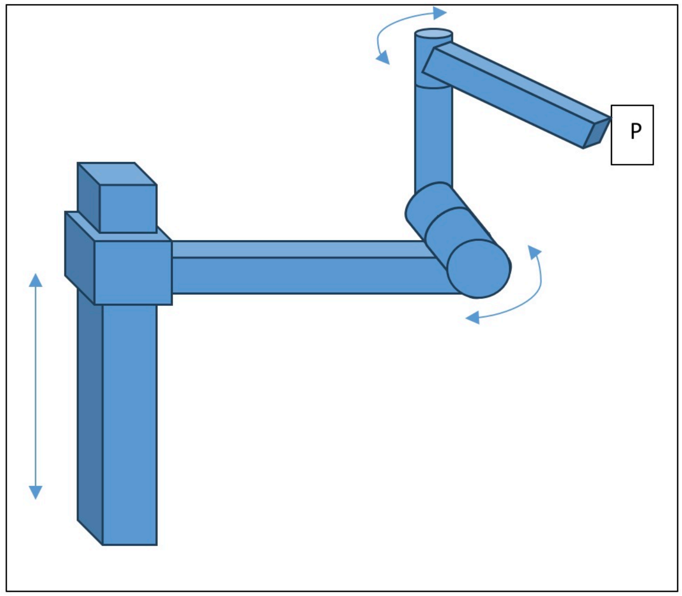
where $\hat{x}_B, \hat{y}_B, \hat{z}_B$ are unit vectors of axes the rotating frame B.

given ${}^{B}_{A}R(t) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ we can deduce:

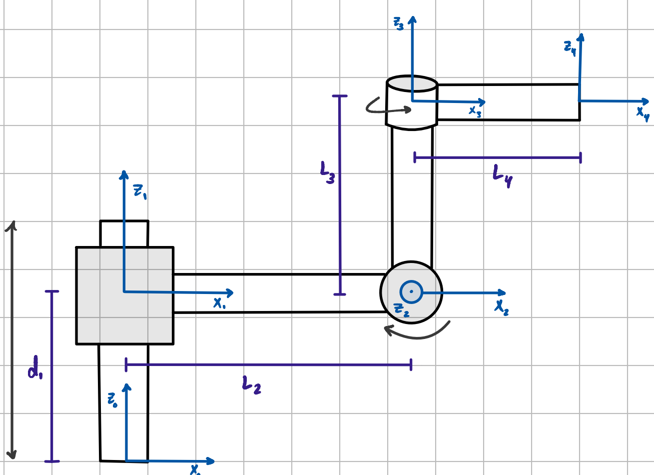
$$ \left.\begin{aligned} \hat{x}_B &= r_{11} \cdot \hat{x}_A + r_{12} \cdot \hat{y}_A + r_{13} \cdot \hat{z}_A \\ \hat{y}_B &= r_{21} \cdot \hat{x}_A + r_{22} \cdot \hat{y}_A + r_{23} \cdot \hat{z}_A \\ \hat{z}_B &= r_{31} \cdot \hat{x}_A + r_{32} \cdot \hat{y}_A + r_{33} \cdot \hat{z}_A \end{aligned}\right\} \hat{x}_B = \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \end{pmatrix}, \quad \hat{y}_B = \begin{pmatrix} r_{21} \\ r_{22} \\ r_{23} \end{pmatrix}, \quad \hat{z}_B = \begin{pmatrix} r_{31} \\ r_{32} \\ r_{33} \end{pmatrix} $$

$$ \rightarrow \quad {}^{B}\Omega_{A,B} = \dot{\phi} \cdot \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \end{pmatrix} + \dot{\theta} \cdot \begin{pmatrix} r_{21} \\ r_{22} \\ r_{23} \end{pmatrix} + \dot{\psi} \cdot \begin{pmatrix} r_{31} \\ r_{32} \\ r_{33} \end{pmatrix} $$

5. For the following robot:



Calculate the linear and rotational velocity of the end effector.



joint	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	0
2	90	L_2	0	θ_2
3	-90	0	L_3	θ_3
4	0	L_4	0	0

$${}^0_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_0T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & L_2 \\ 0 & 0 & -1 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_1T = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) & -L_2\cos(\theta_2) \\ -\sin(\theta_2) & 0 & \cos(\theta_2) & L_2\sin(\theta_2) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & L_3 \\ -\sin(\theta_3) & -\cos(\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_2T = \begin{bmatrix} \cos(\theta_3) & 0 & -\sin(\theta_3) & 0 \\ -\sin(\theta_3) & 0 & -\cos(\theta_3) & 0 \\ 0 & 1 & 0 & -L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{3}_{4}T=\begin{bmatrix}1 & 0 & 0 & L_4\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\qquad {}^{4}_{3}T=\begin{bmatrix}1 & 0 & 0 & -L_4\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$${}^{0}_{2}T=\begin{bmatrix}\cos(\theta_2) & -\sin(\theta_2) & 0 & L_2\\ 0 & 0 & -1 & 0\\ \sin(\theta_2) & \cos(\theta_2) & 0 & d_1\\ 0 & 0 & 0 & 1\end{bmatrix}\qquad {}^{2}_{0}T=\begin{bmatrix}\cos(\theta_2) & 0 & \sin(\theta_2) & \frac{L_2\sin^2(\theta_2)-L_2-\frac{d_1\sin(2\theta_2)}{2}}{\cos(\theta_2)}\\ -\sin(\theta_2) & 0 & \cos(\theta_2) & L_2\sin(\theta_2)-d_1\cos(\theta_2)\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$${}^{0}_{3}T=\begin{bmatrix}\cos(\theta_2)\cos(\theta_3) & -\sin(\theta_3)\cos(\theta_2) & -\sin(\theta_2) & L_2-L_3\sin(\theta_2)\\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0\\ \sin(\theta_2)\cos(\theta_3) & -\sin(\theta_2)\sin(\theta_3) & \cos(\theta_2) & L_3\cos(\theta_2)+d_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$${}^{3}_{0}T=\begin{bmatrix}\cos(\theta_2)\cos(\theta_3) & \sin(\theta_3) & \sin(\theta_2)\cos(\theta_3) & -(L_2\cos(\theta_2)+d_1\sin(\theta_2))\cos(\theta_3)\\ -\sin(\theta_3)\cos(\theta_2) & \cos(\theta_3) & -\sin(\theta_2)\sin(\theta_3) & (L_2\cos(\theta_2)+d_1\sin(\theta_2))\sin(\theta_3)\\ -\sin(\theta_2) & 0 & \cos(\theta_2) & L_2\sin(\theta_2)-L_3-d_1\cos(\theta_2)\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$${}^{0}_{4}T=\begin{bmatrix}\cos(\theta_2)\cos(\theta_3) & -\sin(\theta_3)\cos(\theta_2) & -\sin(\theta_2) & L_2-L_3\sin(\theta_2)+L_4\cos(\theta_2)\cos(\theta_3)\\ \sin(\theta_3) & \cos(\theta_3) & 0 & L_4\sin(\theta_3)\\ \sin(\theta_2)\cos(\theta_3) & -\sin(\theta_2)\sin(\theta_3) & \cos(\theta_2) & L_3\cos(\theta_2)+L_4\sin(\theta_2)\cos(\theta_3)+d_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$${}^{4}_{0}T=\begin{bmatrix}\cos(\theta_2)\cos(\theta_3) & \sin(\theta_3) & \sin(\theta_2)\cos(\theta_3) & -L_2\cos(\theta_2)\cos(\theta_3)-L_4-d_1\sin(\theta_2)\cos(\theta_3)\\ -\sin(\theta_3)\cos(\theta_2) & \cos(\theta_3) & -\sin(\theta_2)\sin(\theta_3) & (L_2\cos(\theta_2)+d_1\sin(\theta_2))\sin(\theta_3)\\ -\sin(\theta_2) & 0 & \cos(\theta_2) & L_2\sin(\theta_2)-L_3-d_1\cos(\theta_2)\\ 0 & 0 & 0 & 1\end{bmatrix}$$

from the tirgul:

Revolute

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\hat{Z}_{i+1}$$

$${}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}$$

Prismatic

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

rotational velocities:

$$^0\omega_0 = 0$$

$$^1\omega_1 = {}^1_0R\,{}^0\omega_0 = 0$$

$$^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \dot\theta_2 \hat{Z}_2 = \dot\theta_2 \hat{Z}_2 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix}$$

$$^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \dot\theta_3 \hat{Z}_3 = \begin{bmatrix} \cos(\theta_3) & 0 & -\sin(\theta_3) \\ -\sin(\theta_3) & 0 & -\cos(\theta_3) \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_3 \end{pmatrix} = \begin{pmatrix} -\dot\theta_2 \sin(\theta_3) \\ -\dot\theta_2 \cos(\theta_3) \\ \dot\theta_3 \end{pmatrix}$$

$$^4\omega_4 = {}^4_3R\,{}^3\omega_3 = I \cdot {}^3\omega_3 = \begin{pmatrix} -\dot\theta_2 \sin(\theta_3) \\ -\dot\theta_2 \cos(\theta_3) \\ \dot\theta_3 \end{pmatrix}$$

$$^0\omega_4 = {}^0_4R\,{}^4\omega_4$$

linear velocities:

$$^0v_0 = 0$$

$$^1v_1 = {}^1_0R\left({}^0v_0 + {}^1\omega_1 \times {}^0P_1\right) + \dot d_1 \hat{Z}_1 = \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix}$$

$$^1v_2 = {}^1v_1 + {}^1\omega_1 \times {}^1P_2 = \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix}$$

$$^2v_2 = {}^2_1R\left({}^1v_1 + \underbrace{{}^1\omega_1}_{0} \times {}^1P_2\right) + \underbrace{\dot d_2}_{0} \hat{Z}_2 = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix} = \begin{pmatrix} \dot d_1 s_{\theta_2} \\ \dot d_1 c_{\theta_2} \\ 0 \end{pmatrix}$$

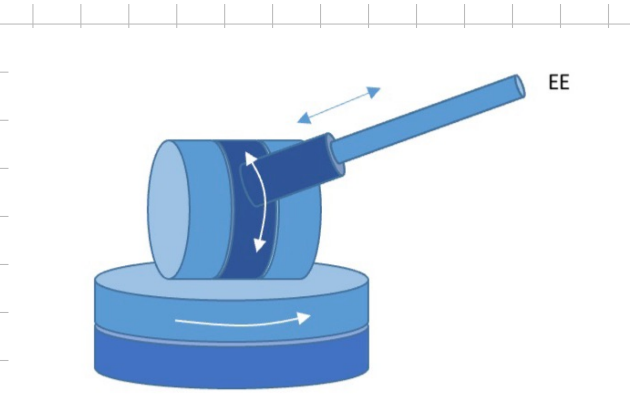
$$^2v_3 = {}^2v_2 + {}^2\omega_2 \times {}^2P_3 = \begin{pmatrix} \dot d_1 s_{\theta_2} \\ \dot d_1 c_{\theta_2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ L_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot d_1 s_{\theta_2} - L_3 \dot\theta_2 \\ \dot d_1 c_{\theta_2} \\ 0 \end{pmatrix}$$

$$^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) = \begin{bmatrix} \cos(\theta_3) & 0 & -\sin(\theta_3) \\ -\sin(\theta_3) & 0 & -\cos(\theta_3) \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \dot d_1 s_{\theta_2} - L_3 \dot\theta_2 \\ \dot d_1 c_{\theta_2} \\ 0 \end{pmatrix} = \begin{pmatrix} \dot d_1 s_{\theta_2} c_{\theta_3} - L_3 \dot\theta_2 c_{\theta_3} \\ \dot d_1 s_{\theta_2} s_{\theta_3} - L_3 \dot\theta_2 s_{\theta_3} \\ \dot d_1 c_{\theta_2} \end{pmatrix}$$

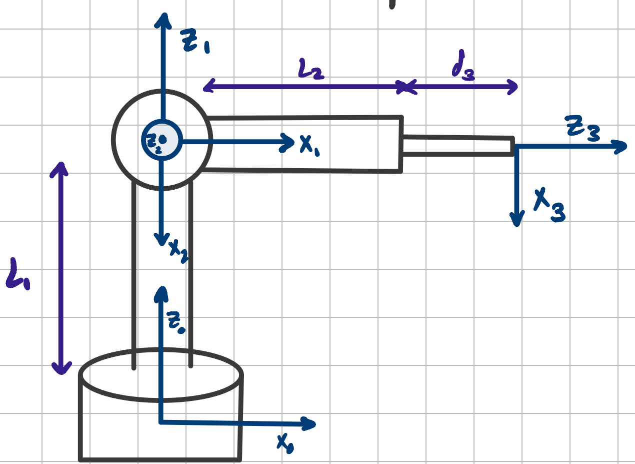
$$^4v_4 = {}^4_3R\left({}^3v_3 + {}^3\omega_3 \times {}^3P_4\right) = I\left({}^3v_3 + \begin{pmatrix} -\dot\theta_2 \sin(\theta_3) \\ -\dot\theta_2 \cos(\theta_3) \\ \dot\theta_3 \end{pmatrix} \times \begin{pmatrix} L_4 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \dot d_1 s_{\theta_2} c_{\theta_3} - L_3 \dot\theta_2 c_{\theta_3} \\ \dot d_1 s_{\theta_2} s_{\theta_3} - L_3 \dot\theta_2 s_{\theta_3} - \theta_2 L_4 \\ \dot d_1 c_{\theta_2} - L_4 \dot\theta_2 c_{\theta_3} \end{pmatrix}$$

$$^0v_4 = {}^0_4R \cdot {}^4v_4$$

6. Compute the linear and rotational velocity for the manipulators in problems 5 and 6 of the inverse kinematics homework.



In the previous homework we calculated the following:



joint i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	L_1	θ_1
2	90	0	0	$-90+\theta_2$
3	-90	0	L_2+d_3	0

we used python to get the matrices multiplication:

$$ {}^{0}_{1}T = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{0}T = \left({}^{0}_{1}T\right)^{-1} = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & -L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\cos(\theta_2) & \sin(\theta_2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{1}T = \left({}^{1}_{2}T\right)^{-1} = \begin{bmatrix} \sin(\theta_2) & 0 & -\cos(\theta_2) & 0 \\ \cos(\theta_2) & 0 & \sin(\theta_2) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_2 + d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{2}T = \left({}^{2}_{3}T\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -L_2 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{0}T = {}^{1}_{0}T\,{}^{2}_{1}T = \begin{bmatrix} \sin(\theta_2)\cos(\theta_1) & \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_2) & L_1\cos(\theta_2) \\ \cos(\theta_1)\cos(\theta_2) & \sin(\theta_1)\cos(\theta_2) & \sin(\theta_2) & -L_1\sin(\theta_2) \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{0}T = {}^{2}_{0}T\,{}^{3}_{2}T = \begin{bmatrix} \sin(\theta_2)\cos(\theta_1) & \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_2) & L_1\cos(\theta_2) \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ \cos(\theta_1)\cos(\theta_2) & \sin(\theta_1)\cos(\theta_2) & \sin(\theta_2) & -L_1\sin(\theta_2) - L_2 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = \begin{bmatrix} \sin(\theta_2)\cos(\theta_1) & -\sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) & (L_2 + d_3)\cos(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1) & \sin(\theta_1)\cos(\theta_2) & (L_2 + d_3)\sin(\theta_1)\cos(\theta_2) \\ -\cos(\theta_2) & 0 & \sin(\theta_2) & L_1 + (L_2 + d_3)\sin(\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

from the tirgul:

Revolute

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\hat{Z}_{i+1} $$

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

Prismatic

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

rotational velocities:

$${}^0\omega_0 = 0$$

$${}^1\omega_1 = \underbrace{\cancel{{}^1_0R \cdot {}^0\omega_0}}_{0} + \dot{\theta}_1 \cdot \hat{Z}_1 = \dot{\theta}_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix}$$

$${}^2\omega_2 = {}^2_1R \cdot {}^1\omega_1 + \dot{\theta}_2 \cdot \hat{Z}_2 = \begin{bmatrix} \sin(\theta_2) & 0 & -\cos(\theta_2) \\ \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -\dot{\theta}_1 \cos(\theta_2) \\ \dot{\theta}_1 \sin(\theta_2) \\ \dot{\theta}_2 \end{pmatrix}$$

$${}^3\omega_3 = {}^3_2R \cdot {}^2\omega_2 + \underbrace{\cancel{\dot{\theta}_3 \cdot \hat{Z}_2}}_{0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} -\dot{\theta}_1 \cos(\theta_2) \\ \dot{\theta}_1 \sin(\theta_2) \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -\dot{\theta}_1 \cos(\theta_2) \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \sin(\theta_2) \end{pmatrix}$$

$${}^0\omega_3 = {}^0_3R \cdot {}^3\omega_3 = \begin{bmatrix} \sin(\theta_2)\cos(\theta_1) & -\sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1) & \sin(\theta_1)\cos(\theta_2) \\ -\cos(\theta_2) & 0 & \sin(\theta_2) \end{bmatrix} \begin{pmatrix} -\dot{\theta}_1 \cos(\theta_2) \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \sin(\theta_2) \end{pmatrix} = \begin{pmatrix} -\dot{\theta}_2 s_{\theta_1} \\ -\dot{\theta}_2 c_{\theta_1} \\ \dot{\theta}_1 \end{pmatrix}$$

linear velocities:

$${}^0v_0 = 0$$

$${}^0v_1 = {}^0v_0 + {}^0\omega_0 \times {}^0P_1 = 0$$

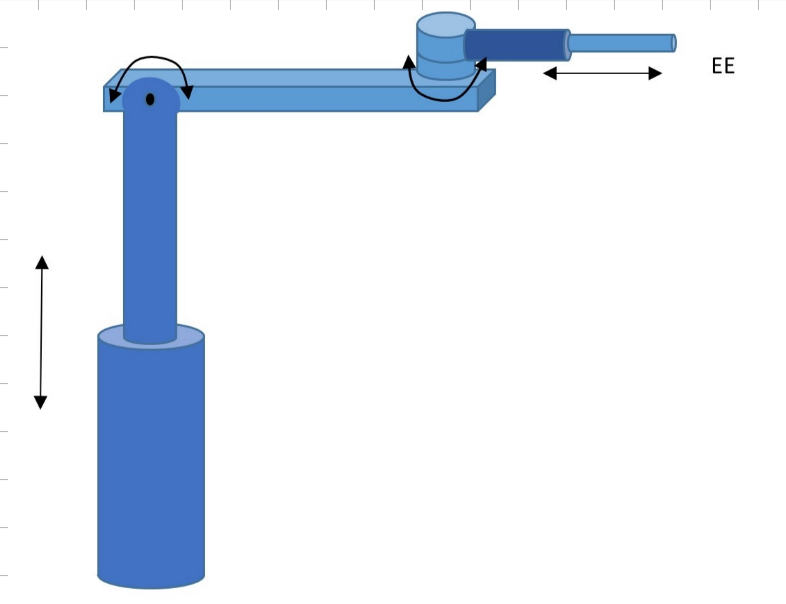
$${}^1v_1 = {}^1_0R(\underbrace{{}^0v_0}_{0} + \underbrace{{}^0\omega_0}_{0} \times {}^0P_1) + \underbrace{\dot{d}_1}_{0} \hat{Z}_1 = 0$$

$${}^1v_2 = {}^1v_1 + {}^1\omega_1 \times {}^1P_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

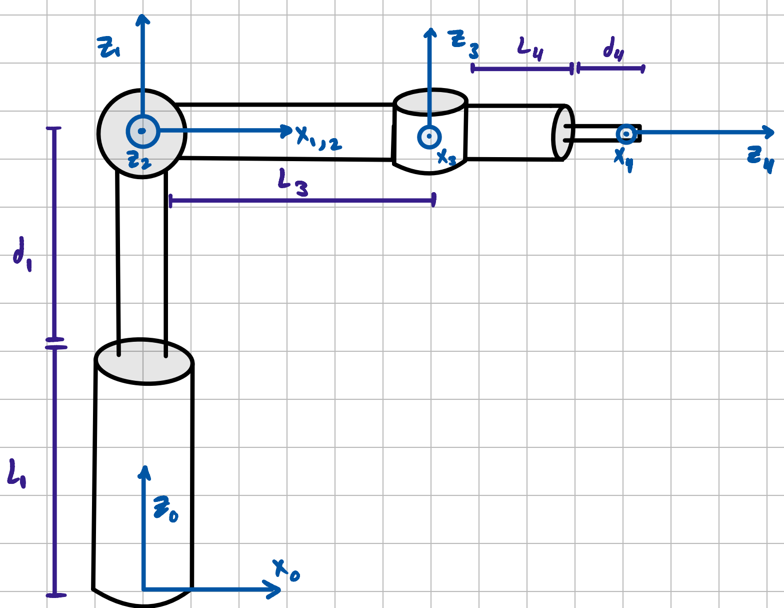
$${}^2v_2 = {}^2_1R(\underbrace{{}^1v_1}_{0} + {}^1\omega_1 \times \underbrace{{}^1P_2}_{0}) + \underbrace{\dot{d}_2}_{0} \hat{Z}_2 = 0$$

$${}^3v_3 = {}^3_2R(\underbrace{{}^2v_2}_{0} + {}^2\omega_2 \times {}^2P_3) + \dot{d}_3 \hat{Z}_3 = \begin{pmatrix} -\dot{\theta}_1 \cos(\theta_2) \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \sin(\theta_2) \end{pmatrix} \times \begin{pmatrix} 0 \\ L_2 + d_3 \\ 0 \end{pmatrix} + \dot{d}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dot{\theta}_1 s_{\theta_2}(L_2 + d_3) \\ 0 \\ \dot{d}_3 - \dot{\theta}_1 c_{\theta_2}(L_2 + d_3) \end{pmatrix}$$

$${}^0v_3 = {}^0_3R \cdot {}^3v_3 = \begin{bmatrix} \sin(\theta_2)\cos(\theta_1) & -\sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1) & \sin(\theta_1)\cos(\theta_2) \\ -\cos(\theta_2) & 0 & \sin(\theta_2) \end{bmatrix} \begin{pmatrix} -\dot{\theta}_1 s_{\theta_2}(L_2 + d_3) \\ 0 \\ \dot{d}_3 - \dot{\theta}_1 c_{\theta_2}(L_2 + d_3) \end{pmatrix} = \text{the rest is math}$$



from the previous homework:



joint	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	L_1+d_1	0
2	90	0	0	θ_2
3	-90	L_3	0	$-90+\theta_3$
4	-90	0	L_4+d_4	0

$${}^0_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_0T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -L_1-d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_1T = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} \sin(\theta_3) & \cos(\theta_3) & 0 & L_3 \\ 0 & 0 & 1 & 0 \\ \cos(\theta_3) & -\sin(\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_2T = \begin{bmatrix} \sin(\theta_3) & 0 & \cos(\theta_3) & -L_3\sin(\theta_3) \\ \cos(\theta_3) & 0 & -\sin(\theta_3) & -L_3\cos(\theta_3) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^3_4T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_4 + d_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^4_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -L_4 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_2T = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & L_1 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^2_0T = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) & (-L_1 - d_1)\sin(\theta_2) \\ -\sin(\theta_2) & 0 & \cos(\theta_2) & (-L_1 - d_1)\cos(\theta_2) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_3T = \begin{bmatrix} \sin(\theta_3)\cos(\theta_2) & \cos(\theta_2)\cos(\theta_3) & -\sin(\theta_2) & L_3\cos(\theta_2) \\ -\cos(\theta_3) & \sin(\theta_3) & 0 & 0 \\ \sin(\theta_2)\sin(\theta_3) & \sin(\theta_2)\cos(\theta_3) & \cos(\theta_2) & L_1 + L_3\sin(\theta_2) + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_0T = \begin{bmatrix} \sin(\theta_3)\cos(\theta_2) & -\cos(\theta_3) & \sin(\theta_2)\sin(\theta_3) & -(L_1\sin(\theta_2) + L_3 + d_1\sin(\theta_2))\sin(\theta_3) \\ \cos(\theta_2)\cos(\theta_3) & \sin(\theta_3) & \sin(\theta_2)\cos(\theta_3) & -(L_1\sin(\theta_2) + L_3 + d_1\sin(\theta_2))\cos(\theta_3) \\ -\sin(\theta_2) & 0 & \cos(\theta_2) & (-L_1 - d_1)\cos(\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_4T = \begin{bmatrix} \sin(\theta_3)\cos(\theta_2) & \sin(\theta_2) & \cos(\theta_2)\cos(\theta_3) & (L_3 + (L_4 + d_4)\cos(\theta_3))\cos(\theta_2) \\ -\cos(\theta_3) & 0 & \sin(\theta_3) & (L_4 + d_4)\sin(\theta_3) \\ \sin(\theta_2)\sin(\theta_3) & -\cos(\theta_2) & \sin(\theta_2)\cos(\theta_3) & L_1 + L_3\sin(\theta_2) + d_1 + (L_4 + d_4)\sin(\theta_2)\cos(\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^4_0T = \begin{bmatrix} \sin(\theta_3)\cos(\theta_2) & -\cos(\theta_3) & \sin(\theta_2)\sin(\theta_3) & -(L_1\sin(\theta_2) + L_3 + d_1\sin(\theta_2))\sin(\theta_3) \\ \sin(\theta_2) & 0 & -\cos(\theta_2) & (L_1 + d_1)\cos(\theta_2) \\ \cos(\theta_2)\cos(\theta_3) & \sin(\theta_3) & \sin(\theta_2)\cos(\theta_3) & -L_1\sin(\theta_2)\cos(\theta_3) - L_3\cos(\theta_3) - L_4 - d_1\sin(\theta_2)\cos(\theta_3) - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

from the tirgul:

Revolute

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\hat{Z}_{i+1} $$

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

Prismatic

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

rotational velocities:

$$^0w_0 = 0$$

$$^1w_1 = 0$$

$$^2w_2 = {}^2_1R\,{}^1w_1 + \dot\theta_2\,\hat z_2 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix}$$

$$^3w_3 = {}^3_2R\,{}^2w_2 + \dot\theta_3\,\hat z_3 = \begin{bmatrix} \sin(\theta_3) & 0 & \cos(\theta_3) \\ \cos(\theta_3) & 0 & -\sin(\theta_3) \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} + \dot\theta_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_3\dot\theta_2 \\ -s_3\dot\theta_2 \\ \dot\theta_3 \end{pmatrix}$$

$$^4w_4 = {}^4_3R\,{}^3w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} c_3\dot\theta_2 \\ -s_3\dot\theta_2 \\ \dot\theta_3 \end{pmatrix} = \begin{pmatrix} c_3\dot\theta_2 \\ -\dot\theta_3 \\ -s_3\dot\theta_2 \end{pmatrix}$$

$$^0w_4 = {}^0_4R\,{}^4w_4$$

linear velocities:

$$^0v_0 = 0$$

$$^1v_1 = {}^1_0R\,(\underbrace{^0v_0}_{0} + \underbrace{^0w_0}_{0} \times {}^0P_1) + \dot d_1\,\hat z_1 = \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix}$$

$$^1v_2 = {}^1v_1 + {}^1w_1 \times {}^1P_2 = \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix}$$

$$^2v_2 = {}^2_1R\,({}^1v_1 + \underbrace{^1w_1}_{0} \times {}^1P_2) + \underbrace{\dot d_2}_{0} \cdot \hat z_2 = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix} = \begin{pmatrix} s_{\theta_2}\dot d_1 \\ c_{\theta_2}\dot d_1 \\ 0 \end{pmatrix}$$

$$^2v_3 = {}^2v_2 + {}^2w_2 \times {}^2P_3 = \begin{pmatrix} s_{\theta_2}\dot d_1 \\ c_{\theta_2}\dot d_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s_{\theta_2}\dot d_1 \\ c_{\theta_2}\dot d_1 - \dot\theta_2 L_3 \\ 0 \end{pmatrix}$$

$$^3v_3 = {}^3_2R\,({}^2v_2 + {}^2w_2 \times {}^2P_3) + \underbrace{\dot d_3}_{0} \cdot \hat z_3 =$$

$$\begin{bmatrix} \sin(\theta_3) & 0 & \cos(\theta_3) \\ \cos(\theta_3) & 0 & -\sin(\theta_3) \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{pmatrix} s_{\theta_2}\dot d_1 \\ c_{\theta_2}\dot d_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} s_{\theta_3} s_{\theta_2}\dot d_1 \\ c_{\theta_3} s_{\theta_2}\dot d_1 \\ c_{\theta_2}\dot d_1 + \dot\theta_2 L_3 \end{pmatrix}$$

$${}^3V_4 = {}^3V_3 + {}^3\omega_3 \times {}^3P_4 = \begin{pmatrix} s_{\theta_3} s_{\theta_2} \dot{d}_1 \\ c_{\theta_3} s_{\theta_2} \dot{d}_1 \\ c_{\theta_2} \dot{d}_1 + \dot{\theta}_2 L_3 \end{pmatrix} + \begin{pmatrix} c_3 \dot{\theta}_2 \\ -s_3 \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ L_4 + d_4 \\ 0 \end{pmatrix} = \begin{pmatrix} s_{\theta_3} s_{\theta_2} \dot{d}_1 - \dot{\theta}_3 (L_4 + d_4) \\ c_{\theta_3} s_{\theta_2} \dot{d}_1 \\ c_{\theta_2} \dot{d}_1 + \dot{\theta}_2 L_3 + c_{\theta_3} \dot{\theta}_2 (L_4 + d_4) \end{pmatrix}$$

$${}^4V_4 = {}^4_3R \left({}^3V_3 + {}^3\omega_3 \times {}^3P_4 \right) + \dot{d}_4 \hat{Z}_4 =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{pmatrix} s_{\theta_3} s_{\theta_2} \dot{d}_1 \\ c_{\theta_3} s_{\theta_2} \dot{d}_1 \\ c_{\theta_2} \dot{d}_1 + \dot{\theta}_2 L_3 \end{pmatrix} + \begin{pmatrix} -\dot{\theta}_3 (L_4 + d_4) \\ 0 \\ c_{\theta_3} \dot{\theta}_2 (L_4 + d_4) \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_4 \end{pmatrix} = \begin{pmatrix} s_{\theta_3} s_{\theta_2} \dot{d}_1 - \dot{\theta}_3 (L_4 + d_4) \\ -\left(c_{\theta_2} \dot{d}_1 + \dot{\theta}_2 L_3 + c_{\theta_3} \dot{\theta}_2 (L_4 + d_4) \right) \\ c_{\theta_3} s_{\theta_2} \dot{d}_1 + \dot{d}_4 \end{pmatrix}$$

$${}^0V_4 = {}^0_4R \begin{pmatrix} s_{\theta_3} s_{\theta_2} \dot{d}_1 - \dot{\theta}_3 (L_4 + d_4) \\ -\left(c_{\theta_2} \dot{d}_1 + \dot{\theta}_2 L_3 + c_{\theta_3} \dot{\theta}_2 (L_4 + d_4) \right) \\ c_{\theta_3} s_{\theta_2} \dot{d}_1 + \dot{d}_4 \end{pmatrix} =$$

$$\begin{bmatrix} \sin(\theta_3)\cos(\theta_2) & -\cos(\theta_3) & \sin(\theta_2)\sin(\theta_3) \\ \sin(\theta_2) & 0 & -\cos(\theta_2) \\ \cos(\theta_2)\cos(\theta_3) & \sin(\theta_3) & \sin(\theta_2)\cos(\theta_3) \end{bmatrix} \cdot \begin{pmatrix} s_{\theta_3} s_{\theta_2} \dot{d}_1 - \dot{\theta}_3 (L_4 + d_4) \\ -\left(c_{\theta_2} \dot{d}_1 + \dot{\theta}_2 L_3 + c_{\theta_3} \dot{\theta}_2 (L_4 + d_4) \right) \\ c_{\theta_3} s_{\theta_2} \dot{d}_1 + \dot{d}_4 \end{pmatrix} =$$ the rest is just algebra