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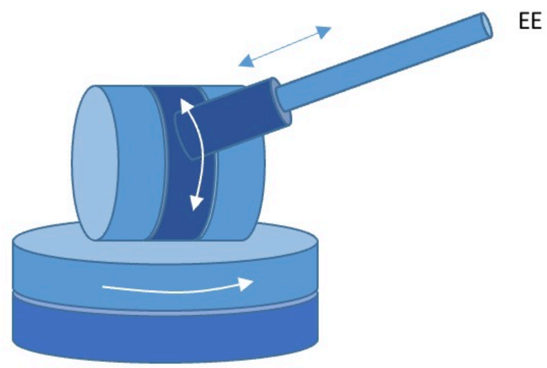
Homework #3

Due 24/12/2024

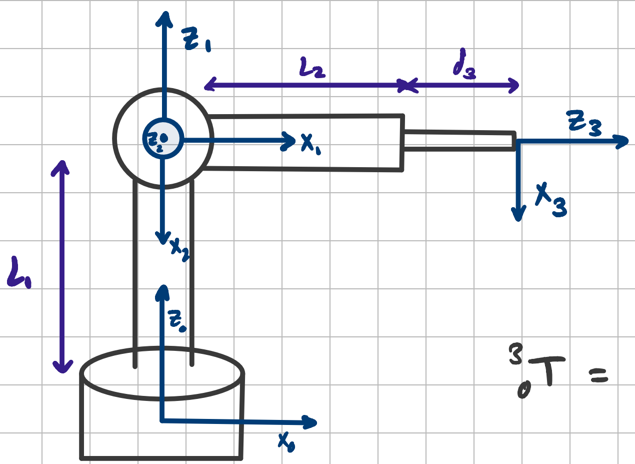
This homework consists of two parts, where the first (problems 1 to 4) is multiple-choice.

1. Given a robot with six degrees of freedom, then:
   - ☐ A close-form solution to the inverse kinematics problem can always be found using geometric methods.
   - ☐ A close-form solution to the inverse kinematics problem can always be found using analytic methods.
   - ☐ A close-form solution to the inverse kinematics problem can always be found using a mixture of geometric and analytic methods.
   - ☑ A close-form solution to the inverse kinematics problem cannot always be found.
2. If a manipulator has more than 6 degrees of freedom, then:
   - ☐ The workspace is a subspace of the same dimension as the number of DOF.
   - ☐ The reachable workspace is equal to the dexterous workspace.
   - ☑ The dexterous workspace may be empty.
   - ☐ None of the above is true.
3. For a serial manipulator the inverse kinematics problem is often harder than the direct one because:
   - ☐ The resulting Denhavit-Harterberg's matrices may be hard to invert.
   - ☐ They combine prismatic and revolution joints.
   - ☐ The dexterous and reachable workspaces may be different.
   - ☑ The problem is usually highly nonlinear.
4. When an inverse kinematics problem becomes singular, then
   - ☐ The inverse kinematics problem cannot be solved.
   - ☐ The number of possible solutions becomes infinite.
   - ☑ The number of possible solutions is different than in the close neighborhood.
   - ☐ A single solution exists.

5. Given the location of the end effector of the next robot, compute a solution to the inverse kinematics problem:



In the previous homework we calculated the following:



$$^3_0T = \begin{pmatrix} c_{\theta_1}s_{\theta_1} & -s_{\theta_1} & c_{\theta_1}c_{\theta_2} & c_{\theta_1}c_{\theta_2}(L_2+d_3) \\ s_{\theta_1}s_{\theta_1} & c_{\theta_1} & s_{\theta_1}c_{\theta_2} & s_{\theta_1}c_{\theta_2}(L_2+d_3) \\ -c_{\theta_2} & 0 & s_{\theta_2} & s_{\theta_2}(L_2+d_3)+L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

given the location of the end-effector (x_e, y_e, z_e):

$$x_e = c_{\theta_1}c_{\theta_2}(L_2+d_3)$$

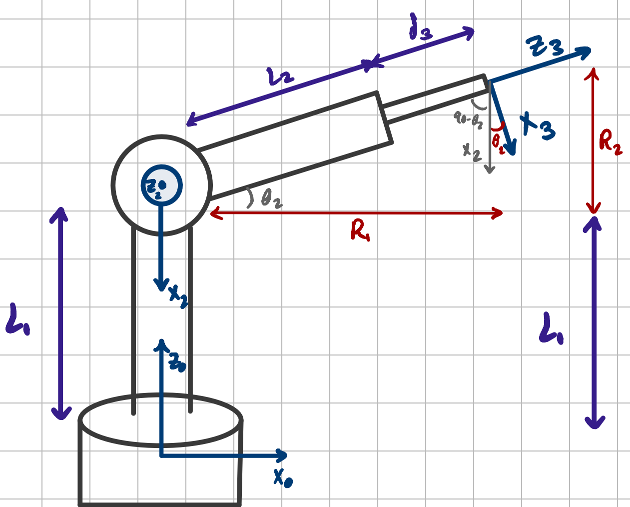
$$y_e = s_{\theta_1}c_{\theta_2}(L_2+d_3)$$

therefore: $$\frac{y_e}{x_e} = \tan(\theta_1)$$

$$\rightarrow \theta_1 = \operatorname{atan2}(y_e, x_e)$$

where atan2 is the function we saw in the lecture

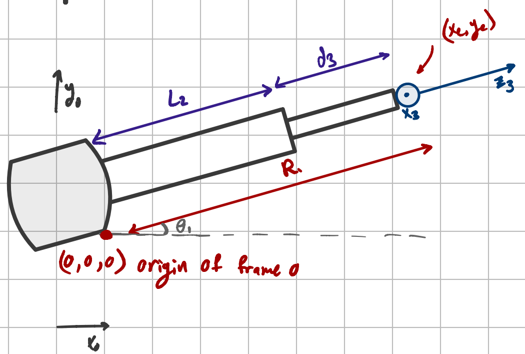
side view:



① $z_e = L_1 + R_2 \quad \rightarrow R_2 = z_e - L_1$

② also : $\cos\theta_2 = \dfrac{R_1}{L_2 + d_3}$

top view:



③ $R_1^2 = x_e^2 + y_e^2$

we get the following equations:

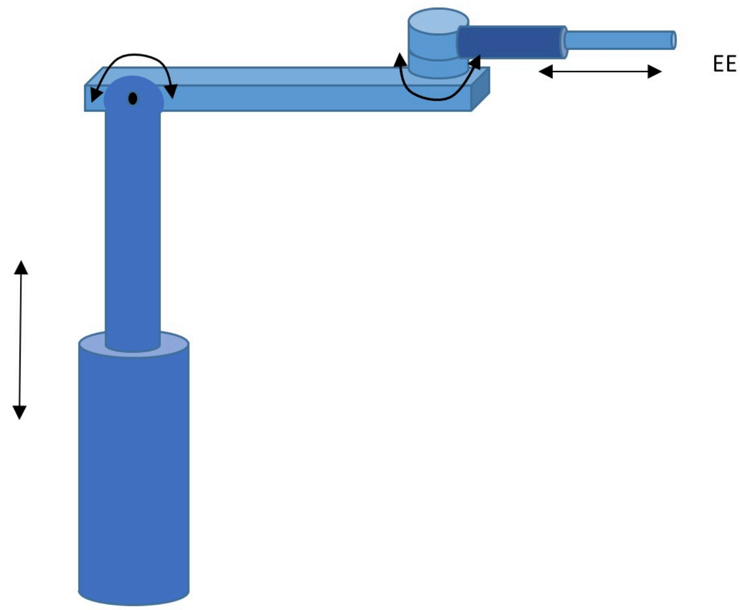
$$\frac{R_2}{R_1} = \tan(\theta_2)$$ from the side view

$$\rightarrow \boxed{\theta_2 = \operatorname{atan2}(R_2, R_1) = \operatorname{atan2}\left(z_e - L_1, +\sqrt{x_e^2 + y_e^2}\right)}$$

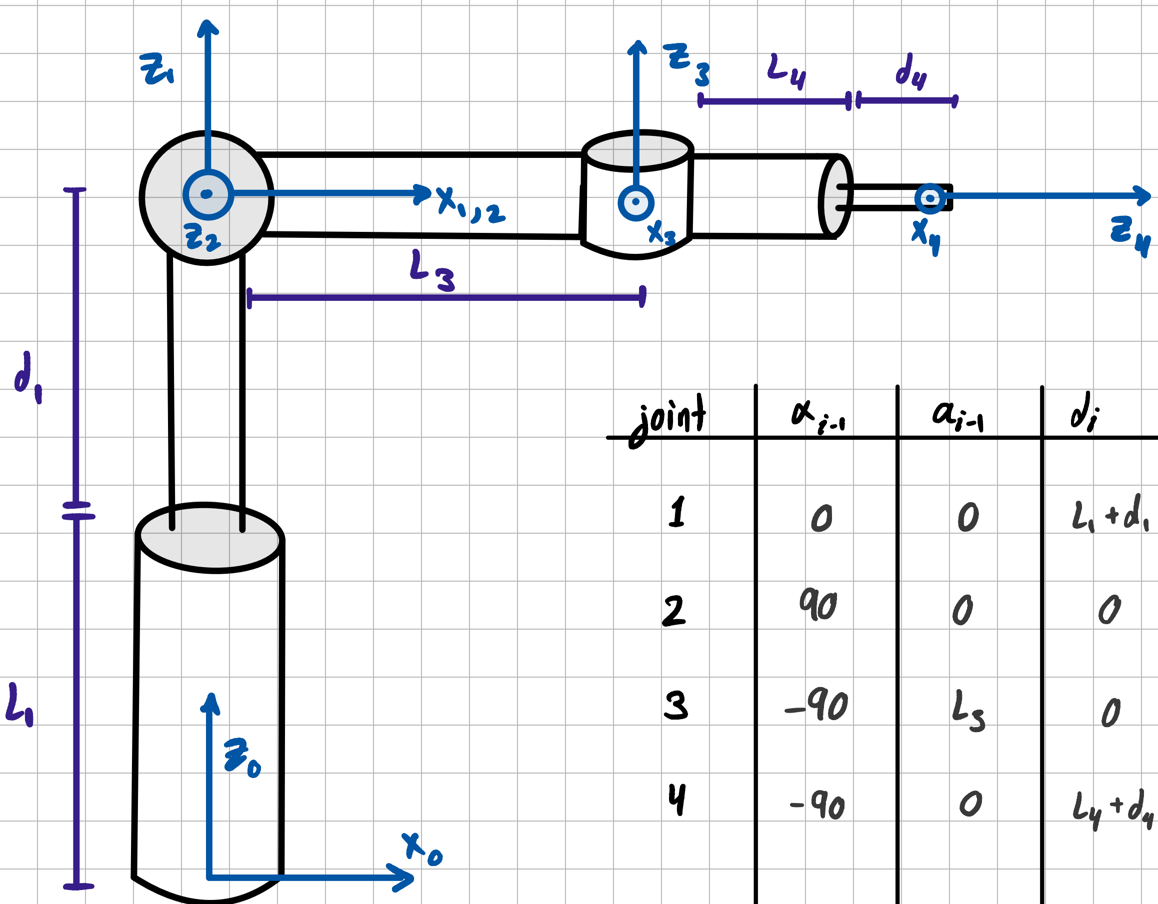
$$\cos(\theta_2) = \frac{R_1}{L_2 + d_3}$$ eq ②

$$\rightarrow \boxed{d_3 = \frac{R_1}{\cos(\theta_2)} - L_2 = \frac{+\sqrt{x_e^2 + y_e^2}}{\cos\left(\operatorname{atan2}\left(z_e - L_1, +\sqrt{x_e^2 + y_e^2}\right)\right)} - L_2}$$

6. Given the position and orientation of the end effector, compute a solution to the inverse kinematics problem of the following robot:



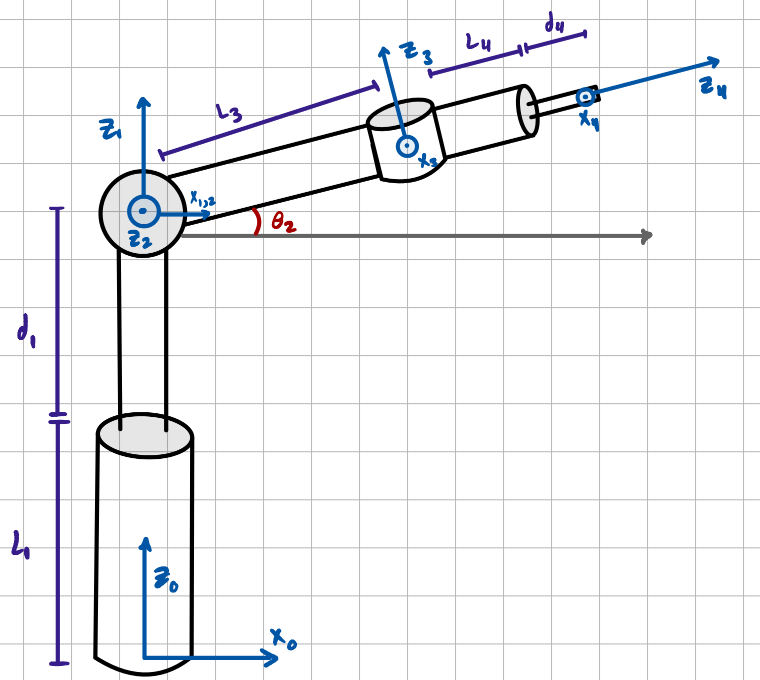
from the previous homework:



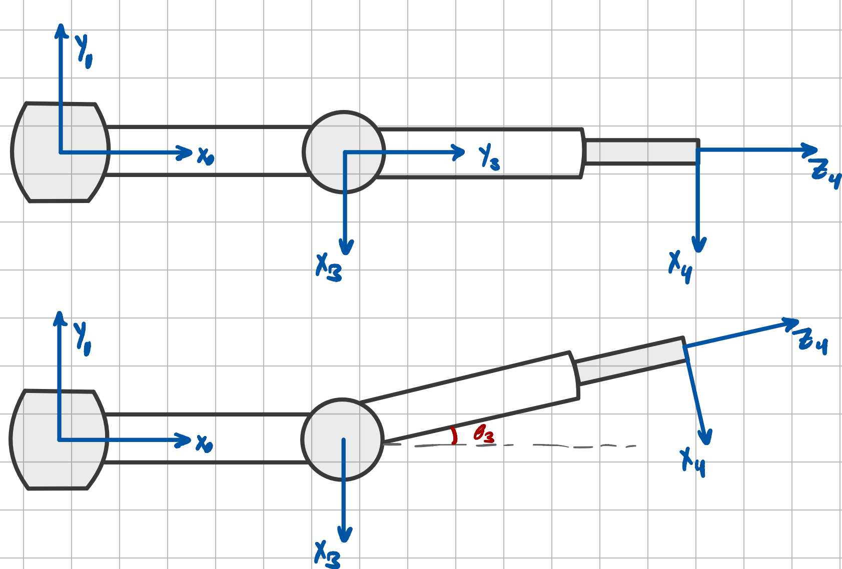
joint	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	$L_1 + d_1$	0
2	90	0	0	θ_2
3	-90	L_3	0	$-90 + \theta_3$
4	-90	0	$L_4 + d_4$	0

$$
{}^{0}_{4}T = \begin{pmatrix} C_{\theta_2} S_{\theta_3} & S_{\theta_2} & C_{\theta_2} C_{\theta_3} & C_{\theta_2}\left(L_3 + C_{\theta_3}(L_4+d_4)\right) \\ -C_{\theta_3} & 0 & S_{\theta_3} & S_{\theta_3}(L_4+d_4) \\ S_{\theta_2} S_{\theta_3} & -C_{\theta_2} & S_{\theta_2} C_{\theta_3} & L_1+d_1+S_{\theta_2}\left(L_3 + C_{\theta_3}(L_4+d_4)\right) \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

side view:



top view:



given both the orientation and the position of the end effector:

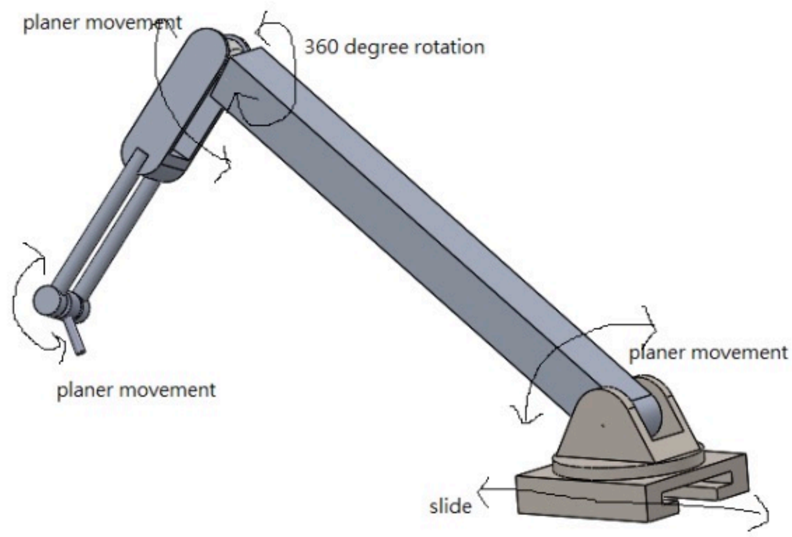
$$\frac{{}^4_0T(1,2)}{-{}^4_0T(3,2)} = \frac{s_{\theta_2}}{--c_{\theta_2}} = \tan(\theta_2) \rightarrow \theta_2 = \operatorname{atan2}\left({}^4_0T(1,2), -{}^4_0T(3,2)\right)$$

$$\frac{{}^4_0T(2,3)}{-{}^4_0T(2,1)} = \frac{s_{\theta_3}}{--c_{\theta_3}} = \tan(\theta_3) \rightarrow \theta_3 = \operatorname{atan2}\left({}^4_0T(2,3), -{}^4_0T(2,1)\right)$$

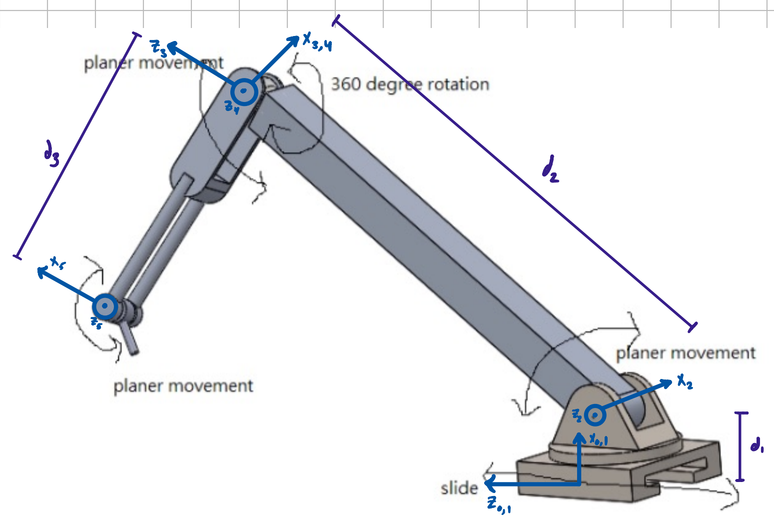
$$y_e = s_{\theta_3}(L_4 + d_4) \rightarrow d_4 = \frac{y_e}{\sin\left(\operatorname{atan2}\left({}^4_0T(2,3), -{}^4_0T(2,1)\right)\right)} - L_4$$

$$z_e = L_1 + d_1 + s_{\theta_2}\left(L_3 + c_{\theta_3}(L_4 + d_4)\right) \rightarrow d_1 = z_e - L_1 + s_{\theta_2}\left(L_3 + c_{\theta_3}(L_4 + d_4)\right)$$

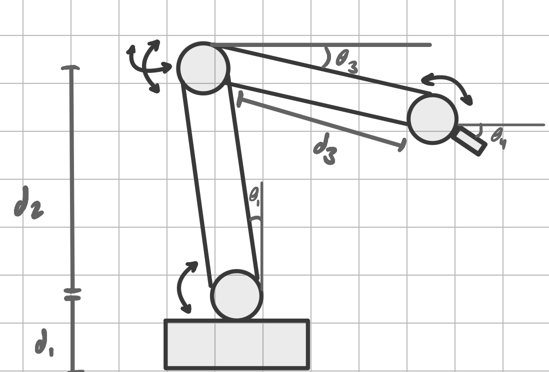
7. Given the position and orientation of the end effector, compute a solution to the inverse kinematics problem of the following robot:



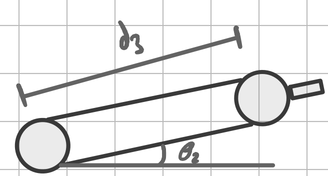
from the previous homework:



side view:



top view:



based on the following DH table:

joint	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	d_1	$-d$	0
2	90	d_1	0	θ_1
3	-90	0	d_2	θ_2
4	90	0	0	θ_3
5	0	d_3	0	$\theta_4 + 90$

we get the following ${}^{5}_{0}T$ matrix: (from matlab)

$a = \theta_1$
$b = \theta_2$
$c = \theta_3$
$d = \theta_4$
$w = d$
$x = d_1$
$y = d_2$
$z = d_3$

```
[
   [sin(a)*sin(c)*sin(d) - cos(c)*cos(d)*sin(a) - cos(a)*cos(b)*cos(c)*sin(d) -
cos(a)*cos(b)*cos(d)*sin(c),
    -sin(c + d)*sin(b),
    cos(a)*cos(c)*cos(d) - cos(a)*sin(c)*sin(d) - cos(b)*cos(c)*sin(a)*sin(d) -
cos(b)*cos(d)*sin(a)*sin(c),
    z*sin(d) - y*cos(c)*cos(d) + y*sin(c)*sin(d) + w*cos(a)*cos(c)*cos(d) + 2*x*cos(c)*cos(d)*sin(a)
- w*cos(a)*sin(c)*sin(d) - 2*x*sin(a)*sin(c)*sin(d) - w*cos(b)*cos(c)*sin(a)*sin(d) -
w*cos(b)*cos(d)*sin(a)*sin(c) + 2*x*cos(a)*cos(b)*cos(c)*sin(d) +
2*x*cos(a)*cos(b)*cos(d)*sin(c)],

   [cos(c)*sin(a)*sin(d) + cos(d)*sin(a)*sin(c) - cos(a)*cos(b)*cos(c)*cos(d) +
cos(a)*cos(b)*sin(c)*sin(d),
    -cos(c + d)*sin(b),
    cos(b)*sin(a)*sin(c)*sin(d) - cos(a)*cos(d)*sin(c) - cos(b)*cos(c)*cos(d)*sin(a) -
cos(a)*cos(c)*sin(d),
    z*cos(d) + y*cos(c)*sin(d) + y*cos(d)*sin(c) - w*cos(a)*cos(c)*sin(d) - w*cos(a)*cos(d)*sin(c) -
2*x*cos(c)*sin(a)*sin(d) - 2*x*cos(d)*sin(a)*sin(c) - 2*x*cos(a)*cos(b)*sin(c)*sin(d) +
w*cos(b)*sin(a)*sin(c)*sin(d) + 2*x*cos(a)*cos(b)*cos(c)*cos(d) -
w*cos(b)*cos(c)*cos(d)*sin(a)],

   [cos(a)*sin(b),
    -cos(b),
    sin(a)*sin(b),
    -sin(b)*(2*x*cos(a) - w*sin(a))],

   [0, 0, 0, 1]
]
```

[disgusting, we know, sorry!]

from the matrix we get:

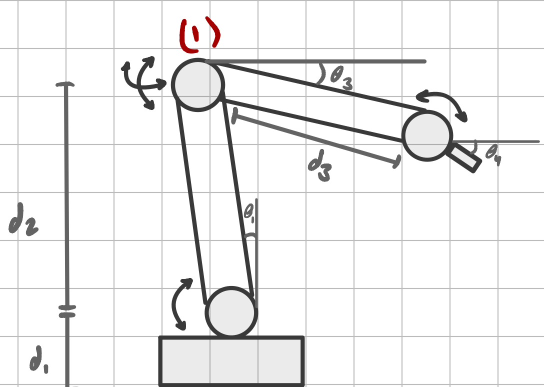
$$\frac{T(4,3)}{T(4,1)} = \frac{\sin(\theta_1)\sin(\theta_2)}{\cos(\theta_1)\sin\theta_2} = \tan(\theta_1) \rightarrow \theta_1 = \operatorname{atan2}(T(4,3), T(4,1))$$

$$T(4,2) = -\cos(\theta_2) \rightarrow \theta_2 = \operatorname{acos}(-T(4,2))$$

$$z_e = -\sin(\theta_2)\left(2 d_1 \cos(\theta) - d \sin\theta_1\right) \rightarrow d = \frac{\cancel{\frac{z_e}{-\sin\theta_2}} - 2 d_1 \cos\theta_1}{\sin\theta_1}$$

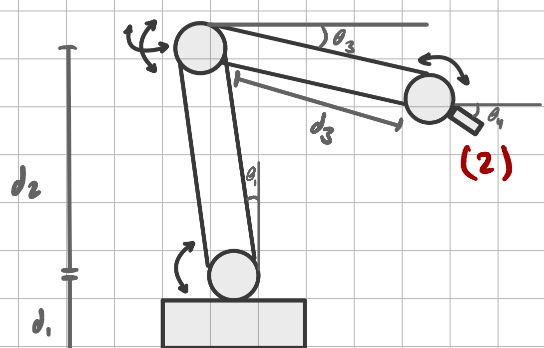
$$\frac{T(1,2)}{T(2,2)} = \frac{-\sin(\theta_3+\theta_4)\sin\theta_2}{-\cos(\theta_3+\theta_4)\sin\theta_2} = \tan(\theta_3+\theta_4) \rightarrow \theta_3+\theta_4 = \operatorname{atan2}(T(1,2), T(2,2))$$

from the side view we get:



$$\rightarrow \quad (1)_x = (d_1+d_2)\sin(\theta_1)$$

$$(1)_y = (d_1+d_2)\cos(\theta_1)$$

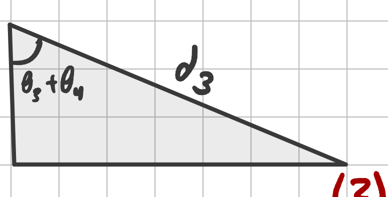


$$\rightarrow \quad (2)_x = x_e - (1)_x$$

$$(2)_y = (1)_y - y_e$$

on the other hand:

assuming the triangle is right



$$\rightarrow \sin(\theta_3+\theta_4) = \frac{(2)_x}{d_3} = \frac{x_e - (d_1+d_2)\sin(\theta_1)}{d_3}$$

$\rightarrow$ $$\theta_3 + \theta_4 = \operatorname{atan2}(T(1,2), T(2,2))$$

and

$$\theta_3 + \theta_4 = \frac{x_e - (d_1 + d_2)\sin(\theta_1)}{d_3}$$

we got 2 equations with 2 variables $\theta_{3,4}$, we can extract both using the matrix.

the rest is just math we're hoping is unnecessary.

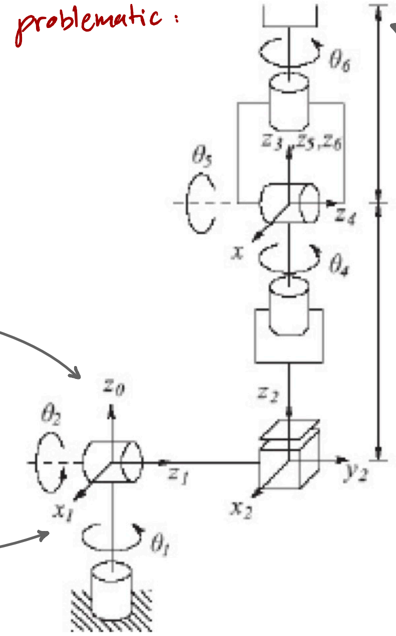
8. Compute a solution to the inverse kinematics problem of the following robot:

the provided sketch is problematic:

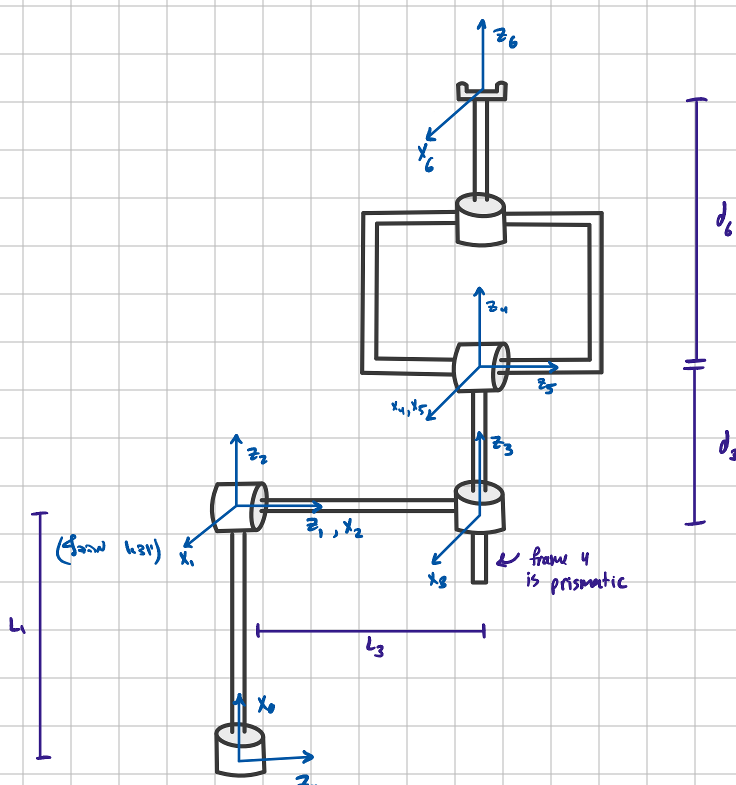
frame 6 isn't where the end-effetor is.

z_0 is supposed to be static.

x_1 is supposed to intersect with z_2.



note: since frame 0 in the given image is w.r.t a rotational joint, and per our understanding frame 0 is supposed to be a static frame for which the position of the end factor is calculated based on, we will not be using the provided frames and will define a few frames of our own, sorry!



joint	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	L_1	0	$\theta_1 + 90$
2	90	0	0	$\theta_2 + 90$
3	0	L_3	0	$\theta_3 - 90$
4	0	0	d_3	0
5	-90	0	0	θ_5
6	90	0	d_6	θ_6

using the matrices from the tirgul :

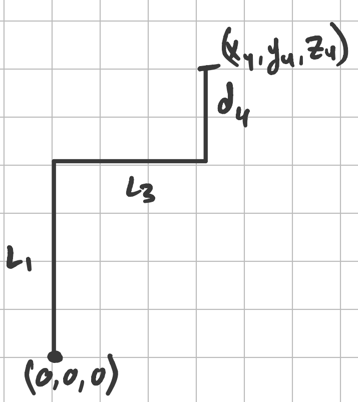
$$\rightarrow \quad {}^{0}_{4}T = \begin{pmatrix} -s_{\theta_1} c_{\theta_2+\theta_3} & s_{\theta_1} s_{\theta_2+\theta_3} & c_{\theta_1} & L_1 + L_3 s_{\theta_1} s_{\theta_2} + c_{\theta_1} d_3 \\ -c_{\theta_1} c_{\theta_2+\theta_3} & c_{\theta_1} s_{\theta_2+\theta_3} & s_{\theta_1} & -L_3 c_{\theta_1} c_{\theta_2} + s_{\theta_1} d_3 \\ s_{\theta_2+\theta_3} & c_{\theta_2+\theta_3} & 0 & L_3 c_{\theta_2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^{6}_{4}T = \begin{pmatrix} c_{\theta_5} c_{\theta_6} & s_{\theta_6} & -s_{\theta_5} c_{\theta_6} & 0 \\ -c_{\theta_5} s_{\theta_6} & c_{\theta_6} & s_{\theta_5} c_{\theta_6} & 0 \\ s_{\theta_5} & 0 & c_{\theta_5} & -d_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^{4}_{6}T = \begin{pmatrix} c_{\theta_5} c_{\theta_6} & -c_{\theta_5} s_{\theta_6} & s_{\theta_5} & d_6 s_{\theta_5} \\ s_{\theta_6} & c_{\theta_6} & 0 & 0 \\ -s_{\theta_5} c_{\theta_6} & s_{\theta_5} c_{\theta_6} & c_{\theta_5} & d_6 c_{\theta_5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^{0}P_{4\text{-org}} = \left({}^{0}P_6 - (0,0,d_6) \right) \cdot {}^{0}_{6}R \; :$$

$$\equiv (x_4, y_4, z_4)$$



$$z_4 = L_3 c_{\theta_2} \rightarrow \theta_2 = a\cos\left(z_4 / L_3 \right)$$

$$X_4 = L_1 + L_3 s_{\theta_1} s_{\theta_2} + c_{\theta_1} d_3 \rightarrow X_4 - L_1 = L_3 s_{\theta_1} s_{\theta_2} + c_{\theta_1} d_3$$

$$Y_4 = -L_3 c_{\theta_1} s_{\theta_2} + s_{\theta_1} d_3$$

we get the following:

$$(X_4 - L_1)^2 = L_3^2 s_{\theta_1}^2 s_{\theta_2}^2 + d_3^2 c_{\theta_1}^2 + 2L_3 s_{\theta_2} s_{\theta_1} d_3 c_{\theta_1}$$

$$(Y_4)^2 = L_3^2 c_{\theta_1}^2 s_{\theta_2}^2 + d_3^2 s_{\theta_1}^2 - 2L_3 c_{\theta_1} s_{\theta_2} s_{\theta_1} d_3$$

$$\rightarrow \quad d_3^2 = (X_4 - L_1)^2 + Y_4^2 - L_3^2 s_{\theta_2}^2$$

$$d_3 = \sqrt{(X_4 - L_1)^2 + Y_4^2 - L_3^2 s_{\theta_2}^2}$$

reminders: $\quad c_{\theta_1}^2 = \dfrac{1}{1 + \tan^2\theta_1}$, $\quad s_{\theta_1}^2 = \dfrac{\tan^2\theta_1}{1 + \tan^2\theta_1}$

$$\rightarrow \quad (d_3^2 - Y_4^2)\tan\theta_1 - 2L_3 d_3 s_{\theta_2} \tan\theta_1 + L_3^2 s_{\theta_2}^2 - Y_4^2 = 0$$

by using the quadratic formula we get:

$$\tan(\theta_1) = \frac{L_3 d_3 s_{\theta_2} \pm Y_4 (X_4 - L_1)}{(d_3^2 - Y_4^2)}$$

$$\Rightarrow \quad \theta_1 = \operatorname{atan2}\left(L_3 d_3 s_{\theta_2} \pm Y_4 (X_4 - L_1) ,\ d_3^2 - Y_4^2 \right)$$

from frame 0 to frame 2 we only have a rotation.

$$ {}^{2}_{0}R = \begin{pmatrix} s_{\theta_1} s_{\theta_2} & -c_{\theta_1} s_{\theta_2} & c_{\theta_2} \\ s_{\theta_1} c_{\theta_2} & -c_{\theta_1} c_{\theta_2} & -s_{\theta_2} \\ c_{\theta_1} & s_{\theta_1} & 0 \end{pmatrix}$$

$${}^{2}_{6}R = {}^{2}_{0}R \; {}^{0}_{6}R = \quad R = {}^{2}_{6}R = \begin{pmatrix} - & - & \sin(t3)\sin(t5) \\ - & - & -\cos(t3)\sin(t5) \\ -s(t5)c(t6) & s(t5)s(t6) & c(t5) \end{pmatrix}$$

$$\frac{R(1,3)}{R(2,3)} = -\tan(\theta_3) \rightarrow \theta_3 = \operatorname{atan2}(-R(1,3),\ R(2,3))$$

$$\frac{R(3,2)}{R(3,1)} = -\tan(\theta_6) \rightarrow \theta_6 = \operatorname{atan2}(-R(3,2),\ R(3,1))$$

$$R(3,3) = c_{\theta_5} \rightarrow \theta_5 = \operatorname{acos}(R(3,3))$$