Analysis2

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7 [illegible]

7.1 $\mathbb{R}$ [illegible]

Definition 7.1.1 $\mathbb{R}^n = \left\{x = (x^1, ...x^n) \,\middle|\, x^i \in \mathbb{R}, i = 1, ..., n\right\}$, [illegible] x [illegible] n [illegible],[illegible] $\mathbb{R}^n$ [illegible],[illegible],$\mathbb{R}^n$ [illegible].

7.1.1 [illegible]

Definition 7.1.1.2 [illegible] $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}(x, y) \mapsto d(x, y)$,[illegible] $d(x, y) = \left[\sum_{i=1}^{n} (x^i - y^i)^2\right]^{\frac{1}{2}}$.

[illegible] d [illegible]:

1. [illegible],$\forall x, y \in \mathbb{R}^n, d(x, y) \geq 0, " = " \Longleftrightarrow x = y$.
2. [illegible], $d(x, y) = d(y, x)$.
3. [illegible], $\forall x, y, z \in \mathbb{R}^n, d(x, y) \leq d(x, z) + d(z, y)$

[illegible] d [illegible],[illegible] d [illegible] $\mathbb{R}^n$ [illegible].

Remark

Definition 7.1.1.3

$$p \geq 1, d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+, (x, y) \mapsto d_p(x, y) = \left(\sum_{i=1}^{n} |x^i - y^i|^p\right)^{\frac{1}{p}},$$

$$d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+, (x, y) \mapsto d_\infty(x, y) = \max_{1 \leq i \leq n} |x^i - y^i|$$

[illegible],d_p, d_∞ [illegible] $\mathbb{R}^n$ [illegible].

Proposition 7.1.1.4 *(Minkowski [illegible])* $d_\infty(x, y) \leq d(x, y) \leq n d_\infty(x, y)$
$C_{p,1} d(x, y) \leq d_p(x, y) \leq C_{p,2} d(x, y)$,[illegible] $C_{p,1}, C_{p,2}$ [illegible] p [illegible].

7.1.2 [illegible],[illegible],[illegible]

Definition 7.1.2.5 [illegible] $a \in \mathbb{R}^n, \delta > 0, B(a; \delta) = \{x \in \mathbb{R}^n \mid d(a, x) < \delta\}$ [illegible] a [illegible],δ [illegible] / δ [illegible].

Definition 7.1.2.6 [illegible] $U \subset \mathbb{R}^n, \forall a \in U, \exists \delta > 0, \text{s,t. } B(a; \delta) \subset U$,[illegible] U [illegible].

Example 7.1.2.7 $B(a; r)$ [illegible](r > 0).

Proposition 7.1.2.8

1. $\mathbb{R}^n, \emptyset$ [illegible].
2. [illegible].
3. [illegible].

Definition 7.1.2.9 $\mathbb{R}^n$ [illegible],[illegible] $\mathbb{R}^n$ [illegible].

$\mathbb{R}^n$ [illegible] d [illegible].

□□□□,□□□□□.

Definition 7.1.2.10 *(□□□□)* □ X □□□, τ □ X □□□□,□□:

1. $\varphi, X \in \tau$
2. $\forall \tau_\alpha, \alpha \in \Lambda, \bigcup_{\{\alpha \in \Lambda\}} \tau_\alpha \in \tau$
3. □ $\tau_1, ..., \tau_m \in \tau$ □ $\bigcap_{\{i=1\}}^{m} \tau_i \in \tau$

□□□ (X, τ) □□□□□.

Definition 7.1.2.11 □ $A \subset \mathbb{R}^n$,□ $A^c = \mathbb{R}^n \setminus A$ □□□,□□□ A □□□.

Example 7.1.2.12

- $\forall x, y \in \mathbb{R}^n, A = \{x, y\}$ □□
- $\overline{B}(a; r) = \{x \in \mathbb{R}^n \mid d(a; x) \leq r\}$ □□
- $\mathcal{S}^{n-1}(a, r) = \{x \in \mathbb{R}^n \mid d(a; x) = r\}$ □□□

□ $\mathcal{S}^{n-1} = \mathcal{S}^{n-1}(0, 1)$

□ De Morgan □□□□:

Proposition 7.1.2.13

1. $\mathbb{R}^n, \emptyset$ □□□
2. □□□□□□□□□□□□
3. □□□□□□□□□□□□

7.1.3 □□,□□,□□□,□□

Definition 7.1.3.14 *(□□,□□,□□□)*

1. □ $x \in \mathbb{R}^n$,□□□□ x □□□ U □□ x □□□,$\mathring{U} = U \setminus \{x\}$ □□ x □□□□□
2. □ $D \subset \mathbb{R}^n$,□ $x \in D, \exists x$ □□□ U, s.t. $U \subset D$, □ x □ D □□□.□□□,□ x □ D^c □□□,□ x □ D □□□.
3. □ $D \subset \mathbb{R}^n$, x □□□□□□□□,□ x □ D □□□□. $\partial D = \{x \in \mathbb{R}^n \mid x□D \text{ □□□□}\}$,□ ∂D □ D □□□,□□□□□ $\partial D = \{x \in \mathbb{R}^n \mid x \text{ □□□□□ } U, U \cap D \neq \emptyset, U \cap D^c \neq \emptyset\}$

Definition 7.1.3.15 *(□□)* □ $D \subset \mathbb{R}^n$,□ D □□□□□□□□ D □□□□□□, □ x □ D □□□□□. $\Longleftrightarrow$ x □□□□□ $U, \mathring{U} \cup D \neq \emptyset$

$D' = \{x \in \mathbb{R}^n \mid x□D \text{ □□}\}$,□□□ D □□□

□ $\overline{D} = D \cup D'$ □ D □□□.

Theorem 7.1.3.16 $D \subset \mathbb{R}^n$ □□□ $\Longleftrightarrow D' \subset D$.

Proof

"$\Longrightarrow$" $\forall a \in D'$ □□□ $a \in D$.

(□□□): □ $a \notin D$ □ $a \in D^c$, □□ D □□,□ D^c □□,$\exists \delta > 0, B(a; \delta) \subset D^c, B(a; \delta) \cap D = \emptyset$,□□ a □□□□□,□ $a \in D$, i.e. $D' \subset D$.

"$\Longleftarrow$" $D' \subset D$ □□□ D^c □□□.

$\forall a \in D^c, a$ □□□□□,□ $\exists \delta > 0$, s.t. $B(a; \delta) \cap D = \emptyset$,□ $B(a; \delta) \subset D^c$,□□ D^c □□□.

7.2 $\mathbb{R}^n$ □□□(□)□

Definition 7.2.1 *([illegible])* [illegible] $A \subset \mathbb{R}^n$ [illegible] A [illegible],[illegible] A [illegible] $\mathbb{R}^n$ [illegible](compet set).

[illegible] Heine-Bored [illegible] $\mathbb{R}$ [illegible].

Definition 7.2.2 *([illegible])* [illegible]
$a, b \in \mathbb{R}^n, a = (a^1, ..., a^n), b = (b^1, ..., b^n), a^i \le b^i, i = 1, ..., n, I_{a,b} = \left\{x \in \mathbb{R}^n \mid a^i \le x^i \le b^i\right\}$ [illegible].

Proposition 7.2.3 $I_{a,b}$ [illegible] $\mathbb{R}^n$ [illegible].

Proof

([illegible]): [illegible] $\{U_\alpha\}_{\alpha\in\Lambda}$ [illegible] $I_{a,b}$ [illegible],[illegible],[illegible] I_1 [illegible] 2^n [illegible],[illegible],[illegible] I_2,[illegible],[illegible] $I_3, ..., I_n, ...$,[illegible] $I_1 \supset I_2 \supset ... \supset I_k \supset ....$

$I_k = \left\{x \in \mathbb{R}^n \mid a_k^i \le x^i \le b_k^i, i = 1, ..., n\right\}$

[illegible] Cauchy-Cantor [illegible],$\exists! x_0^i \in \bigcap_{\{k=1\}}^{\infty} [a_k^i, b_k^i]$,[illegible] $x_0 = (x_0^1, ...x_0^n)$,[illegible] $\exists \alpha_0 \in \Lambda, \text{s.t. } x_0 \in U_{\alpha_0}$,[illegible]

$\operatorname{diam} I_k = \sup_{\{x,y\in I_k\}} d(x, y)$

[illegible] $\lim_{k\to\infty} \operatorname{diam} I_k = 0$,$\exists k \in \mathbb{N}^*, \text{s.t. } k \ge K, x_0 \in I_k \subset U_{\alpha_0}$,[illegible]! □