



Discrete Math Cram Cheat Sheet

Discrete Mathematics

Greenville University

6 pag.

Discrete Math Cram Sheet

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1 Propositional Logic

1.1 Truth Tables

p	T	T	F	F	
q	T	F	T	F	
F	F	F	F	F	contradiction
$p \nabla q$	F	F	F	T	joint denial
$p \nleftarrow q$	F	F	T	F	converse nonimplication
$\neg p$	F	F	T	T	left negation
$p \nrightarrow q$	F	T	F	F	nonimplication
$\neg q$	F	T	F	T	right negation
$p \oplus q$	F	T	T	F	exclusive disjunction
$p \bar{\wedge} q$	F	T	T	T	alternative denial
$p \wedge q$	T	F	F	F	conjunction
$p \leftrightarrow q$	T	F	F	T	biconditional/equivalence
q	T	F	T	F	right projection
$p \rightarrow q$	T	F	T	T	implication
p	T	T	F	F	left projection
$p \leftarrow q$	T	T	F	T	converse implication
$p \vee q$	T	T	T	F	disjunction
T	T	T	T	T	tautology

1.2 Logical Equivalences

Identity	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent	$p \wedge p \equiv p$ $p \vee p \equiv p$
Commutative	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
Associative	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ $p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption	$p \wedge (p \vee q) \equiv p$ $p \vee (p \wedge q) \equiv p$
Negation	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$
Double Negation	$\neg(\neg p) \equiv p$

Involving Biconditionals

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Involving Conditional Statements

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

1.3 Rules of Inference

Modus Ponens	$p \rightarrow q$ p — q
Modus Tollens	$\neg q$ $p \rightarrow q$ — $\neg p$
Associative	$(p \vee q) \vee r$ — $p \vee (q \vee r)$
Commutative	$p \wedge q$ — $q \wedge p$
Biconditional	$p \rightarrow q$ $q \rightarrow p$ — $p \leftrightarrow q$
Exportation	$(p \wedge q) \rightarrow r$ — $p \rightarrow (q \rightarrow r)$
Contraposition	$p \rightarrow q$ — $\neg q \rightarrow \neg p$
Hypothetical Syllogism	$p \rightarrow q$ $q \rightarrow r$ — $p \rightarrow r$
Material Implication	$p \rightarrow q$ — $\neg p \vee q$
Distributive	$(p \vee q) \wedge r$ — $(p \wedge r) \vee (q \wedge r)$
Absorption	$p \rightarrow q$ — $p \rightarrow (p \wedge q)$
Disjunctive Syllogism	$p \vee q$ $\neg p$ — q
Addition	p — $p \vee q$
Simplification	$p \wedge q$ — p
Conjunction	p q — $p \wedge q$
Double Negation	p — $\neg \neg p$
Disjunctive Simplification	$p \vee p$ — p
Resolution	$p \vee q$ $\neg p \vee r$ — $q \vee r$

1.4 Satisfiability

A proposition is *satisfiable* if some setting of the variables makes the proposition true. For example, $p \wedge \neg q$ is satisfiable because the expression is true if p is true or q is false. On the other hand, $p \wedge \neg p$ is not satisfiable because the expression as a whole is false for both settings of p .

2-SAT Problem

(to follow...)

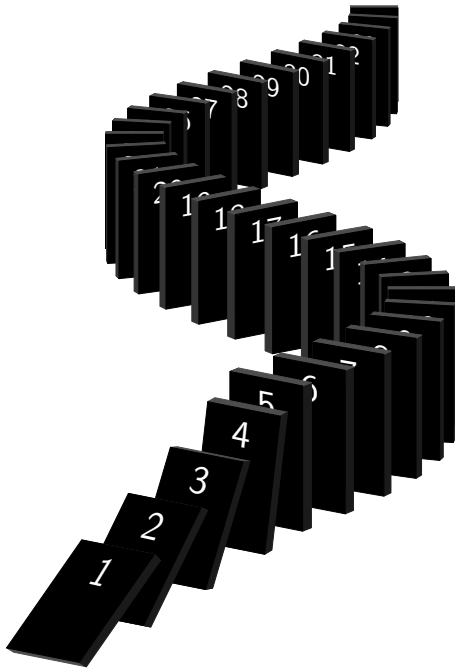
2 Proofs

2.1 Well-Ordering Principle

Every non-empty subset of the natural numbers has a smallest element.

2.2 Mathematical Induction

A statement $P(n)$ involving the positive integer n is true for all positive integer values of n is true if $P(1)$ is true and if $P(k)$ is true for any arbitrary positive integer k , then $P(k+1)$ is true.



The base case need not be for $n = 1$. It can be adjusted to whatever the smallest integer value n assumes.

2.3 Strong Induction

Let $P(n)$ be a predicate defined over all integers n , and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. Base cases: $P(a), P(a+1), \dots, P(b)$ are all true.
2. Inductive step: For any integer $k > b$, if $P(i)$ is true for all integers i with $a \leq i < k$, then $P(k)$ is true.

Then the statement $P(n)$ is true for all integers $n \geq a$.

3 Recurrence Relations

4 Number Theory

4.1 Divisibility

Properties

$$\begin{aligned} a|b &\rightarrow a|bc \quad \forall c \\ (a|b \wedge b|c) &\rightarrow a|c \\ (a|b \wedge a|c) &\rightarrow a|sb + tc \quad \forall s, t \\ \forall c \neq 0 \quad (a|b &\leftrightarrow ca|cb) \end{aligned}$$

4.2 Primes and Factors

Prime Numbers

OEIS A000040: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ...

4.3 Divisors

Greatest Common Divisor

This can be defined by the following recurrence relation:

$$\gcd(a, b) = \begin{cases} a & \text{if } b = 0 \\ \gcd(b, a \bmod b) & \text{else} \end{cases}$$

Bézout's Identity

Let a and b be nonzero integers (i.e., $a, b \in \mathbb{R}^*$) and let $d = \gcd(a, b)$. Then:

$$\exists x, y \in \mathbb{Z} (ax + by = d)$$

In addition,

- the greatest common divisor d is the smallest positive integer that can be written as $ax + by$
- every integer of the form $ax + by$ is a multiple of the greatest common divisor d .

Extended Euclidean Algorithm

4.4 Modular Arithmetic

Basic Rules

(to follow...)

Fermat's Little Theorem

If p is a prime number and a is a natural number, then

$$a^p \equiv a \pmod{p}$$

Chinese Remainder Theorem

Let m_1, m_2, \dots, m_n be pairwise relatively prime positive integers, and a_1, a_2, \dots, a_n be arbitrary integers. Then the system

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \vdots \\ x \equiv a_n \pmod{m_n} \end{cases}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$, where $x = \sum_{k=1}^n a_k M_k y_k$, $M_k = \frac{m}{m_k}$, and y_k is the modular inverse of M_k modulo m_k , i.e. $M_k y_k \equiv 1 \pmod{m_k}$.

5 Graph Theory**5.1 Notation****Fundamental Notation**

G	graph	E	edge set
V	vertex set		

Graph Invariants

$c(G)$	circumference	$\chi'(G)$	chromatic index
$d(u, v)$	distance between two vertices	$\delta(G)$	minimum degree
$\deg(v)$	degree of a vertex	$\Delta(G)$	maximum degree
$\text{gir}(G)$	girth	$\kappa(G)$	vertex connectivity
$\chi(G)$	chromatic number	$\lambda(G)$	edge connectivity

5.2 Definitions

graph an ordered pair (V, E) where V is the set of vertices and E is the set of edges

simple a graph having neither loops nor multiple edges

multigraph a graph with multiple edges but no loops

pseudograph a graph having both loops and multiple edges

digraph a directed graph in which each edge has a direction

adjacency two distinct vertices v and w in a graph are adjacent if the pair $\{v, w\}$ is an edge

incidence a vertex v and an edge e are incident with one another if $v \in e$

degree (of a vertex v , in symbols $\deg(v)$) the number of vertices adjacent to v

walk an alternating sequence $v_0, e_1, v_1, \dots, e_k, v_k$ of vertices v_i and edges e_i for which e_i is incident with v_{i-1} and with v_i for each i

path a walk whose vertices are distinct

trail a walk whose edges are distinct

circuit a trail whose first and last vertices are identical

cycle a circuit where each pair of whose vertices other than the first and the last are distinct

5.3 Properties**Handshaking Lemma**

In any graph the sum of the vertex degrees is equal to twice the number of edges.

$$\sum_{v \in V} \deg(v) = 2|E|$$

6 Linear Algebra**7 Combinatorics****7.1 Permutations and Combinations****Permutation**

A permutation or ranking of n objects is a listing of them in a certain order from first to last.

The number of permutations of length k from n distinct objects where repetition is not allowed is

$${}_n P_k = (n)_k = \frac{n!}{(n-k)!}$$

where $(n)_k$ denotes the falling factorial.

Combination

A combination of k objects taken from a collection of n objects is simply a selection of k of those distinct objects without regard to order.

The number of different combinations of k objects taken from a collection of n distinct objects without repetition is

$${}_n C_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

7.2 Binomial Coefficients

The binomial coefficient $\binom{n}{k}$ can be defined as the coefficient of the x^k term in the polynomial expansion of $(x+1)^n$, which occurs in the binomial formula

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{n-k}$$

Pascal's Triangle

Row 0:																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																				
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7.3 Generalized Permutations and Combinations

Permutations with Repetitions

The number of permutations of length k from n distinct objects where repetition is allowed is n^k .

Permutations with Duplicate Objects

The number of permutations of a multiset of n objects made up of k distinct objects can be expressed as follows:

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$$

where n_i represents the multiplicity of a distinct object i in the multiset.

Combinations with Repetition (Stars and Bars)

The number of combinations of length n using k different kinds of objects is

$${}_n R_k = \binom{n+k-1}{n-1} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

Number of Non-negative Integer Solutions The number of solutions of the equation $x_1 + x_2 + \dots + x_k = n$ in non-negative integers is $\binom{n+k-1}{k-1}$.

Number of Positive Integer Solutions The number of solutions of the equation $x_1 + x_2 + \dots + x_k = n$ in positive integers is $\binom{n-1}{k-1}$.

7.4 Principle of Inclusion-Exclusion

This provides an organized method/formula to find the number of elements in the union of a given group of sets, the size of each set, and the size of all possible intersections among the sets.

Two/Three Sets

Suppose that A, B , and C are finite sets. Then:

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

General Form

For finite sets A_1, \dots, A_n , one has the identity:

$$\begin{aligned} \left| \bigcup_{i=1}^n A_i \right| &= \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &\quad + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\ &\quad - \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{k=1}^n (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}| \right) \end{aligned}$$

7.5 Derangements

A derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of n elements can be determined as follows:

$$!n = (n-1) (! (n-1) + ! (n-2)) = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

OEIS A000166: 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, ...

7.6 Catalan Numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^n \frac{n+k}{k} \quad \text{for } n \geq 0$$

$$= \binom{2n}{n} - \binom{2n}{n+1} = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

OEIS A000108: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, ...

Second Kind (Subsets)

Counts the number of ways to partition a set of n objects into k non-empty subsets.

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \begin{cases} 1 & \text{if } n = k = 0 \\ 0 & \text{if } n \neq k \wedge k = 0 \\ (k-1) \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} & \text{if } n, k > 0 \end{cases}$$

Applications

1. number of expressions containing n pairs of parentheses which are correctly matched
2. number of different ways $n+1$ factors can be completely parenthesized
3. number of full binary trees with $n+1$ leaves
4. number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal
5. number of triangulations of a convex polygon with $n+2$ sides
6. number of permutations of $\{1, \dots, n\}$ that avoid the pattern 123 (or any of the other patterns of length 3)
7. number of noncrossing partitions of the set $\{1, \dots, n\}$
8. number of ways to tile a staircase shape of height n with n rectangles
9. number of ways to form a "mountain range" with n upstrokes and n downstrokes that all stay above the original line
10. number of semiorders on n unlabeled items

8 Probability

7.7 Partitions

The function $p(n, k)$ denotes the number of ways of writing n as a sum of exactly k terms.

$$p(n, k) = \begin{cases} 1 & \text{if } n = k = 0 \\ 0 & \text{if } n < k \\ p(n-1, k-1) + p(n-k, k) & \text{if } n \geq k \end{cases}$$

7.8 Stirling Numbers

First Kind (Cycles)

Counts number of permutations of n elements with k disjoint cycles.

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \begin{cases} 1 & \text{if } n = k = 0 \\ 0 & \text{if } n \neq k \wedge k = 0 \\ (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] & \text{if } n, k > 0 \end{cases}$$