Num meth.

15/1/20 (1)

$\sin(x) \cdot f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$a_0 = \sin(0) = 0$

$a_1 = f'(0) = a_1 + 2a_2 x + \dots$

$= \cos(0) = 1$

$a_2 = f''(0) = 2a_2 + 3 \cdot 2 a_3 x + \dots$

$= -\sin(0) = 0$

$\vdots$

$f(x) = \sin(x) = 0 + x + 0 - \frac{x^3}{3!} + \dots$

Taylor Series

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0) \cdot x^i}{i!}$$

Ex.

$\sin(x) = \underbrace{\left(x - \frac{x^3}{3!}\right)}_{\text{lazy guess}} \pm$ error bound.

Taylor's Thm.

$$f(x) = \underbrace{f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \dots + f^{(n)}(0)\frac{x^n}{n!}}_{\text{Lazy guess}} + \underbrace{\frac{f^{(n+1)}(\xi)x^n}{(n+1)!}}_{\text{Remainder}} \quad \xi \in [0, x]$$

$\sin(0.5) = 0.5 \pm$ something

$= \cancel{f(0)} + f'(0)x + \frac{f''(\xi)(0.5)^2}{2}$ $\quad \xi \in [0, 0.5]$

$f''(x) = -\sin(x)$

$-\sin(x) \leq 1$ worst case $\quad \forall x \in [0, 0.5]$

$= 0.5 \pm \frac{(1)(0.5)^2}{2}$

$= 0.5 \pm 0.125$

e^x

$e^x\Big|_{x=0.5} = \underset{\text{3 terms}}{\underline{\qquad}} \pm \underset{\text{error}}{\underline{\qquad}}$

$= f(0) + f'(0)(0.5) + f''(0)\frac{(0.5)^2}{2}$

$= e^0 + e^0(0.5) + e^0\frac{(0.5)^2}{2}$

$= 1 + 0.5 + \frac{0.25}{2} = 0.5 + 0.125 + 1$

$= 0.625 + 1 = 1.625$

$e^{(0.5)} = 1.625 \pm$ sth

sth $= \frac{f^{(3)}(\xi)(0.5)^3}{3!}$

$f^{(3)}(e^x) = e^x \quad \xi \in [0, 0.5]$

$e^x \leq e^{0.5} \quad \forall x \in [0, 0.5]$

sth $= \frac{(e^{0.5})(0.5)^3}{3!} = 0.0343$

$e^{0.5} = 1.625 \pm 0.0343$

$\sin(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)\, x^i}{i!} + \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$

$\text{Error}_n = \frac{(1)(0.5)^{n+1}}{(n+1)!}$

$\frac{df}{dx} = \frac{f(x+h) - f(x)}{h}$ with "small h"

forward difference.

$g(x) = f(x_0 + h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \dots$

find value of a_n

$f(x_0 + 0) = f(x_0) = a_0$

$a_1 = f'(x_0)$ $\quad a_2 = \frac{f''(x_0)}{2}$ $\quad \xi \in [x_0, x_0+h]$

$$g(x) = f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(\xi)h^2}{2!}$$

$$f(x_0+h) - f(x_0) - \frac{f''(\xi)}{2!}h^2 = f'(x_0)h$$

$$\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{f''(\xi)h}{2}$$

$$\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + O(h)$$

Shift

from * below.

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2) \quad (*)$$

$$g(x) \overset{(1)}{=} f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2!} + \frac{f'''(\xi^+)h^3}{3!}$$

(2)

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)h^2}{2!} - \frac{f'''(\xi)h^3}{3!}$$

From (*), 1-2

$$f(x_0+h) - f(x_0-h) = 2f'(x_0)h + \square\frac{h^3}{3!}$$

$$\frac{f(x_0+h) - f(x_0-h)}{2h} = f'(x_0) + \square h^2 \ *$$

2nd derivative = 1st deriv of 1st deriv

$$\frac{\frac{f(x+2h) - f(x)}{2h} - \frac{f(x) - f(x-2h)}{2h}}{2h}$$

$$= \frac{f(x+2h) - 2f(x) + f(x-2h)}{4h^2}$$

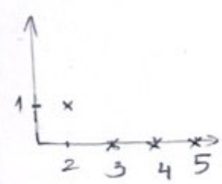
edge-ness = difference of pixel and its neighbor.

Interpolation 22/1/20

Find a polynomial that passes through all the data points

(x,y): $(2,3)$ $(3,1)$ $(4,2)$ $(5,2)$

1) Find polynomial that is $y=0$ when $x = 3, 4, 5$ and $y = 1$ at $x = 2$



$W_2 = (x-3)(x-4)(x-5)$

$W_2(x=2) = (-1)(-2)(-3) = -6$

so $W_2 = \frac{(x-3)(x-4)(x-5)}{-6}$,

$W_3 = 1$ at 3, 0 at 2, 4, 5

$W_4 = 1$ at 4, 0 at 2, 3, 5

$W_5 = 1$ at 5, 0 at 2, 3, 4

Multiply the right value with the corresponding polynomials and add them up

$\Rightarrow 3W_2 + 1W_3 + 2W_4 + 2W_5 \Rightarrow$ Deg 3

Num meth

Need polynomial degree $(n-1)$ to plot for n points.

→ Legendre's Polynomial

---

ex. $(2,3)$ $(3,1)$ $(4,2)$ $(5,2)$

w_2 $\quad y=0$ at $x=3,4,5$

$y=1$ at $x=2$

$$w_2 = \frac{(x-3)(x-4)(x-5)}{-6}$$

w_3 $\quad y=0$ at $x=2,3,5$

$y=1$ at $x=4$

$$w_3 = \frac{(x-2)(x-3)(x-5)}{-2}$$

w_4 $\quad y=0$ at $x=2,4,5$

$y=1$ at $x=3$

$$w_4 = \frac{(x-2)(x-4)(x-5)}{2}$$

w_5 $\quad y=0$ at $x=2,3,4$

$y=1$ at $x=5$

$$w_5 = \frac{(x-2)(x-3)(x-4)}{6}$$

22/1/20 (3)

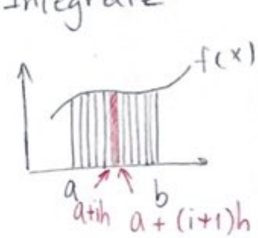
Combine

$$3w_2 + 2w_3 + 1w_4 + 2w_5$$

$$= \left(\frac{1}{-2}\right)(x-3)(x-4)(x-5) + (-1)(x-2)(x-3)(x-5) + \left(\frac{1}{2}\right)(x-2)(x-4)(x-5) + \left(\frac{1}{3}\right)(x-2)(x-3)(x-4)$$

$$= -\frac{1}{6}\left(4x^3 - 45x^2 + 161x - 192\right)$$

Integrate $\quad \int_{x=a}^{x=b} f(x)\,dx$



$h = \frac{b-a}{n}$

Area of a piece

$$A_i = \frac{h}{2}\Big(f(a+ih) + f(a+(i+1)h)\Big)$$

$$\text{Integral} = \sum_{i=0}^{i=n-1} A_i$$

$$= \frac{h}{2}\Big(\underbrace{f(a) + f(a+h)}_{A_0} + \underbrace{f(a+h) + f(a+2h)}_{A_1} + f(a+2h) + f(a+3h) + \dots + f(a+(n-1)h) + f(a+nh)\Big)$$

$$= \frac{h}{2}\Big(f(a) + 2\sum_{i=1}^{i=2n-1} f(a+ih) + f(\overset{a+nh}{b})\Big)$$

Trapezoid Rule

Error $E_T = \frac{1}{12}\frac{(b-a)^3}{n^2} f''(\xi)$; $\xi \in [a,b]$

$\int_1^3 x^3\,dx$ error : $\frac{(3-1)^3}{12n^2}(6\xi)$; $\xi \in [1,3]$

2nd derivative of f(x)

Worst case : $\xi = 3$

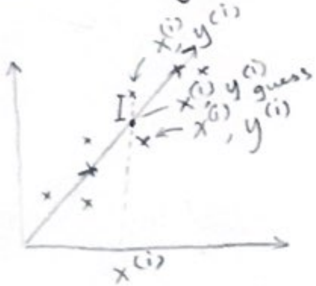
$$E_T \le \frac{(3-1)^3}{12n^2}\cdot 6\times 3$$

29/1/20

$$x+y+z=6$$
$$2x-y+z=3$$
$$x-y-2z=-7$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -7 \end{bmatrix}$$

Linear Regression.



distance → less → better

minimize distance btw line and point

$$\sum_i \left(y^{(i)}_{guess} - y^{(i)}\right)^2$$

$$y^{(i)}_{guess} = m x^{(i)} + c$$

$$\text{cost}(m,c) = \sum_i \left(m x^{(i)} + c - y^{(i)}\right)^2$$

↑ minimize this

$$\frac{\partial \text{cost}}{\partial c} = 0 \qquad \frac{\partial \text{cost}}{\partial c} = 0$$

$$0 = \frac{\partial}{\partial m}\text{cost} = \frac{\partial}{\partial m}\sum_i \left(m x^{(i)} + c - y^{(i)}\right)^2$$

$$= \sum_i \frac{\partial}{\partial m}\left(m x^{(i)} + c - y^{(i)}\right)^2$$

$$= \sum_i 2\left(m x^{(i)} + c - y^{(i)}\right)\left(x^{(i)}\right)$$

$$*(1)\quad 0 = 2\left[m\sum_i x^{(i)^2} + c\sum_i x^{(i)} - \sum_i x^{(i)} y^{(i)}\right]$$

$$0 = \frac{\partial}{\partial c}\text{cost} = \frac{\partial}{\partial c}\sum_i \left(m x^{(i)} + c - y^{(i)}\right)^2$$

$$= \sum 2\left(m x^{(i)} + c - y^{(i)}\right).1$$

$$*(2) = 2\left[m\sum_i x^{(i)} + cn - \sum y^{(i)}\right]$$

From *(1) and *(2)

$$0 = m\sum x^{(i)^2} + c\sum x^{(i)} - \sum y^{(i)} x^{(i)}$$

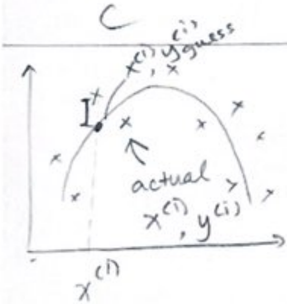
$$0 = m\sum x^{(i)} + cn - \sum y^{(i)}$$

Combine

$$\sum y^{(i)} x^{(i)} = m\sum x^{(i)^2} + c\sum x^{(i)}$$

$$\sum y^{(i)} = m\sum x^{(i)} + c\cdot n$$

$$\underbrace{\begin{bmatrix} \sum y^{(i)} x^{(i)} \\ \sum y^{(i)} \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} \sum x^{(i)^2} & \sum x^{(i)} \\ \sum x^{(i)} & n \end{bmatrix}}_{A}\begin{bmatrix} m \\ c \end{bmatrix}$$



$$y = ax^2 + bx + c$$

$$\sum_i \left(y^{(i)}_{guess} - y^{(i)}\right)^2$$

$$\text{cost}(m,c) = \sum_i \left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)^2$$

$$\frac{\partial \text{cost}}{\partial m} = 0\ ?\ \frac{\partial \text{cost}}{\partial c} = 0.$$

$$0 = \frac{\partial}{\partial a}\text{cost} = \frac{\partial}{\partial a}\left(\sum_i \left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)^2\right)$$

$$= \sum_i \frac{\partial}{\partial a}\left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)^2$$

$$0 = \sum_i (\cancel{2})\left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)\left(x^{(i)^2}\right)$$

$$= a\sum_i x^{(i)^4} + b\sum_i x^{(i)^3} + c\sum x^{(i)^2}$$

$$- \sum y^{(i)} x^{(i)^2}$$

Num meth

$0 = \frac{\partial}{\partial b}\left(\sum_i \left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)^2\right)$

$= \sum_i \left[\cancel{(2)}\left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)\left(x^{(i)}\right)\right]$

$= a\sum_i x^{(i)^3} + b\sum x^{(i)^2} + c\sum x^{(i)}$

$- \sum x^{(i)} y^{(i)}$

$0 = \frac{\partial}{\partial c}\left(\sum_i \left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)^2\right)$

$= \sum_i \left((2)\left(a x^{(i)^2} + b x^{(i)} + c - y^{(i)}\right)(1)\right)$

$= a\sum_i x^{(i)^2} + b\sum x^{(i)} + cn$

$- \sum y^{(i)}$

$\sum y^{(i)} x^{(i)^2} = a\sum_i x^{(i)^4} + b\sum_i x^{(i)^3} + c\sum x^{(i)^2}$

$\sum y^{(i)} x^{(i)} = a\sum x^{(i)^3} + b\sum x^{(i)^2} + c\sum x^{(i)}$

$\sum y^{(i)} = a\sum x^{(i)^2} + b\sum x^{(i)} + cn$

$$\begin{bmatrix} \sum yx^2 \\ \sum yx \\ \sum y \end{bmatrix} = \begin{bmatrix} \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x^3 & \sum x^2 & \sum x \\ \sum x^2 & \sum x & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Optimization problem 5/2/20

$f(x) = (x-2)^2 + 1$

$\arg\min_x f(x)$

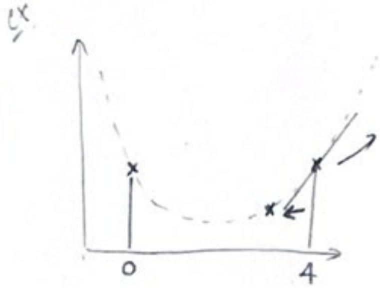
value of x that makes $f(x)$ minimum.

Minimise = Maximise

29/1/20 (5)

Max $g(x)$ = Min $-g(x)$

ex



$f'(x) > 0$

x should be less

go left to reach min

direction $= \frac{-f'(x_n)}{|f'(x_n)|}$

$x_{n+1} = x_n + (\text{direction})(\text{step size})$

$f'(x) = 2(x-2)$

- step size
  - small is bad → takes too long
  - large is bad → sad accuracy

stepsize $= \overset{\text{learning rate}}{\eta}|f'(x_n)|$

$x_{n+1} = x_n - \frac{f'(x_n)}{\cancel{|f'(x_n)|}} \cdot \eta \cancel{|f'(x_n)|}$

$x_{n+1} = x_n - \eta f'(x_n)$

$f(x_n+\Delta x, y_n+\Delta y) = f(x_n, y_n) + \Delta x \left.\frac{\partial f}{\partial x}\right|_{x_n,y_n}$

or $f(\vec{v}_n + \Delta\vec{v})$

$+ \Delta y \left.\frac{\partial f}{\partial y}\right|_{x_n,y_n} + O(\Delta x^2, \Delta y^2, \Delta x\Delta y)$

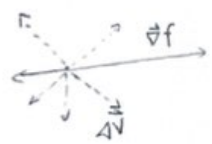
$\Delta f = f(x_n+\Delta x, y_n+\Delta y) - f(x_n, y_n)$

as negative as possible

$= \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} = (\Delta x, \Delta y) \overset{\text{dot product}}{\cdot} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$= \Delta\vec{v} \cdot \underbrace{\vec{\nabla} f}_{\text{gradient}}\Big|_{x,y}$

5/2/20. (6)



→ $\Delta\vec{v}$ opposite to $\vec{\nabla} f$

→ $\Delta\vec{v} = C \cdot \frac{-\vec{\nabla} f}{|\vec{\nabla} f|}$

$\vec{\nabla} f \cdot \Delta\vec{v} = |\vec{\nabla} f||\Delta\vec{v}| \cdot \cos\theta$

$\vec{x}_{n+1} = \vec{x}_n + (\text{Stepsize})\left(\frac{-\vec{\nabla} f}{|\vec{\nabla} f|}\right)$

Stepsize → large when far
→ small when near

Stepsize $\propto |\vec{\nabla} f|$

$\vec{x}_{n+1} = \vec{x}_n + \lambda \cdot \cancel{|\vec{\nabla} f|} \cdot \frac{-\vec{\nabla} f}{\cancel{|\vec{\nabla} f|}}$

$\vec{x}_{n+1} = \vec{x}_n - \lambda \vec{\nabla} f \big|_{\vec{x}_n}$ Gradient Descend.

$cost(m,c) = \sum (mx^{(i)} + c - y)^2$

→ a point is above or below the line

From graph in jupyter

→ blue → above line = good
below = bad

→ red → above line = bad
below = good.

$Cost(m,c) = \sum_i$ if in the right position +1, if in the wrong position −1

Minimize cost(m, c) to get m and c

above line → $y_{point} > y_{line}$

below line → $y_{point} < y_{line}$.