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Second orthogonality relation

$(\chi_i,\chi_j)_G=\delta_{ij}$

$(\chi_i,\chi_j)_G=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)}=\frac{1}{|G|}\sum_{k=1}^{r}\sum_{g\in K_k}\chi_i(g)\overline{\chi_j(g)}=\sum_{k=1}^{r}\frac{|K_k|}{|G|}\chi_i(g)\overline{\chi_j(g)}=\delta_{ij}$

Let $\{K_1,\dots,K_r\}$ are distinct conjugate classes of G, $g_i\in K_i$

$|G|=|K|\cdot|C_G(g)|$, $g\in K$

$\Rightarrow \sum_{k=1}^{r}\frac{\chi_i(g_k)\overline{\chi_j(g_k)}}{|C_G(g_k)|}=\sum_{k=1}^{r}\frac{\chi_i(g_k)}{\sqrt{|C_G(g_k)|}}\cdot\frac{\overline{\chi_j(g_k)}}{\sqrt{|C_G(g_k)|}}=\delta_{ij}$ $(\star)$

Denote $M=\left(\frac{\chi_i(g_k)}{\sqrt{|C_G(g_k)|}}\right)$. $\star\Leftrightarrow M\cdot\overline{M}^T=E \Rightarrow \underline{M \text{ is unitary}}$ matrix $\Leftrightarrow M^T\cdot\overline{M}=E\Leftrightarrow\sum_{i=1}^{r}\frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}\cdot\frac{\overline{\chi_i(g_k)}}{\sqrt{|C_G(g_k)|}}=\delta_{jk}$

$$\Leftrightarrow \boxed{\sum_{i=1}^{r}\chi_i(g)\overline{\chi_i(h)}=\begin{cases}|C_G(g)|, & \text{if } g,h \text{ are conjugate},\\ 0, & \text{otherwise}.\end{cases}}$$

Th. The dimension of any irreducible complex representation $\varphi: G\to GL(V)$ (over $\mathbb{C}$) divides $|G|$.

A complex number z is called algebraic, if there is some polynomial $p(x)=a_0x^n+\dots+a_n$ $(n\geq 1)$, $a_i\in\mathbb{Z}$, $a_0\neq 0$, s.t. $p(z)=0$.

z is algebraic integer, if $a_0=1$.

Denote by $\mathcal{O}$ the set of all algebraic integers.

Lemma 1. $\mathcal{O}$ is ring.

Proof. First show that if $\omega_1,\dots,\omega_m\in\mathbb{C}$, $\omega_j\neq 0$, $j=1,\dots,m$ and $M=\langle\omega_1,\dots,\omega_m\rangle=\mathbb{Z}\omega_1+\dots+\mathbb{Z}\omega_m$ is a ring, then $M\subset\mathcal{O}$.

For any $\alpha\in M$, ω_j, $\alpha\omega_j=\sum_{i=1}^{m}a_{ij}\omega_i$, $a_{ij}\in\mathbb{Z}$ (1)

(1) $\Leftrightarrow (\alpha E-A)\begin{pmatrix}\omega_1\\ \vdots\\ \omega_m\end{pmatrix}=0$, $A=(a_{ij})$, $(\omega_1,\dots,\omega_m)$ is a nonzero solution of this system $\Rightarrow |\alpha E-A|=0$ – it is a polynomial of degree m with integer coeff. and $a_0=1\Rightarrow \alpha\in\mathcal{O}$.

$\forall\alpha$: $\alpha^n+a_1\alpha^{n-1}+\dots+a_n=0$,

β: $\beta^l+b_1\beta^{l-1}+\dots+b_l=0$.

The $M=\{\sum_{i,j\geq 0}c_{ij}\alpha^i\beta^j \mid c_{ij}\in\mathbb{Z}, 0\leq i\leq n-1, 0\leq j<l\}$

M is a ring: $\alpha^p\beta^q$ can be expressed through $\alpha^i\beta^j$, $i<n$, $j<l$.

$\Rightarrow$ all the elements of M are alg. int, especially, $\alpha\pm\beta$, $\alpha\beta\in\mathcal{O}$. q.e.d.

Lemma 2. If $\alpha\in\mathbb{Q}\cap\mathcal{O}\Rightarrow\alpha\in\mathbb{Z}$.

Lemma 3. If $\chi=\chi_\varphi$, then $\forall g\in G$, $\chi(g)$ is alg. int, and $|\chi(g)|\leq\chi(1)$. $|\chi(g)|=\chi(1)\Leftrightarrow\varphi(g)=\lambda E$.

Proof. $\chi(g)=\sum_{j=1}^{n}\lambda_j$, λ_j are characteristic roots of $\varphi(g)$.

$\lambda_j^{|G|}=1\Rightarrow$ they are alg. int. $\Rightarrow$ (by L.1) $\chi(g)$ is alg. int.

$|\chi(g)|\leq\sum_{j=1}^{n}|\lambda_j|=n=\chi(1)$. The equality means, that $\lambda_j=\lambda$, $j=1,\dots,n$ $\Rightarrow\varphi(g)=\lambda E$. q.e.d.

Now use the group algebra $\mathbb{C}G=\{\sum_{g\in G}\alpha_g g\}$ $\{g\in G\}$ is a linear basis. $G\subset\mathbb{C}G$.

Denote with $\overline{K}=\sum_{g\in K}g$, K is a conjugate class of G.

L.4 $\{\overline{K}_1,\dots,\overline{K}_r\}$ is the basis of $Z(\mathbb{C}G)$. Moreover $\forall i,j$, $\overline{K}_i\overline{K}_j=\sum_{l=1}^{r}a_{ij}^l\overline{K}_l$, $a_{ij}^l\geq 0$, $a_{ij}^l\in\mathbb{Z}$.

Proof. Find by what condition $z=\sum_{g\in G}\alpha_g g\in Z(\mathbb{C}G)$.

$\forall h\in G$, $hzh^{-1}=\sum_{g\in G}\alpha_g(\underbrace{hgh^{-1}}_{b})=\sum_{b\in G}\alpha_{h^{-1}bh}b=\sum_{b\in G}\alpha_b b\Leftrightarrow$ α is constant on conj. classes.

$\Rightarrow z=\sum_{i=1}^{r}\alpha_{g_i}\overline{K}_i$, $g_i\in K_i$.

Note that K_iK_j is union of some conjugate classes: $\forall g\in G$, $gxg^{-1}=(gx_ig^{-1})(gx_jg^{-1})\in K_iK_j$ if $x\in K_l$, $x=x_ix_j\in K_iK_j$, $\Rightarrow K_l\subseteq K_iK_j$. $\Rightarrow$ any $x=x_ix_j$ contributes one summand into decomp. of $\overline{K}_i\cdot\overline{K}_j\Rightarrow$

$a_{ij}^l=|\{x\in K_l \mid x=x_ix_j,\ x_i\in K_i, x_j\in K_j\}|\in\mathbb{N}\cup\{0\}$.

L.5. Let χ is irreducible, $\forall g\in G$ the number $(\omega(\chi,g)=)\ |K_g|\frac{\chi(g)}{\chi(1)}$ is algebraic integer. $(g\in K_g)$

Proof. If $\chi=\chi_\varphi$, $\varphi: G\to GL(V)$ irreducible, we may extend it to the representation $\Phi:\mathbb{C}G\to L(V)$,

$\Phi(\sum_{g\in G}\alpha_g g)=\sum_{g\in G}\alpha_g\varphi(g)$.

We get lin. operators $\Phi_j=\Phi(\overline{K}_j)=\sum_{g\in K_j}\varphi(g)$ commute with all $\varphi(g)$, $g\in G$

From L.4 $\Rightarrow\Phi_j$ (Schur's Lemma) $\Rightarrow\Phi_j=\lambda_jE$. Calculate trace:

$\operatorname{tr}\Phi_j=\sum_{g\in K_j}\operatorname{tr}\varphi(g)=|K_j|\chi(g_j)=\lambda_j\chi(1)\Rightarrow\lambda_j=\frac{|K_j|\chi(g_j)}{\chi(1)}$

By L.4, $\Phi_i\Phi_j=\sum_{l=1}^{r}a_{ij}^l\Phi_l$, or $\omega(\chi,g_i)\omega(\chi,g_j)=\sum\alpha_{ij}^l\omega(\chi,g_l)$

As in the proof of L1, we get that $\omega(\chi,g_i)$ is alg. int. q.e.d.

Finish the proof of Th.

As χ is irred. $\Rightarrow |G|(\chi,\chi)_G=\sum_{i=1}^{r}|K_i|\chi(g_i)\overline{\chi(g_i)}=|G|$

$\Rightarrow\sum\frac{|K_i|\chi(g_i)}{\chi(1)}\cdot\overline{\chi(g_i)}=\frac{|G|}{\chi(1)}\in\mathcal{O}\cap\mathbb{Q}=\mathbb{Z}\Rightarrow\chi(1)\,|\,|G|$. q.e.d.