LOMONOSOV MOSCOW STATE UNIVERSITY
FACULTY OF MECHANICS AND MATHEMATICS
CHAIRS OF HIGHER ALGEBRA

SPECIAL COURSE

ON

FINITE GROUP AND IT'S REPRESENTATION

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02.2023

Contents

1 LECTURE 01 (22.02.2023)

1.1 Group, subgroups, cosets, etc

1.1.1 Two varieties of group actions

I. First variety

(1) associativity

(2) $\exists e(left)\ s.t,\ \forall\ g \in G,\ eg = g \Rightarrow G\ is\ a\ group$

(3) $\forall g \in G,\ \exists g'\ inverse\ to\ g,\ g'g = e$

II. Second variety

(1) associativity

(2) $\exists e\ \forall g \in G,\ eg = g$

(3) $\exists g''(right),\ \forall g \in G,\ gg'' = e$

Pr.1 G is not necessarily a group

1. Construct an example
2. Decide such semigroups

If (H -subgroup of G), $H < G$, then

$$G = \bigsqcup_{t \in T} tH$$

tH -left cosets with representative t, T -left transversal.

$$G = \bigsqcup_{s \in S} Hs$$

Hs -right cosets with representative s, S -right transversal.

Pr.2 If $|G| < \infty$, then one may take $S = T$

Prop.1 Let $A, B < G$

(a)

$$A = \bigcup_{r \in R} r(A \cap B) \Rightarrow AB = \bigcup_{r \in R} B$$

(b)

$$AB\ is\ a\ subgroup\ of\ G\ \Leftrightarrow AB = BA$$

(c)

$$If\ |A| < \infty,\ |B| < \infty,\ then\ |AB| = \frac{|A||B|}{|A \cap B|}$$

(a)

$$AB = \bigcup_{r \in R} (A \cap B)B = \bigcup_{r \in R} B$$

suppose:

$$r_1 B \cup r_2 B \neq \varnothing$$

$$r_1(A \cup B) = r_2(A \cup B) \Rightarrow r_1 = r_2$$

Proof. ($\Leftarrow$)

$$(a_1b_1)(a_2b_2) = a_1(b_1a_2)b_2 = a_1(a_2'b_1')b_2 = (a_1a_2')(b_1b_2')$$

$$(ab)^{-1} = b^{-1}a^{-1} = a'b' \in AB \ \Rightarrow AB \ - subgroup$$

($\Rightarrow$)

$$(ab)^{-1} = b^{-1}a^{-1} \in BA \Rightarrow AB \subseteq BA$$

$$(AB)^{-1} = AB, \ (ba)^{-1} = a^{-1}b^{-1} \in AB \ is \ a \ subgroup$$

$$(AB)^{-1} = AB \Rightarrow ba \in AB \Rightarrow BA \subseteq AB$$

$$\Rightarrow AB = BA$$

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(b) From (a)

$$|R| = \frac{|A|}{|A \cap B|} = \frac{|AB|}{|B|}$$

Prop.2 Dedkind's identity

Let $A, B, C \subseteq G$, $A \leqslant C$, $C \leqslant AB$. Then $C = (AB) \cap C = A(B \cap C)$.

Proof. $\forall c \in C$ as $C \subseteq AB \Rightarrow \exists a \in A, \ b \in B$
$c = ab \Rightarrow b = a_{-1}c \in B \cap C$
$\Rightarrow c \in A(B \cap C) \ \Rightarrow C = A(B \cap C)$

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Exercise.3 Let $|G| < \infty$, $AB < G$, s.t. $(|G : A|, |G : B| = 1)$, (coprime = 1)

Prove that $G = AB$

1.2 Double cosets

Let $A, B < G$, take $g \in G$, the double coset defined by g with respect to A and B:

$$AgB = agb$$

Theorem 1.1.

$$G = \bigsqcup_{i \in I} Ag_iB$$

Proof. $\forall g \in G,\ g \in AgB$, If $Ag_1B \cap Ag_2B \neq \varnothing$,
$a_1g_1b_1 = a_2g_2b_2 \Rightarrow (a_1g_1)B = (a_2g_2)B$
$\Rightarrow (a_1g_2)^{-1}(a_2g_2) \in B \Rightarrow g_1 \in Ag_2B, \Rightarrow Ag_1B = Ag_2B$

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Theorem 1.2.

$$|G| =< \infty,\ \Rightarrow |AgB| = \frac{|A||B|}{|A \cap B|}$$

Proof.

$$gg^{-1}|AgB| = |g(g^{-1}Ag)B| = |\underbrace{(g^{-1}Ag)}_{Ag}B|$$

$$= \frac{|g^{-1}Ag||B|}{|(g^{-1}Ag) \cap B|} = \frac{|A||B|}{|(g^{-1}Ag) \cap B|}$$

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1.3 Homomorphism and automorphism

1.3.1 Normal and characteristic subgroups

Definition 1.1. H is **characteristic** in G ($H\ char\ G$) $\Leftrightarrow$ H invariant under all $\alpha \in Aut(G)$.

Definition 1.2. G is called **simple**, if $N \lhd G \Rightarrow N = G\ or\ N = \{e\}$.

Definition 1.3. G is called **characteristically simple** $\Leftrightarrow H\ char\ G \Rightarrow H = G\ or\ H = \{e\}$.

Theorem 1.3 (Main Theorem)**.** $\varphi : G \to H$ *(not necessarily epimorphism)*

$$Im\varphi = \varphi(G) \cong {}^{G}/_{Ker\varphi}$$

Corollary 1.3.1. (Correspondence of subgroups):

Let $\varphi : G \to H$ is surjective, hom = epimorphism. Then there are core bijections.

$$F \leqslant H \leftrightarrow \forall K \leqslant G | Ker\varphi \leqslant K$$

$$F \trianglelefteq H \leftrightarrow \forall K \trianglelefteq G | Ker\varphi$$

$$\varphi(k) := F \leqslant H$$

$$k \leqslant G$$

converse mapping: take any $F \leqslant H$, then, $k := \varphi^{-1}(F) = g \in G | \varphi(g) \in F$
$\varphi(\varphi^{-1}(F)) = F$, $\varphi^{-1}(\varphi(K)) = K$, iff $k \ni Ker\varphi$

1.3.2 Automorphism

Definition 1.4. α is an **automorphism** of the group G, if $\alpha : G \to G$ is isomorphism.

1.3.3 Inner automorphism

$$ig(x) = gxg_{-1}, \forall x \in G$$

$$Aut(G) \trianglerighteq Int(G) \cong {}^{G}/_{Z(G)}$$

$\star$ A long-standing problem: If G is a finite simple group $\Rightarrow ? {}^{Aut(G)}/_{Int(G)}$ is solvable?

(Solved module the classification of the Finite Simple Groups) $N \lhd G \Leftrightarrow N$ invariant under all ig.

2 LECTURE 02 (01.03.2023)

1. $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$

   $\alpha \in Aut(\mathbb{Z}_n)$, $\alpha(k) = k \cdot \varphi(1)$, $(k = \underbrace{1 + \cdots + 1}_{k})$ $|k| = |\alpha(k)|$. $\alpha(1) \leftrightarrow m|(m,n) = 1$

   $\beta(1) = l$, $(\beta\alpha)(1) = (lm) \cdot 1$

   Problem: Prove that if G is not cyclic, then $Aut(G)$ is not Abelian.

2. G is elementary Abelian p-group. $G = \underbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n} = \mathbb{Z}_p^n$.

   $$Aut(\mathbb{Z}_p^n) \cong GL(n,p)$$

   $\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ is a vector space over the field $\mathbb{Z}_p$. Any automorphism is a $\mathbb{Z}_p$-linear operator. $\mathbb{Z}_p$ is characteristically simple. (G is characteristically simple if $H\ char\ G \Rightarrow H = G$ and $H = \{u\}$)

   Proof. $\mathbb{Z}_p$ is characteristically simple.

   Let H be subgroup of $G = {\mathbb{Z}_p}^n$, $H \lhd G$. If $H \neq \{0\}$

   $\Rightarrow$ Take some $h \in H$, $\forall v \in G$, $\exists \alpha \in Aut(G),\ v = \alpha(G)$

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2.1 Characteristically simple group

Theorem 2.1. *G is characteristically simple iff $G = H_1 \times \cdots \times H_r$, where $H_i \cong H_1$, $(i = 1, 2, 3, \ldots, r)$ is a simple group.*

Proof. ($\Rightarrow$) G is characteristically simple $\Rightarrow G = H_1 \times \cdots \times H_r$, consider H -some minimal normal subgroup of G, $(1 < H < G)$. The set of subgroup $\alpha(H)$, $\forall \alpha \in Aut(G)$

$\{H_1 = H, H_1, \ldots, H_r\}$

$M =< H_1, \ldots, H_r >$ is characteristically in G.

$\beta(M) =< H_{i_1}, \ldots, H_{i_r} >=< H_1, \ldots, H_r >$,

$\beta \in Aut(G)$

G is characteristically simple $\Rightarrow M = G$, Show that $G = H_1 \times \cdots \times H_r$, $\forall i,\ H_i \lhd G$, H_i is a minimal subgroup of G. $\Rightarrow G' = H_1 H_2 \cdots H_r$

This product is direct:

$$\Leftrightarrow H_i \cap (\prod_{j \neq 1} H_j) = \{e\}$$

H_i is minimal, $\prod H_i \lhd G \Rightarrow H_i \cap \prod H_j \trianglelefteq G$, $\Rightarrow= \{e\}$.

H_1 is simple if $N \trianglelefteq H_1$, then $N \lhd G$. $g = h_1 \cdots h_r,\ gNg^{-1} = h_1 N_{-1} = N$. ($N$ and $h_i,\ i > 1$, commute elements).

$\Rightarrow H_1$ is minimal normal of G, $\Rightarrow N = H$ or $N = \{e\}$.

$\Rightarrow H_1$ is simple.

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Proof. ($\Leftarrow$) If $G = H_1 \times \cdots \times H_r$, $H_i \cong H_1$ -simple, then G is characteristically simple. If we take some $e \neq N \trianglelefteq G$, that N is not characteristically. Evidently,

$$N = \bigtimes_{i \in J} H_j,\ J \subset \{i, \ldots, j, \ldots, r\}$$

Where $J = \{j | N \cap H_j \neq \{e\}\}$, $\Rightarrow N > H_j$.

We can define such automorphism, that permutes these subgroups cyclic.

$$\{\underbrace{H_1, \ldots, H_s}_{H}, H_{s+1}, \ldots, H_r\} \Rightarrow \alpha(N) \neq N$$

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2.2 Nilpotent group (vs. Solvable group)

Rem: A group is solvable iff $\exists n \in N,\ G^n = \{e\}$.

It follows that G^n is an Abelian characteristic subgroup of a solvable group of G. Consequence of the Theorem: If $N \lhd G$ is a minimal normal subgroup of a solvable group G, then $N \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$, (p is some prime number).

Proof. N is Abelian as N is minimal $\Rightarrow$ N is characteristically simple. $\Rightarrow N \cong \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$.

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Definition 2.1. Define subgroups of a group G: $G_0 = G_1,\ G_1 = [G, G] = G',\ G_2 = [G_1,\ G],\ etc$

If G_2 is defined, then $G_{k+1} = [G_k,\ G]$, $G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots$.

If $\exists m \in \mathbb{N}$, $G_m = \{e\}$, then G is called **nilpotent**.

$ad_x(y) = [x, y]$, The Lie Ring is nilpotent, if ${ad_x}^m = 0$.

Question: Is it true that if G is nilpotent $\Rightarrow$ G is solvable?

$\star$ The converse is not true.

$$G = S_3 = \mathbb{Z}_3 \leftthreetimes \mathbb{Z}_2\ Z(G) = \{e\}$$

Prop.1 If G is nilpotent, $G \neq \{e\}$, then $Z(G) \neq \{e\}$.

Prop.2 All $G_k\ char\ G$.

Prop.3 ${}^{G_k}/_{G_{k+1}} \leqslant Z({}^{G}/_{G_{k+1}})$

It means that the series $G_0 \rhd G \rhd \cdots \rhd G_m = e$ is a descending central series.

Proof. 1. If $m = 1$, then G is Abelian $\Rightarrow$ $Z(G) = G$, If $m > 1$, then $G_{m-1} \leqslant Z(G)$, $G_m = [G_{m-1},\ G] = \{e\}$.

2. Induction on k:

   $G_1 = G' = \{[g_1,\ h_1] \cdots [g_q,\ h_q]\}$

   $\alpha \in Aut(G)$, $\alpha[g_i,\ h_i] = ([\alpha(g_i), \alpha(h_i)]) \Rightarrow \alpha(G') = G'$.

If it is proved that G_k is characteristic in G,

$G_{k+1} =< [g_k,\ g] | g_k \in G_k, g \in G >$
$\alpha(G_{k+1}) =< [\alpha(g_k),\ \alpha(g)] >= G_{k+1}$. ($\alpha(g_k) \in G_k$ -by induction hypothesis.)
${G_k}/{G_{k+1}} \leqslant Z({G}/{G_{k+1}}) \Leftrightarrow [G_k, G] \leqslant G_{k+1}$

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Theorem 2.2. *The following conditions are equivalent:*

1. *G is nilpotent.*
2. *If $H \lneqq G$, then $N_G(H) > H$ (Normalizer condition).*
3. *($|G| < \infty$) $G = G_{p_1} \times G_{p_r}$, the direct product if its Sylow subgroups.*

Proof. ($2 \Rightarrow 3$) Let $|G| = {P_1}^{k_1}, \dots, {P_r}^{k_r}$ and $|G_i| = P_i^{k_i}$
From the Sylow theorems, we know that $H = N_G(G_i)$.
As G has the normalizer properly, $\Rightarrow N_G(G_i) = G \Rightarrow G_i \lhd G \Rightarrow G = G_1 \times \cdots \times G_r$

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3 LECTURE 03 (15.03.2023)

G is characteristically simple, $H \lhd G$, H is a minimal normal subgroup of G, $\{H = H_1 = \alpha_1(H),\ H_2 = \alpha_2(H), \ldots, H_r = \alpha_r(H)\}$ -all the images of H by $Aut(G)$.

$$\{e\} \neq H =< H_1 \cdots H_r > \ char\ G \Rightarrow G =< H_1, \ldots, H_r >$$

Consider $\{F = Hi_1 \times \cdots \times H_{i_k}\} \neq \varnothing$. Let M be the maximal among these subgroups.

$\Rightarrow M = G$. If not, $\exists H_i \lneq M,\ M \lhd G$, then $H_i \cap M = \{e\}$

$\Rightarrow H_i \cdot M = H_i \times M$ -a larger subgroup which is direct product of some of those subgroups. This contradiction means that $M = G$

3.1 Nilpotent group

3.1.1 Lower central series

Definition 3.1. $G_0 = G \geqslant G_1 = G' \geqslant G_2 = [G_1,\ G] \geqslant \cdots \geqslant G_k \geqslant G_{k+1} = [G_k, G] \geqslant \cdots$

If $\exists n \in \mathbb{N} : G_n = e$, then G is called **nilpotent**, n is **nilpotency** class of G, if $G_{n-1} \neq \{e\}$. $\Rightarrow G_0 = G \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_{n-1} \rhd G_n = \{0\}$ -the **lower descending normal series**.

$^{G_k}\big/_{G_{k+1}=Z(G/G_{k+1})}$

3.1.2 Upper central series

Definition 3.2 (The upper central chain of G). $Z_0 = \{e\}$, $Z_1 = \{G\}$. Define that Z_2 such that, $^{Z_2}/_{Z_1} = Z(^G/_{Z_1})$ etc. If Z_i is defined, then Z_{i+1}, $^{Z_{i+1}}/_{Z_i} = Z(^G/_{Z_i})$.

If $\exists H_0 = G \leqslant H_1 \leqslant \cdots \leqslant H_r$, some central chain $\Rightarrow H_i \leqslant Z_i$.

$Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_r \leqslant \cdots$ -**upper central series**.

Theorem 3.1. *The following conditions are equivalent:*

1. $\exists n \in \mathbb{N},\ such\ that\ G_n = \{e\}$
2. $\exists m \in \mathbb{N},\ such\ that\ Z_m = G$
3. $\forall H \lneq G,\ H \lneq N_G(H)$
4. $(|G| < \infty)$, *G is the direct product of its Sylow subgroups.*

Proof. $(1 \Leftrightarrow 2)$

Let n be minimal with condition $G_n = \{e\}$, $Z_m = G$.

For convenience write these series in such way:

$\{e\} = Z_0 < Z_1 < \cdots < Z_{m_1} < Z_m = G$

$\{e\} = G_n < G_{n-1} < \cdots < G_1 < G_0 = G$

Let $G_n = \{e\}$, $(1 \Rightarrow 2)$ Show that $\forall k = 0, 1, \ldots$, $G_{n-k} \leqslant Z_k(*) \Rightarrow (k = n)G_0 = G \leqslant Z_n \Rightarrow Z_n = G$

Use induction on k, for $k = 0$, $G_n = \{e\} = Z_0$ -true.

For $k \geqslant 1$, suppose $(*)$ is true, and show that $G_{n-k-1} \leqslant Z_{k+1}$

As $G \triangleright G_{n-k} \leqslant Z_k \triangleleft G$, $\exists$ epimorphism:

$$G/G_{n-k} \xrightarrow{onto} G/Z_k$$

$$\Rightarrow (G_{n-k-1}Z_k)/Z_k \leqslant Z(G/Z_k)$$

By construction of upper central series

$G_{n-k-1} \cdot Z_k \leqslant Z_{k+1}$, $Z_k \leqslant Z_{k+1}$. $G_{n-k-1} \leqslant Z_{k+1}$ -We proved. Conversely, $(2 \Rightarrow 1)$ Show that $\forall k = 0, 1, \ldots G_k \leqslant Z_{m-k}(**)$.

Induction on k, $k = 0,\ G_0 = Z_m = G$ -true.

If $(**)$ is true for k, show that $G_{k+1} \leqslant Z_{m-k-1}$.

By definition, $G_{k+1} = [G_k, G] \leqslant [Z_{s-k},\ G]$, but

$Z_{m-k}/Z_{m-k-1} = Z(G/Z_{m-k-1}) \Rightarrow [Z_{m-k},\ G] \leqslant Z_{m-k-1} \Rightarrow G_{k+} \leqslant Z_{m-k-1} \Rightarrow$ $(**)$ is valid for all m, if $Z_m = G \Rightarrow (k = m),\ G_M \leqslant Z_0 = \{e\}$.

■

Proof. $(2 \Rightarrow 3)$

Let H be any proper subgroup of G and $Z_0 < Z_1 < \cdots < Z_m = G$ is the upper central series.

Evidently, if $Z(G) \lneq H$, $H < Z(G)$, $H \leqslant N_G(H)$.

Otherwise, $Z(G) \leqslant H$, we have $Z_1 \leqslant H$.

$\Rightarrow \exists j\ Z_j \leqslant H \leqslant$, because H is proper.

$\Rightarrow Zj + 1 \leqslant N_G(H)$, as $Z_{j-1}/Z_j = Z(G/Z_j)$

$H < H \cdot Z_{j+1} \leqslant N_G(H)$, we have proved $2 \Rightarrow 3$.

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Proof. $(3 \Rightarrow 4)$

Let $|G| < \infty$, $|G| = {p_1}^{n_1} \cdot \cdots \cdot {p_r}^{n_r}$, ($p_i$ are primes, $p_i \neq p_j,\ i \neq j$).

We know that (?), if P is some Sylow p-subgroup of G. $H = N_G(P) \Rightarrow N_G(P) = H$. (Lemma)

Take some $a \in N_G(H) \Rightarrow aHa^{-1} \in H$.

$P \leqslant H = N_G(P),\ P \in Syl_P(H),\ aPa^{-1}$ is another Sylow p-subgroup of H,

$\Rightarrow$ by the (2) Sylow Theorem, for H, $\exists h \in H$, $h^{-1}aPa^{-1} = hPh^{-1}h = H \Rightarrow P = (h^{-1}a)P(a^{-1}h) = h^{-1}a \in N_G(P) = H \Rightarrow a \in h \cdot H = H \Rightarrow N_G(H) = H$

If follows from Lemma, that $P \triangleleft G$, if $H = N_G(P) < G$, then $N_G(H) > H$ -a contradiction. $\Rightarrow$ all Sylow subgroups of G are normal in G.

$$G = P_1 \times \cdots \times P_s$$

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Lemma 1. *A p-subgroup is nilpotent.*

Proof. Show that if P is a p-group that it has finite upper central series, $Z_m(P) = P$ for some $m \in \mathbb{N}$

IF P is Abelian $\Rightarrow P = Z(P) = Z_1$ -true.

Otherwise, consider $Z_1 = Z(P) > \{e\}$, use induction of $|P|$.

$|nicefracPZ_1| = |P/Z(G)| < |P|$, $\overline{P} = P/Z(G) \Rightarrow \exists s,\ \overline{Z_S} = Z_S(\overline{P}) = (\overline{P})$.

$\pi : P \xrightarrow{canonical} \overline{P} = P/Z_1$

$\pi(Z_1) = \{e\}$

$\pi^{-1}(\overline{Z_s}) = P \Rightarrow \exists upper\ central\ series.$
$\Rightarrow Z_s = P$

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Lemma 2. *If the groups $P_1, \ldots, P_s$ are nilpotents, then $P_1 \times \cdots \times P_s$ is nilpotent.*

Proof. **(Exercise)**

$$Z(P_1 \times \cdots \times P_k) \overset{?!}{=} Z_k(P_1) \times \cdots \times Z_k(P_k)$$

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Proof. $4 \Rightarrow 2$ ($|G| < \infty$) is evident. ■

4 LECTURE 04 (22.03.2023)

4.1 Minimal non-nilpotent groups (Schmidt's groups)

Definition 4.1. A group is **minimal non-nilpotent**, if G is non-nilpotent, but $\forall H < G$ is nilpotent.

Example 1 $G = \mathbb{Z}_p \leftthreetimes \mathbb{Z}_q$, p, q are primes, $p > q,\ q|(p-1)$

Example 2 $G = (\mathbb{Z}_p \leftthreetimes \mathbb{Z}_p) \leftthreetimes \mathbb{Z}_q$, if $Aut(\mathbb{Z}_p \leftthreetimes \mathbb{Z}_p)$ is divisible by q ($Aut(\mathbb{Z}_p \leftthreetimes \mathbb{Z}_p)| \vdots q$),

$Aut(\mathbb{Z}_p \leftthreetimes \mathbb{Z}_p) \cong GL(2, p)$

Theorem 4.1. *Let G be a finite minimal non-nilpotent group, then*

1. *G is solvable.*
2. *$|G| = p^\alpha q^\beta$, (p, q are distinct primes).*
3. *$G = P \leftthreetimes Q$ ($P \lhd G,\ |P| = p^\alpha,\ |Q| = q^\beta$), Q is cyclic, $d(P) \leqslant 2$, and Q acts on P by automorphisms of order q.*

Proof. (1)

By contradiction, let G be a contrary of minimal order, G is not solvable, but any group of order $< |G|$, that satisfies the contradiction of the theorem is solvable.

Suppose: $\exists 1 \neq N \lhd G,\ \Rightarrow N$ is nilpotent, and ${}^G/_N$ satisfies the condition (every proper subgroup of ${}^G/_N$ is nilpotent). N is nilpotent $\Rightarrow N$ is solvable, $|{}^G/_N| < |G| \Rightarrow {}^G/_N$ solvable, by minimality of G, $\Rightarrow G$ is solvable.

${}^G/_N$ solvable, by minimality of G, $\Rightarrow G$ is solvable, -contradiction. $\Rightarrow G$ is simple.

We claim that any two different maximal subgroup of G, have trivial intersection.

By contradiction, suppose that $\exists M_1 \neq M_2$, maximal subgroup of G, such that $H := M_1 \cap M_2 \neq \{1\}$. (Choose them in such a way that H have minimal possible order).

We have $H \leqslant M_1$, $H \leqslant M_2$, $M_1 \neq G \Rightarrow M_1$ is nilpotent.

By normalizer condition, $H < M_{M_1}(H) := H$.

$N_{M_1}(H) = M_1 \cap H \lneqq G$, N is contained in some maximal subgroup M of G, and $H < M_1 \cap N \leqslant M_1 \cap M$. $|M_1 \cap M| > |H|$, -contradiction $\Rightarrow M_1 \cap M_2 = \{1\}$.

Final contradiction: $\forall g \in G$, g is contained in some (and unique) maximal subgroup of G, $< g > \leqslant m$

$$G = \bigsqcup_k (M_k - \{1\}) \cap \{1\}$$

Let G has s conjugate class of maximal subgroup (with representatives M_i. $1 \leqslant i \leqslant s$).

$$G = \bigsqcup_{i=1}^{s} (M - \{1\}),\ M \in \{e(M_i) \cup \{1\}\}$$

Note that for any maximal subgroup $M < G$, $N_G(M) = M \Rightarrow$ the number of subgroups conjugated with M equals $\frac{|G|}{|N_G(M)|} = \frac{|G|}{|M|}$.

Let's denote that $|G| = n$, $|M_i| = m_i$. $\frac{|G|}{|M_i|} = k_i$,

$$n = |G| = i + \sum_{i=1}^{s}(|M_i| - 1) \cdot \frac{|G|}{|M_i|} = 1 + s|G| - \sum_{i=1}^{s} k_i$$

$$n = 1 + sn - \sum_{i=1}^{s} k_i \geqslant 2n - (k_1 + k_2)$$

Evidently, $s \geqslant 2$

Exercise: No finite group $G \neq \{1\}$ can't be covered with conjugates of a single $H < G$.

$$n < k_1 + k - 2/ : n \Rightarrow \frac{1}{m_1} + \frac{1}{m_2} > 2$$

On the other hand: $m_i \geqslant 2,\ i = 1, 2\ \frac{1}{m_1} + \frac{1}{m_2} \leqslant \frac{1}{2} + \frac{1}{2} = 1$ -contradiction.

This contradiction shows that G is not simple $\Rightarrow$ G is solvable.

■

Proof. (2)

$|G| = p^\alpha q^\beta$, (p, q are different primes, $\alpha \geqslant 1, \beta \geqslant 2$). In general,

$$n = |G| = \prod_{i=1}^{r} {p_i}^{\alpha_i} (p_i, \ldots, p_r are\ distinct\ primes).$$

By contradiction: suppose that $r \geqslant 3$, As G is solvable, we can find a maximal normal subgroup M of a prime index (say. P_1).

$G' \leqslant G$, ${}^{G}/{}_{G'}$ is the finite Abelian group, $\Rightarrow \exists M = G = {}^{G}/{}_{G'}$ of a prime index p_i, $\Rightarrow \pi^{-1}(\overline{M}) = M_{p_i} \lhd G$.

Denote as P_i, -Sylow P_π subgroup of G. All $P_i (i = 2, \ldots, r)$, $P_i \leqslant M$, M is nilpotent.

$$M = P_2 \times \cdots \times P_r \times P_\pi \Rightarrow P_i\ char\ M \lhd G$$

$$P \lhd G,\ i = 2, \ldots, r$$

As $r = 3$, then subgroup $P_1 \times P_2,\ P_1 \times P_3$ are proper subgroups of G, $\Rightarrow$ all products are distinct $\Rightarrow$ the elements of P_i commute with the elements of all P_i $(i \geqslant 2)$.

$G = P_1 \times \cdots \times P_r \Rightarrow G$ is nilpotent. -Contradiction. $\Rightarrow r = 2$.

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Proof. (3)

$|G| = p^\alpha q^\beta$. Denote P, Q -Sylow subgroups of G. We can choose a maximal normal subgroup M of index g.

M is nilpotent $\Rightarrow M = P \times Q \Rightarrow P\ char\ M \Rightarrow P \lhd G$.

Q is cyclic. If not $\forall g \in G,\ < g >< Q$

Consider $< g, P >=< g > P =< g > \times P \Rightarrow$ element $g \in G$ commutes with any $h \in P \Rightarrow G = p \times Q$ -nilpotent. -Contradiction.

Q is cyclic.

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5 LECTURE 05 (29.03.2023)

5.1 Ph. Hall's theorems

Definition 5.1. Let G be a finite group $|G| = n$.

$k \in \mathbb{N}$ is called a **Hall divisor** of n. If $k|n$, and $(k, \frac{n}{k}) = 1$, if G has a subgroup $H < G$, $|H| < K$, H Is called **Hall subgroup** of G. When $k = p_r$ (p is a prime), then H will be a Sylow p-subgroup of G.

Theorem 5.1. *(Hall's Theorems) Let G be a finite solvable group. Then*

1. *$\forall k$ -Hall divisor of $n = |G|$, $|G|$ has Hall's subgroups of order k, (Existence).*
2. *Any two subgroups of order k are conjugates in G (Conjugate).*
3. *If K is some subgroup of G of order $|K| \mid k$ is contained in some subgroup of order k (Inclusion).*

Lemma 1. *If G is a finite solvable group, N is a minimal subgroup of G, then N is elementary Abelian: $N \cong \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r\ times,\ r \geqslant 1}$ (p is some prime number).*

Proof. As N is a minimal normal subgroup M of G, if M were such a subgroup, then $M \lhd G$, $M \lneqq N$, but N is minimal normal subgroup.

$\Rightarrow$ N is characteristically simple $\Rightarrow$ N is a direct product of some isomorphisms of some simple subgroups $\cong \mathbb{Z}_p$

■

Lemma 2. *(Frattini-Argument)*

Let $H \lhd G$. $p \mid |H|$, $P \in Syl_p(H)$. Then $G = H = N_G(P)$

Proof. Take any $g \in G$, and consider the subgroup $gPg^{-1} \leqslant H$, moreover this subgroup is Sylow in $H \Rightarrow$ by 2nd Sylow theorem applied to H. $\exists h \in H$, such that $gPg^{-1} = hPh^{-1}$.

$\Rightarrow (h^{-1}g)P(h^{-1}g)^{-1} = P \Rightarrow h_{-1}g = n \in N_G(P) \Rightarrow g = hn$

■

Proof. (of 1st Hall's theorem)

Use induction on $n = |G|$. (It's clear when G is Abelian)

$n = 1$, it is trivial.

If $n > 1$ inductive hypothesis.**Th.1** is true for all solvable groups of order $< n$ and Hall's divisor of their order.

If $A \lhd G$, A be a minimal normal subgroup of G. By **L.1**, $A \cong \mathbb{Z}_P{}^n,\ r \geqslant 1$.

There are two possibilities.

1. $p|k$, Consider $\overline{G} = {}^{G}\!/_{A}$, it is solvable of order $\frac{n}{p^r} < n \Rightarrow G$ has a subgroup $\overline{H} < \overline{G}$ of order $\frac{k}{p^r} = k'$.

   $H := \pi^{-1}(\overline{H})$, where $\pi : G \to \overline{G} = \frac{G}{A}$

   $\Rightarrow |H| = k$.

2. $p \nmid k \Rightarrow (p, k) = r$

Consider $D \lhd G$, of maximal order $(|D|, k) = 1$ (Set set of such subgroups is not empty:A)

Consider $\overline{\overline{G}} = {}^{G}/_{D} = \overline{e}$. $\overline{\overline{G}}$ is solvable. $\Rightarrow \exists$ a minimal normal subgroup. $\overline{\overline{b}} \lhd \overline{\overline{G}}$, a such $|B| = q^m$. (q is a prime, $q \neq p$, $m > 1$)

$\overline{\overline{B}} = {}^{B}/_{D}$, $|B| = q^m \cdot |D|$, $q|k$, $(|D|, k) = 1$

$|B| < |Q|$, Denote by Q, a Sylow q-subgroup of B.

(2a) $Q \lhd G$, then we could replace A with Q and p with Q and p with q, and get the designed subgroup in G as in **1**.

(2b) $Q \ntriangleleft G \Rightarrow N_G(Q) = B \cdot D \Rightarrow k|B|$, $|B| < |G|$ ($|B| = q^m \cdot |D|$, $|G| = k \cdot k'$)
$\Rightarrow$ by induction B contains a subgroup of order k.

■

Note: Without the condition of solvability of G, neither of theorems 1 2 3 is true.

Example 1 Consider $G = A_5$ -the simple group $|A_5| = 60 = 2^3 \cdot 3 \cdot 5$. A_5 has no subgroup orders 20 and 15.

Any subgroup $H < A_5$, $|H| = 15$ would be cyclic, but A_5 contains no commuting elements of order 3 and 5.

Proposition If a group contains a subgroup H of index $|G : H| \leqslant 4$, Then G cannot be simple.

Idea Consider the action of G on $X = {}^{G}/_{H}$ if left cosets by H by means of left multiplication $\Rightarrow$ it gives a homomorphism.

$$\varphi : G \to S_n \; (n \geqslant 4, \; n = {}^{G}/_{H})$$

$$S_3, \; S_4 \; is \; solvable$$

$$\Rightarrow It \; gives \; a \; non-trivial \; subgroup.$$

Example 2 The simple group $G = PSL(2, \mathbb{Z}_7) \cong GL(3, \mathbb{Z}_2)$ of order 168 has some subgroup o f order 24, that are non-ismorphic. $H_1 \cong S_4$, $H_2 = SL(2, \mathbb{Z}_3)$.

Theorem 5.2. *(Philip Hall, S.A. Chunikhin)*
If p | $|G|$, $|G| = p_k \cdot m$ is solvable $\Rightarrow \exists H < H, |H| = m$ (Hall's p'-subgroups).
If G has p'-Hall's subgroup for any p | $|G|$, then G is solvable.

Theorem 5.3. *If G contains a nilpotent Hall subgroup of H, then all subgroups of G order H are conjugated.*

6 LECTURE 06 (05.04.2023)

6.1 Linear representation

Let G be a group.

Definition 6.1. Linear representation of the group is some homomorphism:

$$\rho : G \to GL(V) \xrightarrow{\sim} GL(n, \mathbb{K}) = A_{n\times n} \mid |A| \neq 0,\ (\dim(V) = n)$$

(Invertible linear operators in a vector space V over some field $\mathbb{K}$)
ρ -matrix representation corresponding to the operator.

Definition 6.2. $U \subset V$, U is $\rho(G)$**-invariant subspace** (or invariant subspace) of the representation ρ, if $\forall u \in U$, $\forall g \in G$, $\rho(g)u \in U$.

As $\rho(g)$ are invertible $\Rightarrow \rho(g)V = U$

Definition 6.3. The representation is (G, V, ρ).
It is **reducible**, if $\exists$ some invariant subspace U, $0 \neq U \neq V$, otherwise it is called **irreducible**.

6.1.1 Linearization of a permutation representation

Let a group G acts on a finite set X, $X = \{x_1, \ldots, x_n\}$, consider a vector space $V_x = < e_x(basis)|x \in X >_{\mathbb{K}}$, and define the action of G on V: $\rho(g)e_x = e_{g\cdot x}$ and then by linearity:

$$\rho(g) = \sum_{x\in X} \alpha_i e_x = \sum \alpha_x e_{g\cdot x}$$

Evidently, this is a linear representation of G.
Evidently, subspaces $V_1 =< e_q + \cdots + e_n >= a$, $(e_i = e_{x_i})$. $\rho(g)$ permutes basic vectors. $\rho(g)a = a$.

$$V_0 = \{\sum_{i=1}^{n} \alpha_i e_i | \sum_{i=1}^{n} \alpha_i = 0\},\ (V_0\ is\ invariant.)$$

Let $\mathbb{K} = \mathbb{C}$, introduce the scalar product $((\alpha_i), (\beta_i)) = \sum \alpha_i \overline{\beta_i}$. $V_0 = {V_1}^{\perp}$ and $V = V_0 \oplus V_1$, V_0, V_1 are invariant.
When $char\mathbb{K} = p \mid n$, then $V_1 \subset V_0$, $V \neq V_0 \oplus V_1$.

Definition 6.4. The representation (G, V, ρ) is called **completely reducible** id for any invariant subspace $U \subset V$, there exists an invariant complement W: $V = U \oplus W$.

Theorem 6.1. *If ρ (or V) is completely reducible,* $\dim V < \infty$*, then* $V = V_1 \oplus \cdots \oplus V_s$. *$V_i$ are invariant subspaces, minimal invariant, if ($U \subseteq V_i \Rightarrow U = \{0\}$ or $U = V_i$). $\rho(G)$ -invariant V_i.*

6.1.2 Subrepresentation

Definition 6.5. If $U \subset V$, U is $\rho(G)$-invariant, we can restrict all the operators $\rho(g)$ to U, $(\rho(g)|_U)$ and get a **subrepresentation** $(\rho|_{(U)}, G, U)$:

$$\rho|_U : G \to GL(U)$$

6.1.3 Factor subspace

Definition 6.6. U is $\rho(G)$-invariant subspace of V, construct the **factor space**: $\overline{U} = {}^{V}\!/_{U} = \{v+U | v \in V\}$

$\forall\lambda \in \mathbb{K}$, define $\lambda \cdot \overline{u} = \lambda v + U = \lambda\overline{v}$

In matrix form, if we choose a basis of V, in correspondence with U, $e_1, \ldots, e_m$ -a basis of U, $e_{m+1}, \ldots, e_n$ -some vectors that $\overline{e_{m+1}}, \ldots, \overline{e_n}$, constitute a basis of $\overline{V}$.

$$\forall g \in G,\ A_{\rho(g)} = \begin{pmatrix} A_{\rho(g)} \cdot t & C \\ 0 & B \end{pmatrix} \cdot B = A_{\overline{\rho}(g)}$$

$$\overline{\rho}(g)(v+U) = \rho(g)v + U$$

When ρ (or V) is decomposed into direct sum of minimal invariant subspaces $V = V_1 \oplus \cdots \oplus V_s \Rightarrow$

$$\Rightarrow A_{\rho(g)} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix},\ A_i = A_{\rho(g)}|_{V_i}$$

6.1.4 Direct sum of representations

Definition 6.7. Internal: if $V = V_1 \oplus \cdots \oplus V_s$, where V_i are invariant subspaces, then $\rho = \rho_1 \oplus \cdots \oplus \rho_s$, where $\rho_i = \rho|_{V_i}$.

ρ is the **direct sum** if subrepresentations.

Definition 6.8. External: Let $W_1, \ldots, W_s$ are arbitrary spaces over the field $\mathbb{K}$, the same $W_1 \oplus \cdots \oplus w_1, \ldots, w_s W_s$ with evident linear operators if there are given liner representations:

$$\varphi_i : G \to GL(W_i)_j,\ (1 \leqslant i \leqslant s)$$

then we can define for any $g \in G$, the linear operator

$$\varphi(g)(w_i, \ldots, w_s) = \varphi_1(g)w_1, \ldots, \varphi_s(g)w_s$$

We get a representation

$$\varphi(G) \to GL(W)$$

Define $V_i = \{0, \ldots, w_i, \ldots, 0\}$ not only as vectors, but also linear representations.

6.1.5 Homomorphism of two linear representations

Definition 6.9. Let (ρ, G, V) and (φ, G, W) are linear representation.

A linear map: $f : V \to W$ is called **homomorphism** of representation ρ and φ, if the diagram is commutative i.e.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \varphi(g)} \\ V & \xrightarrow{f} & W \end{array}$$

Where $\forall g \in G,\ \forall v \in V,\ f(\rho(g))v = \varphi(g)(f(v))$, or simply $f \circ \rho = \varphi \circ f$.

Representations ρ and φ (or V and W) are called **isomorphic**, if $\exists$ a non-degenerated homomorphism: $f : V \to W$, f is called **isomorphism**.

If $\rho \cong \varphi$, then $\dim V = \dim W$.

6.1.6 Tensor product of linear representations

Definition 6.10. Let V_1, V_2 over two vector spaces of the same field $\mathbb{K}$, consider $V =< V_1 \times V_2 >= \{\sum \alpha_{V_1,V_2}(V_1, V_2)\}$ with finite number of summerance:

$$\begin{aligned} U =< &(v_1' + v_2'', v_2) - (v_1', v_2) - (v_1'', v_2), \\ &(v_1, v_2' + v_2'') - (v_1, v_2') - (v_1, v_2''), \\ &(\lambda v_1, v_2) - \lambda(v_1, v_2), (v_1, \lambda v_2) - \lambda(v_1, v_2) > \end{aligned} \tag{1}$$

Call $V_1 \otimes V_2 = {}^{V}/_{U} =< v_1 \otimes v_2 | v_i \in V_i,\ i = 1, 2 >$ the tensor product.

If (G, V_1, ρ_1) and (H, V_2, ρ_2) are linear representations, we define $(G \times H, V_1 \otimes V_2, \rho_1 \otimes \rho_2)$ namely $(\rho_1 \otimes \rho_2)(g \cdot h)(v_1 \otimes v_2) = (\rho_1(g)v_1) \otimes (\rho_2(h)v_2)$ as the **tensor product** of the two linear representations.

If $\dim V_{1,2} < \infty$, then $\dim(V_1 \otimes V_2) = \dim V_1 + \dim V_2$.

7 LECTURE 07 (12.04.2023)

7.1 Schur's Lemma

Lemma 1. *(Schur's Lemma) If $f : U \to W$ is a homomorphism of irreducible representations (G, ρ, V) and (G, ρ, W), then $f = 0$, or f is an isomorphism.*

Proof. Note that $Kerf$ is $\rho(G)$-invariant subspace of V. Take $v \in Kerf,\ g \in G$,

$$f(\rho(g)v) \overset{\text{def}}{=} \varphi(g)(f(v)) = 0$$

As V irreducible, there are two possibilities:

either $Kerf = V \Rightarrow f = 0$

or $Kerf = \{0\} \Rightarrow f$ is monomorphism. (f is injective).

Note that Imf is $\varphi(G)$-invariant subspace, take any $w \in W \Rightarrow \exists v \in V, f(v) = w$, $\forall g \in G, \varphi(g)w = \varphi(g)(f(v)) = f(\underbrace{\rho(g)v}_{\in V}) \in Kerf$

As W is irreducible, then either $f(v) = g$, that is not the case $\Rightarrow f(v) = w \Rightarrow f$ is isomorphism. ■

7.1.1 Multiplicity

Definition 7.1. If a representation V is completely reducible, then $V = V_1 \oplus \cdots \oplus V_r$ -direct sum of irreducible representations. Let U be irreducible representation of G. Say that U is a component of V, if $\exists i,\ 1 \leqslant i \leqslant r;\ U \cong V_i$

We can collect all the summons in V, isomorphic to U,

$$V = \underbrace{V_1 \oplus \cdots \oplus V_m}_{v_1,\ldots,v_m \cong U} \oplus W$$

m is called **multiplicity** if U in V.

Lemma 2. *m doesn't depend on the decomposition of V.*

Proof. Consider $Hom_G(V, U) = Hom_G(V_1 \oplus \cdots \oplus V_m, U) \cong$

$$\cong \bigoplus_{i=1}^{r} HomG(V_i, U) \cong Hom_G(U, U)$$

By the Schur's Lemma:

$$Hom_G(V_i, U) = \begin{cases} 0,\ if\ V_i \neq U \\ \cong Hom_G(U, U),\ if\ V_i \cong U \end{cases}$$

$\Rightarrow \dim Hom_G(V_i, U) = m \cdot \dim End_G(G) \Rightarrow m$ is unique.

■

Take into account:

Theorem 7.1. *(the Machket's Theorem)*

Let V be a finite-dimensional representation of a finite group (G over a field $\mathbb{K}$). If $char\mathbb{K} = 0$ or $char\mathbb{K} = p \mid |G|$, then V is completely reducible.

Such a representation is called common of non-modular. Consider the following situation:

U and W are two representations of a finite group G, and **M. Theorem** is valid. Suppose we are given an epimorphism of representation:

$$f : V \to W,\ then$$

$$V \cong W \oplus Kerf$$

$Kerf$ is an invariant subspace if $V \Rightarrow \exists$ some invariant $W' \subset V$, such that $V = W \oplus Kerf$, but $W = Imf \cong {}^{V}/_{Kerf} \cong W'$.

7.1.2 Regular representation of the group G

Theorem 7.2. *Every irreducible representation U of G is isomorphic with some invariant subspace of the regular representation of G, (i.e. U is a component of the regular representation with a positive multiplicity).*

Let $G = g_1, \ldots, g_n$, $V_G =< e_g | g \in G >$, $reg(g)(e_h) = e_{gh}$ (Linearization of the left regular representation).

$V_1 =< \sum e_g >$, $reg_G|_{V_1}$ is one-dimensional identity if representation.

$V_0 = \{v = \sum x_g e_g | \sum x_g = 0\} = {V_1}^{\perp}$

We may make V_G to an algebra over $\mathbb{K}$. Define $e_g e_h = e_{gh}$

Identify V_G with the algebra $\mathbb{K}G = \{\sum x_g g\}$ with the evident multiplicity. We may consider the action of $\mathbb{K}G$ on itself by left multiplications and reg_G is the restriction of this action on G.

(Proof of the Theorem)

Construct an epimorphism of representations:

$$f : \mathbb{K}G \to U$$

Where U is irreducible representation of $G(G, \rho, U)$.

Pick certain vector $u \in U$, define

$$f(1) = u_0$$

$$f(g) = f(reg(g) = 1) := \rho(g)u_0 \in U$$

$g : G \to \{\rho(g)u_0 | g \in G\}$, and $f : \mathbb{K}G \to< \rho(g)u_0 >\subseteq U$. Evidently, that this subspace is invariant in U. We have constructed an epimorphism f of representation.

$$\mathbb{K}G = (V_G, reg, G)\ onto\ (U, \rho, G)$$

In this situation, $\exists$ some $E \in V_G$, $reg(G)$-invariant, such that $V_G = W \oplus \ker f \Rightarrow U \cong W$, and we are ready.

7.2 Character theory (over $\mathbb{C}$)

Let G be a finite group, ρ: a certain representation.

$$\rho : G \to GL(V) \cong GL(n, \mathbb{C})$$

$$\dim(V) = n$$

matrix representation

Definition 7.2. The **Character** χ_ρ of the representation ρ, is the function:

$$\chi_\rho : G \to \mathbb{C}$$

$$\chi_\rho(G) = tr\rho(G) = trA_{\rho(G)} = \sum_{i=1}^{n} a_{ii}(g)$$

Some properties:

1. $\chi_\rho(1) = \dim(V)$.
2. $\forall g, h \in G, \chi_\rho(h) = \chi_\rho(g^{-1}h)g$.
3. If ρ and φ are two isomorphic representations, then $\chi_\rho(g) = \chi_\varphi(g)$.
4. If $\rho = \rho_1 \oplus \cdots \oplus \rho_s$,-direct sum of some (sub) representations, then $\chi_\rho = \sum \chi_{\rho_i}$
5. $\chi_{\rho\otimes\varphi}(g) = \chi_\rho(g) \cdot \chi_\varphi(g)$

Proof. 1. $\rho(1) = E_n$. (identical operator, $trE_n = n$)

3. $trA_{\rho(g)} = tr(C^{-1}A_{\rho(g)}C) = trA_{\rho(g)}$. ($C$-transition matrix to a new basis)

2. $\chi_\rho(g^{-1}hg) = trA_{\rho(g^{-1})\rho(h)\rho(g)} = trA_{\rho(g^{-1})}trA_{\rho(h)}trA_{\rho(g)} = trA_{\rho(h)}$.

4. $V = V_1 \oplus \cdots \oplus V_s$

In an appropriate basis:

$$A_\rho(g) = \begin{pmatrix} A_{\rho_1(g)} & & 0 \\ & \ddots & \\ 0 & & A_{\rho_s(g)} \end{pmatrix} \Rightarrow trA_\rho = \sum_{i=1}^{s} trA_{\rho(g)}$$

5. Let (ρ, V), (φ, W) are representations of the groups G, H, $(\rho \otimes \varphi, V \otimes W, G \times H)$.

$(\rho \otimes \varphi)(g, h)(v_i \otimes w_j) = (\rho(g)v_i) \otimes (\varphi(h)w_j)$. Note That for $h = g$, we get a representation of G, if:

$$A = A_{\rho(g)},\ B = A_{\varphi(g)} \Rightarrow$$

$$A_{\rho\otimes\varphi(g)} = A \times B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}$$

$$tr(A \times B) = \sum_i tra_{ii}B = trA \cdot trB$$

∎

8 LECTURE 08 (19.04.2023)

8.1 Consequences of Schur's Lemma

If φ is an irreducible representation of G (over $\mathbb{C}$), then its character χ_φ is an irreducible character. (It means that $\chi_\varphi \neq \eta + \theta$, where η, θ are characters.)

Lemma 1. ***Schur's Lemma***

If $\varphi : G \to GL(V), \psi : G \to GL(W)$ are irreducible representations, and $f : V \to W$ is a homomorphism of these representations, then $f = 0$, (if $\varphi = \psi$), or f is isomorphic.

Consequence: If K is alg closed, $V \equiv W$, f is isomorphism $\Rightarrow f\lambda E, \lambda \in K$.

Proof. As K is alg closed, then $f : V \to V \equiv W$ has an eigenvector v with eigenvalue λ, then the subspace:

$$V_\lambda = \{v \in V | f(v) = \lambda v\} \neq 0$$

is an invariant subspace of $V \Rightarrow V_\lambda = V \Rightarrow f = \lambda E$ is a scalar operator. ∎

Lemma 2. *($K = \mathbb{C}$) In the condition of* ***Schur's Lemma****:*

Let $f : V \to W$ be some linear mapping. Consider the average mapping:

$$\widetilde{f} := \frac{1}{|G|} \sum_{g \in G} \psi(g) f \varphi(g)^{-1} = \begin{cases} 0, \ if \ \varphi \neq \psi \\ \lambda, \ if \ V \equiv W, \ \varphi = \psi, \ where \ \lambda = \frac{trf}{\dim V} \end{cases}$$

Proof. It's evident that: $\widetilde{f}\varphi(h) = \psi(h)\widetilde{f} \Rightarrow$ by the Consequence (where $V = W$, $\varphi = \psi$), $\widetilde{f} = \lambda E$, calculate the trace of $\widetilde{f}$:

$$tr\widetilde{f} = trf = \lambda trE = \lambda \dim V$$

$$\lambda = \frac{trf}{\dim V}$$

∎

8.2 Matrix Version of the Lemma

Choose bases in V and W:

$$\{v_i | i \in I \subset V\}, \ \{w_j | j \in J \subset W\}$$

The matrices of operators are denoted with the same letters

$$\varphi(g) = (\varphi_{ii'}(g)), \ \psi(g) = (\psi_{jj'}(g))$$

$$f = (f_{ji}), \ \widetilde{f} = (\widetilde{f}_{ji})$$

By the definition of $\widetilde{f}$, we can write:

$$\widetilde{f}_{ji} = \frac{1}{|G|} \sum_{g, i', j'} \psi_{jj'}(g) f_{j'i'} \varphi_{i'i}(g^{-1}) \tag{2}$$

If we take $f = E_{j_0 i_0}$, $(f_{j_0 i_0} = 1,\ f_{ji} = 0,\ if\ (j,i) \neq (j_0, i_0))$

1. if $\varphi \neq \psi$, then from (2) $\Rightarrow$

$$\frac{1}{|G|}\sum_{g\in G}\psi_{jj_0}(g)\varphi_{ij}(g') = 0, \forall i, j, i_0, j_0 \tag{3}$$

2. when $V \equiv W$, $\varphi = \psi$, then

$$\widetilde{f} = \frac{trf}{\dim V}E$$

$$trf = \sum_i f_{ii} = \sum_{i,i'}\delta_{j'i}\cdot f'_{j'i} \ \Rightarrow$$

$$\frac{1}{|G|}\sum \varphi_{jj'}(g)\cdot f_{j'i}\cdot\varphi_{i'i}(g^{-1}) = \frac{\delta_{ji}}{\dim V}\sum_{j',i'}\delta_{j'i'}\cdot f_{j'i'}$$

Taking $f = E_{j_0 i_0}$, we got:

$$\frac{1}{|G|}\sum_{g\in G}\varphi_{jj_0}(g)\varphi_{i_0 i}(g^{-1}) = \begin{cases}\frac{\delta_{ji}}{\dim V}, if\ i_0 = j_0,\\ 0,\ otherwise\end{cases}$$

8.3 Orthogonality relation of characters

Definition 8.1. Define on the space F_G of all functions $f : G \to \mathbb{C}$, the Hermitian scalar product:

$$(f_1, f_2)_G := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

If χ, θ are characters, then they are constant on conjugate classes, if $G = \bigcup_{i=1}^{r} K_i$, $(K_1 \ldots K_r)$ are all conjugate classes, so

$$(\chi, \theta)_G = \frac{1}{|G|}\sum_{i=1}^{r}|K_i|\chi(g_i)\theta(g_i),\ when\ g_i \in K_i$$

Theorem 8.1. *(The first orthogonality relation) Let φ, ψ are irreducible Representations of G over $\mathbb{C}$*

$$\frac{1}{|G|}\sum_{g\in G}\psi_{jj_0}(g)\varphi_{i_0 i}(g^{-1}) = 0,\ \forall i, j, i_0, j_0$$

then

$$(\chi_\varphi, \chi_\psi) = \delta_{\varphi,\psi} = \begin{cases}1,\ if\ \varphi \neq \psi,\\ 0,\ otherwise.\end{cases}$$

Proof. $\chi_{\varphi(g)} = \sum_i \varphi_{ii}(g)$, $\chi_{\psi(g)} = \sum \psi_{jj}(g)$. Put $i = i_0 j = j_0$ and sum over all $i, j \Rightarrow$

$$\frac{1}{|G|}\sum_{g,i,j}\psi_{ij}(g)\varphi_{ji}(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\psi_{ij}(g)\varphi_{ji}(g^{-1}) = (\chi_\psi, \chi_\varphi)_G$$

■

Consequence 1: The number s of all pairwise non-isomorphic irreducible complex representations of G, $s = r$ -the number of conjugate classes of G.

Proof. The irreducible characters $\chi_1 \ldots \chi_s$ are orthogonal $\Rightarrow$ they are linearly independent, but dim $ZF(G)$ (the space of central class) functions on G, which are constant on conjugate classes.

■

9 LECTURE 9 (26.04.2023)

10 LECTURE 10 (03.05.2023)

A Sylow Theorems

Sylow Theorems [1]

References

[1] James S. Milne. Group theory (v4.00), 2021. Available at www.jmilne.org/math/.