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<u>Th</u>. If $\varphi: G \to GL(V)$, $\psi: H \to GL(U)$ are irreducible representations, then $\varphi\otimes\psi: G\times H \to GL(V\otimes U)$ is irreducible

Suppose: V, U are finite-dimensional, $\overline{K}=K$

<u>Proof</u>. Suppose $0 \neq W \subset V\otimes U$, W is $\varphi\otimes\psi$-invariant.
Consider the representation $\varphi\otimes \mathrm{Id}_H = (\varphi\otimes\psi)\cdot i_1$, $i_1(g)=(g,e_H)$ - natural embedding $i_1: G \to G\times H$.
Obviously, W is invariant under this representation.
It follows that W contains a subspace of form $V\otimes u_0$, $u_0\in U$.
$\forall a \in V\otimes U$ can be written in a form $x_1\otimes f_1+\dots+x_m\otimes f_m$, where $\{f_1,\dots,f_m\}$ is some basis of U. In particular, $\forall w\in W$
$\exists$ vectors $\sigma_1(w),\dots,\sigma_m(w)$, s.t. $w=\sigma_1(w)\otimes f_1+\dots+\sigma_m(w)\otimes f_m$,
and $\sigma_i((\varphi\otimes \mathrm{Id}_H)w)=\varphi(g)\sigma_i(w)$, $\forall g\in G$. $\Rightarrow$ σ_i are homomorphisms of the representation $(\varphi\otimes\mathrm{Id}_H)|_W \to \varphi$.
By the consequence of Schur's Lemma, $\sigma_i = c_i\sigma$, σ some isomorphism (cf. "Th.3, p.46 by Vinberg") $\Rightarrow$ $(c_i\in K^*)$
$w=\sigma(w)\otimes(c_1f_1+\dots+c_mf_m)$, $u_0 = c_1f_1+\dots+c_mf_m$.
$\forall x\in V$ consider the subspace $U_x=\{u\in U \mid x\otimes u\in W\}\subset U$.
It is "evident" that U_x invariant under $\psi(H)$, $U_x\neq\{0\}$.
As U is irreducible, $U_x=U \Rightarrow x\otimes u\in W$, for all $x\in V$, $u\in U$
$\Rightarrow W=V\otimes U$, q.e.d.

Prop. If A is a finite Abelian group, K is algebraically closed field, $\mathrm{char}\,K\nmid|G|$, then the group of characters of A
$\hat{A}=\{\lambda: A\to K^*\}\cong A$.

<u>Proof</u>. $A=\langle a_1\rangle\times\dots\times\langle a_r\rangle$, $|a_i|=n_i$.
$\forall a\in A$: $a=a_1^{k_1}\cdot\ldots\cdot a_r^{k_r}$, $0\le k_i\le n_i-1$.
$\forall\lambda\in\hat{A}$, $\lambda(a)=\lambda(a_1)^{k_1}\cdot\ldots\cdot\lambda(a_r)^{k_r}$; $\lambda(a_i)^{n_i}=1$, $\lambda(a_i)\in\{\sqrt[n_i]{1}\}<K^*$
$|\{\sqrt[n_i]{1}\}|=n_i$, we can choose $\varepsilon_i=\lambda(a_i)$ - any root
$\Rightarrow$ for $\lambda(a)$ we have $n_1\cdot\ldots\cdot n_r=|A|$ possibilities
$\widehat{\langle a_i\rangle}=\langle\lambda_i\rangle$, $\lambda_i(a_i)=\varepsilon_i$,
$\hat{A}=\widehat{\langle a_1\rangle}\times\dots\times\widehat{\langle a_r\rangle}$, $\widehat{\langle a_i\rangle}\cong\langle a_i\rangle$, q.e.d.
where ε_i is fixed primitive root of 1 of degree n_i.

<u>Th</u>. If $|G|<\infty$, K is a field, $\mathrm{char}\,K\nmid|G|$,
then any irreducible representation $\rho: G\to GL(V)$ is isomorphic with some subrepresentation of the regular representation (Λ, G, KG), which is a direct summand of KG.
(Note: if $\overline{K}=K$, then the multiplicity of V in KG equals the dimension of V)

<u>Proof</u>. Construct a homomorphism of representations
$f: KG\longrightarrow V$.
Put $f(1_G)=v_0\in V$, $v_0\neq0$; for any $a\in G$, $a=1_G\cdot a$,
$a=\Lambda(a)(1_G)$. Put $f(a)=\varphi(a)(v_0)$; we can define $f(a)$, $\forall a\in KG$, by linearity. By construction, f is a homomorphism of representations, $V'=\mathrm{Im}\,f$ is invariant in V, as V is irreducible, then $\mathrm{Im}\,f=V \Rightarrow V\cong KG/\mathrm{Ker}\,f$; $\mathrm{Ker}\,f$ is invariant in KG
$\Rightarrow$ by Maschke's th., $\exists W\subset KG$, Λ_G-invariant: $KG=\mathrm{Ker}\,f\oplus W$;
certainly, $W\cong V$ (as representations)

<u>Consequence</u>. 1. Any irreducible representation of a finite group is finite-dimensional.
2. There are only finite number of non-isomorphic irreducible representations of a finite group G $(\mathrm{char}\,K\nmid|G|)$?
1. $\dim V\le\dim KG=|G|$. $(V=\langle\rho(g)v_0, v_0\neq0\rangle)$
2. $KG=\bigoplus_i W_i$, $\forall$ irred. V, $V\cong W_i$, some W_i
We can collect the isomorphic summands: $U=\bigoplus_j W_j$, $j: W_j\cong V$.
$\Rightarrow KG=U_1\oplus\dots\oplus U_s$, so there are only s classes of isomorphism of irreducible representations.