6.05.22.

If $\{\chi_1,\dots,\chi_s\}$ are all irreducible complex characters of a group G (= the characters of irreducible representations), then $(\chi_i,\chi_j)_G=\delta_{ij}=\begin{cases}1, \text{ if } i=j \Leftrightarrow \varphi_i\cong\varphi_j;\\ 0, \text{ if } i\ne j \Leftrightarrow \varphi_i\not\cong\varphi_j.\end{cases}$

Corollary 1. $\{\chi_1,\dots,\chi_s\}$ is an orthonormal system of the space $ZF(G)$ of central functions on G $\Rightarrow s\le r$ = the number of conjugate classes of G.

Proof. Central function f on G is constant on conjugate classes $K_1,\dots,K_r$ of G, $\dim ZF(G)=r$; $\{\chi_1,\dots,\chi_s\}$ are linearly independent $\Rightarrow s\le r$.

Corollary 2. If two representations φ and ψ of G have equal characters: $\chi_\varphi=\chi_\psi$ then $\varphi\cong\psi$.

Proof. $\varphi\cong k_1\varphi_1\oplus\dots\oplus k_s\varphi_s$, k_i are multiplicities ($k_i\ge 0$), and $\{\varphi_i\}$ are irreducible representations.

$\psi\cong l_1\varphi_1\oplus\dots\oplus l_s\varphi_s$,

$\chi_\varphi=\sum_{i=1}^{s}k_i\chi_i \Rightarrow (\chi_\varphi,\chi_j)_G=\sum_i k_i(\chi_i,\chi_j)=k_j(\chi_j,\chi_j)_G=k_j$

$\chi_\varphi=\chi_\psi \Rightarrow k_i=l_i,\ i=1,\dots,s \Rightarrow \varphi\cong\psi$. q.e.d.

Th.1. $r=s$.

Proof. The system $\chi_1,\dots,\chi_s$ is a basis of $ZF(G)$ $\Leftrightarrow$ it is closed, it means: if $f\in ZF(G)$, $(\chi_i,f)_G=0\Rightarrow f\equiv 0$.

For arbitrary representation (φ,G,V) define the following linear operator

$$\varphi^*(f)=\sum_{g\in G}\bar f(g)\varphi(g): V\to V.$$

Note, the $(\varphi\oplus\psi)^*(f)=\varphi^*(f)\oplus\psi^*(f)$.

In matrix form: $(\varphi\oplus\psi)(g)=\left(\begin{array}{c|c}\varphi(g) & 0\\ \hline 0 & \psi(g)\end{array}\right)\Rightarrow(\varphi\oplus\psi)^*(f)=\left(\begin{array}{c|c}\varphi^*(f) & 0\\ \hline 0 & \psi^*(f)\end{array}\right)$

- The operator $\varphi^*(f)$ is endomorphism of the representation φ:

$\forall h\in G$, $\varphi(h)\varphi^*(f)\varphi(h^{-1})=\sum_{g\in G}\bar f(g)\varphi(hgh^{-1})=\sum_{g\in G}\bar f(\underbrace{hgh^{-1}}_{a})\varphi(hgh^{-1})=\sum_{a\in G}\bar f(a)\varphi(a)=\varphi^*(f)$.

For $\varphi=\varphi_i$ - irreducible, by Schur's lemma, $\varphi_i^*(f)=\lambda_i E$, calculate the traces:

$\lambda_i\chi_i(1)=\sum_{g\in G}\bar f(g)\,\mathrm{tr}\,\varphi_i(g)=\sum_{g\in G}\chi_i(g)\bar f(g)=|G|(\chi_i,f)_G=0$, by condition.

$\forall\varphi$ - representation of G, $\chi_\varphi=\sum_{i=1}^{n}k_i\chi_i\Rightarrow(\chi_\varphi,f)_G=0\Rightarrow\varphi^*(f)=0$

Let apply it for the regular representation $(\Lambda_G,G,\mathbb{C}G)$

$\mathbb{C}G=\langle g\mid g\in G\rangle$; $\Lambda_G(g)(x)=gx$, $\forall g\in G$, $\forall x\in\mathbb{C}G$

$\Lambda_G(g)(1)=g\cdot 1=g$.

$0=\Lambda_G^*(f)(1)=\sum_{g\in G}\bar f(g)\Lambda_G(g)(1)=\sum_{g\in G}\bar f(g)g$, but $\{g\mid g\in G\}$ are linearly independent $\Rightarrow f(g)=0$, $\forall g\in G\Rightarrow f\equiv 0$. q.e.d.

Consequence. Any irreducible representation of G enters in the regular representation with multiplicity equal its dimension.

Proof. Recall, that $\forall g\in G$, $\chi_\Lambda(g)$ = the number of fixed points of g in its action. When G acts on itself by left multiplication, it equals 0, if $g\ne 1\Rightarrow \chi_\Lambda(g)=\begin{cases}|G|, & g=1\\ 0, & g\ne 1\end{cases}$

Calculate $(\chi_i,\chi_\Lambda)_G=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\bar\chi_\Lambda(g)=\chi_i(1)$. q.e.d.

The Burnside's formula: $\sum_{i=1}^{s}\chi_i(1)^2=|G|$.

% Rem. If (φ,G,V) is completely reducible representation, then $\varphi\cong\bigoplus_j\varphi_j$, $V=\bigoplus_j V_j$, V_j are minimal invariant subspaces.

If we collect together the summands which are equivalent with the same irreducible representation, we get

$V=W_1\oplus\dots\oplus W_s$, $W_i=\bigoplus_j V_{ij}$, $V_{ij}\cong V_i$ - irreducible, $1\le i\le s$

$W_i\cong\underbrace{V_i\oplus\dots\oplus V_i}_{k_i}\Rightarrow k_i$ is the multiplicity of V_i in V $\Leftrightarrow$ the multiplicity of φ_i in φ.

The multiplicities $k_1,\dots,k_s$ are uniquely determined for the given representation φ, so if $\varphi\cong\bigoplus_{i=1}^{s}k_i\varphi_i$, $\psi\cong\bigoplus_{i=1}^{s}l_i\varphi_i$, then $\varphi\cong\psi\Leftrightarrow k_i=l_i$, $i=1,\dots,s$. %

Character tables $g_i\in K_i$

$$X_G=\begin{array}{c}\\ \chi_1\\ \vdots\\ \chi_i\\ \vdots\\ \chi_r\end{array}\begin{array}{c}K_1\quad K_2\ \dots\ K_j\ \dots\ K_r\\ \left[\begin{array}{ccc} & & \\ \dots & \dots & \chi_i(g_j)\\ & & \\ & & \end{array}\right]\end{array}$$

The I orthogonality relation means that the rows of this table are orthogonal in the sence:

$$(\chi_i,\chi_j)=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)}=\frac{1}{|G|}\sum_{l=1}^{r}\sum_{g\in K_l}\chi_i(g)\overline{\chi_j(g)}=$$

$$=\frac{1}{|G|}\sum_{l=1}^{r}|K_l|\chi_i(g_l)\overline{\chi_j(g_l)}=\sum_{l=1}^{r}\frac{1}{|C_G(g_l)|}\chi_i(g_l)\overline{\chi_j(g_l)}=\delta_{ij}$$

$|G|=|K_l|\cdot|C_G(g_l)|$

Consider the matrix $M=\left(\frac{\chi_i(g_j)}{\sqrt{|C(g_j)|}}\right)$: it is unitary by rows: $M\cdot\bar M^T=E\Rightarrow M^T\cdot\bar M=E\Rightarrow$

$$\sum_{i=1}^{r}\frac{\chi_i(g_j)}{\sqrt{|C(g_j)|}}\cdot\frac{\overline{\chi_i(g_k)}}{\sqrt{|C(g_k)|}}=\delta_{jk}$$

Th.2. (Second orthogonality relation):

$$\sum_{i=1}^{r}\chi_i(g)\overline{\chi_i(h)}=\begin{cases}|C_G(g)| & \text{if } g,h \text{ are conjugate},\\ 0, & \text{otherwise.}\end{cases}$$