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Character Theory.

<u>Def 1.</u> Let $\varphi: G \to GL(V)$ a linear representation of a group G in the space V (over some field K), $\dim V < \infty$.

The character of the representation φ is

$$\chi_\varphi(g) = tr_V\, \varphi(g), \ \forall g \in G.$$

If A_g is a matrix of the operator $\varphi(g)$ then $\chi_\varphi(g) = \sum_{i=1}^{n} a_{ii}$ $(n = \dim V)$

<u>Some properties of characters</u> (K will be algrebraically closed, $char K \nmid |G|$, if $|G| < \infty$)

<u>Proposition.</u> 1) $\chi_\varphi(1) = \dim V$;

2) $\forall g, h \in G$, $\chi_\varphi(h^{-1}gh) = \chi_\varphi(g)$, so χ_φ is constant on conjugate classes of G;

3) If $|g| < \infty$ and $K = \mathbb{C}$, then $\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)}$;

4) $\chi_{\varphi\oplus\psi} = \chi_\varphi + \chi_\psi$;

5) If $\varphi \cong \psi$ (isomorphic) then $\chi_\varphi = \chi_\psi$.

<u>Proof.</u> 1) As $\varphi(1) = E$, then $tr_V\, \varphi(1) = trE = n = \dim V$.

2) $\varphi(h^{-1}gh) = \varphi(h)^{-1}\varphi(g)\varphi(h) \Rightarrow$ the matrices $A_{h^{-1}gh}$ and $A_{\varphi(g)}$ have equal traces.

3) We know, that $trA_g = \sum_{i=1}^{n} \lambda_i$, λ_i are characteristic roots.

If $|g| = m$, $A_g^m = E \Rightarrow \lambda_j^m = 1$, $\lambda_j \in U_m$. If $K = \mathbb{C}$,

$\lambda_j = e^{i\alpha_j} \Rightarrow \lambda_j^{-1} = e^{-i\alpha_j} = \bar{\lambda}_j \Rightarrow \chi_{\varphi(g^{-1})} = \sum_{j=1}^{n} \lambda_j^{-1} = \overline{\sum_{j=1}^{m} \lambda_j} = \overline{\chi_{\varphi(g)}}$.

4) If $V = U \oplus W$, (φ, U), (ψ, W) are representations of G, then in the basis combined of some bases of U and W,

$$A_{(\varphi\oplus\psi)(g)} = \left(\begin{array}{c|c} A_{\varphi(g)} & 0 \\ \hline 0 & A_{\psi(g)} \end{array}\right) \Rightarrow trA_{(\varphi\oplus\psi)(g)} = trA_{\varphi(g)} + trA_{\psi(g)}.$$

5) If $\varphi \cong \psi$, then $A_{\psi(g)} = C^{-1}A_{\varphi(g)}C$, some matrix C, $|C| \neq 0$

$\Rightarrow \chi_{\psi(g)} = \chi_{\varphi(g)}$, $\forall g \in G$.

A function $f: G \to K$ is called central function, if f is constant on conjugate classes of G. So, the characters of representations are central functions.

Denote $\mathcal{F}_G$ the set of all functions on G $(f: G \to K)$, $Z\mathcal{F}_G$ the space of central functions.

If $|G| = n$, the $\dim \mathcal{F}_G = n$;

if the nuber of conjugate classes of G equals r, the $\dim Z\mathcal{F}_G = r$;

<u>Some consequence of Schur's Lemma and its matrix version.</u>

<u>Proposition</u> ($|G| < \infty$, $char K \nmid |G|$, $char K \nmid \dim V$, $\bar{K} = K$)

Let (φ, V) and (ψ, W) be irreducible representations of a group G, $\sigma: V \to W$ some linear mapping.

Then the averaged mapping

$$\tilde{\sigma} = \frac{1}{|G|}\sum_{g\in G} \psi(g)\sigma\varphi(g)^{-1} = \begin{cases} 0, & \text{if } \varphi \not\cong \psi \\ \lambda E & \text{if } V = W, \varphi = \psi, \text{ where } \lambda = \frac{tr\sigma}{\dim V}. \end{cases}$$

<u>Proof.</u> It's easy to check (as in the proof of Maschke's th.) that $\tilde{\sigma}$ is homomorphism of representations: $\tilde{\sigma}\varphi(g) = \psi(g)\tilde{\sigma}$, $\forall g \in G$). By Schur's Lemma, $\tilde{\sigma}$ is 0 if $\varphi \not\cong \psi$, and, as K is alg. closed, in the second case $\tilde{\sigma} = \lambda E$;

$tr\sigma = tr\tilde{\sigma} = \lambda \dim V$, q.e.d.

<u>The matrix variant.</u> Choose some bases in V and W: $\{e_i | i \in I\}$ in V, $\{f_j | j \in J\}$ in W, then $\sigma = (\sigma_{ji})$, $\tilde{\sigma} = (\tilde{\sigma}_{ji})$. By construction,

$\varphi(g) = (\varphi_{ii'}(g))$, $\psi(g) = (\psi_{jj'}(g))$,

$$\tilde{\sigma}_{ji} = |G|^{-1}\sum_{g, i', j'} \psi_{jj'}(g)\sigma_{j'i'}\varphi_{i'i}(g^{-1}) \quad (\mathrm{I})$$

If we take $\sigma_{j_0 i_0} = 1$, $\sigma_{ji} = 0$ otherwise, then we get

1) $(\varphi \not\cong \psi)$ $|G|^{-1}\sum_{g\in G} \psi_{jj_0}(g)\varphi_{i_0 i}(g^{-1}) = 0$, $\forall i, j, i_0, j_0$ (II)

2) $(V = W, \varphi = \psi)$, $\tilde{\sigma} = \frac{tr\sigma}{\dim V}E$, $tr\sigma = \sum_{i\in I}\sigma_{ii} = \sum_{i',j'} \delta_{j'i'}\sigma_{j'i'}$

$\Rightarrow \tilde{\sigma}_{ji} = \delta_{ji}\frac{tr\sigma}{\dim V} = \delta_{ji}(\dim V)^{-1}\sum_{i',j'}\delta_{j'i'}\sigma_{j'i'}$.

$(\mathrm{I}) \Rightarrow |G|^{-1}\sum_{g,i',j'} \varphi_{jj'}\sigma_{j'i'}\varphi_{i'i}(g^{-1}) = (\dim V)^{-1}\sum_{i',j'}\delta_{ji}\delta_{j'i'}\sigma_{j'i'}$.

For $\sigma_{j_0 i_0} = 1$ and 0 in other cases we get

$$|G|^{-1}\sum_{g\in G}\varphi_{jj_0}(g)\varphi_{i_0 i}(g^{-1}) = \begin{cases} \delta_{ji}/\dim V, & \text{if } i_0 = j_0 \\ 0 & \text{otherwise} \end{cases} \quad (\mathrm{III})$$

<u>Orthogonality relations for irreducible characters</u>

$(K = \mathbb{C})$

In the space $\mathcal{F}_G = \{f: G \to \mathbb{C}\}$ introduce the Hermitean form (scalar product)

$$(f_1, f_2)_G := |G|^{-1}\sum_{g\in G} f_1(g)\overline{f_2(g)};$$

$$(f, f)_G = |G|^{-1}\sum_{g\in G} f(g)\overline{f(g)} = |G|^{-1}\sum_{g\in G}|f(g)|^2 = 0 \Rightarrow f \equiv 0.$$

<u>Th.1.</u> (First orthogonality relation)

If φ, ψ are irreducible complex representations of a finite group G, then

$$(\chi_\varphi, \chi_\psi) = \delta_{\varphi,\psi} = \begin{cases} 1, & \text{if } \varphi \cong \psi, \\ 0, & \text{if } \varphi \not\cong \psi. \end{cases}$$

<u>Proof.</u> As previous, $\chi_\varphi(g) = \sum_i \varphi_{ii}(g)$, $\chi_\psi(g) = \sum_i \psi_{ii}(g)$

In equalities (II), (III) take $i_0 = i$, $j_0 = j$ and sum over all i, j, we get

1) $\varphi \not\cong \psi \Rightarrow 0 = |G|^{-1}\sum_{g,i,j}\psi_{jj}(g)\varphi_{ii}(g^{-1}) = (\chi_\psi, \chi_\varphi)_G$ $\quad (\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)})$

2) $\varphi \cong \psi \Rightarrow 1 = (\sum_{i,j}\delta_{ji})/\dim V = |G|^{-1}\sum_{g,i,j}\varphi_{jj}(g)\varphi_{ii}(g^{-1}) =$

$= |G|^{-1}\sum_{g\in G}(\sum_j \varphi_{jj}(g))\cdot(\sum_i \varphi_{ii}(g^{-1})) = (\chi_\varphi, \chi_\varphi) = (\chi_\varphi, \chi_\psi)$. q.e.d.