Final Examination: Magnetic field due to a current loop

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1 Aim

- To model a loop of wire carrying a given current distribution and analyse the magnetic field in the z direction for x,y = 0,0
- To perform curve fitting on the result
- To explain the result

2 Theory and Psuedocode

We are given a loop of wire of radius $a = 10cm$ on the x-y plane with its center at the origin and given current distribution

$$I = \frac{4\pi}{\mu_0} cos(\phi) exp(j\omega t)$$

where ϕ is polar angle in cylinderical coordinates (r, ϕ, z).
We are to find the magnetic field $\vec{B}$ along the z axis from z = 1cm to z = 1000cm. We use the Maxwell's equations:

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{J} + \epsilon \frac{\partial}{\partial t}(\vec{E})$$

$$\nabla \times E = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial(\nabla \times \vec{A})}{\partial t}$$

solving these using the lorentz gauge

$$\nabla \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

and the wave equation

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

A solution using Green's Function of this equation for our case is

$$\vec{A} = \int_V \mu \vec{J}(\vec{r'}) \frac{e^{-j\beta|\vec{r} - \vec{r'}|}}{4\pi|\vec{r} - \vec{r'}|} dv'$$

converting into our coordinate system convention we get:

$$\vec{A}(r, \phi, z) = \frac{\mu_0}{4\pi} \int \frac{I(\phi)\hat{\phi} e^{-jkR} a d\phi}{R}$$

$$\vec{B} = \nabla \times \vec{A}$$

Where $\vec{R} = \vec{r} - \vec{r'}$ and $k = \frac{\omega}{c} = 0.1$. $\vec{r}$ is the position vector of where we want the field, and $\vec{r'}$ is the position vector of a point on the loop.
Now let us solve this analytical problem using numerical methods.
We consider a cuboid with its longest side along the z axis of dimensions $3 \times 3 \times 1000$cm. We slice this cube into 9000 cubelets, each $1 \times 1 \times 1$cm.
We assume the all the parameters in each cubelet are equal to that in the center of the cubelet. We have 9000 sampling points in the space, each representative of its cubelet, with our sampling interval being 1cm. We hence have the vector $\vec{r}$ in cartesian coordinates.

We also split the loop of wire into 100 segments, each being our approximation for $\vec{dl}$, the infinitesimal element on the wire. Now we can find $\vec{r'}$ as follows

$$\vec{r'} = acos\phi\hat{x} + asin\phi\hat{y}$$

The integral now becomes a summation as follows

$$\vec{A}_{ijk} = \sum_{l=0}^{N-1} \frac{cos(\phi_l')exp(-jkR_{ijkl})\vec{dl'}}{R_{ijkl}} \tag{1}$$

Rijk can be calculated by vector subtraction and taking magnitude

$$R_{ijk} = |\vec{r} - \vec{r'}|$$

Now for finding $\vec{B}$, we know

$$\vec{B} = \nabla \times \vec{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\hat{x} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\hat{y} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\hat{z}$$

$$B_z = (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \implies \frac{A_y(\Delta x, 0, z) - A_y(-\Delta x, 0, z)}{2\Delta x} - \frac{A_x(0, \Delta y, z) - A(0, -\Delta y, z)}{2\Delta y}$$

Since $\Delta x = \Delta y = 1cm$

$$B_z = \frac{A_y(\Delta x, 0, z) - A_y(-\Delta x, 0, z) - A_x(0, \Delta y, z) + A_x(0, -\Delta y, z)}{2}$$

Now we have converted this problem to a numerical methods problem. We now devise a psuedocode for this problem.

Algorithm 1 Find-Bz

```
u = number of x samples
v = number of y samples
w = number of z samples
a = radius of the loop of wire
I = current distribution (here Io = 1e7)
A = magnetic vector potential initialized to zero

xCoords = {-(x-1)/2, ... -1, 0, 1, ... , (x-1)/2}
yCoords = {-(y-1)/2, ... -1, 0, 1, ... , (y-1)/2}
zCoords = {1, 2, ... , 1000}

r = (xCoords, yCoords, zCoords)

phi = {0, 2*pi/100, 4*pi/100, .. 198*pi/100}
I = Io*cos(phi)
dphi = 2*pi/100
magPhi = a*dphi
ringpos = (acos(phi), asin(phi))

dl = (-magPhi*sin(phi), magPhi*cos(phi))

J = I*dl

foreach ringpos:
        R = |r - ringpos|
        A += (cos(phi_at_that_ringpos)*exp(-jkR)/R)*dl

Bz = ((A_y[1, 0, all_z] - A_y[-1, 0, all_z]
                        - A_x[0, 1, all_z] + A_x[0, -1, all_z])/2)
```

3 Code and Results

- Defining the control variables

```
# mu0
mu0 = 8.85418782e-12
# I0
I0 = 4*p.pi/mu0
N = 100
# Dimensions
u = 3
v = 3
w = 1000
a = 10
# k = w / c (not to be confused by i, j, k so im calling it wByC)
wByC = 0.1
```

- Breaking the volume into a $3 \times 3 \times 1000$ mesh

```
# Q2
```

```
# dividing the volume into 3 x 3 x 1000
xCoords = p.linspace(-(u - 1)/2, (u - 1)/2, u)
yCoords = p.linspace(-(v - 1)/2, (v - 1)/2, v)
zCoords = p.linspace(1, w, w)
```

- Breaking the loop into 100 sections and finding current elements Idl. We also obtain vectors $\vec{r_l'}$ and $\vec{dl_l}$ where l indexes the segments of the loop.

```
# Q3
# dividing angle into segments to obtain 100 dl segments
phi = p.linspace(0, 2*p.pi, N+1)[:-1]
# initialize magnitude of current I.
# we can ignore exp(jwt) because at no point in the analysis do we involved time

# Q4
# so the exp(jwt) factor will be present in Bz,
# and when we take magnitude, it will vanish
I = I0*p.cos(phi)
ringpos_x = a*p.cos(phi)
ringpos_y = a*p.sin(phi)
# dphi = 2pi/100
dphi = phi[1] - phi[0]
# |dl| = radius * dphi
mag_dl = a*dphi

# dl vector = magnitude * direction
dl_x = -mag_dl*p.sin(phi)
dl_y = mag_dl*p.cos(phi)

# current element J = I*dl
JX = I*dl_x
JY = I*dl_y
```

We are taking current equal to $I_0 cos(\phi)$ because at no point do we take any time dependent operations, therefore the $exp(-j\omega t)$ will be visible as it is in Bz. When we take magnitude, it disappears. So it does not really have any effect on $|Bz|$.

- Plotting JX and JY yeilds the following current element distribution on the x-y plane

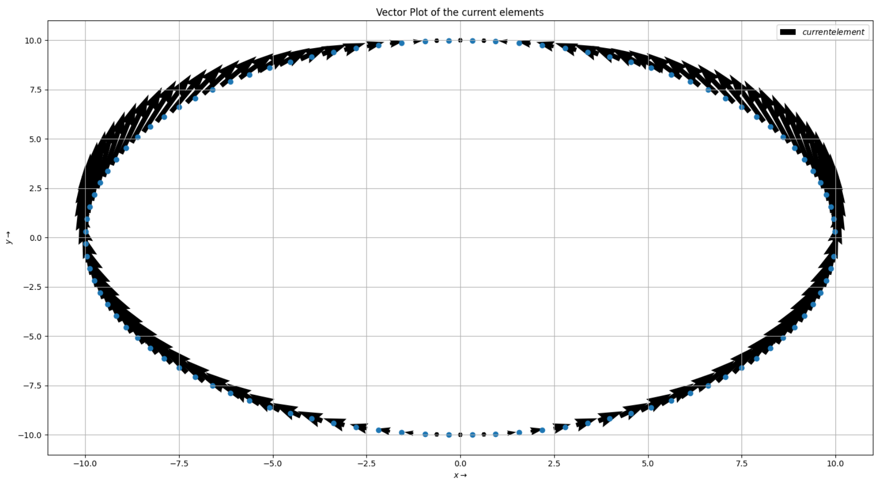

Figure 1: Vector plot of the current elements on the x-y plane

- Defining a vectorized function calc(l) for calculating $R_{ijkl} = |\vec{r_{ijk}} - \vec{r_l'}|$ for all $\vec{r_{ijk}}$. We can vectorize this function using meshgrid.

```
# Q 5 and 6
# calc function
def calc(l):
    # create meshgrid
    xx, yy, zz = p.meshgrid(xCoords, yCoords, zCoords, indexing='ij')
    # find Rijk vector
    xx = xx - ringpos_x[l]
    yy = yy - ringpos_y[l]
    # find |Rijk|
    Rijk = p.sqrt(xx**2 + yy**2 + zz**2)
    # return x component and y component of A vector
    return (I[l]/I0)*p.exp(complex(0, -1)*wByC*Rijk)*dl_x[l]/Rijk,
                        (I[l]/I0)*p.exp(complex(0, -1)*wByC*Rijk)*dl_y[l]/Rijk
```

- Computing $\vec{A_{ijk}}$

```
# initialize A_x and A_y
A_x = 0
A_y = 0
# Q7
# populate all the values of A_x and A_y
# we have to use a loop here because for vectorized operations the dimensions
# of the two arrays have to match, which would be cumbersome,
# because we have 100 points on our loop, but 9000 points in our mesh
for l in range(N):
    dA_x, dA_y = calc(l)
    if l == 0:
        A_x = dA_x
        A_y = dA_y
```

```
    else:
        A_x += dA_x
        A_y += dA_y
```

- Computing Bz

```
# Q8
# B = del x A
# B(z) = (A_y[deltaX, 0, z] - A_y[-deltaX, 0, z]
                              - A_x[0, deltaY, z] + A[0, -deltaY, z])/2
Bz = (A_y[u//2 + 1, v//2, :] - A_y[u//2 - 1, v//2, :]
                    - A_x[u//2, v//2 + 1, :]
                                    + A_x[u//2, v//2 - 1, :])/2
```

We obtain the following plot

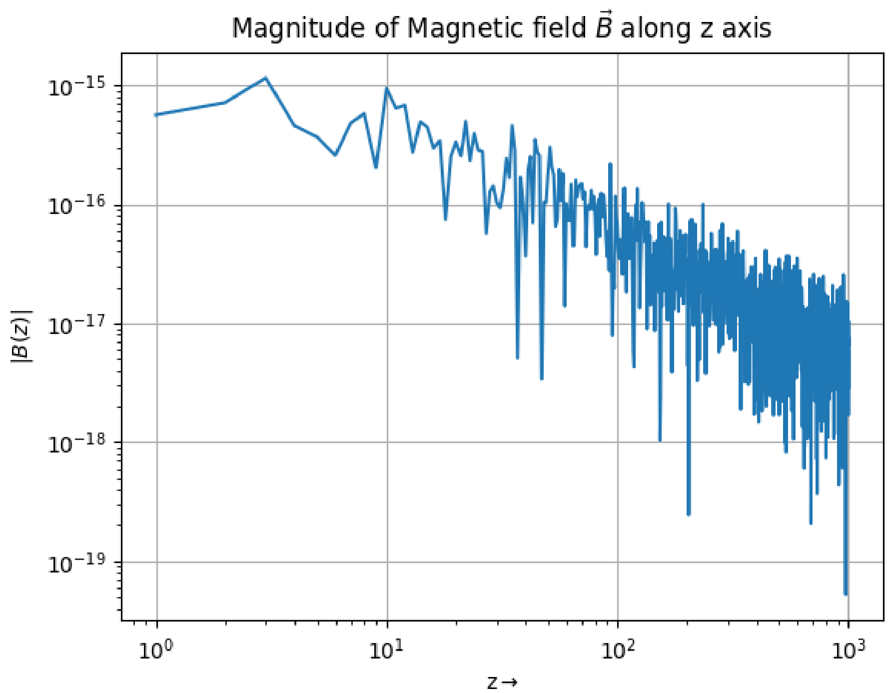

Figure 2: Variation of Magnitude of Magnetic Field $\vec{B}$ along z (loglog)

- We can see it is approximately zero. It has to be zero, because if we notice the current distribution from Figure 1, and choose mirrored elements across the X axis, we can verify by the right hand rule that the z components should cancel. So we expect zero. And the small errors we are getting are due to the precision of our model.

- Using the method of least squares in order to fit the curve cz^b to the data. We use SciPy's lstsq() function

```
# Q10
# curve fitting using lstsq
```

```
M = p.c_[p.log10(zCoords), p.ones(w)]
G = p.log10(abs(Bz))
r = scipy.linalg.lstsq(M, G)[0]
c = 10**(r[1])
b = r[0]
print("Least Squares fitting result")
print("c = " + str(c))
print("b = " + str(b))
fit = c*(zCoords**b)
```

We plot the fit along with the data and we obtain the following fit

```
Least Squares fitting result
c = 4.4167464316159245e-15
b = -0.9855140244629008
```

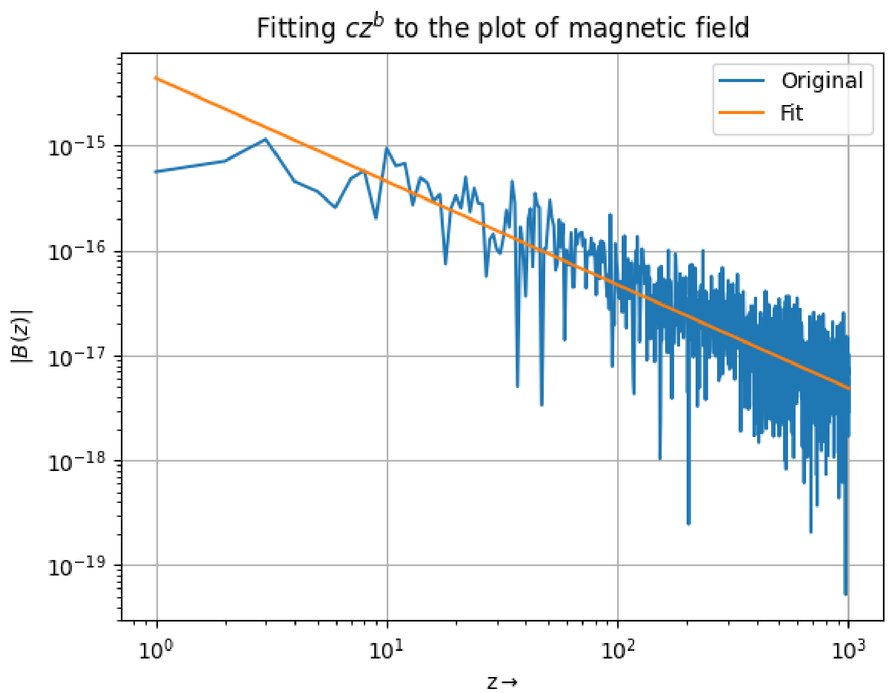

Figure 3: Fitting the curve cz^b to the magnetic field data

- We see that Bz is nonzero, albeit a very small value. We see that approximately $Bz \propto \frac{1}{z}$ for this case. This doesn't provide any real conclusion though, since this is simply the decay rate of the numerical errors the computer made in its calculations. We cannot comment about the magnetic field from this result, since the analytical method gives us zero.

4 Statics case

Now let us see what happens if we do the same for statics case. If the current through the loop is now constant with respect to time, but not space

$$I = \frac{4\pi}{\mu_0} cos(\phi)$$

Now we know from the maxwell's equations in magnetostatics case for line currents

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{r} dl'$$

$$\vec{A}(r, \phi, z) = \frac{\mu_0}{4\pi} \int \frac{I(\phi)\hat{\phi} a d\phi}{R}$$

$$\vec{A}_{ijk} = \sum_{l=0}^{N-1} \frac{cos(\phi_l')\vec{dl}'}{R_{ijkl}} \tag{2}$$

- In statics case we will have to change a couple of things in our code.
- We change $k = \frac{\omega}{c} = 0$. The reason we do this is because we want the exponential term in equation (1) function to disappear to match with the equation (2).

```
# we need our dAx, dAy = dl_x/Rijk, dl_y/Rijk,
#so we remove the exponential part by setting the exponent to zero
# so we do that by setting k = w / c to zero
wByC = 0
```

- Now we get the following plots

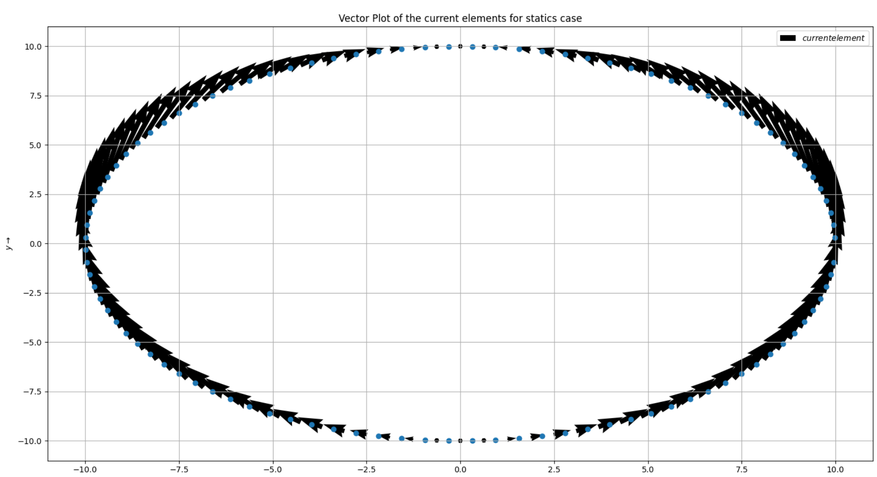

Figure 4: Current element distribution in statics case

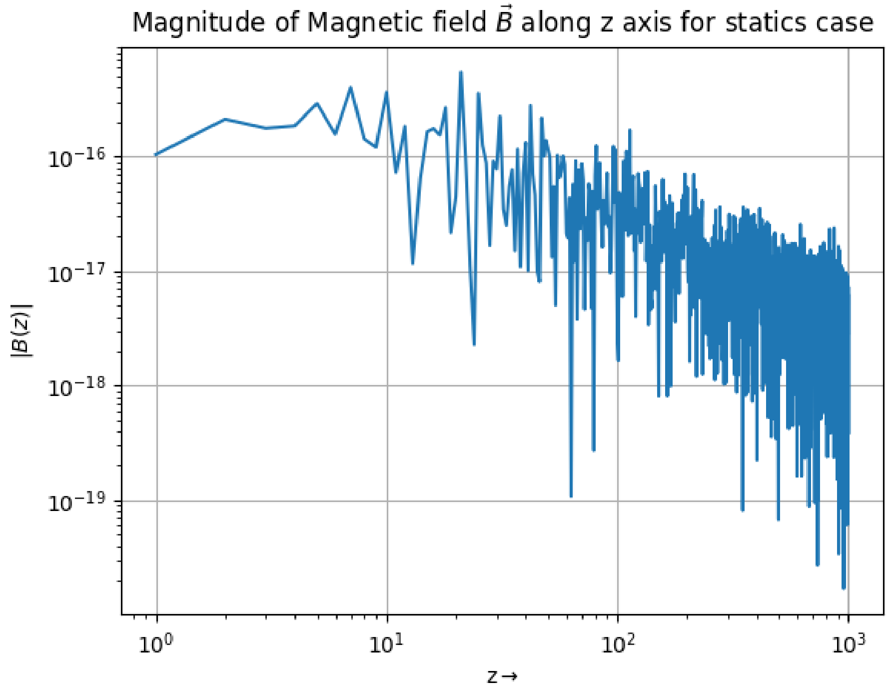

Figure 5: Variation of Magnitude of Magnetic Field $\vec{B}$ along z (loglog) for statics case

We get the following curve fit results

```
Statics case
Least Squares fitting result
c = 1.2004553457040477e-15
b = -0.8685902540971611
```

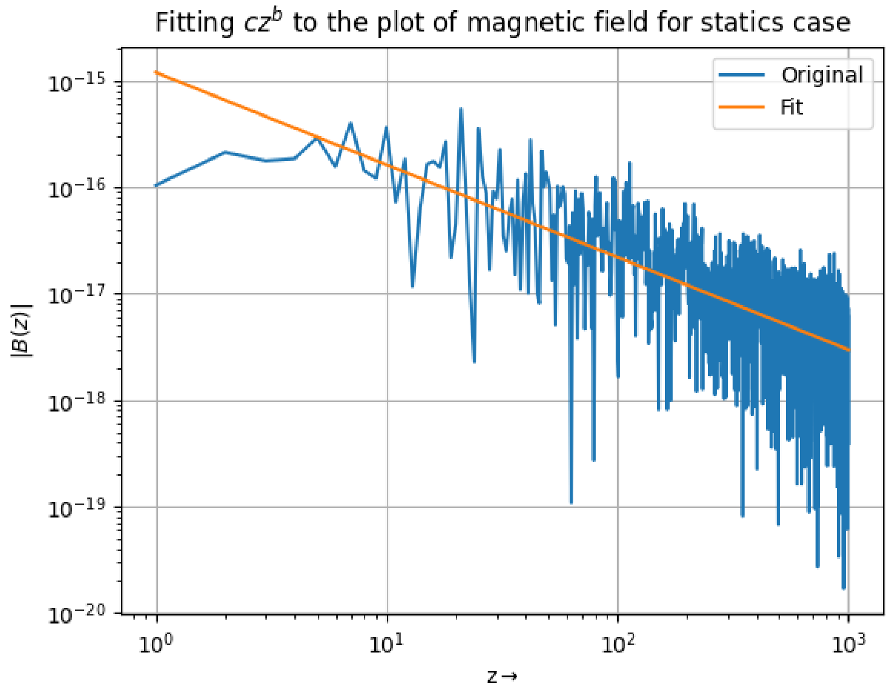

Figure 6: Fitting the curve cz^b to the magnetic field data for statics case

We again get approximately zero, which is as expected again by applying the right hand rule on a pair of current elements as explained earlier. The time dependence of the current is such that $|Bz|$ does not change with time.

5 Constant Current case

If we assume that the current has no dependence on space as well as time

$$I = I_0 = \frac{4\pi}{\mu_0}$$

We use the same procedure as that statics case, except in the summation we make the following change

$$\vec{A}_{ijk} = \sum_{l=0}^{N-1} \frac{dl'}{R_{ijkl}} \tag{3}$$

We now obtain the following plots

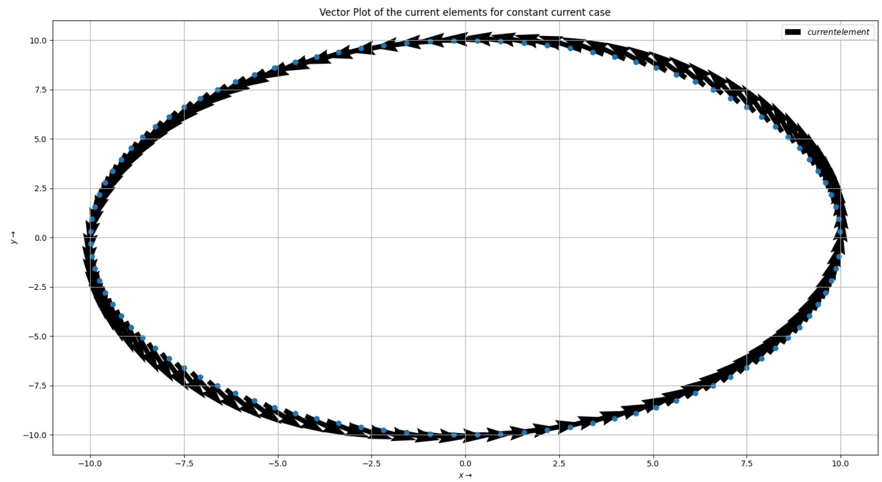

Figure 7: Current element distribution in constant current case

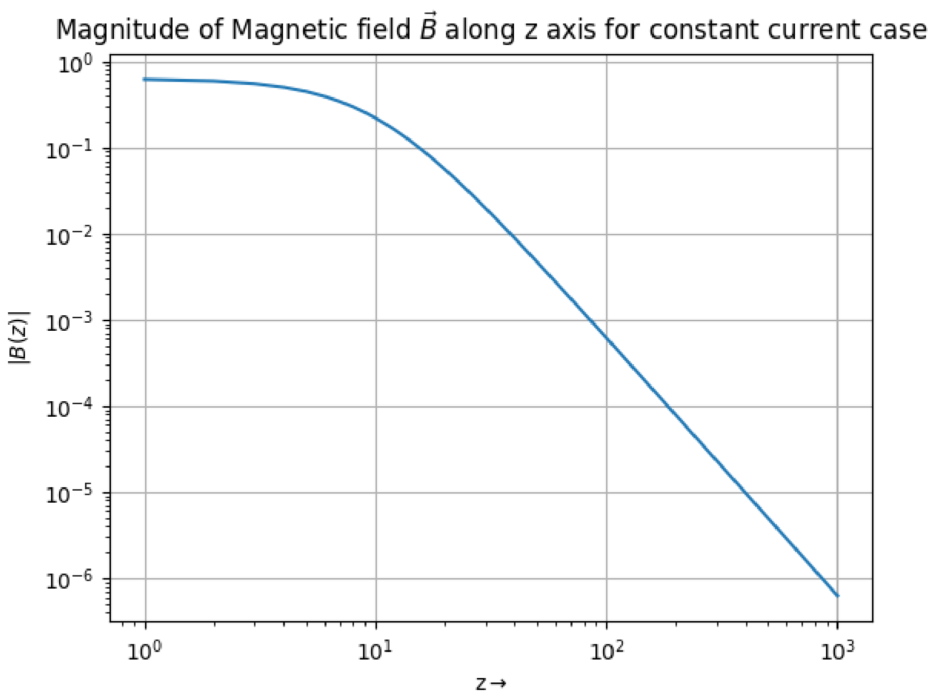

Figure 8: Variation of Magnitude of Magnetic Field $\vec{B}$ along z (loglog) for constant current case

We get the following curve fit results

```
Constant Current case
Least Squares fitting result
c = 215.8579024434782
b = -2.8261920569267023
```

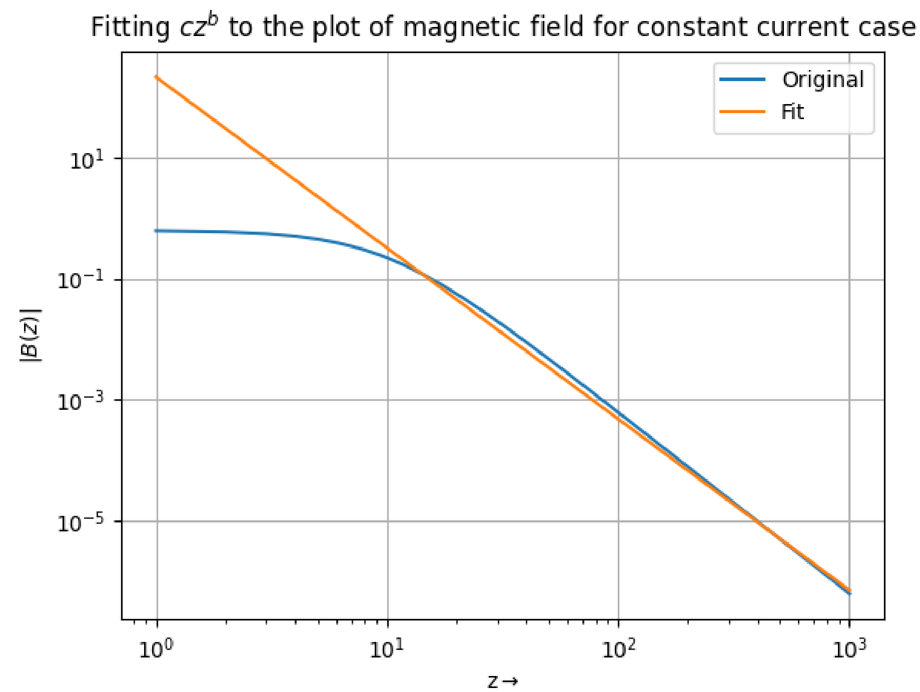

Figure 9: Fitting the curve cz^b to the magnetic field data for constant current case

We can see that the curve is nonzero as expected, since we know that by Biot-Savarts Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int_C \frac{\vec{dl} \times \vec{r'}}{|\vec{r'}|^3}$$

$$Bz = \frac{\mu_0}{4\pi} \frac{2\pi a^2 I}{(z^2 + a^2)^{\frac{3}{2}}}$$

$$Bz = \frac{2\pi a^2}{(z^2 + a^2)^{\frac{3}{2}}}$$

for large z,

$$Bz \propto \frac{1}{z^3}$$

and we see that in our fit, $b \approx -3$, which matches our analysis.

6 Conclusions

- The magnetic field along the z axis for loop of wire for various current configurations, namely time and space varying, only space varying and constant current configurations was modelled as a numerical methods problem
- Vectorized Pylab functions were used in order to speed up the calculations and avoid loops as far as possible
- The resulting data was fitted to the curve cz^b and found the constants c and b for
- The meaning behind the obtained graphs was analysed and compared with the expected result.