K A P I L C H A U D H A R Y

M O D U L E T H E O R Y

U N I V E R S I T Y O F D E L H I

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## *Dedicated to my family and my best friend*

*Neeraj K. Gaud*

#### *Warning :*

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

<https://github.com/sirkapil/module-theory/issues/new>1 1 (may require a github account)

#### *About :*

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my LATEX documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

<https://github.com/sirkapil>

#### *Contribution :*

If you find my work useful and want to contribute then you are wel- come by heart.

Any suitable changes to document repository through pull re- quests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of reposi- tory before creating any pull-request.

<https://github.com/sirkapil/module-theory/compare>

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*Defination of Module*

##### *Left Module:*

Let *R* be a ring with identity and *M* be an abelian group with addi-

tion. We say *M* is a left *R*−module if there exists a mapping2 2 often called as scaler multiplication.

*R* × *M* → *M*

defined by

(*a* , *x*) → *ax*

∀ *a* ∈ *R* and *x* ∈ *M*

satisfying following properties :

and denoted by *R M*

(*a* + *b*)*x* =*ax* + *bx* (1.1)

*a*(*x* + *y*) =*ax* + *ay* (1.2)

(*ab*)*x* =*a*(*bx*) (1.3)

1*x* =*x* (1.4)

∀ *a* , *b* ∈ *R* and *x* , *y* ∈ *M*

##### *Right Module:*

Let *R* be a ring with identity and *M* be an abelian group with addi- tion. We say *M* is a right *R*−module if there exists a mapping

*M* × *R* → *M*

defined by

(*x* , *a*) → *xa*

∀ *a* ∈ *R* and *x* ∈ *M*

satisfying following properties :

and denoted by *MR*.

*x*(*a* + *b*) = *xa* + *xb* (1.5)

(*x* + *y*)*a* = *xa* + *ya* (1.6)

*x*(*ab*) = (*xa*)*b* (1.7)

*x*1 = *x* (1.8)

∀ *a* , *b* ∈ *R* and *x* , *y* ∈ *M*

##### *Examples :*

1. Let *V* be a vector space over a field *F* then *V* is a left as well as right *F*−Module.
2. Let *G* be any abelian group under addition , then *G* is a Z−Module where Z is set of integers.
3. Let *R* be ring and *M* = *R*[*x*] where *R*[*x*] is a group of all poly- Suppose ring *R* is a field then

nomials with coefficents in *R* then *M* is a left as well as a right

*R*−Module with scaler multiplication being usual multiplication.

1. Let *M* be collection of all *m* × *n* matrices over ring *R* , then *M* is left *R*−Module where scaler multiplication being usual multiplica- tion of a scaler to a matrix.

In particular, if *M* is a set of 1 × *n* matrices over *R* or *M* = *Rn*(set of *n*−tuples) then *Rn* is a left *R*−module.

**Remark: 2.** *Let R be a commutative ring then every left R*−*module can be transformed to right R*−*module and vice-versa.*

*Proof.* Let *M* be left *R*−module and *R* be a commutative ring. so, ∃ a mapping

*R*−Module *R*[*x*] is a vector space

over field *R*.

defined by

*R* × *M* → *M*

(*a* , *x*) → *ax*

for each *a* ∈ *R* and *x* ∈ *M* satisfying following properties : ∀ *a* , *b* ∈ *R* and *x* , *y* ∈ *M*

|  |  |  |
| --- | --- | --- |
| (*a* + *b*)*x* | = | *ax* + *bx* |
| *a*(*x* + *y*) | = | *ax* + *ay* |
| (*ab*)*x* | = | *a*(*bx*) |
| 1*x* | = | *x* |

∵ *R* is a commutative ring.

Now, Define an another mapping

*M* × *R* → *M*

defined by

(*x* , *a*) → *x* ∗ *a* = *ax*

To check *M* is a right *R*−Module , we need to verify properties num- ber **??**-(1.8)

1. *Distribuitive Law*

*x* ∗ (*a* + *b*) = (*a* + *b*)*x*

= *ax* + *bx*

= (*x* ∗ *a*) + (*x* ∗ *b*)

1. *Distributive Law*

(*x* + *y*) ∗ *a* = *a*(*x* + *y*)

= *ax* + *ay*

= (*x* ∗ *a*) + (*y* ∗ *a*)

*(iii)*

*x* ∗ (*ab*) = (*ab*)*x*

= (*ba*)*x*

= *b*(*ax*)

= (*ax*) ∗ *b*

*(iv)*

*x* ∗ 1 = 1*x*

= *x*

Thus, *R M* is transformed to *MR*.

Similarly, Converse statement can be verified.

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**Remark: 3.** *Let S be a subring of ring R then S M exists only if R M exists.* by existance means *M* is a valid left

module over mentioned ring or subring.

i.e. satisfying those four properties.

**Remark: 4.** *Same Abelian group can have the structure of a Module for a* For Instance, The field R is

*number of different rings.*

R−module,Q−module and Z−module.

**Remark: 5.** *Let I be left ideal of R then quotient ring R* /*I is a left R-*

*module.* Here scaler multiplication is

*verification:* ‘left to reader’

defined as

*R* × *R* /*I* >-- *R* /*I*

**Hint:** you need to verify those four properties: (1.1)-(1.4) •

(*a* , *x* + *I*) >-- *ax* + *I*

∀ *a* ∈ *R* and ∀ *x* + *I* ∈ *R* /*I*

**Theorem 1.1.** *(Elementry Properties:)*

*Let M be a left R-module . Suppose* 0*m and* 0*r denotes additive identities of M and R respectively. Then, for each x* ∈ *M and r* ∈ *R*

*(i)*

0*m* = 0*r x* = *r* 0*m*

*(ii)*

*r*(−*x*) = (−*r*)*x* = −*rx*

*Proof. (i)* As 0*m* is the additive identity of *M*. so, 0*m* = 0*m* + 0*m*

Consider *r*(0*m* + 0*m* ) = *r* 0*m* = *r* 0*m* + 0*m* ∵ (*r* , 0*m* ) >-- *r* 0*m* ∈ *M*

so, *r* 0*m* = *r* 0*m* + 0*m*

but, *r*(0*m* + 0*m* ) = (*r* 0*m* ) + (*r* 0*m* ) ∵ *M* is a left *R*-module.

so, we have

*r* 0*m* + *r* 0*m* = *r* 0*m* + 0*m*

(using distribuitive property)

as (*M*, +) is an abelian group so left and right cancellation law holds.

✟*r* 0✟ + *r* 0*m*

*m*

= ✟*r* 0✟ + 0*m*

*r* 0*m* = 0*m*

*m*

a similiar argument can be used to prove 0*m* = 0*r x*.

*(ii)* as *M* is a left *R*-module so (*r* , *x*) >-- *rx* ∈ *M*

Now, Consider (−*r*)*x* + *rx*

using distribuitive law

(−*r*)*x* + *rx* = (−*r* + *r*)*x*

= 0*r x*

= 0*m*

i.e. (−*r*)*x* is additive inverse of (*rx*) but additive inverse of (*rx*) is

−*rx* and it is unique for an abelian group(*M* here)

∴ (−*r*)*x* = −*rx*

a similar argument can be used to prove that *r*(−*x*) = −*rx*.

•

**Definition 1.1** (Ring Homomorphism)**.** *Let R and S be two rings with* often called as ring homo

*identities* 1*r* , 1*s respectively then a map(say f )*

*f* : *R* → *S*

*is said to be a ring homomorphism or ring linear map if for every a* , *b* ∈ *R following properties holds*

if *R* = *S* then we call ring homo as ring

endomorphism. For instance , let *f* be ring homo from *R* to *R* . we say *f* is endomorphism of *R* and denoted by *End R*

1. *Preserves Addition*

(*a* + *b*) *f* = (*a*) *f* + (*b*) *f*

1. *Preservers Multiplication*

(*ab*) *f* = (*a*) *f* .(*b*) *f*

1. *Maps identity to identity*

(1*r* ) *f* = 1*s*

**Remark: 6.** *Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.*

**Theorem 1.2.** *Let R be a ring and M be any abelian group with addition. M* is a right R-module

*then M is a right R-module if and only if there exists a map which is ring homomorphism from R to End M*

*Proof. (Forward Part)* Let us suppose that *M* is a right *R*-module.

**Claim:** there exists a map which is ring homomorphism from *R* to

*End M*

∵ *M* is a left *R*-module , so there exist a map

JJ

Ring

∃ *f* : *R* −H−o−m→o *End M*

defined by

*f* : *M* × *R* >-- *M*

(*x* , *a*) >-- *ax*

satisfying following properties:

(*x* + *y*)*a* = (*x*)*a* + (*y*)*ax*(*a* + *b*) = *xa* + *xb x*(*ab*) = (*xa*)*b*

*x*1 = *x*

for each *a* ∈ *R* , define a map(say *φa*)

*φa* : *M* >-- *M*

∀ *x*, *y* ∈ *M* & *a*, *b* ∈ *R*

such that for each *x* ∈ *M*

(*x*)*φa* = *xa* ∈ *M*

Now, we’ll show that *φa* ∈ *End M*

Let *x*, *y* ∈ *M*

Consider (*x* + *y*)*φa*

= (*x* + *y*)*a*

= *xa* + *ya*

= (*x*)*φa* + (*y*)*φa*

using defination of *φa*

using (1.9)

so, *φa* preserves addition and is a group homo from *M* to *M*.

i.e. *φa* ∈ *End M*

Now, we can define a map (say *f* )

defined as

*f* : *R* >-- *End M*

1. *f* >-- *φa* ∀*a* ∈ *R* and *φa* ∈ *End M*

Now, We’ll show that *f* is a ring homomorphism.

*(A)*

(*a* + *b*) *f* = *φa*+*b*

= *φa*

+ *φb*

for each *x* ∈ *M* we have,

*(B)*

= (*a*) *f* + (*b*) *f*

(*x*)*φa*+*b* = *x*(*a* + *b*) = *xa* + *xb*

= (*x*)*φa* + (*y*)*φb*

∴ *φa*+*b* = *φa* + *φb*

(*ab*) *f* = *φab*

= *φa*

* *φb*

for each *x* ∈ *M* we have,

*(C)*

= (*a*) *f* (*b*) *f*

(*x*)*φab* = *x*(*ab*) = (*xa*)*b*

= (*xa*)*φb* = (*x*)*φa* • *φb*

∴ *φab* = *φa* • *φb*

Thus, Forward Part is proved.

* 1. *f* = *φ*1

for each *x* ∈ *M* we have,

(*x*)*φ*1 = *x*(1)

= *x*

∴ *φ*1 is identity of*End M*

*(Converse Part)* Assume that ∃ a ring homo.( say *f* )

Ring

*f* : *R* −H−o−m→o *End M*

for any *a* ∈ *R*,we denote the (*a*) *f* by *fa* ∈ *End M*

**Claim:** *M* is a right *R*-module.

so let’s define a map

*R* × *M* −→ *M*

defined by

(*a*, *x*) >-- *x* ∗ *a* = (*x*) *fa*

to prove *M* is a right *R*-module , we need to verify four properties (1.5)- (1.9) of right *R*-module.

*(i)*

(*x* + *y*) ∗ *a* = (*x* + *y*) *fa*

= (*x*) *fa* + (*y*) *fa*

= *x* ∗

*a* + *y* ∗ *a*

∵ *fa* ∈ *End M*

(ii)

*x* ∗ (*a* + *b*) = (*x*) *fa*+*b*

= (*x*)( *fa* + *fb* )

= (*x*) *fa*

+ (*x*) *fb*

∵ *fa* , *fb* ∈ *End M*

= (*x* ∗ *a*) + (*x* ∗ *b*)

(iii)

*x* ∗ (*ab*) = (*x*) *fab*

= (*x*) *fa* • *fb*

= (*x fa* ) *fb*

= (*x* ∗ *a*) ∗ *b*

(iv)

*x* ∗ 1 = (*x*) *f*1 = *x*

Thus, *M* is a right *R*-module. ∵ *f*1is identity in*End M*

•

**Definition 1.2** (Anti-Ring Homomorphism)**.** *Let R and S be two rings with identities* 1*r and* 1*s respectively. Define a map f*

*f* : *R* >-- *S*

*satisfying following properties, for each a*, *b* ∈ *R (i)*

(*a* + *b*) *f* = (*a*) *f* + (*b*) *f*

*(ii)*

(*ab*) *f* = (*b*) *f* (*a*) *f*

*(iii)*

(1*r* ) *f* = 1*s*

*Then, f is called anti-ring homomorphism.*

**Theorem 1.3.** *Let R be a ring and M be any abelian group with addition. M* is a left R-module

*then M is a left R-module if and only if there exists a map which is anti-ring homomorphism from R to End M.*

*Proof.* Left to reader. •

*Submodules*

**Definition 1.3** (SubModule)**.** *Let M be a left (right) R-module then a subset N of M is called a submodule of M if N is a left (right) R-module under the operation induced from M.*

JJ

Anti-Ring

∃ *f* : *R* −−H−o−m−o→ *End M*

*In other words, A subset N of M is called submodule of M if*

* + 1. *N is subgroup of M.*
    2. *N is closed under induced scaler multiplication from M.*

**Theorem 1.4** (Criterion for Checking Modules)**.** *Let M be a left (right) R-module and N be a subset of M then N is a submodule of M if and only if*

*(i)*

*x* − *y* ∈ *N*

∀ *x*, *y* ∈ *N*

*(ii)*

*ax* ∈ *N*

∀ *a* ∈ *R* & *x* ∈ *N*

*Proof.* Left to reader. •

##### *Examples:*

1. As every Vector Space *V* over a Field *F* is a *F*-module. So, sub- modules of *V* are subspaces of *V*.
2. As every abelian group *G* is a Z-module. So, all subgroups of *G*

are submodules.

1. Let *R* be a ring then *R* is a left as well as right *R*-module then left (right) ideals of *R* are left (right) submodules of *R*.
2. {0} and *M* are trivial submodules of any left (right) *R*-module *M*.

###### Remark: 7.

1. *Union of two submodules need not to be a submodule.* Think an example !
2. *Intersection of any number of submodules is again a submodule.* **Hint:** Verify using criterion for checking

modules.

**Remark: 8. *(Smallest Submodule containing a set)***

*Let M be any left (right) R-module and S be any subset of M. Suppose* F

*be the family of all submodules of M containing S.*

*Let P* = (1 *N*

*N*∈F

*then P is a submodule of M containing S as being intersection of an indexed family of submodules containing S.*

*Moreover, P is the smallest submodule of M containing S. i.e. for any arbitrary submodule K* ∈ F *, we have P* ⊆ *K. Such submodule P of M is said to be generated by set S and is denoted by*

*P* = (*S*) = (*S*)

###### Remark: 9.

*Let S be any subset of left R-module M and* (*S*) *is the smallest submodule of M containing S.*

* 1. *if S is non-empty and finite , S* = {*x*1, *x*2, *x*3, · · · , *xn* }

(*S*) = ({*x*1, *x*2, *x*3, · · · , *xn* }) = (*x*1, *x*2, *x*3, · · · , *xn* )

*is said to be a finitely generated by S and is smallest submodule of M containing S.*

* 1. *if S* = *φ i.e. S is an empty set*

(*S*) = (*φ*) = {0}

* 1. *if S* = {*a*} *i.e. S is singleton then* (*S*) = (*a*) *is said to be a cyclic submodule.*

**Definition 1.4** (Cyclic module)**.** *A module M is said to be a cyclic mod-*

*ule*3 *if it can be generated by a single element.* 3 P. M. Cohn. *Basic Algebra*. Springer, 2

edition, 2005. ISBN 978-1-4471-1060-6

*For Example:* A ring *R* over itself is a module and can be generated by identity element {1} so is a cyclic module.

**Theorem 1.5.** *Let M be left R module and S being any subset of M.*



{0} *if S* = *φ*

(*S*) =

(



∑

*i*∈*Jn*

*aixi* | *ai* ∈ *R* , *xi* ∈ *S*

*otherwise*

i.e. Every submodule of *M* will contain

*Proof. Case-I* Let us suppose that *S* = *φ* ,as (*S*) is the intersection of *S*

all the submodules of *M* containing *S*.

In particular, {0} also contains *S* i.e.

{0} ∈ F

so,

(*S*) = (1 *N*

*N*∈F

= {0}

*Case-II* Suppose *S* is non-empty and let

(

∵ F is a collection of all submodules of *M*

containing *S*

*P* = ∑ *aixi* | *ai* ∈ *R* , *xi* ∈ *S*

*i*∈*Jn*

First , we’ll show that *S* ⊆ *P*

Let *x* ∈ *S* then it can be expressed in following form:

*x* = 1.*x* = ∑ *aixi* with *a*1 = 1 and *x*1 = *x*

*i*∈*J*1

∴ *x* ∈ *P* ⇒ *S* ⊆ *P*

Now ,we’ll show that *P* is a submodule of *M* using submodule criterion.

∵ *x* was chosen arbitirary.

Let *u* , *v* ∈ *P* . so, we need to show *u* + *αv* ∈ *P* for any *α* ∈ *R*

*u* = ∑ *aixi* ∀*xi* ∈ *S* & *ai* ∈ *R*

*i*∈*Jn*

*v* = ∑ *bjyj* ∀*yj* ∈ *S* & *bj* ∈ *R*

*j*∈*Jm*

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# *Stay Tuned for next chapters!*

*Bibliography*

P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471- 1060-6.