For this algorithm we'll use the following DGEs:

$$\begin{gathered}
F_{n,p}(X_{(1)}-1) < U_{(1)} \leqslant F_{n,p}(X_{(1)}) \\
F_{n,p}(X_{(2)}-1) < U_{(2)} \leqslant F_{n,p}(X_{(2)}) \\
\vdots \\
F_{n,p}(X_{(m)}-1) < U_{(m)} \leqslant F_{n,p}(X_{(m)}),
\end{gathered} \tag{1}$$

where $F_{n,p}$ is the CDF of a $Bin(n,p)$ distribution,

$$\begin{gathered}
G_{n-X_{(1)}+1,X_{(1)}}(1-p) < U_{(1)} \leqslant G_{n-X_{(1)},X_{(1)}+1}(1-p) \\
G_{n-X_{(2)}+1,X_{(2)}}(1-p) < U_{(2)} \leqslant G_{n-X_{(2)},X_{(2)}+1}(1-p) \\
\vdots \\
G_{n-X_{(m)}+1,X_{(m)}}(1-p) < U_{(m)} \leqslant G_{n-X_{(m)},X_{(m)}+1}(1-p),
\end{gathered} \tag{2}$$

where $G_{\alpha,\beta}$ is the CDF of a $Beta(\alpha,\beta)$ distribution, and

$$\begin{gathered}
G^{-1}_{X_{(1)},n-X_{(1)}+1}(1-U_{(1)}) < p \leqslant G^{-1}_{X_{(1)}+1,n-X_{(1)}}(1-U_{(1)}) \\
G^{-1}_{X_{(2)},n-X_{(2)}+1}(1-U_{(2)}) < p \leqslant G^{-1}_{X_{(2)}+1,n-X_{(2)}}(1-U_{(2)}) \\
\vdots \\
G^{-1}_{X_{(m)},n-X_{(m)}+1}(1-U_{(m)}) < p \leqslant G^{-1}_{X_{(m)}+1,n-X_{(m)}}(1-U_{(m)})
\end{gathered} \tag{3}$$

Algorithm 1: Produce a posterior fiducial distribution for $\mathrm{Bin}(n, p)$.

Input: $\mathsf{iterations}$: how many particles to draw from the posterior
$\mathsf{starting\_method}$: method for choosing starting n
$\mathsf{X} = X_1 X_2 \ldots X_m$: observed data values
$\mathsf{user\_n}$: optional param to choose first n
σ: tuning param for the MH Logit step

Output: $\mathsf{fiducial\_dictionary}$: The set of fiducial posterior particles: each a set of n and associated μ values.

$\hat{n} \leftarrow$ first starting n chosen using $\mathsf{user\_n}$ method. If no starting n is provided, the DasGupta-Rubin estimator is used;
$\hat{p} \leftarrow \mathrm{mean}\{\mathsf{X}\}/\hat{n}$;
$\mathsf{first\_uniforms} \leftarrow$ output from **Algorithm 2**, a set of uniforms that work for our data $\hat{n}, \hat{p}$;
$\mathsf{X}^* \leftarrow$ all unique data points (without repeats);
$\mathsf{N}^* \leftarrow$ number of occurrences in X of each value in X^*;
$\mathsf{U}_{min} \leftarrow$ minimum uniform associated with each element of X^*;
$\mathsf{U}_{max} \leftarrow$ minimum uniform associated with each element of X^*;
$\mathsf{all\_data} \leftarrow \mathsf{X}^*, \mathsf{N}^*, \mathsf{U}_{min}, \mathsf{U}_{max}$;
$\mathsf{fiducial\_posterior} \leftarrow NULL$;
for $\mathsf{iter} \leftarrow 1, \ldots, \mathsf{iterations}$ **do**
    $\mathsf{updated\_uniforms} \leftarrow$ updated uniform values using **Algorithm 3**;
    $\mathsf{posterior\_draw} \leftarrow$ fiducial posterior particles found using $\mathsf{updated\_uniforms}$ with **Algorithm 4**;
    Add $\mathsf{posterior\_draw}$ to $\mathsf{fiducial\_posterior}$;
end

Algorithm 2: Find a first set of working uniforms given our data.

Input: X: original data, ordered from least to greatest
$\hat{n}$: starting value for n
$\hat{mu}$: starting value for mu (if given)

Output: $\mathsf{first\_uniforms}$: a set of uniforms that work for given inputs
$\mathsf{U\_uppers}$: a set of uniforms that work for given inputs
$\mathsf{U\_lowers}$: a set of uniforms that work for given inputs

if $\hat{mu}$ *is given* **then**
    $\hat{p} \leftarrow \hat{mu}/\hat{n}$;
else
    $\hat{p} \leftarrow \mathrm{mean}(\mathsf{X})/\hat{n}$;
for $i \leftarrow 1, \ldots, m$ **do**
    $\mathsf{U\_upper} = G_{\hat{n}-\mathsf{X},\mathsf{X}+1}(1-\hat{p})$;
    $\mathsf{U\_lower} = G_{\hat{n}-\mathsf{X}+1,\mathsf{X}}(1-\hat{p})$;
    Add $(\mathsf{U\_lower} + \mathsf{U\_upper})/2$ to $\mathsf{first\_uniforms}$;
    Add ($\mathsf{U\_lower}$ to $\mathsf{U\_lowers}$;
    Add ($\mathsf{U\_upper}$ to $\mathsf{U\_upperss}$;
end

Algorithm 3: Draw a new uniform set given that a solution exists.

Input: all_data: see **Algorithm 1**
$\hat{n}$: starting value for n
$\bar{n}$: the largest value for n achieved by previous runs of this algorithm
σ: tuning parameter for the MH step
ϵ: how close to the limiting poisson we allow before we stop searching the n space

Output: all_data, updated
$\bar{n}$, updated
ϵ, updated

for index *in* $1 \ldots nrow($all_data$)$ **do**
  fix_max $\leftarrow$ single draw from Bernoulli(0.5);
  if $\mathsf{N}^*[\mathsf{index}] == 1$ **then**
    temp_data $\leftarrow$ all_data\all_data[index];
    p_lowers, p_uppers $\leftarrow$ output of **Algorithm 4** using temp_data;
    min_unif, max_unif $\leftarrow \min\{G_{\mathsf{N}^*-\mathsf{X}^*[\mathsf{index}]+1,\mathsf{X}^*[\mathsf{index}]}(1 - \mathsf{p\_uppers})\}, \max\{G_{\mathsf{N}^*-\mathsf{X}^*[\mathsf{index}],\mathsf{X}^*[\mathsf{index}]+1}(1 - \mathsf{p\_lowers})\}$;
    new_unif $\leftarrow$ single draw from Unif(min_unif, max_unif);
    U_{min}[index] $\leftarrow$ new_unif ;
    U_{max}[index] $\leftarrow$ new_unif ;
  else if fix_max **then**
    max_unif $\leftarrow$ U_{max}[index];
    // Temporarily remove U_{min} from calculation.
    $\mathsf{U}_{min} \leftarrow \mathsf{U}_{max}$[index];
    p_lowers, p_uppers $\leftarrow$ output of **Algorithm 4**;
    min_unif $\leftarrow \min\{G(1 - \mathsf{p\_uppers})_{\mathsf{N}^*-\mathsf{X}^*[\mathsf{index}]+1,\mathsf{X}^*[\mathsf{index}]}\}$;
    U_{min}[index] $\leftarrow$ single draw from minimum order statistic for these uniforms: $\mathrm{Beta}(1, \mathsf{N}^*[\mathsf{index}] - 1) * (\mathsf{max\_unif} - \mathsf{min\_unif}) + \mathsf{min\_unif}$;
  else
    min_unif $\leftarrow$ U_{min}[index];
    // Temporarily remove U_{max} from calculation.
    $\mathsf{U}_{max} \leftarrow \mathsf{U}_{min}$[index];
    p_lowers, p_uppers $\leftarrow$ output of **Algorithm 4**;
    max_unif $\leftarrow \max\{G(1 - \mathsf{p\_lowers})_{\mathsf{N}^*-\mathsf{X}^*[\mathsf{index}],\mathsf{X}^*[\mathsf{index}]+1}\}$;
    U_{max}[index] $\leftarrow$ single draw from maximum order statistic for these uniforms: $\mathrm{Beta}(\mathsf{N}^*[\mathsf{index}] - 1, 1) * (\mathsf{max\_unif} - \mathsf{min\_unif}) + \mathsf{min\_unif}$;

end

Perform the MH-Logit updating step as outlined by **Algorithm 5**;

Let H^{-1} be the quantile function of a gamma distribution.

Algorithm 4: Given data and uniforms, find solution sets $n \times \{p_l, p_u\}$.

Input: all_data: see **Algorithm 1**
first_iteration: starting value for n
ϵ: how close to the limiting poisson we allow before we stop searching the n space

Output: p_lowers: lower bounds of p for the fiducial sets
p_uppers: upper bounds of p for the fiducial sets
n_hats: n values for the fiducial sets

if first_iteration **then**
    Increase ϵ until we no longer explore the sample space beyond $\hat{n} == 1000$.
end
$\hat{n} \leftarrow \max(\mathsf{X}^*)$;
while *TRUE* **do**
    lower_ps $\leftarrow [G^{-1}(1-\mathsf{U}_{min})_{\mathsf{X}^*,\hat{n}-\mathsf{X}^*+1}, G^{-1}(1-\mathsf{U}_{max})_{\mathsf{X}^*,\hat{n}-\mathsf{X}^*+1}]$;
    upper_ps $\leftarrow [G^{-1}(1-\mathsf{U}_{min})_{\mathsf{X}^*+1,\hat{n}-\mathsf{X}^*}, G^{-1}(1-\mathsf{U}_{max})_{\mathsf{X}^*+1,\hat{n}-\mathsf{X}^*}]$;
    potential_soln $\leftarrow \max(\text{lower\_ps}), \min(\text{upper\_ps}), \hat{n}$;
    if max(lower_ps) $\leqslant \min$(upper_ps) **then**
        Add $\max(\text{lower\_ps}), \min(\text{upper\_ps}), \hat{n}$ to fiducial_particle;
    lower_$\mu \leftarrow$ lower_ps $* \hat{n}$;
    upper_$\mu \leftarrow$ upper_ps $* \hat{n}$;
    poisson_upper $\leftarrow [H^{-1}(1-\mathsf{U}_{min})_{\mathsf{X}^*+1,1}, H^{-1}(1-\mathsf{U}_{max})_{\mathsf{X}^*+1,1}]$;
    poisson_lower $\leftarrow [H^{-1}(1-\mathsf{U}_{min})_{\mathsf{X}^*,1}, H^{-1}(1-\mathsf{U}_{max})_{\mathsf{X}^*,1}]$;
    if poisson_lower, poisson_upper *are* ϵ *close to* lower_μ, upper_μ **then**
        Break;
    $\hat{n} += 1$;
end

Algorithm 5: Performs a MH updating step on the uniform values. Forces a further shift of the uniform values that allows for free movement in the n direction.

Input: all_data: see **Algorithm 1**
Output: all_data, (potentially) updated.
all_uniforms $\leftarrow NULL$;
for $i \leftarrow 1 \dots nrow$(all_data) **do**
  Add N*[i] draws from Unif($\mathsf{U}_{min}, \mathsf{U}_{max}$) to all_uniforms;
end
$m \leftarrow$ length(all_uniforms);
p_lowers, p_uppers, n_estimates $\leftarrow$ output of **Algorithm 4** using all_data;
new_mu $\leftarrow Unif$(p_lowers[1] $*$ n_estimates, p_uppers[1] $*$ n_estimates);
coin_flip $\leftarrow$ single draw from Bernoulli(0.5);
new_n $\leftarrow$ n_estimates[1] $+ (-1)^{\text{coin\_flip}}$;
if new_n $\geqslant \max\{\mathsf{X}^*\}$ **then**
  U_lowers, U_uppers $\leftarrow$ output of **Algorithm 2** using new_n, new_mu;
  new_uniforms $\leftarrow Unif$(U_lowers, U_uppers);
  new_p_lowers, new_p_uppers, new_n_estimates $\leftarrow$ output of **Algorithm 4** using all_data with new_uniforms;
  MH_denominator $\leftarrow \left(\frac{1}{2}\right)\left(\frac{1}{\text{p\_uppers[1]*n\_estimates}-\text{p\_lowers[1]*n\_estimates}}\right)\prod_{i=1}\left(\frac{1}{\text{U\_uppers}-\text{U\_lowers}}\right)$;
  U_lowers, U_uppers $\leftarrow$ output of **Algorithm 2** using n_estimates[1], new_mu;
  MH_numerator $\leftarrow$
  $\left(\frac{1}{2}\right)\left(\frac{1}{\text{new\_p\_uppers[1]*new\_n\_estimates}-\text{new\_p\_lowers[1]*new\_n\_estimates}}\right)\prod_{i=1}\left(\frac{1}{\text{U\_uppers}-\text{U\_lowers}}\right)$;
random_draw $\leftarrow$ a single draw from $Unif(0,1)$;
if random_draw $<$ MH_numerator/MH_denominator **then**
  $\mathsf{U}_{min} \leftarrow$ U_lowers;
  $\mathsf{U}_{max} \leftarrow$ U_uppers;

1 Appendix

The following covers how I derived the DGEs given at the beginning of this document. To derive our DGE, we are using the fact that for a $Unif(0,1)$ distribution and a Binomial CDF F, we have $F^{-1}(U) \sim$ Binomial. So, for an observed set of data $X_1, \dots, X_m$, the fiducial framework insists that there must have been a realization of uniforms $U_1, \dots, U_m$ such that

$$\begin{gathered}
F_{n,p}(X_{(1)}-1) < U_{(1)} \leqslant F_{n,p}(X_{(1)}) \\
F_{n,p}(X_{(2)}-1) < U_{(2)} \leqslant F_{n,p}(X_{(2)}) \\
\vdots \\
F_{n,p}(X_{(m)}-1) < U_{(m)} \leqslant F_{n,p}(X_{(m)}).
\end{gathered}$$

According to *Introduction to probability and random variables* by Wadsworth, the binomial CDF is equivalent to the CDF of a Beta distribution. Specifically, for $Bin(n,p)$, we have

$$F_{n,p}(x) = G_{n-x,x+1}(1-p),$$

where G is the CDF for a $Beta(\alpha,\beta)$. Here $\alpha = n-x$ and $\beta = x+1$. Thus, our DGE can be rewritten to

$$
\begin{gathered}
G_{n-X_{(1)}+1,X_{(1)}}(1-p) < U_{(1)} \leqslant G_{n-X_{(1)},X_{(1)}+1}(1-p) \\
G_{n-X_{(2)}+1,X_{(2)}}(1-p) < U_{(2)} \leqslant G_{n-X_{(2)},X_{(2)}+1}(1-p) \\
\vdots \\
G_{n-X_{(m)}+1,X_{(m)}}(1-p) < U_{(m)} \leqslant G_{n-X_{(m)},X_{(m)}+1}(1-p).
\end{gathered}
$$

A property of the beta distribution is that

$$
G_{n-X_{(i)}+1,X_{(i)}}(1-p) = 1 - G_{X_{(i)},n-X_{(i)}+1}(p).
$$

Thus, the upper bounds of the above inequalities can be written as

$$
\begin{aligned}
U_{(i)} &\leqslant G_{n-X_{(i)},X_{(i)}+1}(1-p), \\
U_{(i)} &\leqslant 1 - G_{X_{(i)}+1,n-X_{(i)}}(p), \\
G_{X_{(i)}+1,n-X_{(i)}}(p) &\leqslant 1 - U_{(i)}, \\
p &\leqslant G^{-1}_{X_{(i)}+1,n-X_{(i)}}(1-U_{(i)}),
\end{aligned}
$$

and similarly the lower bound can be written as

$$
\begin{aligned}
G_{n-X_{(i)}+1,X_{(i)}}(1-p) &< U_{(i)}, \\
1 - G_{X_{(i)},n-X_{(i)}+1}(p) &< U_{(i)}, \\
1 - U_{(i)} &< G_{X_{(i)},n-X_{(i)}+1}(p), \\
G^{-1}_{X_{(i)},n-X_{(i)}+1}(1-U_{(i)}) &< p.
\end{aligned}
$$