On Solving the 2D Incompressible Navier Stokes Equations Using the Projection Method Algorithm

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This project report presents the development of a solution to the 2D transient incompressible Navier Stokes Equations, using the Projection Method algorithm and Finite Difference techniques in MATLAB. The set of equations are solved in a square domain of side length $l = 1$, where the boundary conditions at each of the edges of the domain are given. The equations are solved until steady state is reached, and we are given initial conditions for the velocity components in the x and y directions. CDS is used to discretize the convection term, an Euler explicit time scheme, and all of the variables for pressure and velocity are stored in a collocated grid.

I. Introduction

The transient Navier Stokes equations for incompressible flow describe how the velocity and pressure of an incompressible fluid change through space and time, by the effects of convection and diffusion. These equations form a system of non-linear partial differential equations, where Equation 1 is usually called the continuity equation, Equation 2 the x momentum equation, and Equation 3 the y momentum equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

The domain for this problem is a unit square with size length $L = 1$. The velocities of the fluid, u and v are fixed at the North, South, and East boundaries to be $u = 0, v = 0$. And the velocities are fixed at the West boundary to $u = 0, v = 1$.

It is important to note that the equations we are trying to solve are transient, which means that there is a partial derivative with respect to time in them, and thus, a time marching approach will be required to find a solution to the problem. The steady state solution is then found when the changes in the calculated velocity values stop changing between one time iteration to another.

II. Methodology

The Navier Stokes equations form a nonlinear system of partial differential equations, so the solution to them requires a more nuanced approach than the one followed in the previous projects. Although there are many techniques that could be employed to find a solution to the problem, this report will implement the use of the Projection Method algorithm, which splits the Navier Stokes equations to be able to calculate the pressure values at each time step, and consequentially, the velocity values at each of the given times. Prior to discussing the projection method algorithm, the following intermediate equations, and intermediate variables, u^* and v^* need to be introduced. These equations were obtained during class lectures and are a result of discretization and algebraic manipulation of the Navier Stokes equations.

$$Conv_x = \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \tag{4}$$

$$Diff_x = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{5}$$

$$Conv_y = \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} \tag{6}$$

$$Diff_y = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{7}$$

$$\frac{u^* - u^n}{\Delta t} = -Conv_x^n + Diff_x^n \tag{8}$$

$$\frac{v^* - v^n}{\Delta t} = -Conv_y^n + Diff_y^n \tag{9}$$

$$\frac{u^{n+1}-u^*}{\Delta t}=-\frac{1}{\rho}\frac{\partial P^n}{\partial x} \tag{10}$$

$$\frac{v^{n+1}-v^*}{\Delta t}=-\frac{1}{\rho}\frac{\partial P^n}{\partial y} \tag{11}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial P^n}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial P^n}{\partial y}\right)=\frac{\rho}{\Delta t}\left(\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}\right) \tag{12}$$

Here, it is important to note that Equations 4 through 7 are just the convection and diffusion terms of the Navier Stokes equations. Then, the Projection Method algorithm is the following:

1) Find u^* and v^* using Equations 8 and 9, as well as the values of u^n and v^n.
2) Find P^n by solving the Poisson Equation (Equation 12).
3) Find u^{n+1} and v^{n+1} using Equations 10 and 11.

To complete Step 01 of the algorithm, it was needed to discretize the convection and diffusion terms of the equations, both of which were discretized using a Central Difference Scheme (CDS). The discretization of these resulted in the following algebraic expressions for u^* and v^*.

$$u^*_{i,j}=\left[-\left[\frac{u^2_{i+1,j}-u^2_{i-1,j}}{\Delta x}+\frac{(uv)_{i,j+1}-(uv)_{i,j-1}}{\Delta y}\right]+\nu\left[\frac{u_{i+1,j}+u_{i-1,j}-2u_{i,j}}{\Delta x^2}+\frac{u_{i,j+1}+u_{i,j-1}-2u_{i,j}}{\Delta y^2}\right]\right]*\Delta t+u_{i,j} \tag{13}$$

$$v^*_{i,j}=\left[-\left[\frac{v^2_{i,j+1}-v^2_{i,j-1}}{\Delta y}+\frac{(uv)_{i+1,j}-(uv)_{i-1,j}}{\Delta x}\right]+\nu\left[\frac{v_{i+1,j}+v_{i-1,j}-2v_{i,j}}{\Delta x^2}+\frac{v_{i,j+1}+v_{i,j-1}-2v_{i,j}}{\Delta y^2}\right]\right]*\Delta t+v_{i,j} \tag{14}$$

These Equations are then used to solve for u^* and v^* at all the interior nodes of the domain. It is important to note that there is no need to solve these at any of the boundary nodes, as these are only used as an intermediate variable used for the calculation of P, which will have its own boundary conditions. Then, to complete Step 02 of the algorithm, it is needed to discretize Poisson's Equation (12). Since a collocated grid approach is used in this project, the inner and outer partial derivatives in all of the equations were approximated using either backward and forward difference techniques, respectively. The discretization of Equation 12 is as follows:

$$P_{i,j} = \frac{-\left[\frac{P_{i+1,j} + P_{i-1,j}}{\Delta x^2} + \frac{P_{i,j+1} + P_{i,j-1}}{\Delta y^2}\right] + \frac{\rho}{\Delta t}\left[\frac{u^*_{i+1,j} - u^*_{i,j}}{\Delta x} + \frac{v^*_{i,j+1} - v^*_{i,j}}{\Delta y}\right]}{\left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)} \tag{15}$$

This Equation is not explicit, and thus has to be solved by solving a system of equations on all of the nodes of our domain. This system was solved using the Gauss Seidel iterative method with a tolerance of 10^{-3}. The boundaries of the pressure were solved by using a 0th order interpolation technique, where the governing equation at each of the boundary nodes was Equation 16.

$$P_N = P_{N-1} \tag{16}$$

Lastly, to solve Step 3 in the algorithm, it was needed to discretize the right hand side of equations 10 and 11. Since these partial derivatives are inner ones, a Forward Difference technique was used, which resulted in Equations 17 and 18, which is the algebraic expression that was used to calculate u and v at the new time step.

$$u^{n+1}_{i,j} = \frac{-\Delta t}{\rho}\left[\frac{P_{i,j} - P_{i-1,j}}{\Delta x}\right] + u^*_{i,j} \tag{17}$$

$$u^{n+1}_{i,j} = -\frac{\Delta t}{\rho}\left[\frac{P_{i,j} - P_{i,j-1}}{\Delta y}\right] + v^* \tag{18}$$

This approach was taken iteratively, "time marching", until the solver reached a steady state solution. The solver determined when steady state was reached by using the following error calculations and using a conditional statement to check when this change was below the tolerance.

$$error = \max(\max(abs(u^{n+1} - u^n))) \tag{19}$$

$$error = \max(\max(abs(v^{n+1} - v^n))) \tag{20}$$

III. Results

The developed program was used to solve the Navier Stokes equations on a discretized grid. Before being able to trust the results for any grid size, a grid refinement study was conducted to evaluate whether the results had significant variations with changes on the grid size. For this study, three separate grid sizes were created. The coarse grid had the values $Nx = Ny = 50$, the medium grid had the values $Nx = Ny = 100$, and the fine grid $Nx = Ny = 200$. These values were chosen as each time the grid resolution was doubled in each direction. The results for u, v, P, were plotted against the midline in x and y, and against each other. These plots are presented in Figures 1, 2, and 3.

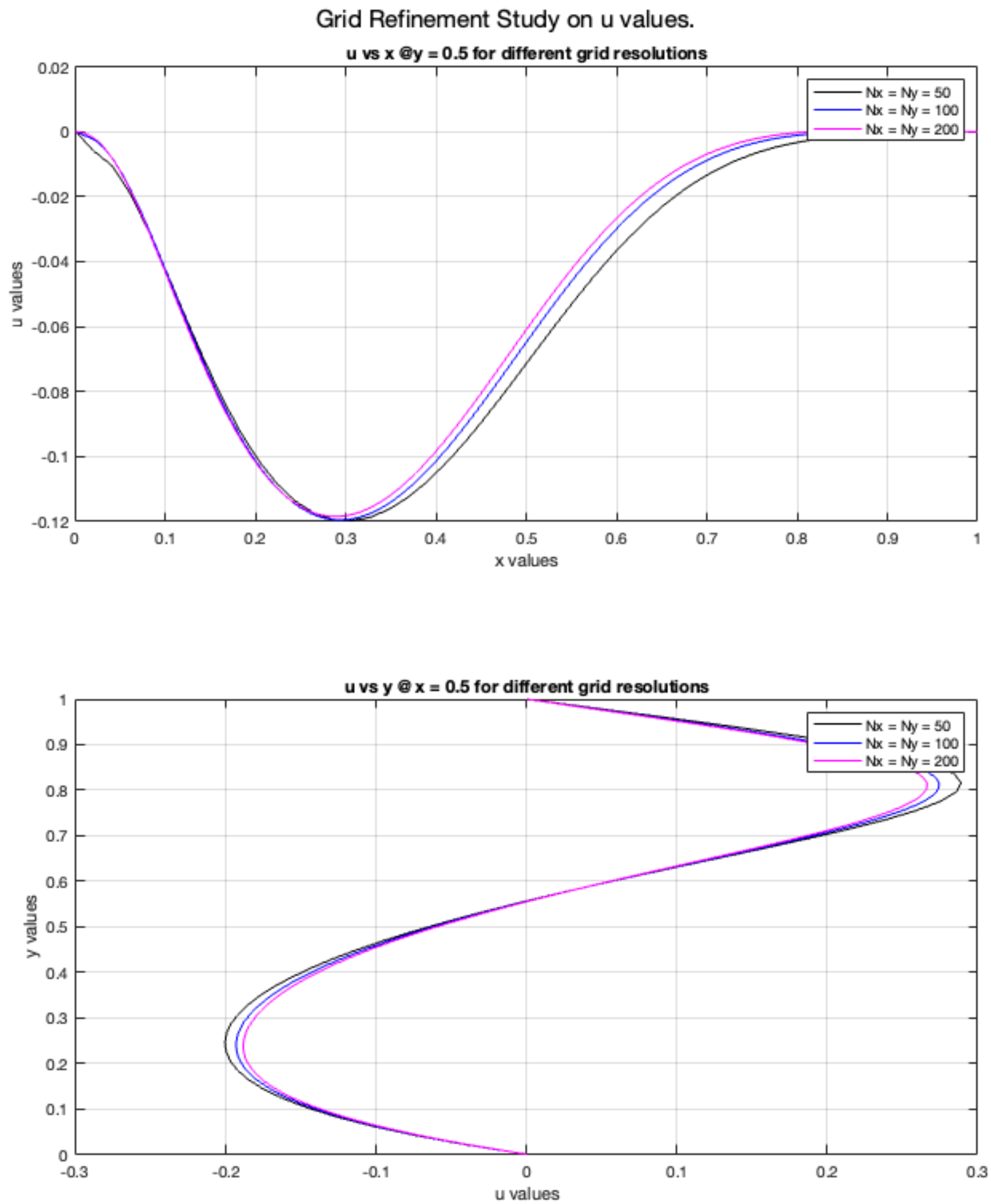

Figure 01: Results of Grid Refinement Study on u values.

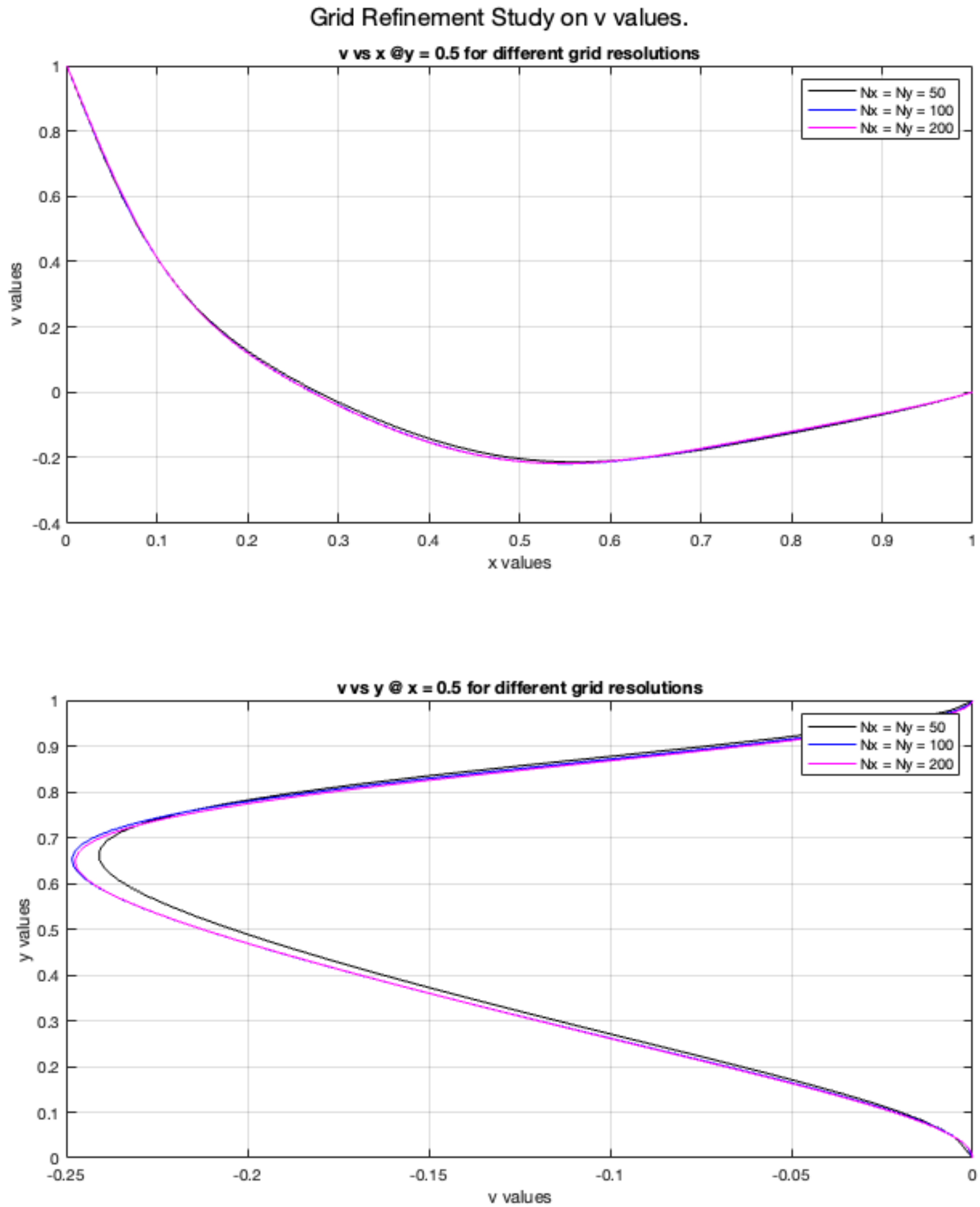

Figure 02: Results of Grid Refinement Study on v values.

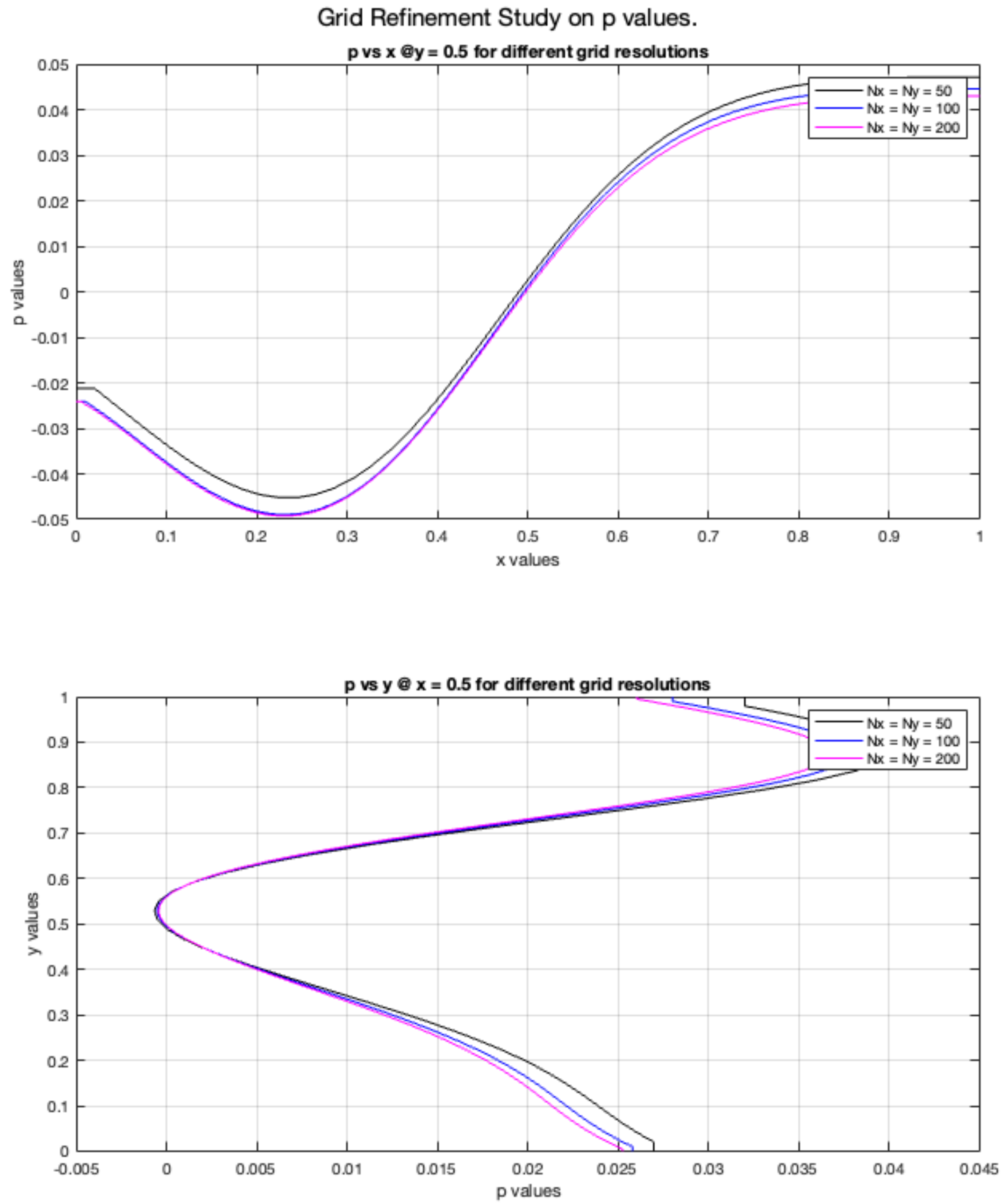

Figure 03: Results of Grid Refinement Study on v values.

These figures show that as the grid resolution is further increased, the change in the converged solution values seems not to change so much, and in fact seems to be converging to a solution.

This solution that the values are converging to is the exact solution of the system of partial differential equations and would theoretically be arrived to as the limit of our grid spacing becomes 0. This study is important as now we can leverage the numerical accuracy and computational speed of using a medium sized grid, whilst also being able to be confident in the accuracy and numerical stability of our approximated results. Then, a grid of size $Nx = Ny = 50$ was used to compute the solutions to the equations, and these were plotted against the midlines of the x and y axis. These results are presented in Figures 4 and 5.

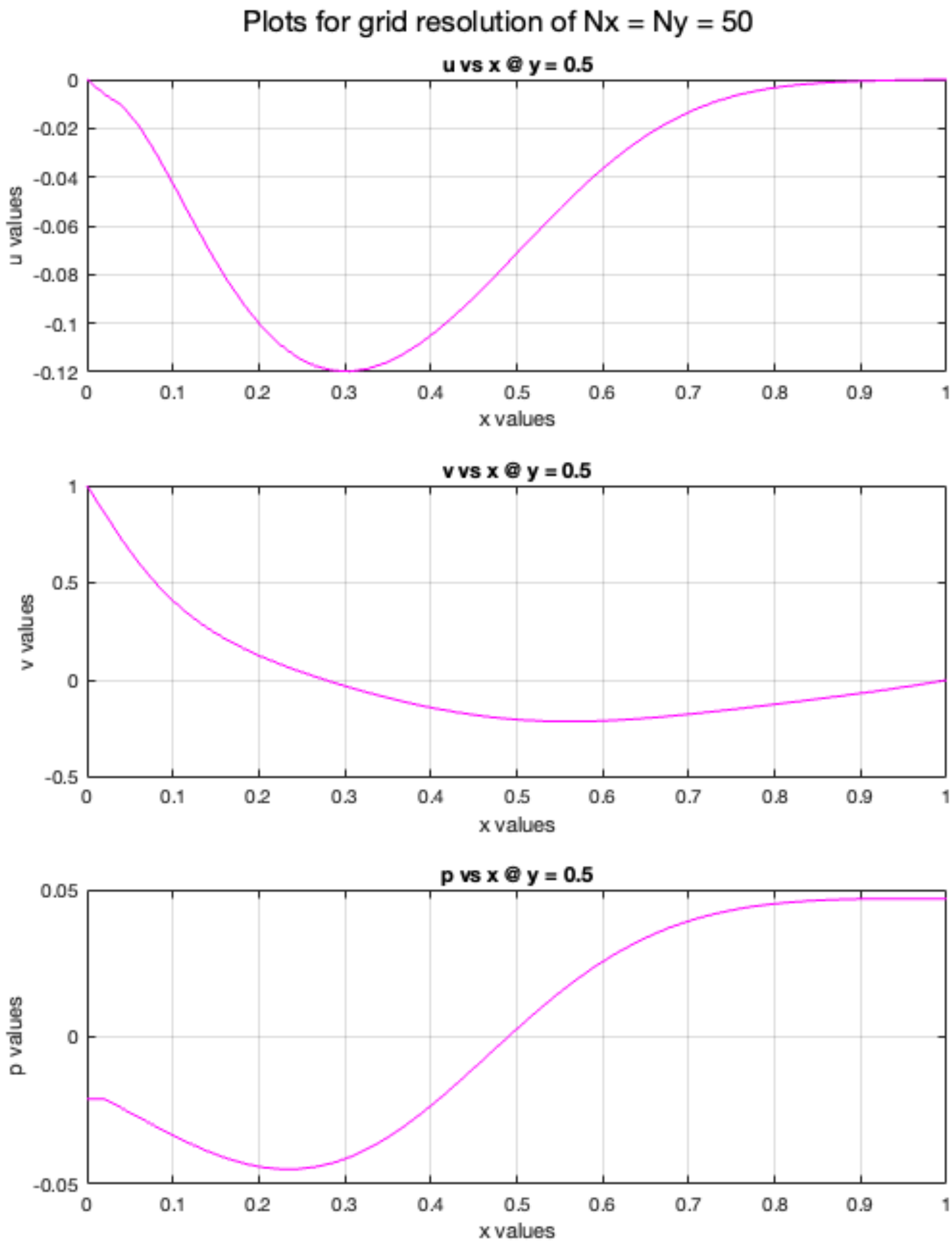

Figure 04: Pressure and velocity components versus x at $y = 0.5$.

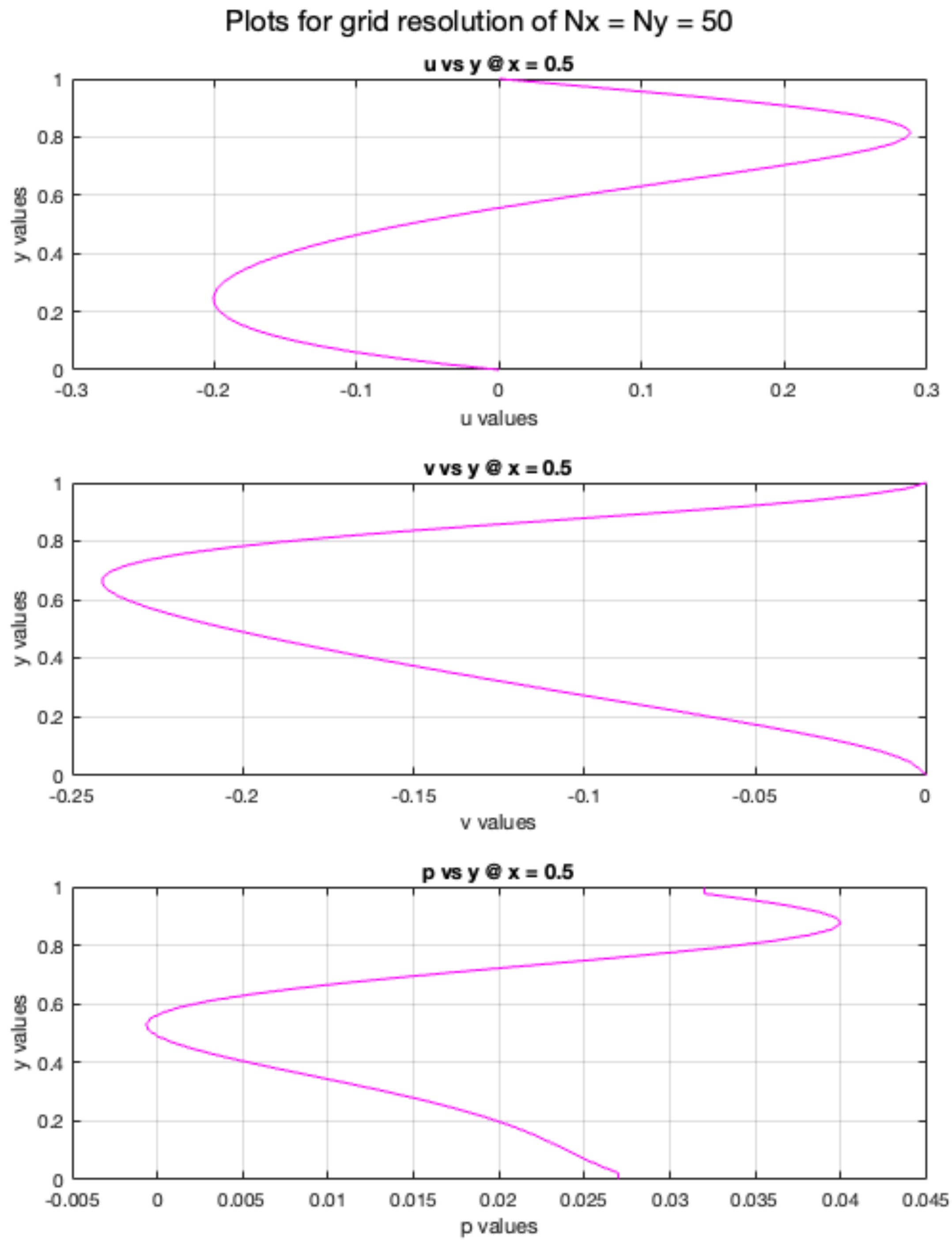

Figure 05: Pressure and velocity components versus y at $x = 0.5$.

IV. Conclusions

The 2D transient Navier Stokes Equations were successfully solved in the unit square domain that represents the lid driven cavity flow with the lid being on the left hand side of the square. The presented results highlight the power of using the projection method algorithm on a collocated grid. Additionally, the grid refinement study allows us to have a level of confidence in the results produced by any sized grid, as they show that as the grid size increases, the changes in the values of the converged solution progressively become less and less relevant, allowing us to reach a level of confidence regardless of the size of the grid.

References

[1] Ferziger. J. H., Peric M., Street R. L., *Computational Methods for Fluid Dynamics,* 4th Ed., Springer, Chap. 3.

[2] Shotorban B, Lecture Notes MAE 623, Fall 2025