MAT1856/APM466 Assignment 1

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February, 2025

Fundamental Questions - 25 points

1.

   (a) Compared with printing money directly, issuing bonds can avoid inflation as much as possible, and at the same time, different maturity dates and coupons can be set according to demand.

   (b) A long-term yield curve might flatten if investors expect slower future economic growth or lower inflation, increasing demand for long-term bonds and reducing their yields relative to short-term bonds.

   (c) Quantitative easing is a monetary policy where central banks purchase financial assets like government bonds to inject liquidity and lower interest rates, which the US Federal Reserve extensively employed during the COVID-19 pandemic by buying trillions of dollars in assets to stabilize markets, encourage lending, and support economic recovery.

2. To construct the "0-5 year" yield and spot curves, I carefully selected the following 10 bonds issued by the Government of Canada based on their coupon rates, maturity dates, and semi-annual coupon structure: "CAN 1.25 Mar 25", "CAN 0.50 Sep 25", "CAN 0.25 Mar 26", "CAN 1.00 Sep 26", "CAN 1.25 Mar 27", "CAN 2.75 Sep 27", "CAN 3.50 Mar 28", "CAN 3.25 Sep 28", "CAN 4.00 Mar 29", "CAN 3.50 Sep 29". Additionally, I included CAN 2.75 Mar 30 for interpolation purposes. These bonds feature a range of coupon rates that reflect the interest rate environment at the time of issuance, ensuring diversity in cash flows. The Government of Canada issues bonds with semi-annual coupon payments, ensuring consistency in cash flow timing. This is essential for accurate yield and spot curve construction since it allows direct comparison of bond yields without adjusting for differing payment structures. The selected bonds are spaced approximately six months apart, ensuring an even distribution across the 0-5-year range. This spacing reduces interpolation errors when bootstrapping the yield and spot curves.

3. Eigenvalues and eigenvectors of the covariance matrix tell us about the main patterns of variation in a set of stochastic processes. The eigenvalues represent the amount of variance captured by each principal component, with larger eigenvalues indicating greater variance. The eigenvectors define the principal components, which are independent directions or combinations of the processes that explain the most variability. For a yield curve, for example, the largest eigenvalue often corresponds to a parallel shift, the second largest to a tilt, and the third largest to curvature (bending).

Empirical Questions - 75 points

4.

   (a) Calculate Dirty Price (DP): For each of the 10 selected bonds, I first calculate the Dirty Price (DP) using the clean price and accrued interest. The steps are as follows:

      i. Compute the semi-annual coupon payment: $C = \frac{\text{coupon rate}}{2}$.

      ii. Calculate the accrued interest (AI): AI = days accrued $\times \frac{\text{coupon rate}}{365}$.

      iii. Add the accrued interest to the clean price to obtain the dirty price: Dirty Price (DP) = Clean Price + Accrued Interest.

   Calculate Yield to Maturity (YTM): For each bond, the yield to maturity ($r(t_N)$) is determined using the bond pricing formula:

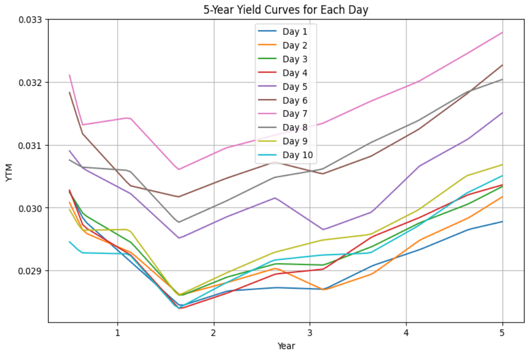
$$PV = \sum_{n=1}^{N-1} Ce^{-r(t_n)t_n} + Fe^{-r(t_N)t_N},$$

   where:

   - PV = Dirty Price (calculated above),
   - C = semi-annual coupon payment,

- $F = 100 + C$ = face value plus the final coupon payment,
- t_n = time in years to each cash flow, and $r(t_n)$ = yield for the corresponding cash flow.

The following figure shows the 5-year yield curve (YTM curve) superimposed for each day of data.



(b) Same as (a), calculate Dirty Price (DP). Then calculate the spot curve: For each bond, the spot rate($r(t_N)$) is determined using the bond pricing formula: $PV = \sum_{n=1}^{N-1} Ce^{-r(t_n)t_n} + Fe^{-r(t_N)t_N}$ where:PV = Dirty Price (calculated above), C = semi-annual coupon payment, $F = 100+C$ = face value plus the final coupon payment, t_n = time in years to each cash flow, and $r(t_n)$ = spot rate for the corresponding cash flow. We use the bootstrapping technique to compute spot rates iteratively. The pseudocode for the process is outlined below:

Procedure: Calculate Spot Rates

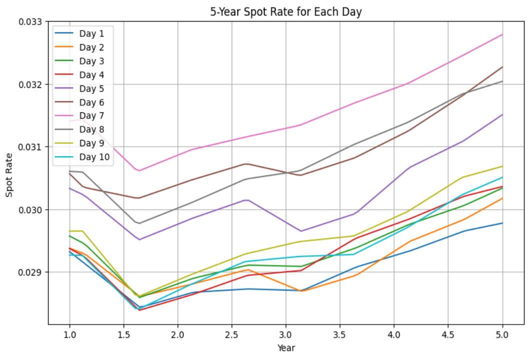
i. Initialize an empty list: *spot_rate_list*.

ii. For each bond i in the list of *bonds*:

   A. **If** $i = 0$: Compute the spot rate using $r(t_1) = -\frac{1}{t_1} \ln \frac{P}{C+F}$ and append $r(t_1)$ to *spot_rate_list*.

   B. **Else**: Solve for $r(t_N)$ using $PV = \sum_{n=1}^{N-1} Ce^{-r(t_n)t_n} + Fe^{-r(t_N)t_N}$ and append $r(t_N)$ to *spot_rate_list*.

iii. Return the *spot_rate_list*.



Algorithm 1 Calculate Spot Rate (Bonds)

procedure CALCULATE SPOT RATE(bonds)
  Assign an empty list to **spot_rate_list**
  for $i = 0$ **to** length(bonds) - 1 **do**
    if $i = 0$ **then**
      Compute the spot rate:

$$r(t_N) = -\frac{1}{t_N} \ln \left(\frac{P}{C+F} \right)$$

      Append $r(t_N)$ to **spot_rate_list**
    else
      Compute **time_difference** := bonds[i].maturity_date - bonds[i-1].maturity_date
      Solve for $r(t_N)$ using:

$$PV = \sum_{n=1}^{N-1} Ce^{-r(t_n)t_n} + Fe^{-r(t_N)t_N}$$

      Append $r(t_N)$ to **spot_rate_list**
    end if
  end for
  return spot_rate_list
end procedure

(c) First, use linear extrapolation [1] to calculate the spot rates for one year, two years, three years, four years, and five years. The formula for linear extrapolation is as follows:

$$r(t^*) = \frac{t_2 - t^*}{t_2 - t_1} \cdot r(t_1) + \frac{t^* - t_1}{t_2 - t_1} \cdot r(t_2),$$

where: t_1 and t_2 are the known maturities surrounding t^*, $r(t_1)$ and $r(t_2)$ are the corresponding spot rates,t^* is the target time for which the spot rate is being calculated.

To derive the one year forward curve, use **equations:** 1)$e^{r_{01} \cdot 1} \cdot e^{f_{1t} \cdot (t-1)} = e^{r_{0t} \cdot t} \rightarrow f_{1t} = \frac{r_{0t} \cdot t - r_{01}}{t-1}$. Where: 1) $\mathbf{r_{0n}}$: the spot rate from year 0 to year n. 2) $\mathbf{f_{1n}}$: the forward rate from year 1 to year n.

The steps involved in deriving the forward curve are as follows:

i. For each day, extract the spot rates for maturities from 1 year to 5 years.

ii. Set the spot rate for 1 year as r_1.

iii. For each maturity T_n (from 2 years to 5 years):

   A. Extract the spot rate r_n for that maturity.

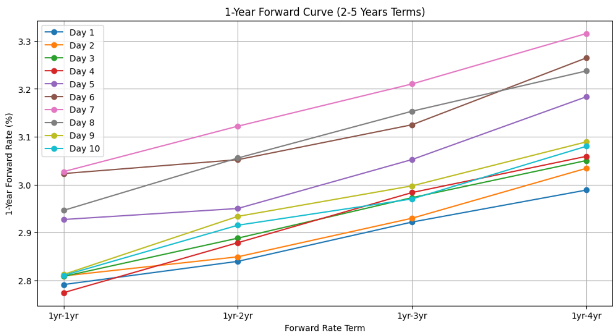
   B. Compute the 1-year forward rate $f_{1,n}$ using the formula:

$$f_{1,n} = \frac{r_n \cdot T_n - r_1 \cdot 1}{T_n - 1}$$

iv. Append the calculated forward rates for the day to the forward curve.

v. Repeat the process for each day in the dataset.

The final result is a forward curve for each day, representing the 1-year forward rates for terms from 2 years to 5 years.



Algorithm 1 Derive 1-Year Forward Curve (2-5 Years)

```
procedure DERIVE 1-YEAR FORWARD CURVE(spot_rates)
    forward_curve = []
    for each day_spot_rate in spot_rates do
        spot_rate_1 = day_spot_rate[1]
        for n = 2 to 5 do
            spot_rate_n = day_spot_rate[n]
            forward_rate_1_n = (spot_rate_n*n - spot_rate_1) / (n-1)
            Append forward_rate_1_n to forward_curve
        end for
    end for
    return forward_curve
end procedure
```

5. We begin by constructing a 10×5 matrix, where each row represents a day and each column corresponds to a year. From this, we derive a 9×5 matrix X, where each element $X_{i,j}$ is given by $X_{i,j} = \log\left(\frac{r_{i,j+1}}{r_{i,j}}\right)$. Next, we calculate the 5×5 covariance matrix for the random variables X_i using the formula $\text{Cov}(X_i, X_k) = \frac{1}{N-1}\sum_{j=1}^{N}(X_{i,j} - \bar{X}_i)(X_{k,j} - \bar{X}_k), \quad i, k = 1, \ldots, 5.$
The resulting covariance matrix is as follows:

$$\begin{bmatrix} 0.00044393 & 0.00048941 & 0.00045226 & 0.00046795 & 0.00049312 \\ 0.00048941 & 0.00059422 & 0.00052104 & 0.00052641 & 0.00059113 \\ 0.00045226 & 0.00052104 & 0.00050543 & 0.00052151 & 0.00053618 \\ 0.00046795 & 0.00052641 & 0.00052151 & 0.00055738 & 0.00055799 \\ 0.00049312 & 0.00059113 & 0.00053618 & 0.00055799 & 0.00061409 \end{bmatrix}.$$

Similar to the analysis for YTM, we calculate the covariance matrix for the forward rates. The covariance matrix for the log-returns of the forward rates (1yr-1yr, 1yr-2yr, 1yr-3yr, 1yr-4yr) is given by:

$$\begin{bmatrix} 0.00087671 & 0.00062082 & 0.00060415 & 0.00073064 \\ 0.00062082 & 0.00057074 & 0.00057879 & 0.00058198 \\ 0.00060415 & 0.00057879 & 0.00062300 & 0.00060383 \\ 0.00073064 & 0.00058198 & 0.00060383 & 0.00067592 \end{bmatrix}.$$

6. **Yield Log-Returns:** For the yield log-returns, the eigenvalues and their associated eigenvectors are listed as below:
$2.61520609e^{-03}$, $[-0.40152441\ -0.05565701\ -0.84830665\ -0.30611364\ -0.14950155]^T$
$5.64016086e^{-05}$, $[-0.46665155\ -0.63220889\ 0.04519935\ 0.35563749\ 0.50401116]^T$
$2.91307032e^{-05}$, $[-0.4343921\ 0.31972634\ 0.02289365\ 0.68626633\ -0.4874351]^T$
$1.18508493e^{-05}$, $[-0.45060199\ 0.65871902\ 0.15014906\ -0.21059395\ 0.544199]^T$
$2.46479859e^{-06}$, $[-0.47882756\ -0.24714072\ 0.50523656\ -0.51430062\ -0.43574823]^T$

Forward Rate Log-Returns: For the forward rate log-returns, the eigenvalues and their associated eigenvectors are listed as below:
$2.56128181e^{-03}$, $[0.55776647\ 0.71087059\ 0.35371069\ -0.24176087]^T$
$1.53737508e^{-04}$, $[0.45893337\ -0.34819097\ -0.47741102\ -0.66349222]^T$
$5.41462851e^{-06}$, $[0.46917854\ -0.60412257\ 0.61187267\ 0.20129394]^T$
$2.59292784e^{-05}$, $[0.50808288\ 0.09198898\ -0.52209128\ 0.67883025]^T$

The largest eigenvalue (λ_1) in each matrix indicates the direction of maximum variance in the data, with the associated eigenvector showing the contribution of each component to this principal direction.

References and GitHub Link to Code

References

[1] Will, Kenton (2018). *Interpolation*. Retrieved from `https://www.investopedia.com/terms/i/interpolation.asp`.

GitHub Link to Code:
`https://github.com/siru1366/APM466_A1`