Notes on Anderson localization

1 Maximal clusters for site percolations

Let $S_n = \{1, 2, \cdots, n\}$ and let $S_n^d = S_n \times \cdots \times S_n$ be the d-dimensional lattice. Suppose that each vertex i of the lattice S_n^d is associated with a Bernoulli random variable X_i with

$$\mathbb{P}(X_i = 0) = p, \quad \mathbb{P}(X_i = 1) = 1 - p, \quad 0 < p < 1,$$

and suppose that all these random variables are independent. Then we obtain a random graph whose vertices are labeled as either 0 or 1. This random graph is called a *site percolation*. Let M_n be the size of the maximal cluster composed of 0 for the site percolation. According to the percolation theory, the site percolation has a critical value p_c depending on the dimension d, which is given by

$$p_c = \begin{cases} 1 & \text{when } d = 1, \\ 0.59 & \text{when } d = 2, \\ 0.31 & \text{when } d = 3, \end{cases}$$

and it can be proved that

$$\lim_{d \to \infty} p_c = \frac{1}{2d}.$$

In regards to the typical size of the maximal cluster for site percolations, it can be proved that in the subcritical case of $p < p_c$, we have [1]

$$\lim_{n \to \infty} \frac{M_n}{\log(n)} = c_1(p, d), \quad a.s.$$

and in the supercritical case of $p > p_c$, we have [1]

$$\lim_{n \to \infty} \frac{M_n}{\log(n)^{\frac{d}{d-1}}} = c_2(p, d), \quad a.s.$$

where $c_1(p, d)$ and $c_2(p, d)$ are positive constants.

We next focus on the one-dimensional case. Since $p_c = 1$ when $d = 1$, the one-dimensional site percolation is always subcritical and thus

$$\lim_{n \to \infty} \frac{M_n}{\log(n)} = c_1(p, 1), \quad a.s.$$

Moreover, it can be proved that

$$c_1(p, 1) = -\frac{1}{\log p}.$$

In fact, we can obtain a better estimation of M_n when n is not very large, which is given by

$$M_n \approx -\frac{\log(np(1-p))}{\log p}.$$

In particular, when $p = 1/2$, we have

$$M_n \approx \frac{\log(n)}{\log 2} - 2.$$

2 Probability of multiple maximal clusters

Consider a one-dimensional site percolation as above. Let u_n be the probability of having at least two different maximal clusters. Then it can be prove that

$$\lim_{n\to\infty} u_n = 1 - \frac{1-p}{\log(1/p)}.$$

In particular, when $p = 1/2$, we have

$$\lim_{n\to\infty} u_n \approx 0.28.$$

3 Solutions for an eigenvalue problem

Consider the following eigenvalue problem

$$-u'' + MV(x)u = \lambda u, \quad u(0) = u(1) = 0.$$

where $M \gg 1$ is a constant and the potential V is given by

$$V(x) = \begin{cases} 0, & 0 \le x \le 1/2, \\ 1, & 0 < x \le 1, \end{cases}$$

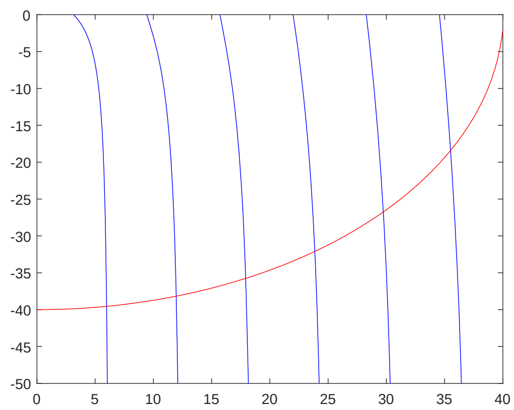
It can be proved that the eigenvalues $\lambda = k^2$ are discrete with k being the solution of the equation

$$k \cot \frac{k}{2} = -\alpha \coth \frac{\alpha}{2},$$

where

$$\alpha = \sqrt{M - k^2}.$$

is a function of k. The following figure illustrates the graphs of the functions $k\cot(k/2)$ (blue curve) and $-\alpha\coth(\alpha/2)$ (red curve) when $M = 1600$, respectively, and their intersections give the positions of possible k.



Since M is very large, the red curve is flat when k is small. Thus the minimum possible k and the associated principal eigenvalue are approximately given by

$$k \approx 2\pi, \quad \lambda \approx (2\pi)^2.$$

Moreover, it can be proved that the associated eigenfunction is given by

$$f_k(x) = \begin{cases} \sin(kx), & 0 \le x \le 1/2, \\ De^{-\alpha x}(1 - e^{2\alpha(x-1)}), & 0 < x \le 1, \end{cases}$$

where the constant D satisfies the boundary layer condition

$$\sin \frac{k}{2} = De^{-\alpha/2}(1 - e^{-\alpha}).$$

For the principal eigenvalue, we have $f_k(1/2) \approx 0$ and $\alpha \gg 1$. This explains why localization occurs.

4 Principal eigenvalues of elliptic operators

Consider the following diffusion operator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{d} b_i(x)\partial_i,$$

where a_{ij} and b_i are smooth functions on $\mathbb{R}^d$ and the matrix $a(x) = (a_{ij}(x))$ is positive definite for each $x \in \mathbb{R}^d$. Let $V = V(x)$ be a continuous function on $\mathbb{R}^d$ and let Ω be a bounded region in $\mathbb{R}^d$ with smooth boundary. Consider the following eigenvalue problem

$$Lu + V(x)u = \lambda u, \quad u|_{\partial\Omega} = 0.$$

Then the principal eigenvalue is given by [2]

$$\lambda_{V,\Omega} = \sup_{\mu(\bar{\Omega})=1} \left[\int_{\mathbb{R}^d} V(x)\mu(dx) - I(\mu) \right],$$

where μ is a probability measure on $\mathbb{R}^d$ with compact support and

$$I(\mu) = -\inf_{\substack{u \in C^\infty(\mathbb{R}^d) \\ u>0}} \int_{\mathbb{R}^d} \frac{Lu(x)}{u(x)}\mu(dx).$$

References

[1] van Der Hofstad, R. & Redig, F. Maximal clusters in non-critical percolation and related models. *J. Stat. Phys.* **122**, 671–703 (2006).

[2] Donsker, M. & Varadhan, S. S. On the principal eigenvalue of second-order elliptic differential operators. *Commun. Pure Appl. Math.* **29**, 595–621 (1976).