Anderson localization for nonsymmetric elliptic operators

Let

$$L = \frac{1}{2}\nabla\cdot(A(x)\nabla) + b(x)\nabla = \frac{1}{2}\sum_{i,j=1}^{d}\partial_i(a^{ij}(x)\partial_j) + \sum_{i=1}^{d} b^i(x)\partial_i \tag{1}$$

be a second-order elliptic differential operator in $\mathbb{R}^d$, where $A(x) = (a^{ij}(x))$ is a semi-positive definite matrix-valued smooth function and let $b(x) = (b_i(x))$ be a vector-valued smooth function. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Consider the following eigenvalue problem with *Robin boundary condition*:

$$\begin{cases} -Lu + KVu = \lambda u \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + hu = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $V = V(x) \geq 0$ is called the *disordered potential*, $K \gg 1$ is a constant called the *degree of randomness*, and $n = n(x)$ is the inward pointing unit normal vector field on $\partial\Omega$. If $h = 0$, the Robin boundary condition reduces to the *Neumann boundary condition*.

Recall that the elliptic operator L is called *symmetric* if there exists a function $\rho \in C_b^\infty(\mathbb{R}^d)$ such that

$$(Lf, g)_\rho = (f, Lg)_\rho, \quad f, g \in C_c^\infty(\mathbb{R}^d), \tag{3}$$

where

$$(f, g)_\rho = \int_{\mathbb{R}^d} f(x)g(x)\rho(x)dx. \tag{4}$$

It is a well-known result that L is symmetric if and only if $2A^{-1}b$ has a potential function $U \in C^\infty(\mathbb{R}^d)$, namely,

$$2A^{-1}(x)b(x) = -\nabla U(x). \tag{5}$$

Note that if L is symmetric, then all eigenvalues of the operator $-L + KV$ must be nonnegative real numbers; if L is nonsymmetric, then the eigenvalues may be complex numbers with nonnegative real parts.

For *Anderson localization*, $\Omega = [0, 1]^d$ is often chosen as the unit hypercube and each side of the hypercube is divided uniformly into n intervals. In this way, the hypercube is divided into n^d smaller hypercubes of the same size. In each smaller hypercube, the potential V is a constant with its value being Bernoulli distributed:

$$V(x) = \begin{cases} 0 & \text{with probability } p, \\ 1 & \text{with probability } 1 - p. \end{cases} \tag{6}$$

Let G be an arbitrary compact subset of Ω. Let $w_1 = w_1(x)$ be the solution to the following partial differential equation:

$$\begin{cases} -Lw_1 + KVw_1 = 1 \quad \text{in } \ \Omega \setminus G, \\ \dfrac{\partial w_1}{\partial n} = 0 \quad \text{on } \ \partial\Omega, \\ w_1 = 0 \quad \text{on } \ \partial G, \end{cases}$$

let $w_2 = w_2(x)$ be the solution to the following partial differential equation:

$$\begin{cases} -Lw_2 + KVw_2 = 1 \quad \text{in } \ \Omega \setminus G, \\ \dfrac{\partial w_2}{\partial n} = 0 \quad \text{on } \ \partial\Omega, \\ w_2 = \dfrac{1}{\alpha} \quad \text{on } \ \partial G, \end{cases}$$

and let $w_3 = w_3(x)$ be the solution to the following partial differential equation:

$$\begin{cases} -Lw_3 + KVw_3 = 1 \quad \text{in } \ \Omega \setminus G, \\ \dfrac{\partial w_3}{\partial n} + hf(w_3) = 0 \quad \text{on } \ \partial\Omega, \\ w_3 = 0 \quad \text{on } \ \partial G, \end{cases}$$

where $\alpha > 0$ is a constant and $f \in C^\infty(\mathbb{R}^d)$ is an arbitrary nonnegative function. Then we have the following important inequality:

$$|u(x)| \leq |\lambda| w_1(x) + \alpha w_2(x) + \frac{w_3(x)}{\min_{x\in\bar{\Omega}} f(w_3(x))}, \quad x \in \Omega \setminus G.$$

where the eigenmode u is normalized so that $\|u\|_\infty = 1$. If $f(x) = x/\beta$ for some $\beta > 0$, then we have

$$|u(x)| \leq |\lambda| w_1(x) + \alpha w_2(x) + \frac{\beta w_3(x)}{\min_{x\in\bar{\Omega}} w_3(x)}, \quad x \in \Omega \setminus G.$$

Thus the *localization landscape* can be defined as

$$w(x) = \max\left\{ w_1(x), w_2(x), \frac{w_3(x)}{\min_{x\in\bar{\Omega}} w_3(x)} \right\}.$$

With this definition, we have the inequality

$$|u(x)| \leq (|\lambda| + \alpha + \beta) w(x), \quad x \in \Omega \setminus G.$$

This inequality shows that the eigenmode u must be small at those points where the landscape of w is small. The *valley lines* of the landscape w separates the domain Ω into several subregions. All eigenmodes are localized inside these subregions and are small at the boundary of each subregion.