# Anderson localization under Robin boundary conditions

Let

$$L=\sum_{i=1}^{d}b^{i}(x)\partial_i+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(x)\partial_{ij} \tag{1}$$

be a second-order elliptic differential operator in $\mathbb{R}^d$ and let $\Omega\subset\mathbb{R}^d$ be a bounded domain. Consider the following eigenvalue problem with *Robin boundary condition*:

$$\begin{cases} -Lu+KVu=\lambda u \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}+hu=0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $V=V(x)$ is a disordered potential, $K\gg 1$ is a large constant called the *degree of randomness*, and $n=n(x)$ is the inward pointing unit normal vector field on $\partial\Omega$. If $h=0$, the Robin boundary condition reduces to the *Neumann boundary condition*.

For *Anderson localization*, $\Omega=[0,1]^d$ is often chosen as the unit hypercube and each side of the hypercube is divided uniformly into $n$ intervals. In this way, the hypercube is divided into $n^d$ smaller hypercubes of the same size. In each smaller hypercube, the potential $V$ is a constant with its value being Bernoulli distributed:

$$V(x)=\begin{cases} 0 & \text{with probability } p, \\ 1 & \text{with probability } 1-p. \end{cases} \tag{3}$$

Let $G$ be an arbitrary compact subset of $\Omega$. We define the landscape of localization $w=w(x)$ as the solution to the following Dirichlet problem with mixed boundary condition:

$$\begin{cases} -Lw+KVw=1 \quad \text{in } \Omega\setminus G, \\ \dfrac{\partial w}{\partial n}=-\dfrac{h}{\beta} \quad \text{on } \partial\Omega, \\ w=\dfrac{1}{\alpha} \quad \text{on } \partial G, \end{cases}$$

where $\alpha,\beta>0$ are two constants. Then we have the following important inequality:

$$|u(x)|\le(\lambda+\alpha+\beta)|w(x)|,\quad x\in\Omega\setminus G. \tag{4}$$

where the eigenmode $u$ is normalized so that $\|u\|_\infty=1$. This inequality shows that the eigenmode $u$ must be small at those points where the landscape of $w$ is small. The *valley lines* of the landscape $w$ separates the domain $\Omega$ into several subregions. All eigenmodes are localized inside these subregions and are small at the boundary of each subregion.

We emphasize here that the landscape $w$ is a function defined in $\Omega \setminus G$ and thus the inequality (4) becomes invalid for $x \in G$. To avoid this invalidity, we can take $G = \{x_0\}$ with $x_0$ being an arbitrary point in $\Omega$ and define the landscape $w = w(x)$ as the solution to the following Dirichlet problem:

$$\begin{cases} -Lw + KVw = 1 \quad \text{in } \ \Omega \setminus \{x_0\}, \\ \dfrac{\partial w}{\partial n} = -h \quad \text{on } \ \partial\Omega, \\ w(x_0) = 1, \end{cases}$$

where the eigenmode $u$ is normalized so that $\|u\|_\infty = 1$. Then we have the following inequality which holds for the entire domain:

$$|u(x)| \leq (\lambda + 2)|w(x)|, \quad x \in \Omega.$$