Anderson localization under Robin boundary conditions

1 Probabilistic representation

Let

$$L=\sum_{i=1}^d b^i(x)\partial_i+\frac{1}{2}\sum_{i,j=1}^d a^{ij}(x)\partial_{ij} \tag{1}$$

be a second-order elliptic differential operator in $\mathbb{R}^d$ and let $\Omega\subset\mathbb{R}^d$ be a bounded domain. Consider the following eigenvalue problem with *Robin boundary condition*:

$$\begin{cases} -Lu+KVu=\lambda u \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}+hu=0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $V=V(x)$ is a disordered potential, $K\gg 1$ is a large constant called the *degree of randomness*, and $n=n(x)$ is the inward pointing unit normal vector field on $\partial\Omega$. If $h=0$, the Robin boundary condition reduces to the *Neumann boundary condition*.

For *Anderson localization*, $\Omega=[0,1]^d$ is often chosen as the unit hypercube and each side of the hypercube is divided uniformly into n intervals. In this way, the hypercube is divided into n^d smaller hypercubes of the same size. In each smaller hypercube, the potential V is a constant with its value being Bernoulli distributed:

$$V(x)=\begin{cases} 0 & \text{with probability } p, \\ 1 & \text{with probability } 1-p. \end{cases} \tag{3}$$

From the perspective of stochastic processes, the elliptic operator L is the infinitesimal generator of a *reflecting diffusion* X_t with drift $b=(b^i)$ and diffusion matrix $a=(a^{ij})$, which is the solution to the *Skorokhod stochastic differential equation*:

$$dX_t=b(X_t)dt+a^{1/2}(X_t)dB_t+n(X_t)dL_t,$$

where B_t is a d-dimensional standard Brownian motion and L_t is a continuous nondecreasing process that increases only when $X_t\in\partial\Omega$. Then the solution of the eigenvalue problem (2) has the following probabilistic representation:

$$\begin{aligned} u(x)&=\mathbb{E}_x u(X_\tau)e^{-K\int_0^\tau V(X_s)ds}+\lambda\mathbb{E}_x\int_0^\tau u(X_t)e^{-K\int_0^t V(X_s)ds}dt \\ &\quad+\mathbb{E}_x\int_0^\tau u(X_t)h(X_t)e^{-K\int_0^t V(X_s)ds}dL_t, \end{aligned} \tag{4}$$

where $\mathbb{E}_x$ is the expectation conditioned on $X_0=x$ and τ is any stopping time of X_t.

2 Landscape of localization

Let G be an arbitrary compact subset of Ω. We define the landscape of localization $w = w(x)$ as the solution to the following Dirichlet problem with mixed boundary condition:

$$\begin{cases} -Lw + KVw = 1 \quad \text{in } \Omega \setminus G, \\ \dfrac{\partial w}{\partial n} = -\dfrac{h}{\beta} \quad \text{on } \partial\Omega, \\ w = \dfrac{1}{\alpha} \quad \text{on } \partial G, \end{cases}$$

where $\alpha, \beta > 0$ are two constants. The solution of this partial differential equation has the following probabilistic representation:

$$w(x) = \frac{1}{\alpha}\mathbb{E}_x e^{-K\int_0^{\tau_G} V(X_s)ds} + \mathbb{E}_x \int_0^{\tau_G} e^{-K\int_0^t V(X_s)ds}dt + \frac{1}{\beta}\mathbb{E}_x \int_0^{\tau_G} h(X_t)e^{-K\int_0^t V(X_s)ds}dL_t, \quad (5)$$

where

$$\tau_G = \inf\{t \geq 0 : X_t \in G\}$$

is the hitting time of X to the subset G. Comparing (4) with (5), we obtain the following important inequality:

$$|u(x)| \leq (\lambda + \alpha + \beta)|w(x)|, \quad x \in \Omega \setminus G. \quad (6)$$

where the eigenmode u is normalized so that $\|u\|_\infty = 1$. This inequality shows that the eigenmode u must be small at those points where the landscape of w is small. The *valley lines* of the landscape w separates the domain Ω into several subregions. All eigenmodes are localized inside these subregions and are small at the boundary of each subregion.

We emphasize here that the landscape w is a function defined in $\Omega \setminus G$ and thus the inequality (6) becomes invalid for $x \in G$. To avoid this invalidity, we can take $G = \{x_0\}$ with x_0 being an arbitrary point in Ω and define the landscape $w = w(x)$ as the solution to the following Dirichlet problem:

$$\begin{cases} -Lw + KVw = 1 \quad \text{in } \Omega \setminus \{x_0\}, \\ \dfrac{\partial w}{\partial n} = -h \quad \text{on } \partial\Omega, \\ w(x_0) = 1, \end{cases}$$

where the eigenmode u is normalized so that $\|u\|_\infty = 1$. Then we have the following inequality which holds for the whole domain:

$$|u(x)| \leq (\lambda + 2)|w(x)|, \quad x \in \Omega.$$

3 Limit of large degree of randomness

In this section, we focus on the limit of large degree of randomness. Let $D = \{x \in \Omega : V(x) = 0\}$ denote the collection of points with zero potential. We first consider the case when $x \in D^c$. In this case, we have $V(X_s) = 1$ when s is small, which implies that

$$\int_0^t V(X_s)ds > 0, \quad \forall\, t > 0.$$

It thus follows from the probabilistic representation (4) and the dominated convergence theorem that

$$\lim_{K \to \infty} u(x) = 0, \quad \forall\, x \in D^c.$$

This shows that all eigenmodes must vanish in D^c in the limit of large degree of randomness.

We next focus on the behavior of eigenmodes in D. Clearly, D can be decomposed as the disjoint union of several connected components:

$$D = D_1 \cup D_2 \cup \cdots \cup D_N. \tag{7}$$

In the limit of $K \to \infty$, the eigenmode u must satisfy the following local eigenvalue problem in each subregion D_k:

$$\begin{cases} -Lu = \lambda u \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + hu = 0 \quad \text{on } \partial D_k \cap \partial\Omega, \\ u = 0 \quad \text{on } \partial D_k \setminus \partial\Omega. \end{cases} \tag{8}$$

Therefore, in the limit of large degree of randomness, the spectrum of the Hamiltonian $H = -L + KV$ is composed of the local eigenvalues of the operator $-L$ in each subregion D_k. If an eigenvalue of H coincides with one of the local eigenvalues of $-L$ in D_k, then the corresponding eigenmode will be localized in D_k. Conversely, if an eigenvalue of H coincides with neither one of the local eigenvalues of $-L$ in D_k, then the corresponding eigenmode will not be localized in D_k.

According to the above discussion, if multiple subregions $D_{k_1}, \cdots, D_{k_r}$ share a common local eigenvalue, then the corresponding eigenmode will have multiple peaks. In other words, an eigenmode has multiple peaks if and only if the eigenmode is a collection of local eigenmodes of subregions sharing a common eigenvalue.