Supporting Information: Mathematical proofs

Preliminaries Let Ω be an arbitrary bounded open set in $\mathbb{R}^n$ and let L be any elliptic differential operator associated to a symmetric positive bilinear form B (an energy integral). Essentially all elliptic operators governing wave propagation, whether in acoustics, mechanics, electromagnetism, or quantum physics, are associated with an energy and fall into this category. The most prominent examples include:

$$\text{the Laplacian} \qquad L = -\Delta, \qquad B[u,v] = \int_\Omega \nabla u\, \nabla v\, dx, \tag{1}$$

$$\text{the Hamiltonian} \qquad L = -\Delta + V(x),\ 0 \leq V(x) \leq C, \qquad B[u,v] = \int_\Omega \nabla u\, \nabla v + Vuv\, dx, \tag{2}$$

$$\text{the bilaplacian} \qquad L = \Delta^2 = -\Delta(-\Delta), \qquad B[u,v] = \int_\Omega \Delta u\, \Delta v\, dx, \tag{3}$$

and finally, any second order divergence form elliptic operator

$$L = -\mathrm{div}\, A(x)\nabla, \qquad B[u,v] = \int_\Omega A(x)\nabla u \nabla v\, dx, \tag{4}$$

where A is an elliptic real symmetric $n \times n$ matrix with bounded measurable coefficients, that is,

$$A(x) = \{a_{ij}(x)\}_{i,j=1}^n,\ x \in \Omega, \qquad a_{ij} \in L^\infty(\Omega), \qquad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,\ \forall\, \xi \in \mathbb{R}^n, \tag{5}$$

for some $\lambda > 0$, and $a_{ij} = a_{ji},\ \forall i, j = 1, ..., n$.

In general, L is a differential operator of order $2m$, $m \in \mathbb{N}$, defined in the weak sense:

$$\int_\Omega Lu\, v\, dx := B[u,v], \qquad \text{for every} \qquad u, v \in \mathring{H}^m(\Omega), \tag{6}$$

where B is a bounded positive bilinear form and $\mathring{H}^m(\Omega)$ is the Sobolev space of functions given by the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathring{H}^m(\Omega)} := \|\nabla^m u\|_{L^2(\Omega)}, \tag{7}$$

and $\nabla^m u$ denotes the vector of all partial derivatives of u of order m. Recall that the Lax-Milgram Lemma ascertains that for every $f \in (\mathring{H}^m(\Omega))^* =: H^{-m}(\Omega)$ the boundary value problem

$$Lu = f, \quad u \in \mathring{H}^m(\Omega), \tag{8}$$

has a unique solution understood in the weak sense.

Remark. The weak solution formalism is necessary to treat the Dirichlet boundary problem on an *arbitrary* bounded domain. When the boundary, the data, and the coefficients of the equation are sufficiently smooth, the weak solution coincides with the *classical solution* and (8) can be written for a second order operator as

$$Lu = f, \qquad u|_{\partial\Omega} = 0, \tag{9}$$

where $u|_{\partial\Omega}$ is the usual pointwise limit at the boundary. For a $2m$-th order operator the derivatives up to the order $m-1$ must vanish as well, e.g.,

$$\Delta^2 u = f, \qquad u|_{\partial\Omega} = 0, \qquad \partial_\nu u|_{\partial\Omega} = 0, \tag{10}$$

where ∂_ν stays for normal derivative at the boundary. In any context, the condition $u \in \mathring{H}^m(\Omega)$ automatically prescribes zero Dirichlet boundary data. On rough domains the definitions akin to (9), (10) might not make sense, i.e., a pointwise boundary limit might not exist (the solution might be discontinuous at the boundary), and then the Dirichlet data can only be interpreted in the sense of (8).

The classification of domains in which all solutions to Laplace's equation with nice data are continuous up to the boundary is available due to the celebrated 1924 Wiener criterion [1]. Over the years, Wiener test has been extended to a variety of operators. We shall not concentrate on this issue, let us just mention the results covering all divergence form second order elliptic operators [2], and the bilaplacian in dimension three [3].

Here we shall impose no additional restriction on $\partial\Omega$ or on the coefficients and work in the general context of weak solutions.

For later reference, we also define the Green function of L, as conventionally, by

$$L_x G(x,y) = \delta_y(x), \quad \text{for all}\, x, y \in \Omega, \quad G(\cdot, y) \in \mathring{H}^m(\Omega) \quad \text{for all}\, y \in \Omega, \tag{11}$$

in the sense of (8), so that

$$\int_{\mathbb{R}^n} L_x G(x,y) v(x)\, dx = v(y), \quad y \in \Omega, \tag{12}$$

for every $v \in \mathring{H}^m(\Omega)$. It is not difficult to show that for a self-adjoint elliptic operator the Green function is symmetric, i.e., $G(x,y) = G(y,x)$, $x, y \in \Omega$.

Control of the eigenfunctions by the solution to the Dirichlet problem: the landscape Let us now turn to the discussion of the eigenfunctions of L. Unless otherwise stated, we assume that L is an elliptic operator in the weak sense described above and that the underlying bilinear form is symmetric, i.e., that L is self-adjoint.

The Fredholm theory provides a framework to consider the eigenvalue problem:

$$L\varphi = \lambda\varphi, \quad \varphi \in \mathring{H}^m(\Omega), \tag{13}$$

where $\lambda \in \mathbb{R}$. If there exists a non-trivial solution to (13), interpreted, as before, in the weak sense then the corresponding $\lambda \in \mathbb{R}$ is called an eigenvalue and $\varphi \in \mathring{H}^m(\Omega)$ is an eigenvector. In fact, under the current assumptions on the operator all eigenvalues are positive real numbers.

Proposition 0.1. *Let Ω be an arbitrary bounded open set, L be a self-adjoint elliptic operator on Ω, and assume that $\varphi \in \mathring{H}^m(\Omega)$ is an eigenfunction of L and λ is the corresponding eigenvalue, i.e.,* (13) *is satisfied. Then*

$$\frac{|\varphi(x)|}{\|\varphi\|_{L^\infty(\Omega)}} \leq \lambda u(x), \quad \textit{for all}\, x \in \Omega, \tag{14}$$

provided that $\varphi \in L^\infty(\Omega)$, with

$$u(x) = \int_\Omega |G(x,y)|\, dy, \qquad x \in \Omega. \tag{15}$$

If, in addition, the Green function is non-negative (in the sense of distributions), then u is the solution of the boundary problem

$$Lu = 1, \quad u \in \mathring{H}^m(\Omega). \tag{16}$$

Remark. The Green function is positive in Ω and eigenfunctions are bounded for the Laplacian (1), the Hamiltonian (2), all second order elliptic operators (4) in all dimensions due to the strong maximum principle (see, e.g., [4], Section 8.7). Hence, for all such operators (14), (16) are valid.

The situation for the higher order PDEs is more subtle. In fact, even for the bilaplacian the positivity in general fails, and then one has to operate directly with (15).

Proof. By (13) and (11) (with the roles of x and y interchanged) and self-adjointness of L, for every $x \in \Omega$

$$\varphi(x) = \int_\Omega \varphi(y)\, L_y G(x,y)\, dy = \int_\Omega L_y \varphi(y)\, G(x,y)\, dy = \int_\Omega \lambda\, \varphi(y)\, G(x,y)\, dy, \tag{17}$$

and hence,

$$|\varphi(x)| \leq \lambda\, \|\varphi\|_{L^\infty(\Omega)} \int_\Omega |G(x,y)|\, dy, \quad x \in \Omega, \tag{18}$$

as desired. Moreover, if the Green function is positive,

$$\int_\Omega |G(x,y)|\, dy = \int_\Omega G(x,y) \cdot 1\, dy, \quad x \in \Omega, \tag{19}$$

which is by definition a solution of (16). □

Vaguely speaking, the inequality (14) provides the “landscape of localization”, as the map of u in (15) – (16) draws lines separating subdomains which will “host” localized eigenmodes. We now discuss in details the situation on these subdomains.

Analysis of localized modes on the subdomains The gist of the forthcoming discussion is that, roughly speaking, a mode of Ω localized to a subdomain $D \subset \Omega$ must be fairly close to an eigenmode of this subdomain, and an eigenvalue of Ω for which localization takes place, must be close to some eigenvalue of D.

To this end, let φ be one of the eigenmodes of Ω, which exhibits localization to D, a subdomain of Ω. This means, in particular, that the boundary values of φ on ∂D are small. In fact, in the present context the correct way to interpret "smallness" of φ on the boundary of D is in terms of the smallness of an L-harmonic function, with the same data as φ on ∂D.

To be rigorous, let us define $\varepsilon = \varepsilon_\varphi > 0$ as

$$\begin{aligned}
&\varepsilon = \|v\|_{L^2(D)}, \text{ where } v \in H^m(D) \text{ is such that} \\
&w := \varphi - v \in \mathring{H}^m(D) \text{ (that is, the boundary values of } \varphi \text{ and } v \text{ on } \partial D \text{ coincide),} \\
&\text{and } Lv = 0 \text{ on } D \text{ in the sense of distributions.}
\end{aligned} \tag{20}$$

Then the following Proposition holds.

Proposition 0.2. *Let Ω be an arbitrary bounded open set, L be a self-adjoint elliptic operator on Ω, and $\varphi \in \mathring{H}^m(\Omega)$ be an eigenmode of L. Suppose further that D is a subset of Ω and denote by ε the norm of the boundary data of φ on ∂D in the sense of* (20).

Denote by λ the eigenvalue corresponding to φ. Then either λ is an eigenvalue of L in D or

$$\|\varphi\|_{L^2(D)} \leq \left(1 + \max_{\lambda_k(D)} \left\{ \left| 1 - \frac{\lambda_k(D)}{\lambda} \right|^{-1} \right\} \right) \varepsilon = \left(1 + \frac{\lambda}{\min\limits_{\lambda_k(D)} \{|\lambda - \lambda_k(D)|\}} \right) \varepsilon, \tag{21}$$

where the maximum is taken over all eigenvalues of L in D.

Proof. First of all, note that (20) implies

$$(L - \lambda) w = \lambda v \quad \text{on } D, \tag{22}$$

as usually, in the sense of distributions. If λ is an eigenvalue of D, there is nothing to prove. If λ is not an eigenvalue of D, we claim that

$$\|w\|_{L^2(D)} \leq \max_{\lambda_k(D)} \left\{ \frac{1}{|\lambda - \lambda_k(D)|} \right\} \|\lambda v\|_{L^2(D)}. \tag{23}$$

Indeed, in our setup, the eigenvalues of L are real, positive, at most countable, and moreover, there exists an orthonormal basis of $L^2(D)$ formed by the eigenfunctions of L on D, $\{\psi_{k,D}\}_k$. In particular, for every $f \in \mathring{H}^m(D) \subset L^2(D)$ we can write

$$f = \sum_k c_k(f) \psi_k, \qquad c_k(f) = \int_D f\, \psi_k \, dx, \tag{24}$$

with the convergence in $L^2(D)$, and $\|f\|_{L^2(D)} = \left(\sum_k c_k(f)^2\right)^{1/2}$. Moreover, such a series $\sum_k c_k(f)\psi_k$ converges in $\mathring{H}^m(D)$ as well and $\{\psi_{k,D}\}_k$ form an orthogonal basis of $\mathring{H}^m(D)$. These considerations follow from ellipticity and self-adjointness of L in a standard way using the machinery of functional analysis (see, e.g., [5], pp. 355–358 treating the case of the second order operator of the type (4)).

Therefore, for every λ not belonging to the spectrum of L on D and $w \in \mathring{H}^m(D)$ with $(L-\lambda)w \in L^2(D)$ (cf. (22)) we have

$$\|(L-\lambda)w\|_{L^2(D)} = \left\|\sum_k c_k((L-\lambda)w)\psi_k\right\|_{L^2(D)}, \tag{25}$$

where

$$\begin{aligned} c_k((L-\lambda)w) &= \int_D (L-\lambda)w\,\psi_k\,dx = \int_D w\,(L-\lambda)\psi_k\,dx && (26)\\ &= (\lambda_k(D)-\lambda)\int_D w\,\psi_k\,dx = (\lambda_k(D)-\lambda)c_k(w). && (27)\end{aligned}$$

Hence,

$$\begin{aligned} \|(L-\lambda)w\|_{L^2(D)} &= \left\|\sum_k(\lambda_k(D)-\lambda)c_k(w)\psi_k\right\|_{L^2(D)} = \left(\sum_k(\lambda_k(D)-\lambda)^2 c_k(w)^2\right)^{1/2} \\ &\geq \min_{\lambda_k(D)}|\lambda_k(D)-\lambda|\left(\sum_k c_k(w)^2\right)^{1/2} = \min_{\lambda_k(D)}|\lambda_k(D)-\lambda|\,\|w\|_{L^2(D)}, \end{aligned}$$

which leads to inequality (23).

Going further, (23) yields

$$\|w\|_{L^2(D)} \leq \max_{\lambda_k(D)}\left\{\left|1-\frac{\lambda_k(D)}{\lambda}\right|^{-1}\right\}\|v\|_{L^2(D)} \leq \max_{\lambda_k(D)}\left\{\left|1-\frac{\lambda_k(D)}{\lambda}\right|^{-1}\right\}\varepsilon, \tag{28}$$

and therefore,

$$\|\varphi\|_{L^2(D)} \leq \left(1+\max_{\lambda_k(D)}\left\{\left|1-\frac{\lambda_k(D)}{\lambda}\right|^{-1}\right\}\right)\varepsilon, \tag{29}$$

as desired. □

References

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