Notes on Anderson localization

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1 Probabilistic representation

Let

$$L = \sum_{i=1}^{d} b^i \partial_i + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij} \partial_{ij} \tag{1}$$

be a second-order elliptic differential operator and let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Consider the following eigenvalue problem with Dirichlet boundary condition:

$$\begin{cases} -Lu + KVu = \lambda u \quad \text{in } \Omega, \\ \qquad\qquad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $K \gg 1$ is a large constant and $V = V(x)$ is a potential. For Anderson localization, $\Omega = [0,1]^n$ is often chosen as the unit hypercube and each side of the hypercube is divided into n intervals of the same length. In this way, the hypercube is divided into n^d smaller hypercubes. In each smaller hypercube, the potential V is a constant with its value being Bernoulli distributed:

$$V(x) = \begin{cases} 0 & \text{with probability } p, \\ 1 & \text{with probability } 1-p. \end{cases} \tag{3}$$

From the perspective of stochastic processes, the operator L is the infinitesimal generator of a diffusion process $X = \{X_t : t \geq 0\}$ with drift $b = (b^i)$ and diffusion matrix $a = (a^{ij})$, which is the solution to the following stochastic differential equation:

$$dX_t = b(X_t)dt + a^{1/2}(X_t)dB_t,$$

where $B = \{B_t : t \geq 0\}$ is a d-dimensional standard Brownian motion. Then the solution of the eigenvalue problem (2) has the following probabilistic representation:

$$u(x) = \lambda \mathbb{E}_x \int_0^{\tau_\Omega} u(X_t) e^{-K\int_0^t V(X_s)ds} dt, \tag{4}$$

where $\mathbb{E}_x$ is the expectation conditioned on $\{X_0 = x\}$ and

$$\tau_\Omega = \inf\{t \geq 0 : X_t \notin \Omega\}$$

is the first exit time of X from Ω.

2 Landscape and the Filochea-Mayboroda inequality

In a previous paper [1], Filochea and Mayboroda introduced the landscape of localization as the solution to the following Dirichlet problem:

$$\begin{cases} -Lw + KVw = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

The solution of this Dirichlet problem has the following probabilistic representation:

$$w(x) = \mathbb{E}_x \int_0^{\tau_\Omega} e^{-K\int_0^t V(X_s)ds} dt. \tag{5}$$

Comparing (4) with (5), we immediately obtain the following Filochea-Mayboroda inequality:

$$|u(x)| \leq \lambda |w(x)|,$$

where the eigenmode u is normalized so that $\|u\|_\infty = 1$. The Filochea-Mayboroda inequality shows that the eigenmode u must be small at those points where the landscape of w is small. The valley lines of the landscape w separates the domain Ω into several subregions. All eigenmodes are localized in exactly one of these subregions and are small at the boundary of each subregion [1].

3 Limit of large structural disorder

Next we focus on the case when $K \gg 1$. Let $D = \{x \in \Omega : V(x) = 0\}$ be the collection of points where the potential V vanishes. Recall that both the eigenmode $u(x)$ and the landscape $w(x)$ can be represented by the diffusion process X starting from a fixed point x.

We first consider the case when $x \notin D$. In this case, we have $V(X_s) = 1$ when s is small, which implies that

$$\int_0^t V(X_s)ds > 0, \quad \text{for any } t > 0.$$

It thus follows from the dominated convergence theorem that

$$\lim_{K\to\infty} w(x) = \lim_{K\to\infty} \mathbb{E}_x \int_0^{\tau_\Omega} e^{-K\int_0^t V(X_s)ds} dt = 0,$$
$$\lim_{K\to\infty} u(x) = \lim_{K\to\infty} \lambda\mathbb{E}_x \int_0^{\tau_\Omega} u(X_t) e^{-K\int_0^t V(X_s)ds} dt = 0.$$

This shows that in the limit of large structural disorder, both $u(x)$ and $w(x)$ vanish when $x \notin D$.

We next consider the case when $x \in D$. Let τ_D be the first exit time of X from D. Then

$$\int_0^{\tau_\Omega} e^{-K\int_0^t V(X_s)ds} dt = \int_0^{\tau_D} e^{-K\int_0^t V(X_s)ds} dt + \int_{\tau_D}^{\tau_\Omega} e^{-K\int_0^t V(X_s)ds} dt.$$

Obviously, $V(X_s) = 0$ whenever $s \leq \tau_D$. This shows that

$$\int_0^{\tau_D} e^{-K\int_0^t V(X_s)ds} dt = \tau_D.$$

On the other hand, it is easy to see that

$$\int_0^t V(X_s)ds > 0, \quad \text{for any } t > \tau_D.$$

This shows that

$$\lim_{K\to\infty}\int_{\tau_D}^{\tau_\Omega} e^{-K\int_0^t V(X_s)ds}dt = 0.$$

It thus follows from the dominated convergence theorem that

$$\lim_{K\to\infty} w(x) = \mathbb{E}_x\tau_D.$$

Similarly, we can prove that

$$\lim_{K\to\infty} u(x) = \lambda\mathbb{E}_x\int_0^{\tau_D}\lim_{K\to\infty} u(X_t)dt.$$

Summarizing the above discussion, we obtain the following theorem.

Theorem 3.1. The landscape w has the following limit:

$$\lim_{K\to\infty} w(x) = \begin{cases} \mathbb{E}_x\tau_D & x\in D, \\ 0 & x\notin D. \end{cases}$$

Moreover, the eigenmode u has the following limit:

$$\lim_{K\to\infty} u(x) = \begin{cases} \lambda\mathbb{E}_x\int_0^{\tau_D}\lim\limits_{K\to\infty} u(X_t)dt & x\in D, \\ 0 & x\notin D. \end{cases}$$

The above theorem is important in two different ways. First, it provides an meaningful expression of the landscape in the limit of large structural disorder. Second, it reveals the essence of Anderson localization. When $K\gg 1$, both the eigenmode and the landscape almost vanish in the subregion D^c of high potential. Moreover, the subregion D of low potential can be decomposed as the disjoint union of several connected components:

$$D = D_1\cup D_2\cup\cdots\cup D_N. \tag{6}$$

In each connected subregion D_k, the eigenmode u approximately satisfies

$$u(x) = \lambda\mathbb{E}_x\int_0^{\tau_{D_k}} u(X_t)dt.$$

This is the probabilistic representation of the following local eigenvalue problem associated with the elliptic operator L in the subregion D_k:

$$\begin{cases} -Lu = \lambda u & \text{in } D_k, \\ \quad u = 0 & \text{on } \partial D_k. \end{cases} \tag{7}$$

Consequently, in the limit of large structural disorder, the spectrum of the operator $H = -L + KV$ is composed of the local eigenvalues of the operator $-L$ in each subregion D_k. If the eigenvalue λ of the operator H coincides with one of the local eigenvalues of the operator $-L$ in D_k, then the corresponding eigenmode u will be localized in D_k. Conversely, if the eigenvalue λ of the operator H coincides with neither one of the local eigenvalues of the operator $-L$ in D_k, then the corresponding eigenmode u will vanish in D_k.

According to the above discussion, if multiple subregions $D_{k_1},\cdots,D_{k_r}$ share a common local eigenvalue, then the corresponding eigenmode will have multiple peaks. In other words, an eigenmode has multiple peaks if and only if the eigenmode is a collection of local eigenmodes of subregions sharing a common eigenvalue.

4 Neumann and Robin boundary conditions

5 Size of the maximal subregion

In statistical physics, the graph of the potential V is called a site percolation. The graph of a typical two-dimension potential V is depicted in Fig. 1, where the brown region has a low potential and the green region has a high potential.

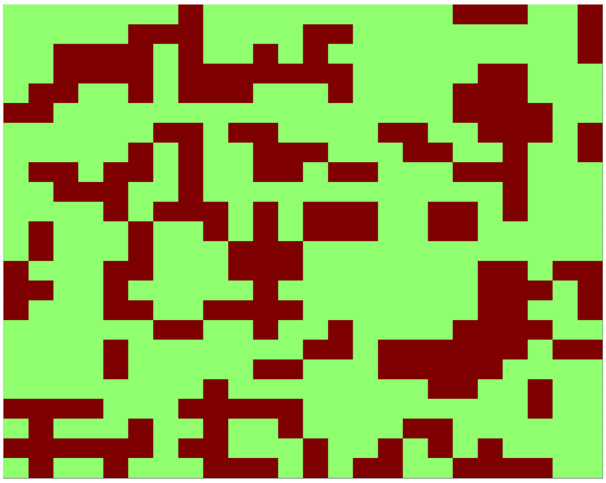

Figure 1. Graph of a two-dimensional potential.

Let M_n be the size of a maximal connected subregion with low potential (maximum connected component of the brown region). According to the percolation theory, the site percolation has a critical value p_c depending on the dimension d, which is given by

$$p_c = \begin{cases} 1 & \text{when } d = 1, \\ 0.59 & \text{when } d = 2, \\ 0.31 & \text{when } d = 3, \end{cases}$$

and it can be proved that

$$\lim_{d\to\infty} p_c = \frac{1}{2d}.$$

In the subcritical case of $p < p_c$, we have [2]

$$\lim_{n\to\infty} \frac{M_n}{\log(n)} = c_1(p,d), \quad \text{a.s.}$$

and in the supercritical case of $p > p_c$, we have [2]

$$\lim_{n\to\infty} \frac{M_n}{\log(n)^{\frac{d}{d-1}}} = c_2(p,d), \quad \text{a.s.}$$

where $c_1(p,d)$ and $c_2(p,d)$ are positive constants. This shows that when p is small, the typical size of a maximal connected subregion has the order of $\log(n)$; when p is large, the typical size of a maximal connected subregion has the order of $\log(n)^{\frac{d}{d-1}}$.

We next focus on the one-dimensional case. Since $p_c = 1$ when $d = 1$, the one-dimensional site percolation is always subcritical and thus

$$\lim_{n\to\infty} \frac{M_n}{\log(n)} = c_1(p,1), \quad \text{a.s.}$$

Moreover, it can be proved that

$$c_1(p, 1) = -\frac{1}{\log p}.$$

Thus, the typical size of a one-dimensional maximal connected subregion is roughly $-\log(n)/\log(p)$.

We next focus on the case when $L = \triangle$ is the Laplace operator. By Weyl's law, the Laplacian eigenvalues λ_m in each subregion D_k has the following asymptotic behavior:

$$\lambda_m \sim \frac{4\pi^2}{(\omega_d \mu_d(D_k))^{2/d}} m^{2/d}, \quad m \to \infty,$$

where $\mu_d(D_k)$ is the Lebesgue measure of D_k and

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}$$

is the volume of the d-dimensional unit ball. Weyl's law shows that larger connected subregions tend to have smaller eigenvalues. This explains why low-frequency eigenmodes are often localized in large connected subregions and why the principal eigenmode is often localized in the maximal connected subregion.

6 Probability of multiple peaks

In this section, we focus on the one-dimensional case. Let p_n be the probability of having only one maximal connected subregion. When $n \gg 1$, it can be proved that

$$p_n \approx \frac{n}{4}(1-p)\sum_{k=1}^{\infty} p^k (1-p^k)^{n/4-1}.$$

Therefore, the probability of having at least two maximal connected subregions is given by

$$1 - p_n \approx 1 - \frac{n}{4}(1-p)\sum_{k=1}^{\infty} p^k (1-p^k)^{n/4-1}.$$

In particular, when $p = 1/2$, we have $p_n \approx 0.72$.

References

[1] Filoche, M. & Mayboroda, S. Universal mechanism for Anderson and weak localization. *Proc. Natl. Acad. Sci. USA* **109**, 14761–14766 (2012).

[2] van Der Hofstad, R. & Redig, F. Maximal clusters in non-critical percolation and related models. *J. Stat. Phys.* **122**, 671–703 (2006).