Numerical Algorithms (MU4IN910)

Practical 2

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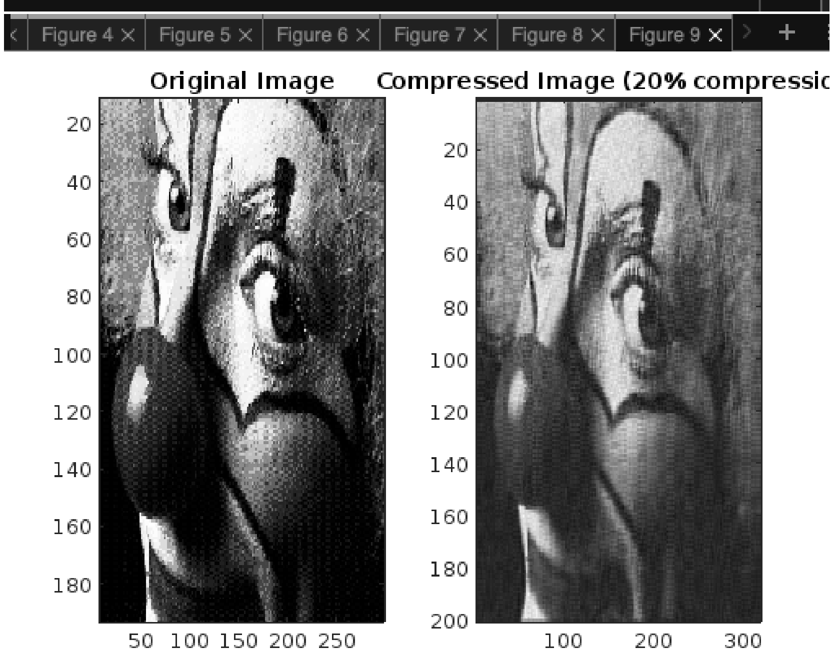
Exercise 6

1-

```
load clown.mat;

% Display the original image
figure;
subplot(1, 2, 1);
imagesc(X);
colormap gray;
title('Original Image');

[U, S, V] = svd(X);

% Set a compression factor
compression_factor = 0.2;
k = round(compression_factor * min(size(X)));

% Compress the image by keeping only the first k singular values
X_compressed = U(:, 1:k) * S(1:k, 1:k) * V(:, 1:k)';

% Display the compressed image
subplot(1, 2, 2);
imagesc(X_compressed);
colormap gray;
title(['Compressed Image (', num2str(compression_factor * 100), '% compression)']
```



2- Compression ratio CR to measure the quality of the compression:

$$CR = \frac{\text{Original Data size}}{\text{Compressed Data size}}$$

Exercise 7

1- The objective is to minimize the function $\|g - Kf\|^2$, where $K = U\Sigma V^T$. We aim to find the solution f^*.

The function to minimize is given by:

$$\|g - Kf\|^2 = \|g - U\Sigma V^T f\|^2$$

By using the properties of the norm operator and the singular value decomposition, we can rewrite this as:

$$\|g - Kf\|^2 = \|U^T g - \Sigma V^T f\|^2 = \|U^T g - \Sigma w\|^2$$

where $w = V^T f$.

Now, the problem becomes minimizing the norm of $U^T g - \Sigma w$, and the solution for w that minimizes this norm is obtained by choosing $w^* = \Sigma^{-1} U^T g$.

Revisiting the relation $w = V^T f$, we have:

$$V^T f^* = \Sigma^{-1} U^T g$$

Multiplying both sides by $V\Sigma^{-1}$, we get:

$$f^* = V\Sigma^{-1} U^T g$$

Given the expression $f^* = V\Sigma^{-1}U^T g$,we want now to show that it is equivalent to $f^* = \sum_{i=1}^{n} \frac{u_i^T g}{\sigma_i} v_i$, where σ_i are the singular values.

Starting with the expression $f^* = V\Sigma^{-1}U^T g$, we can rewrite it using the definition of matrix multiplication:

$$f^* = V\Sigma^{-1}U^T g = \left(\sum_{i=1}^{n} v_i \sigma_i^{-1} u_i^T \right) g$$

Here, we utilize the property of the inverse of a diagonal matrix Σ^{-1}, where $\Sigma_i^{-1} = \frac{1}{\sigma_i}$ Then, we get:

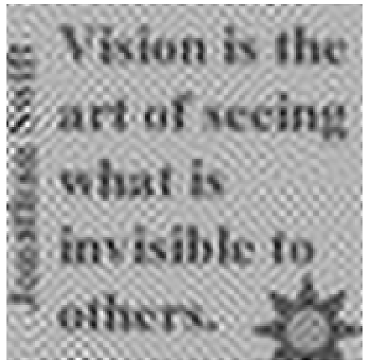
$$f^* = \sum_{i=1}^{n} v_i \sigma_i^{-1} u_i^T g = \sum_{i=1}^{n} \frac{u_i^T g}{\sigma_i} v_i$$

This confirms that the two expressions are equivalent.

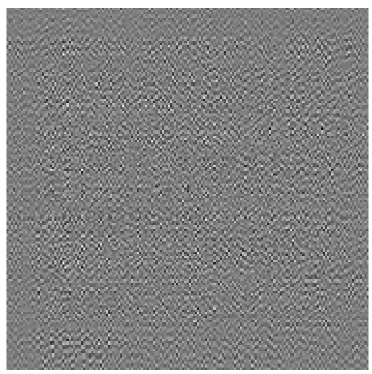
2-

```
url = 'https://www-pequan.lip6.fr/~graillat/teach/anum/defloutage.mat';
websave('defloutage.mat', url);

% Load the matrices
load('defloutage.mat');

% Compute the SVD for A and B
[U_A, S_A, V_A] = svd(A);
[U_B, S_B, V_B] = svd(B);

% Choose the number of singular values to use
p = 55;

G_prime = U_B' * G * U_A;

% Compute the inverse of the first p singular values for S_A and S_B
S_A_inv = diag(1 ./ diag(S_A(1:p, 1:p)));
S_B_inv = diag(1 ./ diag(S_B(1:p, 1:p)));

% Multiply G_prime by the inverses in the truncated singular value space
F_prime = S_B_inv * G_prime(1:p, 1:p) * S_A_inv;

% Multiply by V_B and V_A^T to get the deblurred image F
F = V_B(:, 1:p) * F_prime * V_A(:, 1:p)';

% Display the deblurred image
imshow(F, []);
```

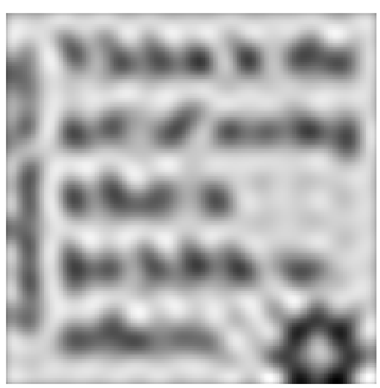
Result with p=55 :



Result with p=100 :



Result with p=20 :



After testing with many different p, we can conclude that the p which gives the best result is equal to 55

Exercise 8

1-

To show that A is a stochastic matrix, we must show that each column sums to 1 and each term is positive.

a) Column Sum Property: If $c_j \neq 0$

$$\sum_{i=1}^{n}(A)_{ij} = \sum_{i=1}^{n}\left(\frac{pg_{ij}}{c_j} + \delta\right) = \frac{p}{c_j}\sum_{i=1}^{n} g_{ij} + n\delta = p + n \times \frac{1-p}{n} = 1.$$

If $c_j = 0$

$$\sum_{i=1}^{n}\frac{1}{n} = 1.$$

Thus, the column sum property is verified.

b) The matrix A is defined as follows if $a \neq b$ 0:

$$(A)_{ij} = \frac{pg_{ij}}{c_j} + \delta$$

Let's analyze the terms of this expression:

- The term $\frac{pg_{ij}}{c_j}$ is non-negative because p, g_{ij}, and c_j are all positive.

- The term δ is also non-negative.

Therefore, each term of the matrix A is non-negative due to the sum of two non-negative terms. This ensures that all elements of the matrix A are non-negative. We can conclude that A is a stochastic matrix, so A^T is also a stochastic matrix. Therefore, we can conclude that the largest eigenvalue of A is 1.

2-

Let's start with the stochastic matrix A^T, where each element is nonnegative and the columns sum to 1. The Perron-Frobenius theorem states that there exists a positive eigenvalue ρ such that all other eigenvalues have absolute values less than or equal to ρ, for a nonnegative and irreducible matrix.

Since A^T is a stochastic matrix, $\rho = 1$ is an eigenvalue, and we can choose the corresponding eigenvector, denoted as x, to be nonnegative.

Now, consider the equation $Ax = \lambda x$, where $\lambda = 1$ (our positive eigenvalue) so the equation becomes $Ax = x$.

Thus, we have our nonnegative vector $x \in \mathbb{R}^n$ such that $Ax = x$. Additionally, since x is an eigenvector corresponding to the eigenvalue 1, it also satisfies $\sum_{i=1}^{n} x_i = 1$.

3-

```
imshow(P, []);


[U, G] = surfer('http://www.sorbonne-universite.fr/', 15);

% Define initial guess for page ranks
x0 = ones(size(G, 1), 1);

% Set tolerance and maximum number of iterations
tol = 1e-6;
max_iter = 1000;

% Compute page ranks using power method
[la, x] = power_iteration(G, x0, tol, max_iter);

% Display page ranks
disp('Page ranks:');
disp(x);
disp(la);

function [dominant_eigenvalue, dominant_eigenvector] = power_iteration(A, x0, tol, max_iter)
    % Initialize variables
    xi = x0;
    i = 0;

    % Iterate until convergence or maximum iterations reached
    while true
        yi = A * xi;            % Calculate y_i+1 = A * x_i
        xi = yi / norm(yi);     % Normalize xi+1 = y_i+1 / ||y_i+1||
        lambda_i = xi' * A * xi; % Approximate eigenvalue λ_i+1 = x_i+1^T * A * x_i+1
        % Check convergence
        if i >= max_iter || norm(A * xi - lambda_i * xi) < tol
            break;
        end

        % Update iteration count
        i = i + 1;
    end

    % Output dominant eigenvalue and eigenvector
    dominant_eigenvalue = lambda_i;
    dominant_eigenvector = xi;
end
```

```
>> TP2
open 1 http://www.sorbonne-universite.fr/
     skip http://purl.org/rss/1.0/modules/content/  dc: http://purl.org/dc/terms/  foaf: http://xmlns.com/foaf/0.1/  og: http://ogp.me/ns#  rdfs: http://www.w3.org/2000/01/rdf-schema#  schema: http:/
     skip http://purl.org/dc/terms/  foaf: http://xmlns.com/foaf/0.1/  og: http://ogp.me/ns#  rdfs: http://www.w3.org/2000/01/rdf-schema#  schema: http://schema.org/  sioc: http://rdfs.org/sioc/ns#
     skip http://xmlns.com/foaf/0.1/  og: http://ogp.me/ns#  rdfs: http://www.w3.org/2000/01/rdf-schema#  schema: http://schema.org/  sioc: http://rdfs.org/sioc/ns#  sioct: http://rdfs.org/sioc/types
     skip http://ogp.me/ns#  rdfs: http://www.w3.org/2000/01/rdf-schema#  schema: http://schema.org/  sioc: http://rdfs.org/sioc/ns#  sioct: http://rdfs.org/sioc/types#  skos: http://www.w3.org/2004/
     skip http://www.w3.org/2000/01/rdf-schema#  schema: http://schema.org/  sioc: http://rdfs.org/sioc/ns#  sioct: http://rdfs.org/sioc/types#  skos: http://www.w3.org/2004/02/skos/core#  xsd: http:
     skip http://schema.org/  sioc: http://rdfs.org/sioc/ns#  sioct: http://rdfs.org/sioc/types#  skos: http://www.w3.org/2004/02/skos/core#  xsd: http://www.w3.org/2001/XMLSchema#
     skip http://rdfs.org/sioc/ns#  sioct: http://rdfs.org/sioc/types#  skos: http://www.w3.org/2004/02/skos/core#  xsd: http://www.w3.org/2001/XMLSchema#
     skip http://rdfs.org/sioc/types#  skos: http://www.w3.org/2004/02/skos/core#  xsd: http://www.w3.org/2001/XMLSchema#
     skip http://www.w3.org/2004/02/skos/core#  xsd: http://www.w3.org/2001/XMLSchema#
     skip http://www.w3.org/2001/XMLSchema#
     link 2 http://analytics.sorbonne-universite.fr/matomo
open 2 http://analytics.sorbonne-universite.fr/matomo
     link 3 http:\/\/example.org\/offer.html'.
     link 4 http:\/\/example.com\/path' and 'http:\/\/good.example.com', tracking requests for 'http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
     link 5 http:\/\/good.example.com', tracking requests for 'http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
     link 6 http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
     link 7 http:\/\/bad.example.com' are ignored.
open 3 http:\/\/example.org\/offer.html'.
fail 3 http:\/\/example.org\/offer.html'.
open 4 http:\/\/example.com\/path' and 'http:\/\/good.example.com', tracking requests for 'http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
fail 4 http:\/\/example.com\/path' and 'http:\/\/good.example.com', tracking requests for 'http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
open 5 http:\/\/good.example.com', tracking requests for 'http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
fail 5 http:\/\/good.example.com', tracking requests for 'http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
open 6 http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
fail 6 http:\/\/example.com\/otherpath' or 'http:\/\/bad.example.com' are ignored.
open 7 http:\/\/bad.example.com' are ignored.
fail 7 http:\/\/bad.example.com' are ignored.
open 8
fail 8
open 9
fail 9
open 10
```

```
open 10
fail 10
open 11
fail 11
open 12
fail 12
open 13
fail 13
open 14
fail 14
open 15
fail 15
Page ranks:
         0
         0
    0.4472
    0.4472
    0.4472
    0.4472
    0.4472
         0
         0
         0
         0
         0
         0
         0
         0

     0
```