1. Tutorial

Exercise 1 (Some algebraic properties of the SVD). Let $A = U\Sigma V^T$ be the SVD of the $m \times n$ matrix A, where $m \geqslant n$.

1. Show that if A has full rank, then the solution of $\min_x \|Ax - b\|_2$ is $x = V\Sigma^{-1}U^T b$.

2. Show that $\|A\|_2 = \sigma_1$ and that if A is also square and nonsingular then $\|A^{-1}\|_2^{-1} = \sigma_n$ and $\|A\|_2 \cdot \|A^{-1}\|_2 = \sigma_1/\sigma_n$.

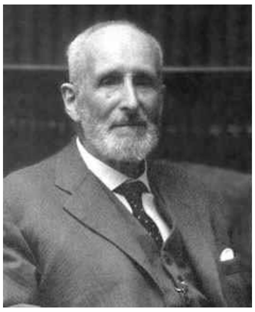
3. Write $V = [v_1, v_2, \ldots, v_n]^t$ and $U = [u_1, u_2, \ldots, u_n]^t$ so $A = U\Sigma V^T = \sum_{i=1}^n \sigma_i u_i v_i^T$. Show that a matrix of rank $k < n$ closest to A (measured with $\| \cdot \|_2$) is $A_k = \sum_{i=1}^k \sigma_i u_i v_i^T$ and $\|A - A_k\|_2 = \sigma_{k+1}$.

Exercise 2 (Perron-Frobenius theorem). A vector $x \in \mathbb{R}^n$ is *nonnegative*, and we write $x \geqslant 0$ if its coordinates are nonnegative. It is *positive*, and we write $x > 0$, if its coordinates are (strictly) positive. Furthermore, a matrix $A \in \mathbb{R}^{n\times m}$ (not necessarily square) is nonnegative (respectively, positive) if its entries are nonnegative (respectively, positive); we again write $A \geqslant 0$ (respectively, $A > 0$). More generally, we define an order relation $x \leqslant y$ whose meaning is $y - x \geqslant 0$. Given $x \in \mathbb{R}^n$, we let $|x|$ denote the nonnegative vector whose coordinates are the numbers $|x_j|$. Likewise, if $A \in \mathbb{R}^{n\times n}$, the matrix $|A|$ has entries $|a_{ij}|$.

1. Show that $|Ax| \leqslant |A|\,|x|$.

2. Show that a matrix A is nonnegative if and only if $x \geqslant 0$ implies $Ax \geqslant 0$.

3. Let A be a nonnegative matrix. Show that $\rho(A)$ is an eigenvalue of de A associated with a nonnegative eigenvector.
Hint: One can use the Brouwer theorem: a continuous function from a compact convex subset of $\mathbb{R}^N$ into itself has a fixed point.


Oskar Perron (1880-1975)


Georg Frobenius (1849-1917)

Exercise 3 (Stochastic matrices). A matrix $P \in \mathbb{R}^{n\times n}$ is said to be *stochastic* if $P \geqslant 0$ and one has:

$$\sum_{j=1}^{n} P_{ij} = 1 \qquad \text{all } i = 1, 2, \ldots, n.$$

1. Show that 1 is an eigenvalue of P. Give an eigenvector associated to this eigenvalue.

2. Show that $\rho(P) = 1$.

Exercise 4 (Power method)**.** The aim is this exercise is to study an algorithm for computing the the largest eigenvalue of a matrix in absolute value and the corresponding eigenvector.

```
i = 0
repeat
    y_{i+1} = A x_i
    x_{i+1} = y_{i+1} / ||y_{i+1}||
    λ̃_{i+1} = x_{i+1}^T A x_{i+1}
    i = i + 1
until convergence
```

$$
\begin{aligned}
& i = 0 \\
& \texttt{repeat} \\
& \quad y_{i+1} = Ax_i \\
& \quad x_{i+1} = y_{i+1}/\|y_{i+1}\| \\
& \quad \widetilde{\lambda}_{i+1} = x_{i+1}^T A x_{i+1} \\
& \quad i = i + 1 \\
& \texttt{until convergence}
\end{aligned}
$$

1. Show that if $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $|\lambda_1| > |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$ then x_i converges to $\pm e_1$ a,d $\widetilde{\lambda}_i$ converges to λ_1.

2. We now assume that $A = S\Lambda S^{-1}$ is diagonalizable with $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $|\lambda_1| > |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$. Show that x_i converges to $\pm s_1$ (where s_1 is the first column of S) and $\widetilde{\lambda}_i$ converges to λ_1.

Exercise 5 (Householder transformation)**.** In this exercise, we consider a third QR algorithm, based on Householder transformations.

1. Suppose we are given an $n \times 1$ vector z. Represent its first coordinate in polar notation as $z_1 = \zeta e^{i\theta}$, where ζ is real. Define $v = z - \alpha e_1$ where $\alpha = -e^{i\theta}\|z\|$ and let $u = v/\|v\|$. Define

$$Q = I - 2uu^*.$$

   Verify that Q is a unitary matrix and that $Qz = \alpha e_1$. The matrix Q is one version of a Householder transformation.

2. Given an $m \times n$ matrix A, we can determine a Householder transformation Q_1 so that $A_1 = Q_1 A$ has zeros in its first column below the main diagonal. Show how to determine Q_2 in the form

$$Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & I - 2u_2u_2^* \end{pmatrix},$$

   so that $A_2 = Q_2 A_1$ has zeros in its first and second column below the main diagonal.

3. Continuing this process, write an algorithm to reduce a matrix A to upper-triangular form by multiplying by a series of Householder transformations.

4. How many floating-point multiplications does your algorithm take?

2. Practical

Exercise 6 (Image compression using the SVD)**.** Figure 1 represents a 320×200 pixel image corresponding to a 320×200 matrix X.

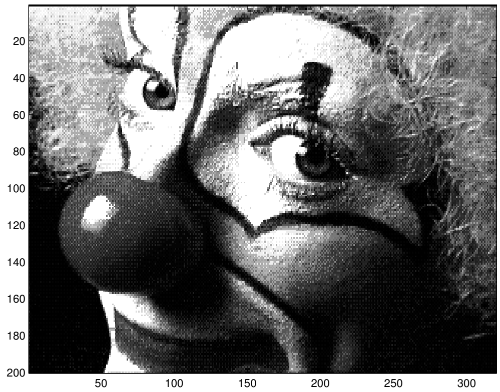


Figure 1: 320 × 200 pixel image

These images were produced by the following commands in MATLAB:

```
load clown.mat;
imagesc(X);
colormap gray;
```

1. Using question 3 of exercise 1, define an algorithm for image compression. Test your algorithm on figure 1.
2. Define a compression ratio to measure the quality of the compression.

Exercise 7 (Deblurring images using the SVD)**.** The aim of this exercise is to deblur image 2. It is an example of *linear inverse problem*. Given a blurred image and a linear model for the blurring, we want to reconstruct the original image. Figure 2 represents a blurred image.

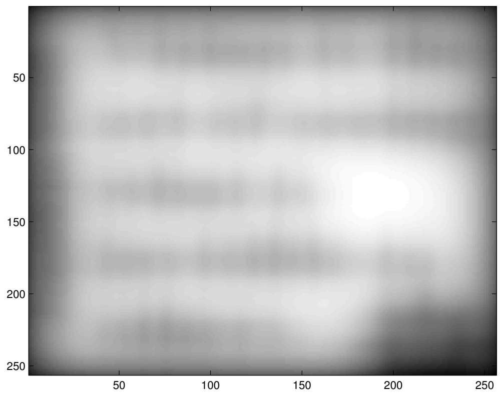


Figure 2: 250 × 250 pixel blurred image

Suppose we have a blurred, noisy image G, as in Figure 2, and some knowledge of the blurring operator, and we want to reconstruct the true original image F. We will use the following operator vec that transforms a matrix X into a column vector $x = \text{vec}(X)$ by stacking the columns of X. Let us denote $g = \text{vec}(G)$ and $f = \text{vec}(F)$. The blurring model is defined by $g = Kf + \eta$ where K is a matrix and η is a vector representing (unknown) noise or measurement errors. We know K and g. To find f, we want to solve the following problem:

$$\min_f \|g - Kf\|_2^2. \tag{1}$$

The Kronecker product $A \otimes B$ where A is an $m \times m$ matrix is defined to be

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}.$$

We assume that the matrix K can be factorized as $K = A \otimes B$.

1. Show that the solution of Equation (1) can be written as

$$f^* = V\Sigma^{-1}U^T g = \sum_{i=1}^{n} \frac{u_i^T g}{\sigma_i} v_i,$$

where $K = U\Sigma V^T$, u_i is the ith column of U and v_i the ith column of V. In fact, we can only look at a truncated expansion

$$f_p^* = \sum_{i=1}^{p} \frac{u_i^T g}{\sigma_i} v_i,$$

for some value of $p < n$.

Let $A = U_A\Sigma_A V_A^T$ and $B = U_B\Sigma_B V_B^T$ be the SVD of A and B. One can show that $A \otimes B = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)$ and that one also has

$$F = B^{-1}GA^{-T} = V_B\Sigma_B^{-1}U_B^T G U_A \Sigma_A^{-1} V_A^T.$$

If we denote $\widehat{G} = U_B^T G U_A$, then

$$\Sigma_B^{-1}\widehat{G}\Sigma_A^{-1} = \widehat{G}./S,$$

where $S = \mathrm{diag}(\Sigma_B)\,\mathrm{diag}(\Sigma_A)^T$.

2. Write a MATLAB program that takes matrices A and B and an image G and computes an image F. Experiment to find the value of the parameter p that gives the clearest image. Sample data (i.e., a blurred image G, and the matrices A and B) can be found in the file `defloutage.mat`[1]. One has to load the data using:

```
load defloutage.mat;
imshow(G);
```

For more information, see for example:

- Per Christian Hansen, James G. Nagy et Dianne P. O'Leary, *Deblurring Images: Matrices, Spectra, and Filtering*, SIAM, 2006

Exercise 8 (Google PageRank algorithm)**.** Let W be the set of Web pages that can be reached by following a chain of hyperlinks starting at some root page, and let n be the number of pages in W. Let G be the $n \times n$ connectivity matrix of a portion of the Web, that is, $g_{ij} = 1$ if there is a hyperlink to page i from page j and $g_{ij} = 0$ otherwise. The matrix G can be huge, but it is very sparse. Its jth column shows the links on the jth page. The number of nonzeros in G is the total number of hyperlinks in W. Let c_j be the column sums of G:

$$c_j = \sum_{i=1}^{n} g_{ij}.$$

The quantity c_j is the *out-degree* of the jth page. Let p be the probability that the random walk follows a link. A typical value is $p = 0.85$. Then $1 - p$ is the probability that some arbitrary page is chosen and $\delta = (1-p)/n$ is the probability that a particular random page is chosen.
Let $A \in \mathbb{R}^{n\times n}$ a matrix whose elements are

$$A_{ij} = \begin{cases} p g_{ij}/c_j + \delta & \text{if } c_j \neq 0, \\ 1/n & \text{if } c_j = 0. \end{cases}$$

[1]available at `http://www-pequan.lip6.fr/~graillat/teach/anum/defloutage.mat`

1. Show that A^T is a stochastic matrix. What can we say about the largest eigenvalue of A?

2. Using the Perron-Frobenius theorem, show there exists a nonnegative vector $x \in \mathbb{R}^n$ such that $Ax = x$ and $\sum_{i=1}^{n} x_i = 1$.

3. Compute x using the power method presented in the tutorial (exercise 4). The page-rank of the page i is x_i. Provide a ranking of the pages according to their page-rank.

You can use the file `surfer.m`[2] to generate the matrix G.

For more information, see for example:

- L. Page, S. Brin, R. Motwani et T. Winograd. ‘‘The PageRank citation ranking: Bringing order to the web’’, technical report, Stanford University, 1998.
- A.M. Langville et C.D. Meyer. *Google's PageRank and Beyond: The Science of Search Engine Rankings*, Princeton University Press, 2006.

[2]available at `http://www-pequan.lip6.fr/~graillat/teach/anum/surfer.m`