The Climate in Climate Economics: Model Summary

Detailed Formulation, Calibration and DEQN implementation

Simon Scheidegger
Department of Economics, University of Lausanne, Switzerland
May 24th, 2024 — 11:00 - 12:30 — University of Leipzig

Introduction to CDICE Model

- Merged calibration of DICE-2016 economic part and re-calibrated CDICE climate part.
- Referred to as CDICE, with detailed calibration in online Appendix B in the article.
- DICE-2016 features a single representative consumer and firm, with a time-separable utility function.

Key Components of the DICE-2016 Model

- Single, infinitely lived representative consumer and firm.
- Equilibrium allocation described as a solution to a planner's problem.
- Maximizes a time-separable utility function over per capita consumption $\left(\frac{C_t}{L_t}\right)_{t=0}^{\infty}$ with a constant intertemporal elasticity of substitution ($\psi > 0$) and time preference parameter ($0 < \beta < 1$).

Planner's Problem

$$V_0 = \max_{\{C_t,\mu_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t}{L_t}\right)^{1-1/\psi} \frac{1-1/\psi}{1-1/\psi} L_t \tag{1}$$

subject to:

$$K_{t+1} = (1-\Theta(\mu_t) - \Omega(T_{AT,t}))K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t \tag{2}$$

Constraints

$$0 \leq K_{t+1} \tag{3}$$

$$0 \leq \mu_t \leq 1 \tag{4}$$

$$E_t = \sigma_t Y_t^{\text{Gross}}(1 - \mu_t) + E_t^{\text{Land}} \tag{5}$$

CO2 Emissions and Output

- E_t: Total CO2 emissions, consisting of non-industrial emissions E_t^{Land} and industrial emissions $\sigma_t Y_t^{\text{Gross}}$.
- σ_t: Emission intensity.
- $\mu_t \geq 0$: Mitigation effort.
- Output produced using Cobb-Douglas technology with capital K_t and labor L_t.
- Mitigation reduces output at a rate $\Theta(\mu_t)$.
- Higher temperatures reduce output at a rate $\Omega(T_{AT,t})$.

Optimal Carbon Tax and Social Cost of Carbon

- The SCC is the marginal cost of atmospheric carbon in terms of the numeraire good.
- Following the literature, SCC is the planner's marginal rate of substitution between atmospheric carbon concentration and capital stock.

$$\mathrm{SCC}_t = -\frac{\partial V_t / \partial M_{AT,t}}{\partial V_t / \partial K_t} \tag{6}$$

$$CT_t = \frac{\theta_{1,t} \theta_2 \mu_t^{\theta_2 - 1}}{\sigma_t} \tag{7}$$

Optimal Mitigation and Investment Path

- Planner first solves the problem without recognizing that higher mitigation reduces damages from temperature increase.
- In BAU scenario, investment path is chosen, and mitigation is set to zero.
- In optimal scenario, planner chooses both mitigation and investment path.

Sensitivity Analysis and General Equilibrium

- Model uncertainty can significantly affect SCC values.
- Sensitivity to the discount rate and emissions scenario.
- General equilibrium analysis compares SCC and optimal mitigation for CDICE and DICE-2016.

Time Step and Parameters

- Explicit inclusion of time step as Δ_t.
- Annual baseline calibration for all parameters.
- Applies to main equations and exogenous parameters.

Labor Law of Motion

$$L_t = L_0 + (L_\infty - L_0)\left(1 - \exp(-\Delta_t \delta^L t)\right), \tag{8}$$

$$g_t^L = \frac{\frac{dL_t}{dt}}{L_t} = \frac{\Delta_t \delta^L}{\frac{L_\infty}{L_\infty - L_0}\exp(\Delta_t \delta^L t) - 1}. \tag{9}$$

Labor Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
Annual rate of convergence	δ^L	0.035	0.0268
World population at starting year [millions]	L_0	6514	7403
Asymptotic world population [millions]	L_∞	8600	11500
Time step of a model	Δ_t	10	5/1

Table 1: Generic parameterization for the evolution of labor.

Total Factor Productivity

$$A_t = A_0 \exp\left(\frac{\Delta_t g_0^A (1 - \exp(-\Delta_t \delta^A t))}{\Delta_t \delta^A}\right), \tag{10}$$

$$g_t^A = \frac{\frac{dA_t}{dt}}{A_t} = \Delta_t g_0^A \exp(-\Delta_t \delta^A t). \tag{11}$$

TFP Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
Initial growth rate for TFP per year	g_0^A	0.01314	0.0217
Decline rate of TFP growth per year	δ^A	0.001	0.005
Initial level of TFP	A_0	0.0058	0.010295
Time step of a model	Δ_t	10	5/1

Table 2: Generic parameterization for the evolution of TFP.

Carbon Intensity

$$\sigma_t = \sigma_0 \exp\left(\frac{\Delta_t g_0^\sigma (1 - \exp(-\Delta_t \delta^\sigma t))}{\Delta_t \delta^\sigma} \right), \tag{12}$$

$$\sigma_t = \sigma_0 \exp\left(\frac{\Delta_t g_0^\sigma}{\log(1 + \Delta_t \delta^\sigma)} \left((1 + \Delta_t \delta^\sigma)^t - 1\right) \right). \tag{13}$$

Carbon Intensity Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
Initial growth of carbon intensity per year	g_0^{σ}	-0.0073	-0.0152
Decline rate of decarbonization per year	δ^{σ}	0.003	0.001
Initial carbon intensity (1000 GtC)	σ_0	0.00013418	0.00009556
Time step	Δ_t	10	5/1

Table 3: Generic parameterization for the carbon intensity evolution.

Abatement Cost Function

$$\theta_{1,t} = \frac{p_0^{\text{back}}(1 + \exp(-g^{\text{back}} t))1000\sigma_t}{\theta_2}, \tag{14}$$

$$\theta_{1,t} = \frac{p_0^{\text{back}} \exp(-g^{\text{back}} t)1000 \cdot \text{c2co2} \cdot \sigma_t}{\theta_2}. \tag{15}$$

Abatement Cost Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
Cost of backstop 2005 thUSD per tC 2005	p_0^{back}	0.585	-
Cost of backstop 2010 thUSD per tCO2 2015	p_0^{back}	-	0.55
Initial cost decline backstop cost per year	g^{back}	0.005	0.005
Exponent of control cost function	θ_2	2.8	2.6
Transformation coefficient from C to CO2	c2co2	-	3.666

Table 4: Generic parameterization for the abatement cost.

Non-Industrial Emissions

$$E_{\mathrm{Land},t} = E_{\mathrm{Land},0} \exp(-\Delta_t \delta^{\mathrm{Land}} t). \tag{16}$$

Non-Industrial Emissions Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
Emissions from land 2005 (1000 GtC per year)	$E_{\mathrm{Land},0}$	0.0011	-
Emissions from land 2015 (1000 GtC per year)	$E_{\mathrm{Land},0}$	-	0.000709
Decline rate of land emissions (per year)	δ^{Land}	0.01	0.023
Time step	Δ_t	10	5/1

Table 5: Generic parameterization for the emissions from land.

Exogenous Radiative Forcings

$$F_t^{EX} = F_0^{EX} + \frac{1}{T/\Delta_t}(F_1^{EX} - F_0^{EX}) \min(t, T/\Delta_t). \tag{17}$$

Exogenous Forcings Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
2000 forcings of non-CO2 GHG (Wm^{-2})	F_0^{EX}	-0.06	-
2015 forcings of non-CO2 GHG (Wm^{-2})	F_0^{EX}	-	0.5
2100 forcings of non-CO2 GHG (Wm^{-2})	F_1^{EX}	0.3	1.0
Number of years before 2100	T	100	85
Time step	Δ_t	10	5/1

Table 6: Generic parameterization for the exogenous forcing.

Mass of Carbon Equations

$$M_{t+1}^{\text{AT}} = (1 - \Delta_t b_{12}) M_t^{\text{AT}} + \Delta_t b_{12} \frac{M_{\text{EQ}}^{\text{AT}}}{M_{\text{EQ}}^{\text{UO}}} M_t^{\text{UO}} + \Delta_t E_t, \tag{18}$$

$$M_{t+1}^{\text{UO}} = \Delta_t b_{12} M_t^{\text{AT}} + (1 - \Delta_t b_{12} \frac{M_{\text{EQ}}^{\text{AT}}}{M_{\text{EQ}}^{\text{UO}}} - \Delta_t b_{23}) M_t^{\text{UO}} + \Delta_t b_{23} \frac{M_{\text{EQ}}^{\text{UO}}}{M_{\text{EQ}}^{\text{LO}}} M_t^{\text{LO}}, \tag{19}$$

$$M_{t+1}^{\text{LO}} = \Delta_t b_{23} M_t^{\text{UO}} + (1 - \Delta_t b_{23} \frac{M_{\text{EQ}}^{\text{UO}}}{M_{\text{EQ}}^{\text{LO}}}) M_t^{\text{LO}}, \tag{20}$$

$$E_t = \sigma_t Y_t^{\text{Gross}} (1 - \mu_t) + E_t^{\text{Land}}. \tag{21}$$

Carbon Mass Parameters

Calibrated parameter	Symbol	2007	2016	CDICE
Carbon cycle, annual value	b_{12}	0.0189288	0.024	0.054
Carbon cycle, annual value	b_{23}	0.005	0.0014	0.0082
Time step	Δ_t	10	5	1
Equilibrium concentration in atmosphere (1000 GtC)	$M_{\mathrm{EQ}}^{\mathrm{AT}}$	0.587473	0.588	0.607
Equilibrium concentration in upper strata (1000 GtC)	$M_{\mathrm{EQ}}^{\mathrm{UO}}$	1.143894	0.360	0.489
Equilibrium concentration in lower strata (1000 GtC)	$M_{\mathrm{EQ}}^{\mathrm{LO}}$	18.340	1.720	1.281
Concentration in atmosphere 2015 (1000 GtC)	$M_{\mathrm{INI}}^{\mathrm{AT}}$	0.8089	0.851	0.851
Concentration in upper strata 2015 (1000 GtC)	$M_{\mathrm{INI}}^{\mathrm{UO}}$	1.255	0.460	0.628
Concentration in lower strata 2015 (1000 GtC)	$M_{\mathrm{INI}}^{\mathrm{LO}}$	18.365	1.740	1.323

Table 7: Generic parameterization for the mass of carbon.

Temperature Equations

$$
\begin{aligned}
T_{t+1}^{\mathrm{AT}} &= T_t^{\mathrm{AT}} + \Delta_t c_1 F_t - \Delta_t c_1 \frac{F_{2\mathrm{XCO2}}}{T_{2\mathrm{XCO2}}} T_t^{\mathrm{AT}} - \Delta_t c_1 c_3 (T_t^{\mathrm{AT}} - T_t^{\mathrm{OC}}), && (22)\\
T_{t+1}^{\mathrm{OC}} &= T_t^{\mathrm{OC}} + \Delta_t c_4 (T_t^{\mathrm{AT}} - T_t^{\mathrm{OC}}), && (23)\\
F_t &= F_{2\mathrm{XCO2}} \frac{\log(M_t^{\mathrm{AT}} / M_{\mathrm{base}}^{\mathrm{AT}})}{\log(2)} + F_t^{EX}. && (24)
\end{aligned}
$$

Temperature Parameters

Calibrated parameter	Symbol	2007	2016	CDICE
Temperature coefficient, annual value	c_1	0.022	0.0201	0.137
Temperature coefficient, annual value	c_3	0.3	0.088	0.73
Temperature coefficient, annual value	c_4	0.01	0.005	0.00689
Forcings of equilibrium CO2 doubling (Wm^{-2})	F_{2XCO2}	3.8	3.6813	3.45
Eq temperature impact (°C per doubling CO2)	T_{2XCO2}	3.0	3.1	3.25
Eq concentration in atmosphere (1000 GtC)	M^{AT}_{base}	0.5964	0.588	0.607
Atmospheric temp change (°C) from 1850	T^{AT}_0	0.7307	0.85	1.1
Lower stratum temp change (°C) from 1850	T^{OC}_0	0.0068	0.0068	0.27
Time step	Δ_t	10	5	1

Table 8: Generic parameterization for the temperature.

Economic Equations

$$K_{t+1} = (1 - \delta^K)^{\Delta_t} K_t + \Delta_t I_t, \tag{25}$$

$$Y_t^{\text{Gross}} = (A_t L_t)^{1-\alpha} K_t^{\alpha}. \tag{26}$$

Damages and Output (DICE-2007)

$$\Omega_t = \frac{1}{1 + \psi_1 T_t^{\mathrm{AT}} + \psi_2 (T_t^{\mathrm{AT}})^2}, \tag{27}$$

$$\Theta_t = \theta_{1,t} \mu_t^{\theta_2}, \tag{28}$$

$$Y_t^{\mathrm{Net}} = Y_t^{\mathrm{Gross}} \cdot \Omega_t = \frac{Y_t^{\mathrm{Gross}}}{1 + \psi_1 T_t^{\mathrm{AT}} + \psi_2 (T_t^{\mathrm{AT}})^2}, \tag{29}$$

$$Y_t = Y_t^{\mathrm{Gross}} \cdot \Omega_t \cdot (1 - \Lambda_t) = \frac{Y_t^{\mathrm{Gross}} (1 - \theta_{1,t} \mu_t^{\theta_2})}{1 + \psi_1 T_t^{\mathrm{AT}} + \psi_2 (T_t^{\mathrm{AT}})^2}. \tag{30}$$

Damages and Output (DICE-2016)

$$\Omega_t = \psi_1 T_t^{\mathrm{AT}} + \psi_2 (T_t^{\mathrm{AT}})^2, \tag{31}$$

$$\Theta_t = \theta_{1,t} \mu_t^{\theta_2}, \tag{32}$$

$$Y_t^{\mathrm{Net}} = Y_t^{\mathrm{Gross}} \cdot (1 - \Omega_t) = Y_t^{\mathrm{Gross}} (1 - \psi_1 T_t^{\mathrm{AT}} - \psi_2 (T_t^{\mathrm{AT}})^2), \tag{33}$$

$$Y_t = Y_t^{\mathrm{Gross}} \cdot (1 - \Lambda_t - \Omega_t) = Y_t^{\mathrm{Gross}} (1 - \theta_{1,t} \mu_t^{\theta_2} - \psi_1 T_t^{\mathrm{AT}} - \psi_2 (T_t^{\mathrm{AT}})^2). \tag{34}$$

Consumption and Utility

$$C_t = Y_t - I_t, \tag{35}$$

$$U_t = \sum_{t=0}^{T} \beta^t \cdot \Delta_t \cdot \frac{\left(\frac{C_t}{L_t}\right)^{1-1/\psi} - 1}{1 - 1/\psi} L_t, \tag{36}$$

$$\beta = \frac{1}{(1+\rho)^{\Delta_t}}. \tag{37}$$

Economic Parameters

Calibrated parameter	Symbol	Value 2007	Value 2016/CDICE
Capital annual depreciation rate	δ^K	0.1	0.1
Elasticity of capital	α	0.3	0.3
Damage parameter	ψ_1	0.0	0.0
Damage quadratic parameter	ψ_2	0.0028388	0.00236
Exponent of control cost function	θ_2	2.8	2.6
Risk aversion	ψ	0.5	0.69
Time preferences	ρ	0.015	0.015
Initial capital (trillions USD 2015)	K_0	-	223.0
Initial capital (trillions USD 2005)	K_0	137.0	-
Time step	Δ_t	10	5/1

Table 9: Generic parameterization for the parameters of economy.

Optimal Solutions

- Maximization of utility with respect to consumption and mitigation.
- Numerical solution using Deep Equilibrium Nets (DEQNs).
- Error statistics for validation.

Deep Equilibrium Nets (DEQNs)

- Simulation-based solution method using deep neural networks.
- Approximation of optimal policy function.
- Iterative procedure with gradient descent updates.

Loss Function for DEQNs

$$\ell_{\boldsymbol{\nu}} := \frac{1}{N_{\text{path length}}} \sum_{\mathbf{x}_t \text{on sim. path}} \sum_{m=1}^{N_{eq}} \left(G_m(\mathbf{x}_t, \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t)) \right)^2 , \tag{38}$$

where $G_m(\mathbf{x}_t, \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t))$ represent all the first-order equilibrium conditions of the model.

Introduction

- Mapping CDICE and non-stationary models onto DEQN framework.
- Detailed mathematical manipulations are required for replicability.

CDICE Model

$$\max_{\{C_t,\mu_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{C_t}{L_t}\right)^{1-1/\psi} - 1}{1-1/\psi} L_t \tag{39}$$

$$\text{s.t.} \quad K_{t+1} = \left(1 - \Omega\left(T_{\mathrm{AT},t}\right) - \Theta\left(\mu_t\right)\right) K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} + (1-\delta)K_t - C_t \quad (\lambda_t) \tag{40}$$

$$M_{t+1}^{\mathrm{AT}} = \left(1 - b_{12}\right) M_t^{:AT} + b_{12}\frac{M_{\mathrm{EQ}}^{\mathrm{AT}}}{M_{\mathrm{EQ}}^{\mathrm{UO}}} M_t^{\mathrm{UO}} + \sigma_t(1-\mu_t)K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} + E_t^{\mathrm{Land}} \quad \left(\nu_t^{\mathrm{AT}}\right) \tag{41}$$

CDICE Model (contd.)

$$M_{t+1}^{\mathrm{UO}} = b_{12} M_t^{\mathrm{AT}} + \left(1 - b_{12} \frac{M_{\mathrm{EQ}}^{\mathrm{AT}}}{M_{\mathrm{EQ}}^{\mathrm{UO}}} - b_{23}\right) M_t^{\mathrm{UO}} + b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}} M_t^{\mathrm{LO}} \quad \left(\nu_t^{\mathrm{UO}}\right) \tag{42}$$

$$M_{t+1}^{\mathrm{LO}} = b_{23} M_t^{\mathrm{UO}} + \left(1 - b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}}\right) M_t^{\mathrm{LO}} \quad \left(\nu_t^{\mathrm{LO}}\right) \tag{43}$$

Temperature Equations

$$T_{t+1}^{\text{AT}} = T_t^{\text{AT}} + c_1 \left(F_{\text{2XCO2}} \frac{\log(M_t^{\text{AT}} / M_{\text{base}}^{\text{AT}})}{\log(2)} + F_t^{EX} \right) - c_1 \frac{F_{\text{2XCO2}}}{T_{\text{2XCO2}}} T_t^{\text{AT}} - c_1 c_3 \left(T_t^{\text{AT}} - T_t^{\text{OC}} \right) \tag{44}$$

$$T_{t+1}^{\text{OC}} = T_t^{\text{OC}} + c_4 \left(T_t^{\text{AT}} - T_t^{\text{OC}} \right) \tag{45}$$

$$0 \leq \mu_t \leq 1 \tag{46}$$

State of the Economy

$$\mathbf{x}_t \in \mathbb{R}^6 := \left(k_t, M_t^{\text{AT}}, M_t^{\text{UO}}, M_t^{\text{LO}}, T_t^{\text{AT}}, T_t^{\text{OC}}, t\right)^T \tag{47}$$

Time Transformation

$$\tau = 1 - \exp(-\vartheta t) \tag{48}$$

$$t = -\frac{\ln(1 - \tau)}{\vartheta} \tag{49}$$

Normalization and Scaling

$$c_t \stackrel{\text{def}}{=} \frac{C_t}{A_t L_t}, k_t \stackrel{\text{def}}{=} \frac{K_t}{A_t L_t} \tag{50}$$

$$\begin{gathered}\hat{\lambda}_t \stackrel{\text{def}}{=} \frac{\lambda_t}{A_t^{1-\frac{1}{\psi}} L_t}, \hat{\lambda}_t^\mu \stackrel{\text{def}}{=} \frac{\lambda_t^\mu}{A_t^{1-\frac{1}{\psi}} L_t}, \hat{\nu}_t^{\text{AT}} \stackrel{\text{def}}{=} \frac{\nu_t^{\text{AT}}}{A_t^{1-\frac{1}{\psi}} L_t}, \hat{\nu}_t^{\text{UO}} \stackrel{\text{def}}{=} \frac{\nu_t^{\text{UO}}}{A_t^{1-\frac{1}{\psi}} L_t}, \hat{\nu}_t^{\text{LO}} \stackrel{\text{def}}{=} \frac{\nu_t^{\text{LO}}}{A_t^{1-\frac{1}{\psi}} L_t} \\ \hat{\eta}_t^{\text{AT}} \stackrel{\text{def}}{=} \frac{\eta_t^{\text{AT}}}{A_t^{1-\frac{1}{\psi}} L_t}, \hat{\eta}_t^{\text{OC}} \stackrel{\text{def}}{=} \frac{\eta_t^{\text{OC}}}{A_t^{1-\frac{1}{\psi}} L_t}\end{gathered} \tag{51}$$

Effective Discount Factor

$$\hat{\beta}_t \stackrel{\text{def}}{=} \exp\left(-\rho + \left(1 - \frac{1}{\psi}\right) g_t^A + g_t^L\right) \tag{52}$$

Policy Function

$$\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) \in \mathbb{R}^9 := \left(k_{t+1}, \mu_t, \hat{\lambda}_t, \hat{\lambda}_t^{\mu}, \hat{\nu}_t^{\mathsf{AT}}, \hat{\nu}_t^{\mathsf{UO}}, \hat{\nu}_t^{\mathsf{LO}}, \hat{\eta}_t^{\mathsf{AT}}, \hat{\eta}_t^{\mathsf{OC}}\right) \tag{53}$$

Lagrangian Formulation

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \hat{\beta}_t \Big[& \frac{c_t^{1-1/\psi} - A_t^{1/\psi-1}}{1-1/\psi} + \hat{\lambda}_t \left\{ \left(1 - \Omega\left(T_{\mathrm{AT},t}\right) - \Theta\left(\mu_t\right)\right) k_t^{\alpha} + (1-\delta)\, k_t - c_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} \right\} \\
& + \hat{\lambda}_t^{\mu} \left\{1 - \mu_t\right\} \\
& + \hat{\nu}_t^{\mathrm{AT}} \left\{ (1 - b_{12})\, M_t^{\mathrm{AT}} + b_{12} \frac{M_{\mathrm{EQ}}^{\mathrm{AT}}}{M_{\mathrm{EQ}}^{\mathrm{UO}}} M_t^{\mathrm{UO}} + \sigma_t (1-\mu_t) A_t L_t k_t^{\alpha} + E_t^{\mathrm{Land}} - M_{t+1}^{\mathrm{AT}} \right\} \\
& + \hat{\nu}_t^{\mathrm{UO}} \left\{ b_{12} M_t^{\mathrm{AT}} + \left(1 - b_{12} \frac{M_{\mathrm{EQ}}^{\mathrm{AT}}}{M_{\mathrm{EQ}}^{\mathrm{UO}}} - b_{23}\right) M_t^{\mathrm{UO}} + b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}} M_t^{\mathrm{LO}} - M_{t+1}^{\mathrm{UO}} \right\} \\
& + \hat{\nu}_t^{\mathrm{LO}} \left\{ b_{23} M_t^{\mathrm{UO}} + \left(1 - b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}}\right) M_t^{\mathrm{LO}} - M_{t+1}^{\mathrm{LO}} \right\} \\
& + \hat{\eta}_t^{\mathrm{AT}} \left\{ T_t^{\mathrm{AT}} + c_1 \left(F_{\mathrm{2XCO2}} \frac{\log(M_t^{\mathrm{AT}} / M_{\mathrm{base}}^{\mathrm{AT}})}{\log(2)} + F_t^{EX} \right) - c_1 \frac{F_{\mathrm{2XCO2}}}{T_{\mathrm{2XCO2}}} T_t^{\mathrm{AT}} - c_1 c_3 \left(T_t^{\mathrm{AT}} - T_t^{\mathrm{OC}}\right) - T_{t+1}^{\mathrm{AT}} \right\} \\
& + \hat{\eta}_t^{\mathrm{OC}} \left\{ T_t^{\mathrm{OC}} + c_4 \left(T_t^{\mathrm{AT}} - T_t^{\mathrm{OC}}\right) - T_{t+1}^{\mathrm{OC}} \right\} \Big]
\end{aligned}
\tag{54}
$$

First-Order Conditions

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow \exp\left(g_t^A + g_t^L\right)\hat{\lambda}_t - \hat{\beta}_t\Big[\hat{\lambda}_{t+1}\Big(\left(1-\Omega\left(T_{\mathrm{AT},t+1}\right) - \Theta\left(\mu_t\right)\right)\alpha k_{t+1}^{\alpha-1} + (1-\delta)\Big) + \hat{\nu}_{t+1}^{\mathrm{AT}}\sigma_{t+1}(1-\mu_{t+1})A_{t+1}L_{t+1}\alpha k_{t+1}^{\alpha-1}\Big] = 0 \tag{55}$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow c_t^{-1/\psi}A_t^{1-1/\psi}L_t - \hat{\lambda}_t = 0 \tag{56}$$

First-Order Conditions (contd.)

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = 0 \Leftrightarrow \hat{\lambda}_t \Theta' \left(\mu_t\right) k_t^{\alpha} + \lambda_t^{\mu} + \hat{\nu}_t^{\mathsf{AT}} \sigma_t A_t L_t k_t^{\alpha} = 0 \tag{57}$$

$$\frac{\partial \mathcal{L}}{\partial M_{\mathsf{AT},t+1}} = 0 \Leftrightarrow \hat{\nu}_t^{\mathsf{AT}} - \hat{\beta}_t \left[\hat{\nu}_{t+1}^{\mathsf{AT}} \left(1 - b_{12}\right) + \hat{\nu}_{t+1}^{\mathsf{UO}} b_{12} + \hat{\eta}_{t+1}^{\mathsf{AT}} c_1 F_{2\mathsf{XCO2}} \frac{1}{\ln 2 M_{\mathsf{AT},t+1}}\right] = 0 \tag{58}$$

$$\frac{\partial \mathcal{L}}{\partial M_{\mathsf{UO},t+1}} = 0 \Leftrightarrow \hat{\nu}_t^{\mathsf{UO}} - \hat{\beta}_t \left[\hat{\nu}_{t+1}^{\mathsf{AT}} b_{12} \frac{M_{\mathsf{EQ}}^{\mathsf{AT}}}{M_{\mathsf{EQ}}^{\mathsf{UO}}} + \hat{\nu}_{t+1}^{\mathsf{UO}} \left(1 - b_{12} \frac{M_{\mathsf{EQ}}^{\mathsf{AT}}}{M_{\mathsf{EQ}}^{\mathsf{UO}}} - b_{23}\right) + \hat{\nu}_{t+1}^{\mathsf{LO}} b_{23}\right] = 0 \tag{59}$$

First-Order Conditions (contd.)

$$\frac{\partial \mathcal{L}}{\partial M_{\mathrm{LO},t+1}} = 0 \Leftrightarrow \hat{\nu}_t^{\mathrm{LO}} - \hat{\beta}_t \left[\hat{\nu}_{t+1}^{\mathrm{UO}} b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}} + \hat{\nu}_{t+1}^{\mathrm{LO}} \left(1 - b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}} \right) \right] = 0 \tag{60}$$

$$\frac{\partial \mathcal{L}}{\partial T_{\mathrm{AT},t+1}} = 0 \Leftrightarrow \hat{\eta}_t^{\mathrm{AT}} - \hat{\beta}_t \left[-\lambda_{t+1} \Omega'(T_{\mathrm{AT},t+1}) k_{t+1}^{\alpha} + \hat{\eta}_{t+1}^{\mathrm{AT}} (1 - c_1 \frac{F_{\mathrm{2XCO2}}}{\mathrm{T2xco2}} - c_1 c_3) + \hat{\eta}_{t+1}^{\mathrm{OC}} c_4 \right] = 0 \tag{61}$$

First-Order Conditions (contd.)

$$\frac{\partial \mathcal{L}}{\partial T_{\mathsf{OC},t+1}} = 0 \Leftrightarrow \hat{\eta}_t^{\mathsf{OC}} - \hat{\beta}_t \left[\hat{\eta}_{t+1}^{\mathsf{AT}} c_1 c_3 + \hat{\eta}_{t+1}^{\mathsf{OC}} (1 - c_4) \right] = 0 \tag{62}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{\lambda}_t} = 0 \Leftrightarrow \left(1 - \Omega\left(T_{\mathsf{AT},t}\right) - \Theta\left(\mu_t\right)\right) k_t^{\alpha} + (1 - \delta)\, k_t - c_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0 \tag{63}$$

First-Order Conditions (contd.)

$$\frac{\partial \mathcal{L}}{\partial \hat{\nu}_t^{\mathsf{AT}}} = 0 \Leftrightarrow (1 - b_{12})\, M_t^{\mathsf{AT}} + b_{12} \frac{M_{\mathsf{EQ}}^{\mathsf{AT}}}{M_{\mathsf{EQ}}^{\mathsf{UO}}} M_t^{\mathsf{UO}} + \sigma_t (1 - \mu_t) A_t L_t k_t^{\alpha} + E_t^{\mathsf{Land}} - M_{t+1}^{\mathsf{AT}} = 0 \tag{64}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{\nu}_t^{\mathsf{UO}}} = 0 \Leftrightarrow b_{12} M_t^{\mathsf{AT}} + \left(1 - b_{12} \frac{M_{\mathsf{EQ}}^{\mathsf{AT}}}{M_{\mathsf{EQ}}^{\mathsf{UO}}} - b_{23}\right) M_t^{\mathsf{UO}} + b_{23} \frac{M_{\mathsf{EQ}}^{\mathsf{UO}}}{M_{\mathsf{EQ}}^{\mathsf{LO}}} M_t^{\mathsf{LO}} - M_{t+1}^{\mathsf{UO}} = 0 \tag{65}$$

First-Order Conditions (contd.)

$$\frac{\partial \mathcal{L}}{\partial \hat{\nu}_t^{\text{LO}}} = 0 \Leftrightarrow b_{23} M_t^{\text{UO}} + \left(1 - b_{23} \frac{M_{\text{EQ}}^{\text{UO}}}{M_{\text{EQ}}^{\text{LO}}}\right) M_t^{\text{LO}} - M_{t+1}^{\text{LO}} = 0 \tag{66}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{\eta}_t^{\text{AT}}} = 0 \Leftrightarrow T_t^{\text{AT}} + c_1 \left(F_{\text{2XCO2}} \frac{\log(M_t^{\text{AT}} / M_{\text{base}}^{\text{AT}})}{\log(2)} + F_t^{EX}\right) - c_1 \frac{F_{\text{2XCO2}}}{T_{\text{2XCO2}}} T_t^{\text{AT}} - c_1 c_3 \left(T_t^{\text{AT}} - T_t^{\text{OC}}\right) - T_{t+1}^{\text{AT}} = 0 \tag{67}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{\eta}_t^{\text{OC}}} = 0 \Leftrightarrow T_t^{\text{OC}} + c_4 \left(T_t^{\text{AT}} - T_t^{\text{OC}}\right) - T_{t+1}^{\text{OC}} = 0 \tag{68}$$

KKT Condition

$$1 - \mu_t \geq 0 \quad \perp \quad \lambda_t^{\mu} \geq 0 \tag{69}$$

Fischer-Burmeister Function

$$\Psi^{\mathsf{FB}}\left(\hat{\lambda}_t^{\mu}, 1-\mu_t\right) = \hat{\lambda}_t^{\mu} + (1-\mu_t) - \sqrt{\left(\hat{\lambda}_t^{\mu}\right)^2 + (1-\mu_t)^2} \tag{70}$$

$$\hat{\lambda}_t^{\mu} \stackrel{\mathsf{def}}{=} -\hat{\lambda}_t \Theta'\left(\mu_t\right) k_t^{\alpha} - \hat{\nu}_t^{\mathsf{AT}} \sigma_t A_t L_t k_t^{\alpha} \tag{71}$$

Loss Function Components

$$l_1 := \exp\left(g_t^A + g_t^L\right) \hat{\lambda}_t - \hat{\beta}\Big[\hat{\lambda}_{t+1}\left(\left(1 - \Omega\left(T_{\mathrm{AT},t+1}\right) - \Theta\left(\mu_t\right)\right) \alpha k_{t+1}^{\alpha-1} + (1-\delta)\right) + \hat{\nu}_{t+1}^{\mathrm{AT}}(1-\mu_{t+1})\sigma_{t+1} A_{t+1} L_{t+1} \alpha k_{t+1}^{\alpha-1}\Big] \tag{72}$$

$$l_2 := \left(1 - \Omega\left(T_{\mathrm{AT},t}\right) - \Theta\left(\mu_t\right)\right) k_t^{\alpha} + (1-\delta) k_t - c_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0 \tag{73}$$

Loss Function Components (contd.)

$$l_3 := \hat{\nu}_t^{\mathrm{AT}} - \beta \left[\hat{\nu}_{t+1}^{\mathrm{AT}} (1 - b_{12}) + \hat{\nu}_{t+1}^{\mathrm{UO}} b_{12} + \hat{\eta}_{t+1}^{\mathrm{AT}} c_1 F_{2\mathrm{XCO2}} \frac{1}{\ln 2 M_{\mathrm{AT},t+1}} \right] = 0 \quad (74)$$

$$l_4 := \hat{\nu}_t^{\mathrm{UO}} - \beta \left[\hat{\nu}_{t+1}^{\mathrm{AT}} b_{12} \frac{M_{\mathrm{EQ}}^{\mathrm{AT}}}{M_{\mathrm{EQ}}^{\mathrm{UO}}} + \hat{\nu}_{t+1}^{\mathrm{UO}} \left(1 - b_{12} \frac{M_{\mathrm{EQ}}^{\mathrm{AT}}}{M_{\mathrm{EQ}}^{\mathrm{UO}}} - b_{23} \right) + \hat{\nu}_{t+1}^{\mathrm{LO}} b_{23} \right] = 0 \quad (75)$$

$$l_5 := \hat{\nu}_t^{\mathrm{LO}} - \beta \left[\hat{\nu}_{t+1}^{\mathrm{UO}} b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}} + \hat{\nu}_{t+1}^{\mathrm{LO}} \left(1 - b_{23} \frac{M_{\mathrm{EQ}}^{\mathrm{UO}}}{M_{\mathrm{EQ}}^{\mathrm{LO}}} \right) \right] = 0 \quad (76)$$

Loss Function Components (contd.)

$$l_6 := \hat{\eta}_t^{\mathrm{AT}} - \beta \left[-\hat{\lambda}_{t+1} \Omega'(T_{\mathrm{AT},t+1}) k_{t+1}^{\alpha} + \hat{\eta}_{t+1}^{\mathrm{AT}} (1 - c_1 \frac{F_{\mathrm{2XCO2}}}{\mathrm{T2xco2}} - c_1 c_3) + \hat{\eta}_{t+1}^{\mathrm{OC}} c_4 \right] = 0 \tag{77}$$

$$l_7 := \hat{\eta}_t^{\mathrm{OC}} - \beta \left[\hat{\eta}_{t+1}^{\mathrm{AT}} c_1 c_3 + \hat{\eta}_{t+1}^{\mathrm{OC}} (1 - c_4) \right] = 0 \tag{78}$$

$$l_8 := \hat{\lambda}_t^{\mu} + (1 - \mu_t) - \sqrt{\left(\hat{\lambda}_t^{\mu} \right)^2 + (1 - \mu_t)^2} = 0 \tag{79}$$

Total Loss Function

$$\ell_{\boldsymbol{\nu}} := \frac{1}{N_{\text{path length}}} \sum_{\mathbf{x}_t \text{on sim. path}} \sum_{m=1}^{N_{eq}=8} \left(I_m(\mathbf{x}_t, \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t))\right)^2 \tag{80}$$

State Evolution

$$\mathbf{x}_{t+1} = \left(k_{t+1}, M^{\mathsf{AT}}_{t+1}, M^{\mathsf{UO}}_{t+1}, M^{\mathsf{LO}}_{t+1}, T^{\mathsf{AT}}_{t+1}, T^{\mathsf{OC}}_{t+1}, t+1\right)^{T} \tag{81}$$