An Example with an Analytical Solution

Evaluation of Solution Methods in Economic Models

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Introduction

- **An Analytical Test Case**
  - **Objective:** Evaluate the performance of solution methods in economic models.
  - **Importance:** Allows precision evaluation beyond Euler equation errors.
  - **Model:** Adapted from Krueger and Kubler (2004) and Huffman (1987).

The Model - Households

- **Households:**
  - **Lifespan:** N periods
  - **Utility:** Log utility
  - **Labor Supply:** Only in the first period ($l_t^s = 1$ for $s = 1$, $l_t^s = 0$ otherwise)
  - **Aggregate Labor Supply:** $L_t = 1$
  - **Savings:** Risky capital

The Model - Household Problem

$$\max_{\{c^i_{t+i}, a^i_{t+i}\}_{i=0}^{N-1}} \mathbb{E}_t \left[\sum_{i=0}^{N-1} \log(c^i_{t+i}) \right] \tag{1}$$

$$c^h_t + a^h_t = r_t k^h_t + l^h_t w_t \tag{2}$$

$$k^{h+1}_{t+1} = a^h_t \tag{3}$$

$$a^N_t \geq 0 \tag{4}$$

where c^h_t is consumption, a^h_t is saving, k^h_t is capital, and r_t is the price of capital.

The Model - Firms

- **Firms:**
  - **Production Function:** Cobb-Douglas
  - **Function:**

$$f(K_t, L_t, z_t) = \eta_t K_t^{\alpha} L_t^{1-\alpha} + K_t(1-\delta_t) \tag{5}$$

where K_t is aggregate capital, L_t is aggregate labor, α is capital share, η_t is TFP, and δ_t is depreciation.

Equilibrium

- **Competitive Equilibrium:**
  - Given initial conditions $z_0, \{k_0^s\}_{s=1}^N$
  - Collection of choices for households $\{(c_t^s, a_t^s)_{s=1}^N\}_{t=0}^\infty$
  - For the representative firm $(K_t, L_t)_{t=0}^\infty$
  - Prices $(r_t, w_t)_{t=0}^\infty$

Functional Rational Expectations Equilibrium

Definition
A FREE consists of equilibrium functions $\theta^a : Z \times \mathbb{R}^N \to \mathbb{R}^{N-1}$ where $\theta^a : Z \times \mathbb{R}^N \to \mathbb{R}^{N-1}$ denotes the capital investment functions, such that for all states $x := [z, k^T]^T \in Z \times \mathbb{R}^N$, where $z \in Z$ denotes the exogenous shock and $k = [k_1, \ldots, k_N]^T$ denotes the endogenous state (i.e., the distribution of capital) with $k_1 = 0$:

$$u'(c_i(x)) = \beta \mathbb{E}_z \left[r(x^+) u'(c_{i+1}(x^+)) \right] \tag{6}$$

Computing the Equilibrium

- **Exact Solution:**

$$\theta^a(x) = \beta \left[\frac{1-\beta^{N-1}}{1-\beta^N} w(x), \frac{1-\beta^{N-2}}{1-\beta^{N-1}} r(x) k_2, \ldots, \frac{1-\beta^1}{1-\beta^2} r(x) k_{N-1} \right]^T \tag{7}$$

- Aggregate capital evolves as:

$$K_t = \gamma_1 w(x_{t-1}) + \gamma_2 w(x_{t-2}) r(x_{t-1}) + \ldots + \gamma_{N-1} w(x_{t-(N-1)}) r(x_{t-1}) \ldots r(x_{t-(N-2)}) \tag{8}$$

where $\gamma_i := \frac{\beta^i - \beta^N}{1-\beta^N}$

Approximating the Solution with DEQNs

- Parameters ρ of a deep neural network N_ρ
- Approximates the true equilibrium policies:

$$\theta^a(x) \approx N_\rho(x) \tag{9}$$

- Minimize errors in the equilibrium conditions using mini-batch gradient descent

Errors in Equilibrium Conditions

$$e^i_{EE}(x^j) := -u'(c_i(x^j)) + \beta \mathbb{E}_z \left[r(x^+) u'(c_{i+1}(x^+)) \right], \quad i = 1, \ldots, N-1 \tag{10}$$

$$x^+ = \left[z^+, 0, (\theta^a(x))^T \right]^T \tag{11}$$

$$r(x) = f_K \left(\sum_{i=1}^{N} x_{1+i}, 1, x_1 \right) \tag{12}$$

$$w(x) = f_L \left(\sum_{i=1}^{N} x_{1+i}, 1, x_1 \right) \tag{13}$$

Relative Errors in Euler Equations

$$e^i_{REE}(x^j) := u'^{-1}\left(\beta\mathbb{E}_{z^j}\left[r(x^j_+)u'(c_{i+1}(x^j_+))\right]\right)c_i(x^j)^{-1} - 1 \tag{14}$$

$$\mathcal{L}_{\text{train}}(\rho) := \frac{1}{|\mathcal{D}_{\text{train}}|}\frac{1}{N-1}\sum_{x^j \in \mathcal{D}_{\text{train}}}\sum_{i=1}^{N-1}(e^i_{REE}(x^j))^2 \tag{15}$$

where $\mathcal{D}_{\text{train}}$ is a set of states collected by simulating the economy using the current approximation of the policy functions.

Parameterization and Hyperparameters

- **Parameters:**
  - Number of Age Groups $N = 6$
  - Discount Factor $\beta = 0.7$
  - Relative Risk Aversion $\gamma = 1$
  - Capital Share $\alpha = 0.3$
  - TFP $\eta = \{0.95, 1.05\}$
  - Depreciation $\delta = \{0.5, 0.9\}$
  - Persistence TFP $P(\eta_{t+1} = 1.05 | \eta_t = 1.05) = 0.5$
  - Persistence Depreciation $P(\delta_{t+1} = 0.5 | \delta_t = 0.5) = 0.5$

Assessing the Quality of the Solution

- Train the neural network for 10,000 episodes
- Evaluate relative errors in Euler equations

	Mean	Max	0.1	10	50	90	99.9
Rel EE capital [log10]	-3.4	-2.4	-6.4	-4.4	-3.6	-3.0	-2.5