A Tutorial on Physics-Informed Neural Networks (PINNs)

Exercises

Simon Scheidegger

Department of Economics, University of Lausanne

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Exercise Overview

- **ODE:**

$$u''(x) + u(x) = 0, \quad x \in \left[0, \frac{\pi}{2}\right]$$

- **Boundary Conditions:**

$$u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 1$$

- **Analytical Solution:**

$$u(x) = \sin(x)$$

Boundary Enforcement Methods

- **Soft Boundary Conditions:** Enforce the boundary conditions by adding penalty terms to the loss function.
- **Hard Boundary Conditions:** Enforce the boundary conditions by constructing a trial solution that automatically satisfies them.

Boundary Enforcement Methods

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  Example Ansatz: Define

  $$u(x) = \frac{2x}{\pi} + x\left(\frac{\pi}{2} - x\right)N(x),$$

  where $N(x)$ is the neural network output. This guarantees $u(0) = 0$ and $u(\pi/2) = 1$.

Exercise Instructions

Task

- Implement a PINN to approximate the solution of the ODE.
- **Part 1 (Soft Boundary):** Enforce the boundary conditions by adding their residuals as penalty terms in the loss function.
- **Part 2 (Hard Boundary):** Modify the network's output by embedding a trial solution that satisfies the boundary conditions by design.

Files Provided

- **Skeleton Code:** A Python file with holes (for both soft and hard boundary methods).
- **Solution Code:** A complete Python implementation demonstrating both methods.

Problem Formulation

- **Differential Equation:**

$$u''(x) + u(x) = 0$$

- **Domain:**

$$x \in \left[0, \frac{\pi}{2}\right]$$

- **Boundary Conditions:**

$$u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 1$$

- **Analytical Solution:**

$$u(x) = \sin(x)$$

Evaluation and Comparison

- After training, evaluate the PINN on a fine grid.
- Compare the PINN prediction $u_{\text{PINN}}(x)$ with the analytical solution $u(x) = \sin(x)$.
- Provide a visual plot for both approaches (soft and hard boundary methods).

Summary

- **Objective:** Solve $u''(x) + u(x) = 0$ with non-trivial boundary conditions using PINNs.
- **Boundary Enforcement:** Implement both soft and hard boundary condition methods.
- **Files:** Use the provided Python files (one with holes and one complete solution).
- **Outcome:** Compare numerical results to the analytical solution $\sin(x)$.