Test Cases for PINN

March 2025

Abstract

JEL codes:

Keywords:

1 Example #1: consumption-savings in partial equilibrium

The economy is populated by a continuum of households. Time is continuous and indexed by $t \geq 0$. We abstract from aggregate uncertainty.

Preferences. A generic household's instantaneous utility function over consumption is $u(c_t)$, where c_t denotes her consumption at date t and follows a stochastic process. The household's lifetime utility given a stochastic process c_t is defined as

$$v_0(\{c_t\}_{t\geq 0}) = \mathbb{E}\int_0^\infty e^{-\rho t}u(c_t)dt,$$

where ρ is the discount rate.

Budget and borrowing constraints. Households can accumulate and trade capital. We denote the stock of the household's capital at date t by a_t. The household's return on capital is r_t, which we take as given in partial equilibrium. Household wealth evolves as

$$\dot{a}_t = r_t a_t + w_t z_t - c_t,$$

where we assume that the household supplies one unit of labor inelastically, earning the wage rate w_t for effective units of work z_t. We will capture idiosyncratic earnings risk through z_t, which follows a stochastic process as described below.

Households cannot borrow beyond a borrowing limit given by

$$a_t \geq \underline{a}$$

which must remain satisfied at all dates t.

Earnings risk. In this first example, we assume that z_t follows a continuous-time AR(1) process. That is, we model it as an OU diffusion process

$$dz_t = \theta(\bar{z} - z_t)dt + \sigma dB_t.$$

We assume that z_t remains bounded on an interval $[\underline{z}, \overline{z}]$ with reflecting boundaries.

HJB. We can give a recursive representation to this problem as follows. We assume that the interest rate and wage are constant, $r_t = r$ and $w_t = w$, implying a stationary problem. The problem of a household with wealth a and labor productivity z admits the recursive representation

$$\rho V(a,z) = \max_c \Big\{ u(c) + \Big[ra + wz - c\Big]\partial_a V(a,z)\Big\} + \theta(\bar{z} - z)\partial_z V(a,z) + \frac{\sigma^2}{2}\partial_{zz}V(a,z)$$

together with the state constraint boundary condition

$$\partial_a V(\underline{a}, z) \geq u'(r\underline{a} + wz)$$

and reflecting boundaries in the z dimension

$$\partial_z V(a, \underline{z}) = 0 \qquad \text{and} \qquad \partial_z V(a, \overline{z}) = 0$$

The resulting consumption policy function is therefore characterized by the FOC

$$u'(c(a,z)) = \partial_a V(a,z).$$

Notice that plugging in for the consumption policy function $c(a,z) = (u')^{-1}\partial_a V(a,z)$ gives rise to a highly non-linear HJB equation

$$\begin{aligned}
\rho V(a,z) &= u(c(a,z)) + \Big[ra + wz - c(a,z)\Big]\partial_a V(a,z) + \theta(\bar{z} - z)\partial_z V(a,z) + \frac{\sigma^2}{2}\partial_{zz} V(a,z) \\
c(a,z)) &= (u')^{-1}\partial_a V(a,z).
\end{aligned}$$

In the finite-difference methods approach, solving this non-linear equation is extremely difficult. That's why the usual approach is instead to solve for a time forward-marching scheme. See Ben's documentation for further details. But roughly speaking, we solve for a sequence of linear equations in $\{V^n(a,z)\}$ such that

$$\begin{aligned}
\frac{V^{n+1}(a,z) - V^n(a,z)}{\Delta} + \rho V^{n+1}(a,z) &= u(c^n(a,z)) + \Big[ra + wz - c^n(a,z)\Big]\partial_a V^{n+1}(a,z) \\
&\quad + \theta(\bar{z} - z)\partial_z V^{n+1}(a,z) + \frac{\sigma^2}{2}\partial_{zz} V^{n+1}(a,z) \\
c^n(a,z)) &= (u')^{-1}\partial_a V^n(a,z)
\end{aligned}$$

where V^{n+1} is now linear in V^n. We add the leftmost term now as if we were solving for a non-stationary equation — see e.g. LeVeque for a discussion.

2 Example #2: discrete earnings risk

Earnings risk. We now model the stochastic process z_t as a two-state Markov chain with transition rates λ.

Formally, the household is endowed with an income transition process $\{N_t\}_{t\geq 0}$, which follows a Poisson process with rate λ. The value $N_t \in \mathbb{N}$ represents the number of earnings state transitions the household has drawn by time t, with $N_0 = 0$. We denote the household's current income state as $z_t \in \{z^1, z^2\}$. Upon drawing an income transition at date t, the household switches her income state. And we take her initial state z_0 as given. In that case, the counting process jumps by 1, so $dN_t \equiv N_t - N_{t^-} = 1$, where N_{t^-} is defined as the left limit.

HJB. We can give a recursive representation to this problem as follows. We assume that the interest rate and wage are constant, $r_t = r$ and $w_t = w$, implying a stationary problem. The problem of a household with wealth a and labor productivity z admits the recursive representation

$$\rho V(a, z^j) = \max_c \left\{ u(c) + \left[ra + wz^j - c \right] \partial_a V(a, z^j) \right\} + \lambda \left[V(a, z^{-j}) - V(a, z^j) \right] \Big\}$$

together with the state constraint boundary condition

$$\partial_a V(\underline{a}, z^j) \geq u'(r\underline{a} + wz^j).$$

The resulting consumption policy function is therefore characterized by the FOC

$$u'(c(a, z^j)) = \partial_a V(a, z^j).$$

References