Boundedness of derivatives and anti-derivatives of holomorphic functions as a rare phenomenon

Maria Siskaki

Abstract

In this article we prove a general result which in particular suggests that, on a simply connected domain Ω in $\mathbb{C}$, all the derivatives and anti-derivatives of the generic holomorphic function are unbounded. A similar result holds for the operator $\widetilde{T}_N$ of partial sums of the Taylor expansion with center $\zeta \in \Omega$ at $z = 0$, seen as functions of the center ζ. We also discuss a universality result of these operators $\widetilde{T}_N$.

2010 Mathematics Subject Classification: primary 30K99, secondary 30K05.

Keywords: Baire's Theorem, generic property, Differentiation and Integration operators, Taylor expansion, partial sums, universal Taylor series

1 Introduction

Let Ω be a domain in the complex plane and consider the space $\mathcal{H}ol(\Omega)$ of all the functions that are holomorphic on Ω with the topology of uniform convergence on compacta. In the first section of this article we show that, for a function $f \in \mathcal{H}ol(\Omega)$, the phenomenon of its k-th derivative or k-th anti-derivative being bounded on Ω is a rare phenomenon in the topological sense, provided that Ω is simply connected. We do this by using Baire's Theorem and we prove that the set $\mathcal{D}$ of all the functions $f \in \mathcal{H}ol(\Omega)$ with the property that, the derivatives and the anti-derivatives of f of all orders are unbounded on Ω, is a dense G_δ set in $\mathcal{H}ol(\Omega)$.

If a function f is holomorphic in an open set containing ζ, then $S_N(f, \zeta)(z)$ denotes the N-th partial sum of the Taylor expansion of f with center ζ evaluated at z. If Ω is a simply connected domain and $\zeta \in \Omega$, we define the class $U(\Omega, \zeta)$ as follows:

Definition 1.1. *Let $U(\Omega, \zeta)$ denote the set of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, with K^{c} connected, and for every function h which is continuous on K and holomorphic in the interior of K, there exists a sequence $\{\lambda_n\} \in \{0, 1, 2, ...\}$ such that*

$$\sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad n \to \infty.$$

Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. It is shown in [6] that $U(\mathbb{D}, 0)$ is a dense G_δ set in $\mathcal{H}ol(\mathbb{D})$. More generally, in [5] it is shown that $U(\Omega, \zeta)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$, where Ω is any simply connected domain and $\zeta \in \Omega$. Next, for Ω as above, we define the set $U(\Omega)$:

Definition 1.2. *Let $U(\Omega)$ denote the set of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, with K^{c} connected, and every function h which is continuous on K and holomorphic in the interior of K, there exists a sequence $\{\lambda_n\} \in \{0, 1, 2, ...\}$ such that, for every compact set $L \subset \Omega$,*

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad n \to \infty.$$

In [5] it is shown that $U(\Omega)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$. Furthermore, in [3] it is shown that $U(\Omega, \zeta) = U(\Omega)$, provided that Ω is contained in a half-plane. This result is generalized in [4], where it is shown that $U(\Omega, \zeta) = U(\Omega)$ for any simply connected domain Ω and $\zeta \in \Omega$.

In the second section of this article, we fix a $\zeta_0 \in \Omega$ and, for $N \geq 1$, we consider the function

$$\begin{aligned} S_N(f, \zeta_0) &: \mathbb{C} \to \mathbb{C} \\ &z \mapsto \sum_{n=0}^{N} \frac{f^{(n)}(\zeta_0)}{n!}(z - \zeta_0)^n = S_N(f, \zeta_0)(z) \end{aligned}$$

V. Nestoridis suggested that, contrary to the functions in $U(\Omega, \zeta)$, whose Taylor partial sums are considered as functions of z with the center ζ fixed, we fix $z = 0$ and let the center ζ vary in Ω. Thus, for $N \geq 0$, we obtain an operator

$$\begin{aligned} \widetilde{T}_N &: \mathcal{H}ol(\Omega) \to \mathcal{H}ol(\Omega) \\ &f \mapsto \widetilde{T}_N(f) \end{aligned}$$

where

$$\begin{aligned} \widetilde{T}_N(f) &: \Omega \to \mathbb{C} \\ &\zeta \mapsto \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(-\zeta)^n = \widetilde{T}_N(f)(\zeta) \end{aligned}$$

for any $f \in \mathcal{H}ol(\Omega)$ and $N \geq 0$. The set of functions $f \in \mathcal{H}ol(\Omega)$ such that $\widetilde{T}_N(f)$ is unbounded on Ω for all $N \geq 0$ is residual in $\mathcal{H}ol(\Omega)$. This led V.Nestoridis to conjecture that, if $0 \notin \Omega$, then the class $\mathcal{S}(\Omega)$ of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, the set $\{\widetilde{T}_N(f) : N = 0, 1, 2, ...\}$ is dense in $\mathcal{H}ol(\Omega)$, is a dense G_δ set in $\mathcal{H}ol(\Omega)$. In this article we show that either $\mathcal{S}(\Omega) = \emptyset$ or $\mathcal{S}(\Omega)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$. The question of whether $\mathcal{S}(\Omega) \neq \emptyset$ will be examined in a future article. However, we do show that, if $0 \notin \Omega$, then the set $\mathcal{S}_t(\Omega)$ of the functions $f \in \mathcal{H}ol(\Omega)$ with he property that, the closure of the set $\{\widetilde{T}_N(f)\}$ contains the constant functions on Ω, is residual in $\mathcal{H}ol(\Omega)$. We do this by proving that $\mathcal{S}_t(\Omega)$ contains the set $U(\Omega)$, which is already proven to be a dense G_δ set in $\mathcal{H}ol(\Omega)$ ([5]).

In the last part of the article, answering a question by T. Chatziafratis, we prove that, for a countable set $E \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the generic holomorphic function on $\mathbb{D}$ has unbounded derivatives and anti-derivatives on each ray $[0, z)$, $z \in E$. We also obtain a more general result, where in fact we do not use Baire's Theorem and, therefore, the topological vector space used need not be a Fréchet space.

2 Preliminaries

Regarding the terminology used, a *set of first category* in $\mathcal{H}ol(\Omega)$ is a set that can be expressed as a countable union of nowhere dense sets in $\mathcal{H}ol(\Omega)$. Because the space $\mathcal{H}ol(\Omega)$ is metrizable complete, Baire's theorem implies that a subset of $\mathcal{H}ol(\Omega)$ is G_δ dense iff it is the countable intersection of open and dense subsets of $\mathcal{H}ol(\Omega)$. A subset of $\mathcal{H}ol(\Omega)$ is called residual if it contains a G_δ dense set. Equivalently, if its complement is contained in an F_σ set of first category.

Let Ω_1, Ω_2 be two domains in $\mathbb{C}$ and $T : \mathcal{H}ol(\Omega_1) \to \mathcal{H}ol(\Omega_2)$ be a linear operator with the property that for every $z \in \Omega_2$, the function $f \mapsto T(f)(z)$ is continuous in $\mathcal{H}ol(\Omega_1)$. Observe that this latter property is weaker than T being continuous. Define

$$\mathcal{U}_T = \big\{ f \in \mathcal{H}ol(\Omega_1) : T(f) \text{ is unbounded on } \Omega_2 \big\}.$$

Proposition 2.1. *If Ω_1, Ω_2 are two domains in $\mathbb{C}$ and T is as above, then either $\mathcal{U}_T = \emptyset$ or $\mathcal{U}_T$ is a dense G_δ set in $\mathcal{H}ol(\Omega_1)$.*

Proof. If $\mathcal{U}_T \neq \emptyset$, for $m \geq 1$ define

$$U_m = \big\{ f \in \mathcal{H}ol(\Omega_1) : |T(f)(z)| \leq m \quad \text{for all } z \in \Omega_2 \big\}.$$

Then

$$\mathcal{U}_T = \Big(\bigcup_{m=1}^{\infty} U_m \Big)^{\mathsf{c}} = \bigcap_{m=1}^{\infty} U_m^{\mathsf{c}}.$$

We will show that U_m is closed and nowhere dense in $\mathcal{H}ol(\Omega_1)$ for each $m \geq 1$.

To see that it is closed, take a sequence $\{f_n\}$ in U_m such that $f_n \longrightarrow f$ uniformly on compact subsets of Ω_1 for some function f. Then $f \in \mathcal{H}ol(\Omega_1)$ and, for $z \in \Omega_2$ we have

$$\begin{aligned} |T(f)(z)| &\leq |T(f)(z) - T(f_n)(z)| + |T(f_n)(z)| \\ &\leq |T(f - f_n)(z)| + m. \end{aligned}$$

Taking $n \to \infty$ we get that $|T(f)(z)| \leq m$ because of the continuity of $f \mapsto T(f)(z)$, i.e. $f \in U_m$. Thus, U_m is closed.

To see that U_m is nowhere dense, it suffices to show that $U_m^\circ = \emptyset$. Suppose $f \in U_m^\circ$. Since $\mathcal{U}_T \neq \emptyset$, there exists a function $g \in \mathcal{H}ol(\Omega_1)$ such that $T(g)$ is unbounded on Ω_2. Then $\{f + \frac{1}{n}g\}_n$ is a sequence in $\mathcal{H}ol(\Omega_1)$ and, if K is a compact subset of Ω_1, we have

$$\begin{aligned}\|(f + \tfrac{1}{n}g) - f\|_K &= \sup_{z\in K} |f(z) + \frac{1}{n}g(z) - f(z)| \\ &= \sup_{z\in K} |\frac{1}{n}g(z)| = \frac{1}{n}\|g\|_K.\end{aligned}$$

By taking $n \to \infty$ and observing that $\|g\|_K < \infty$, g being holomorphic on $\Omega_1 \supset K$, we obtain that $f + \frac{1}{n}g \longrightarrow f$ uniformly on K. Since K was an arbitrary compact subset of Ω_1, $f + \frac{1}{n}\,g \longrightarrow f$ uniformly on compact subsets of Ω_1.
Since $f \in U_m^\circ$, there exists an n_0 such that $f + \frac{1}{n_0}\,g \in U_m$. By the linearity of $f \mapsto T(f)$ this means that

$$\begin{aligned}\tfrac{1}{n_0}\,|T(g)(z)| &\leq |T(f)(z) + \tfrac{1}{n_0}\,T(g)(z)| + |T(f)(z)| \\ &\leq m + m\end{aligned}$$

or $|T(g)(z)| \leq 2mn_0$, for all $z \in \Omega_2$, which is contradictory to the fact that $T(g)$ is unbounded on Ω_2. Thus, $U_m^\circ = \emptyset$ and the proof is complete. □

Proposition 2.2. *For $n \in \mathbb{Z}$, let $T_n : \mathcal{H}ol(\Omega_1) \to \mathcal{H}ol(\Omega_2)$ be linear and such that for every $z \in \Omega_2$, the function $f \mapsto T_n(f)(z)$ is continuous in $\mathcal{H}ol(\Omega_1)$. If $\mathcal{U}_{T_n} \neq \emptyset$ for all $n \in \mathbb{Z}$ then the set $\bigcap\limits_n \mathcal{U}_{T_n}$ is dense G_δ in $\mathcal{H}ol(\Omega_1)$.*

Proof. The space $\mathcal{H}ol(\Omega_1)$ with the metric of uniform convergence on compacta is a complete metric space, so by Baire's Theorem any countable intersection of dense G_δ sets in $\mathcal{H}ol(\Omega_1)$ is again a dense G_δ set in $\mathcal{H}ol(\Omega_1)$. Since $\mathcal{U}_{T_n} \neq \emptyset$, it is a dense G_δ set in $\mathcal{H}ol(\Omega)$ by Proposition 2.1, $n \in \mathbb{Z}$, and the desired result follows immediately. □

Observe that Propositions 2.1 and 2.2 still hold if we replace $\mathcal{H}ol(\Omega_2)$ by $\mathbb{C}^X$, where X is any non-empty set and $\mathbb{C}^X$ is the set of all functions from X to $\mathbb{C}$.

3 Boundedness of derivatives and anti-derivatives as a rare phenomenon

For $f \in \mathcal{H}ol(\Omega)$, we denote by $f^{(k)}$ the k-th derivative of f, $k \geq 1$. By $f^{(0)}$ we denote f itself.

Proposition 3.1. *Let $\Omega \subset \mathbb{C}$ be open and non-empty and $k \in \mathbb{N} \cup \{0\}$. The set $\mathcal{A}_k$ of all functions $f \in \mathcal{H}ol(\Omega)$ such that $f^{(k)}$ is bounded on Ω is a set of first category in $\mathcal{H}ol(\Omega)$.*

Proof. For $m \in \mathbb{N}$, define

$$A_m = \Big\{ f \in \mathcal{H}ol(\Omega) : |f^{(k)}(z)| \leq m, \text{ for all } z \in \Omega \Big\}.$$

It is obvious that

$$\mathcal{A}_k = \bigcup_{m=1}^{+\infty} A_m.$$

We will show that each A_m is closed and has empty interior in $\mathcal{H}ol(\Omega)$.

To see that it is closed, take a sequence $\{f_n\}$ in A_m and a function f on Ω such that $f_n \longrightarrow f$ uniformly on compact subsets of Ω. By the Weierstrass theorem we have that $f \in \mathcal{H}ol(\Omega)$ and $f_n^{(k)} \longrightarrow f^{(k)}$ uniformly on compact subsets of Ω. Therefore, for any $z \in \Omega$ we have that

$$|f^{(k)}(z)| = \lim_{n\to\infty} |f_n^{(k)}(z)| \leq m$$

i.e. $f \in A_m$. Thus, A_m is closed.

To see that $A_m^{\circ} = \emptyset$, first observe that there exists a function $g \in \mathcal{H}ol(\Omega)$ such that $g^{(k)}$ is unbounded on Ω. Indeed, if Ω is unbounded take $g(z) = z^{k+1}$ and if Ω is bounded take $\zeta_0 \in \partial\Omega$ and $g(z) = \frac{1}{z-\zeta_0}$.
Now assume that there exists $f \in A_m^{\circ}$. Then $\{f + \frac{1}{n}g\}_n$ is a sequence in $\mathcal{H}ol(\Omega)$ and $f + \frac{1}{n}g \longrightarrow f$ uniformly on compact subsets of Ω, $n \to \infty$. But $f \in A_m^{\circ}$, hence there exists an $n_0 \in \mathbb{N}$ such that $f + \frac{1}{n_0}g \in A_m^{\circ}$. This means that

$$|f^{(k)}(z) + \frac{1}{n_0}g^{(k)}(z)| \leq m, \text{ for all } z \in \Omega$$

where the linearity of the derivative operator is used. But then, for any $z \in \Omega$ we would have

$$\begin{aligned}
|\frac{1}{n_0}g^{(k)}(z)| &= |f^{(k)}(z) + \frac{1}{n_0}g^{(k)}(z) - f^{(k)}(z)| \\
&\leq |f^{(k)}(z) + \frac{1}{n_0}g^{(k)}(z)| + |f^{(k)}(z)| \\
&\leq m + m,
\end{aligned}$$

Thus $|g^{(k)}(z)| \leq 2mn_0$ for all $z \in \Omega$, which is contradictory to the fact that $g^{(k)}$ is unbounded on Ω. Thus, $A_m^{\circ} = \emptyset$ and the proof is complete. □

At this point observe that the preceding result can be viewed as a corollary of Proposition 2.1: Consider the operators $\Lambda_k : \mathcal{H}ol(\Omega) \mapsto \mathcal{H}ol(\Omega)$, defined by $\Lambda_k(f) = f^{(k)}$, $k \in \mathbb{N} \cup \{0\}$. These operators are linear and continuous (by Weierstrass Theorem). It just suffices to observe that $\mathcal{U}_{\Lambda_n} \neq \emptyset$ for each n, which follows from the argument provided in the proof of the above proposition.

Also observe that for $k = 0$ we get that the set of holomorphic functions that are unbounded on Ω is a dense G_δ set in $\mathcal{H}ol(\Omega)$. In [2] it is shown that, for each domain $\Omega \neq \mathbb{C}$ and $\zeta \in \Omega$, all functions in $U(\Omega, \zeta)$ are unbounded (if they exist). Thus, for each domain Ω for which $U(\Omega, \zeta)$ is a dense G_δ subset of $\mathcal{H}ol(\Omega)$ (for some $\zeta \in \Omega$), we can immediately deduce that the set of unbounded holomorphic functions on Ω is a residual set. For example, this is the case when Ω is simply connected or when it is the complement of a non-degenerate continuum.

Proposition 3.2. *Let $\Omega \subset \mathbb{C}$ be open and non-empty. The set $\mathcal{E}$ of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that $f^{(k)}$ is unbounded on Ω, for all $k \in \mathbb{N} \cup \{0\}$, is a dense G_δ set in $\mathcal{H}ol(\Omega)$.*

Proof. In view of the remark preceding this proposition, the result follows immediately by Proposition 2.2 □

From now on, and throughout the remainder of this section, consider an $\Omega \subset \mathbb{C}$ which is non-empty, open and simply connected. Fix $\zeta_0 \in \Omega$ and, for $f \in \mathcal{H}ol(\Omega)$ define

$$\begin{aligned} \Lambda_{-1}(f)(z) &= \int_{\gamma_z} f(\xi) d\xi, && \text{for all } z \in \Omega \\ \Lambda_k(f)(z) &= \int_{\gamma_z} \Lambda^{(k+1)}(f)(\xi) d\xi, && \text{for all } z \in \Omega, \;\; k \leq -2 \end{aligned}$$

where γ_z is any polygonal line in Ω that starts at ζ_0 and ends at z. Since Ω is assumed to be simply connected, for each $k \leq -1$, Λ_k is well-defined and holomorphic in Ω and its $|k|$-th derivative is f.

Proposition 3.3. *The operator*

$$\begin{aligned} \Lambda_{-1} : \mathcal{H}ol(\Omega) &\longrightarrow \mathcal{H}ol(\Omega) \\ f &\mapsto \Lambda_{-1}(f) \end{aligned}$$

is linear and continuous on $\mathcal{H}ol(\Omega)$.

Proof. The linearity of Λ_{-1} is obvious from the linearity of the integral. For the continuity, take a sequence $\{f_n\}$ in $\mathcal{H}ol(\Omega)$ and a function f on Ω such that $f_n \longrightarrow f$ uniformly on compact subsets of Ω. By the Weierstrass theorem we have that $f \in \mathcal{H}ol(\Omega)$. We must show that $\Lambda_{-1}(f_n) \longrightarrow \Lambda_{-1}(f)$ on compact subsets of Ω.

Let K be a compact subset of Ω. Either $\Omega = \mathbb{C}$ or $\Omega \neq \mathbb{C}$.

In the first case, i.e. $\Omega = \mathbb{C}$, for $z \in K$ we take γ_z to be the line segment $[\zeta_0, z]$. Set $M = \max\{|\zeta_0|, \max_{z \in K} |z|\}$ and observe that M is well defined and finite because K is compact in $\mathbb{C}$. Define $L = \overline{D(0, M)} = \{z \in \mathbb{C} : |z| \leq M\}$. Then L is compact in $\mathbb{C}$, $K \subset L$ and $\gamma_z \subset L$, for all $z \in K$. Therefore, for $z \in K$ we have

$$\begin{aligned} |\Lambda_{-1}(f_n)(z) - \Lambda_{-1}(f)(z)| &= \Big| \int_{\gamma_z} f_n(\xi) d\xi - \int_{\gamma_z} f(\xi) d\xi \Big| \\ &= \Big| \int_{\gamma_z} (f_n(\xi) - f(\xi)) d\xi \Big| \\ &\leq \|f_n - f\|_L \;\, |z - \zeta_0| \\ &\leq 2M \|f_n - f\|_L. \end{aligned}$$

Thus $\|\Lambda_{-1}(f_n) - \Lambda_{-1}(f)\|_K \leq 2M \|f_n - f\|_L \longrightarrow 0$, $n \to \infty$.

In the second case, i.e. $\Omega \neq \mathbb{C}$, since Ω is a simply connected domain, by the Riemann Mapping Theorem there exists an analytic function $\phi : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \longrightarrow \mathbb{C}$ such that ϕ is univalent and $\phi(\mathbb{D}) = \Omega$. Obviously ϕ is a homeomorphism between $\mathbb{D}$ and Ω. Since the set $\{\zeta_0\} \cup K \subset \Omega$ is compact, the set $\phi^{-1}(\{\zeta_0\} \cup K) \subset \mathbb{D}$ is also compact. Therefore, there exists an r, with $0 < r < 1$, such that $\phi^{-1}(\{\zeta_0\} \cup K) \subset \overline{D(0,r)} = \{z \in \mathbb{C} : |z| \leq r\}$. Define $L = \phi(\overline{D(0,r)}) \subset \phi(\mathbb{D}) = \Omega$. Then L is compact and $K \subset L$. For $z \in K$ we have that $\phi^{-1}(\zeta_o), \phi^{-1}(z) \in \overline{D(0,r)}$, hence the line segment $[\phi^{-1}(\zeta_o), \phi^{-1}(z)] \subset \overline{D(0,r)}$. Therefore, if $\sigma : [0,1] \longrightarrow \mathbb{C}$ is a parametrization of $[\phi^{-1}(\zeta_o), \phi^{-1}(z)]$, then $Length(\sigma) \leq 2r$. Take $\gamma_z = \phi([\phi^{-1}(\zeta_o), \phi^{-1}(z)]) \subset \phi(\overline{D(0,r)}) = L$ and observe that γ_z is rectifiable: $\phi \circ \sigma : [0,1] \longrightarrow \Omega$ is a parametrization of γ_z and

$$
\begin{aligned}
Length(\gamma_z) &= \int_0^1 |\gamma_z^{'}(t)| dt \\
&= \int_0^1 |(\phi \circ \sigma)^{'}(t)| dt \\
&= \int_0^1 |(\phi^{'}(\sigma(t))| \, |\sigma^{'}(t)| dt \\
&\leq \max \left\{ |\phi^{'}(z)| : z \in \overline{D(0,r)} \right\} \ Length(\sigma) \\
&\leq \max \left\{ |\phi^{'}(z)| : z \in \overline{D(0,r)} \right\} \ 2r
\end{aligned}
$$

which is of course finite because $\phi^{'}$ is continuous on the compact set $\overline{D(0,r)}$.

We then have

$$
\begin{aligned}
|\Lambda_{-1}(f_n)(z) - \Lambda_{-1}(f)(z)| &= \Big| \int_{\gamma_z} f_n(\xi) d\xi - \int_{\gamma_z} f(\xi) d\xi \ \Big| \\
&= \Big| \int_{\gamma_z} (f_n(\xi) - f(\xi)) d\xi \ \Big| \\
&\leq \|f_n - f\|_L \ Length(\gamma_z) \\
&\leq \|f_n - f\|_L \ \max \left\{ |\phi^{'}(z)| : z \in \overline{D(0,r)} \right\} \ 2r.
\end{aligned}
$$

Thus $\|\Lambda_{-1}(f_n) - \Lambda_{-1}(f)\|_K \leq \|f_n - f\|_L \ \max \left\{ |\phi^{'}(z)| : z \in \overline{D(0,1)} \right\} \ 2r \longrightarrow 0$, $n \rightarrow \infty$.

In any case we have shown that $\Lambda_{-1}(f_n) \longrightarrow \Lambda_{-1}(f)$ uniformly on K. Since K was an arbitrary compact subset of Ω, the continuity of Λ_{-1} follows. □

Corollary 3.4. *Let $k \leq -2$. The operator*

$$
\begin{aligned}
\Lambda_k : \mathcal{H}ol(\Omega) &\longrightarrow \mathcal{H}ol(\Omega) \\
f &\mapsto \Lambda_k(f)
\end{aligned}
$$

is linear and continuous on $\mathcal{H}ol(\Omega)$.

Proof. We have that $\Lambda_k = \Lambda_{-1} \circ \Lambda_{-1} \circ ... \circ \Lambda_{-1}$, the composition of Λ_{-1} k times. Therefore linearity and continuity both follow by Proposition 3.3. □

Corollary 3.5. *If $f_n \longrightarrow f$ uniformly on compact subsets of Ω and $k \leq -1$, then $\Lambda_k(f_n) \longrightarrow \Lambda_k(f)$ pointwise in Ω.*

Proof. By the Weierstrass Theorem, $f \in \mathcal{H}ol(\Omega)$. By Corollary 3.4 we have that $\Lambda_k(f_n) \longrightarrow \Lambda_k(f)$ uniformly on compact subsets of Ω and therefore $\Lambda_k(f_n) \longrightarrow \Lambda_k(f)$ pointwise in Ω. $\square$

Proposition 3.6. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $k \leq -1$. The set $\mathcal{B}_k$ of all $f \in \mathcal{H}ol(\Omega)$ such that $\Lambda_k(f)$ is bounded on Ω is a set of first category in $\mathcal{H}ol(\Omega)$.*

Proof. Consider the operator Λ_k as defined above and observe that $\mathcal{U}_{\Lambda_k} \neq \emptyset$: Indeed, if Ω is unbounded take $g(z) = 1$, $z \in \Omega$, and if Ω is bounded take $\zeta \in \partial\Omega$ and $g(z) = \frac{1}{(z-\zeta)^{-k+1}}$, $z \in \Omega$. Now use Proposition 2.1. $\square$

For $f \in \mathcal{H}ol(\Omega)$, where $\Omega \subset \mathbb{C}$ is a simply connected domain, consider the functions $\Lambda_k(f)$, $k \in \mathbb{Z}$, as were defined after Proposition 3.1 and before Proposition 3.3. Collecting all the above results together we get

Theorem 3.7. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. The set $\mathcal{D}$ of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, $\Lambda_k(f)$ is unbounded on Ω for all $k \in \mathbb{Z}$, is a dense G_δ subset of $\mathcal{H}ol(\Omega)$.*

Proof. For $k \in \mathbb{Z}$ define

$$D_k = \left\{ f \in \mathcal{H}ol(\Omega) : \Lambda_k(f) \text{ is unbounded on } \Omega \right\}$$

Then $\mathcal{D} = \bigcap\limits_{k \in \mathbb{Z}} D_k$. By Propositions 3.1 and 3.6 we have that each D_k is a dense G_δ set in $\mathcal{H}ol(\Omega)$, because its complement is a countable union of closed, nowhere dense sets in $\mathcal{H}ol(\Omega)$. Since $\mathcal{H}ol(\Omega)$ is a complete metric space, Baire's Theorem gives that any countable intersection of dense G_δ sets is again a dense G_δ set. $\square$

4 Universality of operators related to the partial sums

Now assume that Ω is a domain in $\mathbb{C}$. For $N \geq 0$ we define:

$$\begin{aligned} S_N : \mathcal{H}ol(\Omega) &\to \mathcal{H}ol(\Omega \times \mathbb{C}) \\ f &\mapsto S_N(f, \cdot)(\cdot) = S_N(f) \end{aligned}$$

where

$$S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n, \;\; \zeta \in \Omega, z \in \mathbb{C}$$

Then S_N is obviously linear. By the Weierstrass Theorem it is also continuous; indeed suppose $K = K_1 \times K_2$ is a compact subset of $\Omega \times \mathbb{C}$, where K_1, K_2 are compact subsets of Ω and $\mathbb{C}$ respectively, and $f_k \longrightarrow f$ uniformly on compact subsets of Ω. Set $M = \max\limits_{(\zeta,z)\in K} |z-\zeta|$. Then, for $(\zeta, z) \in K$ we have that

$$\begin{aligned}
|S_N(f_k,\zeta)(z) - S_N(f,\zeta)(z)| &= \Big| \sum_{n=0}^{N} \frac{f_k^{(n)}(\zeta) - f^{(n)}(\zeta)}{n!}(z-\zeta)^n \Big| \\
&\leq \sum_{n=0}^{N} \frac{|f_k^{(n)}(\zeta) - f^{(n)}(\zeta)|}{n!}|z-\zeta|^n \\
&\leq \sum_{n=0}^{N} \frac{\|f_k^{(n)} - f^{(n)}\|_{K_1}}{n!} M^n
\end{aligned}$$

which means that

$$\|S_N(f_k) - S_N(f)\|_K \leq \sum_{n=0}^{N} \frac{\|f_k^{(n)} - f^{(n)}\|_{K_1}}{n!} M^n$$

and therefore $S_N(f_k) \longrightarrow S_N(f)$ uniformly on K, for each $N = 0, 1, 2, ...$

Now fix $\zeta_0 \in \Omega$ and, for $N \geq 0$, define

$$\begin{aligned}
T_N : \mathcal{H}ol(\Omega) &\to \mathcal{H}ol(\mathbb{C}) \\
f &\mapsto S_N(f,\zeta_0)(\cdot)
\end{aligned}$$

Then each T_N is linear and continuous in $\mathcal{H}ol(\Omega)$ and

$$\mathcal{U}_{T_N} = \big\{ f \in \mathcal{H}ol(\Omega) : S_N(f,\zeta_0) \text{ is unbounded in } \mathbb{C} \big\}.$$

Observe that $S_N(f,\zeta_0)$ is a polynomial, so it is bounded in $\mathbb{C}$ if and only if it is constant in $\mathbb{C}$. Therefore

$$\mathcal{U}_{T_N} = \big\{ f \in \mathcal{H}ol(\Omega) : S_N(f,\zeta_0) \text{ is non-constant in } \mathbb{C} \big\}.$$

For $N = 0$ we have that $S_N(f,\zeta_0)(z) = f(\zeta_0)$, $z \in \mathbb{C}$, so $\mathcal{U}_{T_N} = \emptyset$.
For $N \geq 1$, we have that

$$S_N(f,\zeta_0)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta_0)}{n!}(z-\zeta_0)^n$$

is constant if and only if $f^{'}(\zeta_0) = f^{''}(\zeta_0) = ... = f^{(N)}(\zeta_0) = 0$. But there always exists a function $f \in \mathcal{H}ol(\Omega)$ such that $f^{(k)}(\zeta_0) \neq 0$, for all $k \in \mathbb{N}$, for example $f(z) = e^z$. Therefore, $\mathcal{U}_{T_N} \neq \emptyset$, for all $N \geq 1$. By Proposition 2.2 we have that the set $\bigcap\limits_{N=1}^{\infty} \mathcal{U}_{T_N}$ of all the functions $f \in \mathcal{H}ol(\Omega)$ with the property that the function $S_N(f,\zeta_0)$ is unbounded in $\mathbb{C}$ for all $N \geq 1$, is a dense G_δ set in $\mathcal{H}ol(\Omega)$.

We mention that $\mathcal{U}_{T_1}$ is an open dense set in $\mathcal{H}ol(\Omega)$ because $\mathcal{U}_{T_1} = \{f \in \mathcal{H}ol(\Omega) : f^{'}(\zeta_0) \neq 0\}$. Similarly, $\mathcal{U}_{T_N}$ is also an open dense set in $\mathcal{H}ol(\Omega)$, so $\bigcap\limits_{N=1}^{\infty} \mathcal{U}_{T_N}$ is G_δ dense in $\mathcal{H}ol(\Omega)$. So this corollary of Proposition 2.2 is well known and obvious. A similar result holds if we replace $\mathbb{C}$ by any unbounded domain Ω_2.

Now fix $z = 0$ and, for $N \geq 0$, define

$$\begin{aligned} \widetilde{T}_N : \mathcal{H}ol(\Omega) &\to \mathcal{H}ol(\Omega) \\ f &\mapsto S_N(f, \cdot)(0) \end{aligned}$$

Each $\widetilde{T}_N$ is linear and continuous in $\mathcal{H}ol(\Omega)$.

For $N = 0$, we have that $S_0(f, \zeta)(0) = f(\zeta)$, $\zeta \in \Omega$, and therefore

$$\mathcal{U}_{\widetilde{T}_N} = \left\{ f \in \mathcal{H}ol(\Omega) : f \text{ is unbounded in } \Omega \right\}$$

which is a dense G_δ set in $\mathcal{H}ol(\Omega)$ by Proposition 3.1.

For $N \geq 1$, if $\Omega = \mathbb{C}$, take $f(z) = e^z$, $z \in \mathbb{C}$. Then

$$S_N(f, \zeta)(0) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(0 - \zeta)^n = \sum_{n=0}^{N} \frac{e^\zeta}{n!}(-\zeta)^n,$$

from which we deduce that the function $S_N(f, \zeta)(0)$ is unbounded in $\mathbb{C}$. If $\Omega \neq \mathbb{C}$, take $\zeta_0 \in \partial\Omega$ and $f(z) = \frac{1}{z - \zeta_0}$, $z \in \Omega$. Then $f \in \mathcal{H}ol(\Omega)$ and

$$S_N(f, \zeta)(0) = \sum_{n=0}^{N} \frac{\zeta^n}{(\zeta - \zeta_0)^{n+1}}, \ \ \zeta \in \Omega$$

which is a rational function with poles only at $z = \zeta_0$. Hence $\lim\limits_{\zeta \to \zeta_0} |S_N(f, \zeta)(0)| = \infty$ and $S_N(f, \cdot)(0)$ is unbounded in Ω.

Therefore, $\mathcal{U}_{\widetilde{T}_N} \neq \emptyset$ for all $N \geq 0$, so by Proposition 2.2 we have that the set $\bigcap\limits_{N=0}^{\infty} \mathcal{U}_{\widetilde{T}_N}$ of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that $S_N(f, \cdot)(0)$ is unbounded in Ω for all $N \geq 0$, is a dense G_δ set in $\mathcal{H}ol(\Omega)$.

Next we consider the following class $\mathcal{S}(\Omega)$ of functions on Ω:

Definition 4.1. *Let Ω be an open, non-empty subset of $\mathbb{C}$. We define $\mathcal{S}(\Omega)$ to be the set of all functions $f \in \mathcal{H}ol(\Omega)$ such that $\left\{\widetilde{T}_N(f)\right\}_{N \geq 0}$ is dense in $\mathcal{H}ol(\Omega)$.*

From now on and unless otherwise stated we assume that Ω is a simply connected domain in $\mathbb{C}$. Our goal is to show that either $\mathcal{S}(\Omega) = \emptyset$ or $\mathcal{S}(\Omega)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$. To this end, first observe that, $\mathcal{H}ol(\Omega)$ is separable: the set $\{p_j\}_j$ of all polynomials with coefficients having rational coordinates is dense in $\mathcal{H}ol(\Omega)$ by Runge's Theorem. Now consider an exhaustive sequence $\{K_m\}_m$ of compact subsets of Ω, i.e. a sequence $\{K_m\}_m$ of compact subsets of Ω such that

1. $\Omega = \bigcup\limits_{m=1}^{\infty} K_m$

2. K_m lies in the interior of K_{m+1}, for $m = 1, 2, ...$

3. Every compact subset of Ω lies in some K_m

4. Every component of K_m^{c} contains a component of Ω^{c}, $m = 1, 2, ...$

(See [8].)

Now we can show that $\mathcal{S}(\Omega)$ can be expressed as a G_δ set in $\mathcal{H}ol(\Omega)$:

Proposition 4.2. $\mathcal{S}(\Omega) = \bigcap\limits_{s,j,m=1}^{\infty} \bigcup\limits_{N=0}^{\infty} \{f \in \mathcal{H}ol(\Omega) : \sup\limits_{\zeta \in K_m} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{s}\}$

Proof. That $\mathcal{S}(\Omega)$ is a subset of the set

$$\bigcap_{s,j,m=1}^{\infty} \bigcup_{N=0}^{\infty} \{f \in \mathcal{H}ol(\Omega) : \sup_{\zeta \in K_m} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{s}\}$$

is an immediate consequence of the definition of $\mathcal{S}(\Omega)$.

Consider now a function f in the set

$$\bigcap_{s,j,m=1}^{\infty} \bigcup_{N=0}^{\infty} \{f \in \mathcal{H}ol(\Omega) : \sup_{\zeta \in K_m} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{s}\},$$

a function $g \in \mathcal{H}ol(\Omega)$, a compact subset K of Ω and an $\epsilon > 0$. There exists an $m \geq 1$ such that $K \subset K_m$ and an $s \geq 1$ such that $\frac{1}{s} < \epsilon$. For these g, K_m and s, there exists a $j \geq 1$ such that

$$\sup_{\zeta \in K} |p_j(\zeta) - g(\zeta)| \leq \sup_{\zeta \in K_m} |p_j(\zeta) - g(\zeta)| < \tfrac{1}{2s}$$

For these K_m, s and j, there exists an $N \geq 0$ such that

$$\sup_{\zeta \in K} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| \leq \sup_{\zeta \in K_m} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| < \tfrac{1}{2s}$$

By the triangle inequality, for $z \in K$, we have

$$\begin{aligned} |\widetilde{T}_N(f)(z) - g(z)| &\leq |\widetilde{T}_N(f)(z) - p_j(z)| + |p_j(\zeta) - g(\zeta)| \\ &\leq \sup_{\zeta \in K} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| + \sup_{\zeta \in K} |p_j(\zeta) - g(\zeta)| \\ &< \frac{1}{2s} + \frac{1}{2s}. \end{aligned}$$

Therefore, $\sup\limits_{\zeta \in K} |\widetilde{T}_N(f)(\zeta) - g(\zeta)| \leq \frac{1}{s} < \epsilon$, so $\{\widetilde{T}_N(f)\}$ is dense in $\mathcal{H}ol(\Omega)$. $\square$

Proposition 4.3. *$\mathcal{S}(\Omega)$ is a G_δ set in $\mathcal{H}ol(\Omega)$.*

Proof. By Proposition 4.2, it suffices to show that, for $j, s, m \geq 1$ and $N \geq 0$, the set

$$E_{j,s,m,N} := \big\{ f \in \mathcal{H}ol(\Omega) : \sup_{\zeta \in K_m} |\widetilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{s} \big\}$$

is open in $\mathcal{H}ol(\Omega)$.

To this end, denote by $C(K_m)$ the space of continuous functions on K_m, endowed with the supremum norm. The mapping $\widetilde{T}_N : \mathcal{H}ol(\Omega) \to C(K_m)$ with $\widetilde{T}_N(f) = S_N(f, \cdot)(0)\big|_{C(K_m)}$ is continuous. The set $E_{j,s,m,N}$ is the inverse image of the open ball in $C(K_m)$ centered at p_j, with radius $1/s$, of the continuous mapping $\widetilde{T}_N$; therefore it is open. □

Proposition 4.4. *Let Ω be a simply connected domain in $\mathbb{C}$. Either $\mathcal{S}(\Omega) = \emptyset$ or $\mathcal{S}(\Omega)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$.*

Proof. If $\mathcal{S}(\Omega) \neq \emptyset$, by Proposition 4.3 it suffices to show that $\mathcal{S}(\Omega)$ is dense in $\mathcal{H}ol(\Omega)$.

Let $f \in \mathcal{S}(\Omega)$. Observe that, if p is a polynomial, then $f + p \in \mathcal{S}(\Omega)$. Indeed, $f + p \in \mathcal{H}ol(\Omega)$ and, for all $N > \deg p$, we have that $\widetilde{T}_N(f + p) = \widetilde{T}_N(f) + q_p$, where

$$q_p(\zeta) = \sum_{n=0}^{N} \frac{(-1)^n p^{(n)}(\zeta)}{n!} \zeta^n, \quad \zeta \in \Omega$$

is a polynomial which is independent of N, when $N > \deg p$. For a function $g \in \mathcal{H}ol(\Omega)$, we have that $g - q_p \in \mathcal{H}ol(\Omega)$, and therefore there exists a sequence $\{\lambda_n\}$ in $\mathbb{N}$ such that $\widetilde{T}_{\lambda_n}(f) \longrightarrow g - q_p$ uniformly on compact subsets of Ω. But then $\widetilde{T}_{\lambda_n}(f + p) = \widetilde{T}_{\lambda_n}(f) + q_p \longrightarrow g$ uniformly on compact subsets of Ω, i.e. $\{\widetilde{T}_N(f + p)\}$ is dense in $\mathcal{H}ol(\Omega)$ and $f + p \in \mathcal{S}(\Omega)$.
Now the density of $\mathcal{S}(\Omega)$ in $\mathcal{H}ol(\Omega)$ follows easily because by Runge's Theorem the polynomials are dense in $\mathcal{H}ol(\Omega)$. □

At this point observe that, if $0 \in \Omega$, then $\mathcal{S}(\Omega) = \emptyset$. Indeed, for $f, g \in \mathcal{H}ol(\Omega)$ such that $f(0) \neq g(0)$, we have that, for any $N \in \mathbb{N}$ and any compact subset L of Ω such that $0 \in L$,

$$\sup_{\zeta \in L} |\widetilde{T}_N(f)(\zeta) - g(\zeta)| \geq |\widetilde{T}_N(f)(0) - g(0)| = |f(0) - g(0)| > 0$$

so there is no subsequence of $\{\widetilde{T}_N(f)\}$ that converges to g uniformly on compact subsets of Ω.

Definition 4.5. *Let Ω be open in $\mathbb{C}$. The set $\mathcal{S}_t(\Omega)$ is the set of all $f \in \mathcal{H}ol(\Omega)$ with the property that, for every $c \in \mathbb{C}$ there exists a sequence $\{\lambda_n\}$ in $\mathbb{N}$ such that, for every $L \subset \Omega$ compact,*

$$\sup_{\zeta \in L} |\widetilde{T}_{\lambda_n}(f)(\zeta) - c| \longrightarrow 0, \quad n \to \infty.$$

Proposition 4.6. *The set $\mathcal{S}_t(\Omega)$ is a G_δ set in $\mathcal{H}ol(\Omega)$.*

Proof. Let $\{z_j\}_{j\in\mathbb{N}}$ be an enumeration of the points in the complex plane with rational coordinates. Following the proof of Propositions 4.2 and 4.3, we get that

$$\mathcal{S}_t(\Omega) = \bigcap_{s,j,m=1}^{\infty} \bigcup_{N=0}^{\infty} \{f \in \mathcal{H}ol(\Omega) : \sup_{\zeta\in K_m} |\widetilde{T}_N(f)(\zeta) - z_j| < \frac{1}{s}\}$$

and that the set

$$\{f \in \mathcal{H}ol(\Omega) : \sup_{\zeta\in K_m} |\widetilde{T}_N(f)(\zeta) - z_j| < \frac{1}{s}\}$$

is open in $\mathcal{H}ol(\Omega)$, $m, j, s \geq 1$, $N \geq 0$. $\square$

Observe again that, if $0 \in \Omega$, then $\mathcal{S}_t(\Omega) = \emptyset$. Indeed, for $f \in \mathcal{H}ol(\Omega)$, $c \in \mathbb{C}$ with $f(0) \neq c$ and $L \subset \Omega$ compact, we have that

$$\sup_{\zeta\in L} |\widetilde{T}_N(f)(\zeta) - c| \geq |\widetilde{T}_N(f)(0) - c| = |f(0) - c| > 0$$

for all $N \in \mathbb{N}$. However, we can show that $\mathcal{S}_t(\Omega)$ is dense in $\mathcal{H}ol(\Omega)$ if Ω is a simply connected domain and $0 \notin \Omega$:

Theorem 4.7. *Let Ω be a simply connected domain with $0 \notin \Omega$. Then $\mathcal{S}_t(\Omega)$ contains a dense G_δ set in $\mathcal{H}ol(\Omega)$.*

Proof. Since Ω is a simply connected domain, the class $U(\Omega)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$. We will show that $U(\Omega) \subset \mathcal{S}_t(\Omega)$.

Let $f \in U(\Omega)$ and $c \in \mathbb{C}$. Take $K = \{0\}$, which is disjoint from Ω because $0 \notin \Omega$. Then K is a compact set in $\mathbb{C}$, $K \cap \Omega = \emptyset$, K^c is connected, and the function $h(z) = c$, $z \in K$, is continuous on K and (trivially) analytic in the interior of K. By definition of the class $U(\Omega)$, there exists a sequence $\{\lambda_n\}$ in $\mathbb{N}$ such that, for every compact set $L \subset \Omega$,

$$\sup_{\zeta\in L} \sup_{z\in K} |S_{\lambda_n}(f,\zeta)(z) - h(z)| \longrightarrow 0, \quad n \to \infty$$

or

$$\sup_{\zeta\in L} |S_{\lambda_n}(f,\zeta)(0) - c| \longrightarrow 0, \quad n \to \infty$$

or

$$\sup_{\zeta\in L} |\widetilde{T}_{\lambda_n}(f)(\zeta) - c| \longrightarrow 0, \quad n \to \infty.$$

Therefore, $f \in \mathcal{S}_t(\Omega)$. This completes the proof. $\square$

5 A more general statement

In [1] it is shown that, for each function $f \in U(\mathbb{D}, 0)$, there exists a residual subset G of the unit circle, such that for every positive integer n, the derivative $f^{(n)}$ is unbounded on all radii with endpoints in the set G. Thus the generic function in $\mathcal{H}ol(\mathbb{D})$ has this property. During a seminar on the topics of this paper, T. Chatziafratis posed the following question: Let E be a countable dense subset of the unit circle. Is it true that, for the generic function $f \in \mathcal{H}ol(\mathbb{D})$, all the derivatives and anti-derivatives of f are unbounded on every radius joining 0 to a point of E?

The answer to this question is affirmative. To see this, we examine a more general case:

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ be an open set, X a non-empty subset of Ω. If $T : \mathcal{H}ol(\Omega) \to \mathcal{H}ol(\Omega)$ is a linear operator with the property that, for every $z \in \Omega$, the mapping $\mathcal{H}ol(\Omega) \ni f \mapsto T(f)(z) \in \mathbb{C}$ is continuous, and*

$$S = S(T, \Omega, X) = \{f \in \mathcal{H}ol(\Omega) : T(f) \text{ is unbounded on } X\},$$

then either $S = \emptyset$ or S is a dense G_δ set in $\mathcal{H}ol(\Omega)$.

Proof. To show that S is a G_δ set, for $m \geq 1$, define

$$S_m = \{f \in \mathcal{H}ol(\Omega) : \exists z \in X \text{ such that } |T(f)(z)| > m\}$$

Then $S = \bigcap_{m=1}^{\infty} S_m$. Since the mapping $f \mapsto T(f)(z)$ is continuous, the set S_m is open in $\mathcal{H}ol(\Omega)$, for each $m \geq 1$. Hence, S is a G_δ set in $\mathcal{H}ol(\Omega)$.

To show that S is dense in $\mathcal{H}ol(\Omega)$ if it is not empty, let $g \in S$, i.e. $g \in \mathcal{H}ol(\Omega)$ and $T(g)$ is unbounded on X, and let $f \in \mathcal{H}ol(\Omega)$. If $T(f)$ is unbounded on X, then $f \in S$ and f is (trivially) the limit in $\mathcal{H}ol(\Omega)$ of a sequence of functions in S. If $T(f)$ is bounded on X by, say, M_1, then, for a fixed $n \geq 1$, the function $T(f + \frac{1}{n}\, g)$ is unbounded on X. Indeed, suppose it is bounded on X by a positive number M_2. Then, if $z \in X$, by the linearity of T we would have

$$\begin{aligned}
|T(g)(z)| &= n\,|T(\frac{1}{n}\, g)(z)| \\
&= n\,|T(f + \frac{1}{n}\, g)(z) - T(f)(z)| \\
&\leq n\,|T(f + \frac{1}{n}\, g)(z)| + n\,|T(f)(z)| \\
&\leq n\, M_2 + n\, M_1.
\end{aligned}$$

This means that $T(g)$ is bounded on X by $n\,(M_1 + M_2)$, which is contradictory to the fact that $T(g)$ is unbounded on X. Therefore, $T(f + \frac{1}{n}\, g)$ is unbounded on X for every $n \geq 1$; in other words $f + \frac{1}{n}\, g \in S$, for every $n \geq 1$. But $f + \frac{1}{n}\, g \longrightarrow f$, $n \to \infty$, uniformly on compact subsets of Ω, so f is again the limit in $\mathcal{H}ol(\Omega)$ of a sequence of functions in S. Since f was an arbitrary function in $\mathcal{H}ol(\Omega)$, S is dense in $\mathcal{H}ol(\Omega)$ and the proof is complete. $\square$

Consider now countable $T^{(k)}$ and X_m such that $S(T^{(k)}, \Omega, X_m) \neq \emptyset$, for all k, m. Then Baire's Theorem gives that $\bigcap_{k,m} S(T^{(k)}, \Omega, X_m)$ is a dense G_δ set in $\mathcal{H}ol(\Omega)$. This answers the aforementioned question in the affirmative, because if $E = \{\zeta_m : m \in \mathbb{Z}\}$ and X_m is the radius joining 0 to ζ_m, then the function $g(z) = \frac{1}{z-\zeta_m}$, $z \in \mathbb{D}$, belongs to $S(T^{(k)}, \mathbb{D}, X_m)$ for all $k \geq 0$, where T is the differentiation operator.

More generally, we can replace $\mathbb{D}$ with any open non-empty set Ω in $\mathbb{C}$, T being the differentiation operator and $X_m \subset \Omega$ having at least one accumulation point in $\partial\Omega$. If Ω is simply connected, then we obtain the analogous result for both the integration operator and the operator related to Taylor partial sums $\widetilde{T}_N$ that was defined before.

Observing that in the proof of Proposition 5.1 no properties of $\mathcal{H}ol(\Omega)$ were used other than those of a topological vector space, we can obtain a generalization of our result, where completeness is not assumed and the proof does not use Baire's Theorem:

Proposition 5.2. *Let $\mathcal{V}$ be a topological vector space over the field $\mathbb{R}$ or $\mathbb{C}$ and X a non-empty set. Denote by $F(X)$ the set of all complex-valued functions on X and consider a linear operator $T : \mathcal{V} \to F(X)$ with the property that, for all $x \in X$, the mapping $\mathcal{V} \ni \alpha \mapsto T(\alpha)(x) \in \mathbb{C}$ is continuous. Let $S = \{\alpha \in \mathcal{V} : T(\alpha)$ is unbounded on $X\}$. Then either $S = \emptyset$ or S is a dense G_δ set in $\mathcal{V}$.*

Proof. That S is a G_δ set follows from the fact that $S = \bigcap_{m=1}^{\infty} \bigcup_{x \in X} \{\alpha \in \mathcal{V} : |T(\alpha)(x)| > m\}$ and the continuity of $\alpha \mapsto T(\alpha)(x)$. The proof that S is dense if it is non-empty is identical to the proof of Proposition 5.1. □

Acknowledgements— The topics discussed in this article were suggested by V. Nestoridis. I would like to thank him for the guidance and the insightful suggestions offered. I would also like to thank T. Chatziafratis for his interest in the topics discussed. In addition I would like to thank the referee for their helpful suggestions which improved this paper.

References

[1] D. H. Armitage and G. Costakis. Boundary behavior of universal taylor series and their derivatives. *Constructive approximation*, 24(1):1–15, 2006.

[2] S. J. Gardiner and M. Manolaki. Boundary behaviour of universal taylor series on multiply connected domains. *Constructive Approximation*, 40(2):259–279, 2014.

[3] A. Melas and V. Nestoridis. Universality of Taylor series as a generic property of holomorphic functions. *Advances in Mathematics*, 157(2001)(no. 2):pages 138–176.

[4] J. Müller, V. Vlachou, and A. Yavrian. Universal overconvergence and Ostrowski-gaps. *Bulletin of the London Mathematical Society*, 38(2006)(no. 4):pages 597–606.

[5] V. Nestoridis. An extension of the notion of universal Taylor series. *CMFT 1997(Nicosia) 421-430. Ser. Approx. Decompos., II, World Sci. Publ. River Edge, NI. 1999.*

[6] V. Nestoridis. Universal Taylor series. *Annales de l'institut Fourier (Grenoble)*, 46 (1996)(no. 5):pages 1293–1306.

[7] V. Nestoridis. Non extendable holomorphic functions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 139(2):351–360, 2005.

[8] W. Rudin. *Real and Complex Analysis, 3rd Ed.* McGraw-Hill, Inc., New York, NY, USA, 1987.

Current Address:
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W Green St, Urbana, IL 61801
USA

e-mail: siskaki2@illinois.edu