

# Assignment 3

EMATM0061: Statistical Computing and Empirical Methods, TB1, 2022

## Introduction

The submission deadline for this assignment is 17 October 2022. Note that this assignment will not count towards your final grade. However, it is recommended that you try to answer the questions to gain a better understanding of the concepts and submit your work before the deadline to receive feedback.

## Create an R Markdown for the assignment

It is a good practice to use R Markdown to organize your code and results.

You may also want to use R Markdown to organize your solutions so that you can submit them later. If you are considering submitting your solutions, please generate a PDF file. You can either choose the “PDF” option when creating the R Markdown file (note that this option may require Tex being installed on your computer), or use R Markdown to output an HTML and print it as a PDF file in a browser. We can only accept PDF files in the submission of this assignment.

## Load packages

We need to load two packages, namely Stat2Data and tidyverse, before answering the questions. If they haven't been installed on your computer, please use `install.packages()` to install them first.

1. Load the tidyverse package:

```
library(tidyverse)
```

2. Load the Stat2Data package and then the dataset Hawks:

```
library(Stat2Data)
data("Hawks")
```

## 1. Exploratory data analysis

This section covers some of the concepts from Lecture 7 on Exploratory Data Analysis.

We will use the Hawks dataset that you have loaded.

```
head(Hawks)
```

##	Month	Day	Year	CaptureTime	ReleaseTime	BandNumber	Species	Age	Sex	Wing
## 1	9	19	1992	13:30		877-76317	RT	I		385
## 2	9	22	1992	10:30		877-76318	RT	I		376
## 3	9	23	1992	12:45		877-76319	RT	I		381
## 4	9	23	1992	10:50		745-49508	CH	I	F	265
## 5	9	27	1992	11:15		1253-98801	SS	I	F	205
## 6	9	28	1992	11:25		1207-55910	RT	I		412
##	Weight	Culmen	Hallux	Tail	StandardTail	Tarsus	WingPitFat	KeelFat	Crop	
## 1	920	25.7	30.1	219		NA	NA	NA	NA	NA
## 2	930	NA	NA	221		NA	NA	NA	NA	NA

```
## 3    990    26.7    31.3    235          NA          NA          NA          NA          NA
## 4    470    18.7    23.5    220          NA          NA          NA          NA          NA
## 5    170    12.5    14.3    157          NA          NA          NA          NA          NA
## 6   1090    28.5    32.2    230          NA          NA          NA          NA          NA
```

## 1.1 Location estimators

(Q1) Let's start by computing some location estimators for Hawks' Tail.

First, create a vector called `HawksTail`, the elements of which are from the `Tail` column of `Hawks` data frame.

Answer:

```
HawksTail = Hawks$Tail
head(HawksTail)
```

```
## [1] 219 221 235 220 157 230
```

Second, use the `mean` and `median` functions to compute the sample mean and sample median from the vector `HawksTail`.

Answer:

```
print(mean(HawksTail))
```

```
## [1] 198.8315
```

```
print(median(HawksTail))
```

```
## [1] 214
```

## 1.2 Combining location estimators with the summarise function

(Q1) Use a combination of the `summarise()`, `mean()` and `median()` to compute the sample mean, sample median and trimmed sample mean (with  $q = 0.5$ ) of the Hawk's wing length and Hawk's weight (i.e., the `Wing` and `Weight` columns). You may need to remove the NA values. What can you say by comparing the results of the median and the trimmed mean that you obtain?

Answer:

```
summarise(Hawks, Wing_mean=mean(Wing, na.rm=TRUE), Wing_t_mean=mean(Wing, trim=0.5, na.rm=TRUE),
          Wing_med=median(Wing, na.rm=TRUE), Weight_mean=mean(Weight, na.rm=TRUE),
          Weight_t_mean=mean(Weight, trim=0.5, na.rm=TRUE), Weight_med=median(Weight, na.rm=TRUE))
```

```
##   Wing_mean Wing_t_mean Wing_med Weight_mean Weight_t_mean Weight_med
## 1   315.6375         370       370    772.0802         970         970
```

(Q2) Combine them with the `group_by()` function to obtain a breakdown by species.

Answer:

```
group_by(Hawks, Species) %>%
  summarise(Wing_mean=mean(Wing, na.rm=TRUE), Wing_t_mean=mean(Wing, trim=0.5, na.rm=TRUE),
            Wing_med=median(Wing, na.rm=TRUE), Weight_mean=mean(Weight, na.rm=TRUE),
            Weight_t_mean=mean(Weight, trim=0.5, na.rm=TRUE), Weight_med=median(Weight, na.rm=TRUE))
```

```
## # A tibble: 3 x 7
##   Species Wing_mean Wing_t_mean Wing_med Weight_mean Weight_t_mean Weight_med
##   <fct>     <dbl>     <dbl>     <dbl>     <dbl>     <dbl>     <dbl>
## 1 CH       244.         240       240       420.         378.       378.
```

## 2 RT	383.	384	384	1094.	1070	1070
## 3 SS	185.	191	191	148.	155	155

### 1.3 Location and dispersion estimators under linear transformations

(Q1) Suppose that a variable of interest  $X$  has values  $X_1, \dots, X_n$ . Suppose that  $X_1, \dots, X_n$  has a sample mean  $A$ . Let  $a, b \in \mathbb{R}$  be real numbers and define a new variable  $\tilde{X}$  with  $\tilde{X}_1, \dots, \tilde{X}_n$  defined by  $\tilde{X}_i = aX_i + b$  for  $i = 1, 2, \dots, n$ . What is the sample mean of  $\tilde{X}_1, \dots, \tilde{X}_n$  as a function of  $a$ ,  $b$  and  $A$ ?

Answer  $a * A + b$

Now using the vector `HawksTail` that you created in Section 1.1 as data and letting  $a = 2$  and  $b = 3$ , verify your conclusion using R codes: Compute the mean of `HawksTail*a+b` and then compare it with the one obtained from the mean of `HawksTail` and your conclusion.

Answer

```
mean(HawksTail)*2+3
```

```
## [1] 400.663
```

```
mean(HawksTail*2+3)
```

```
## [1] 400.663
```

(Q2) Suppose further that  $X_1, \dots, X_n$  has sample variance  $p$  and standard deviation  $q$ . What is the sample variance of  $\tilde{X}_1, \dots, \tilde{X}_n$ ? What is the sample standard deviation of  $\tilde{X}_1, \dots, \tilde{X}_n$ ?

Now using the vector `HawksTail` that you created in Section 1.1 as data and letting  $a = 2$  and  $b = 3$ , verify your result using R codes again.

Answer: Variance:  $a * a * p$  and standard deviation  $a * q$ .

Answer

```
var(HawksTail)*4
```

```
## [1] 5424.147
```

```
var(HawksTail*2+3)
```

```
## [1] 5424.147
```

Answer

```
sd(HawksTail)*2
```

```
## [1] 73.64881
```

```
sd(HawksTail*2+3)
```

```
## [1] 73.64881
```

### 1.4 Robustness of location estimators

In this exercise we shall investigate the robustness of several location estimators: The sample mean, sample median and trimmed mean.

We begin by extracting a vector called “hal” consisting of the talon lengths of all the hawks with any missing values removed.

```
hal<-Hawks$Hallux # Extract the vector of hallux lengths
hal<-hal[!is.na(hal)] # Remove any nans
```

To investigate the effect of outliers on estimates of location we generate a new vector called "corrupted\_hal" with 10 outliers each of value 100 created as follows:

```
outlier_val<-100
num_outliers<-10
corrupted_hal<-c(hal,rep(outlier_val,times=num_outliers))
```

We can then compute the mean of the original sample and the corrupted sample as follows.

```
mean(hal)

## [1] 26.41086
mean(corrupted_hal)

## [1] 27.21776
```

Now let's investigate what happens as the number of outliers changes from 0 to 1000. The code below generates a vector called "means\_vect" which gives the sample means of corrupted samples with different numbers of outliers. More precisely, means\_vect is a vector of length 1001 with the  $i$ -th entry equal to the mean of a sample with  $i - 1$  outliers.

```
num_outliers_vect <- seq(0,1000)
means_vect <- c()
for(num_outliers in num_outliers_vect){
  corrupted_hal <- c(hal,rep(outlier_val,times=num_outliers))
  means_vect <- c(means_vect, mean(corrupted_hal))
}
```

#### (Q1) Sample median:

Copy and modify the above code to create an additional vector called "medians\_vect" of length 1001 with the  $i$ -th entry equal to the median of a sample "corrupted\_hal" with  $i - 1$  outliers.

Answers:

```
num_outliers_vect <- seq(0,1000)
medians_vect <- c()
for(num_outliers in num_outliers_vect){
  corrupted_hal <- c(hal,rep(outlier_val,times=num_outliers))
  medians_vect <- c(medians_vect, median(corrupted_hal))
}
```

#### (Q2) Sample trimmed mean:

Amend the code further to add an additional vector called "t\_means\_vect" of length 1001 with the  $i$ -th entry equal to the trimmed mean of a sample with  $i - 1$  outliers, where the trimmed mean has a trim fraction  $q = 0.1$ .

Answers:

```
num_outliers_vect <- seq(0,1000)
t_means_vect <- c()
for(num_outliers in num_outliers_vect){
  corrupted_hal <- c(hal,rep(outlier_val,times=num_outliers))
  t_means_vect <- c(t_means_vect, mean(corrupted_hal, trim=0.1))
}
```

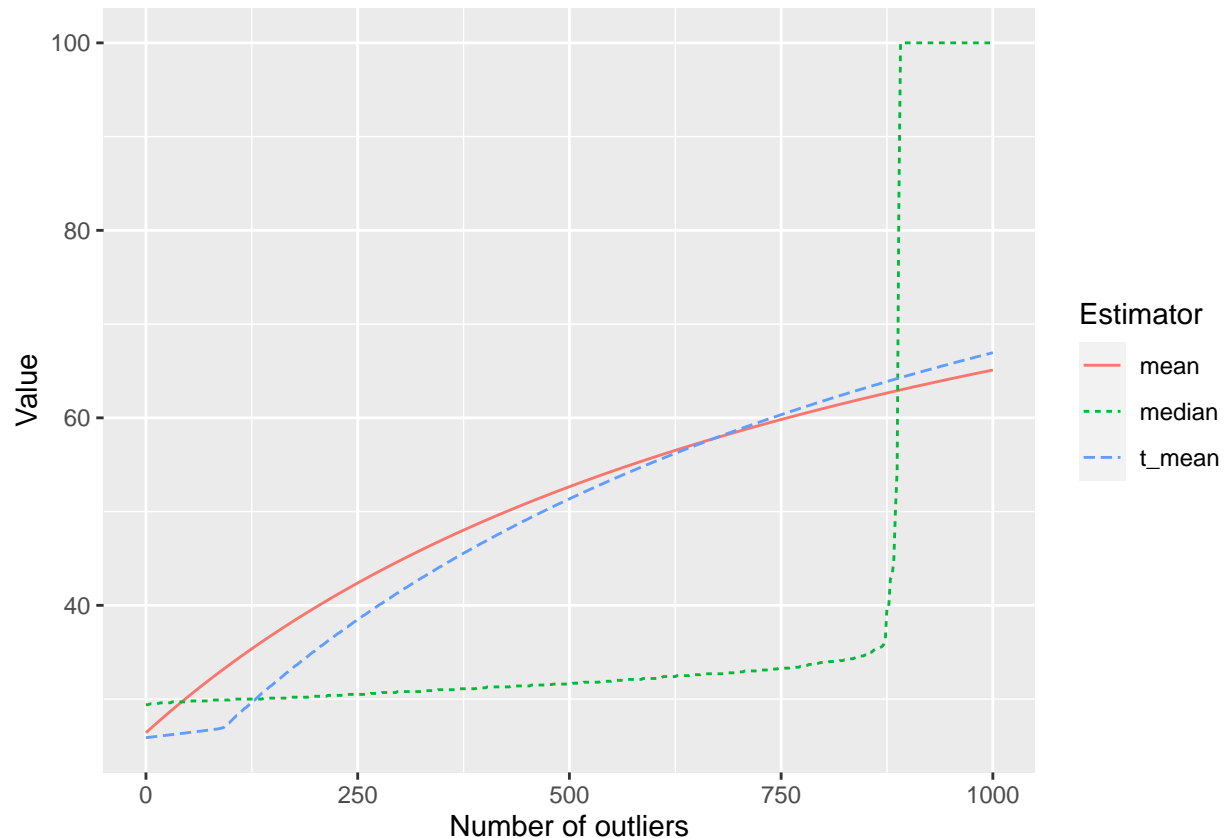
#### (Q3) Visualisation

Now you should have the vectors "num\_outliers\_vect", "means\_vect", "medians\_vect" and "t\_means\_vect". Combine these vectors into a data frame with the following code.

```
df_means_medians <- data.frame(num_outliers=num_outliers_vect, mean=means_vect,
                                t_mean=t_means_vect, median=medians_vect)
```

Now use the code below to reshape and plot the data. Recall that the function `pivot_longer()` below is used to reshape the data. Your result should look like:

```
df_means_medians %>%
  pivot_longer(!num_outliers, names_to = "Estimator", values_to = "Value") %>%
  ggplot(aes(x=num_outliers,color=Estimator, linetype=Estimator,y=Value)) +
  geom_line()+xlab("Number of outliers")
```



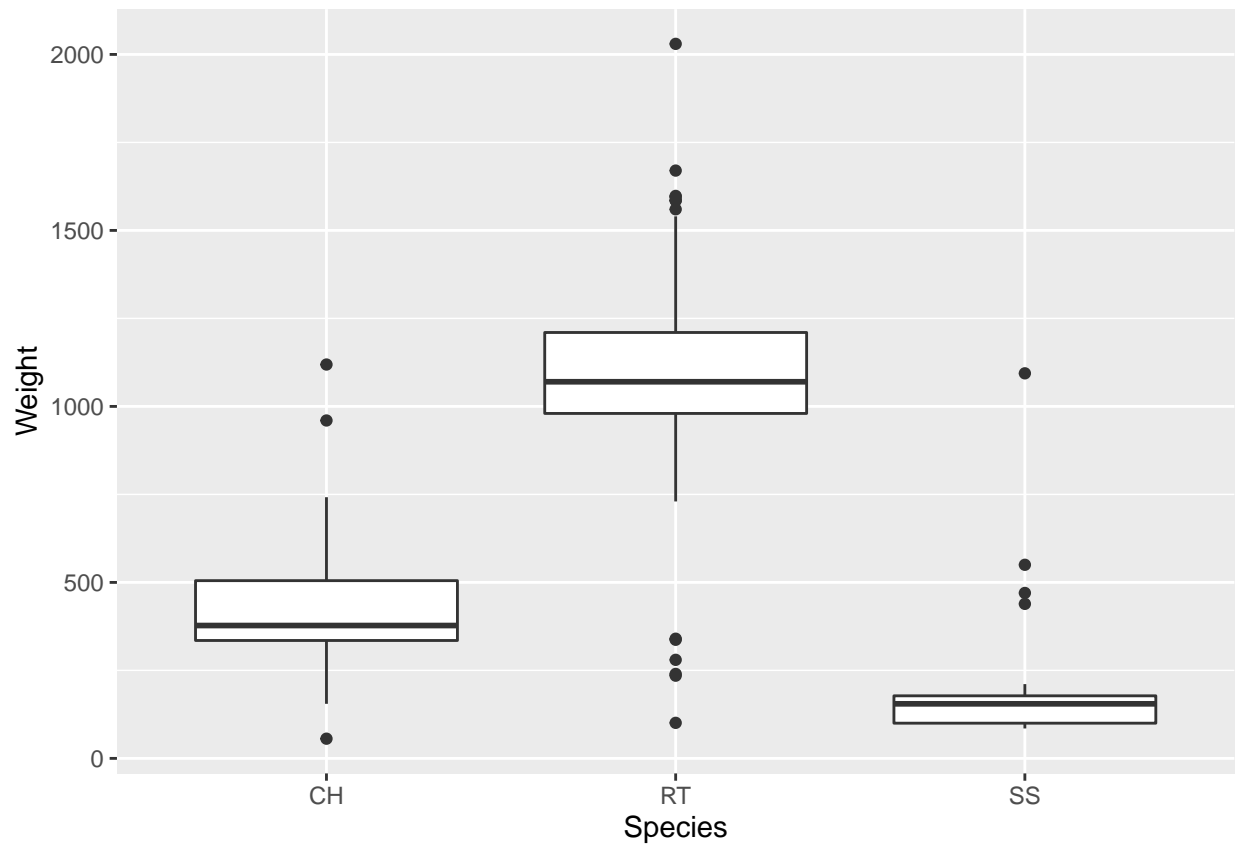
Which quantity is the most robust when the number of outliers is small? (Note that, in this experiment, the term outliers simply means the artificial data used to corrupt the vector. It is not related to the outliers computed in 1.4).

Answer: When the number of outliers is small, the sample median is the most robust one.

## 1.5 Box plots and outliers

(Q1) Use the functions `ggplot()` and `geom_boxplot()` to create a box plot which summarises the distribution of hawk weights broken down by species. Your plot should look as follows: Answers

```
ggplot(Hawks, aes(x=Species, y=Weight)) + geom_boxplot()
```



Note the outliers are displayed as individual dots.

### (Q2) quantile and boxplots

Compute the 0.25-quantile, 0.5-quantile, 0.75-quantile of the `Weight` grouped by `Species`. Your results should look like this

```
group_by(Hawks, Species) %>%
  summarise(quantile025=quantile(Weight, probs=0.25, na.rm=TRUE),
            quantile050=quantile(Weight, probs=0.5, na.rm=TRUE),
            quantile075=quantile(Weight, probs=0.75, na.rm=TRUE))
```

```
## # A tibble: 3 x 4
##   Species quantile025 quantile050 quantile075
##   <fct>         <dbl>         <dbl>         <dbl>
## 1 CH           335           378.          505
## 2 RT           980          1070          1210
## 3 SS           100           155           178.
```

Now compare these values with the boxplot above. Can you explain which parts of the boxplot these numbers correspond to?

Answer: The 0.25 – *quantile* and 0.75 – *quantile* correspond to the bottom and the top of each box in the plot, respectively. The 0.5 – *quantile* corresponds to the bar at the middle of the box.

### (Q3) outliers

Suppose we have a sample  $X_1, \dots, X_n$ . Let “q25” denote the 0.25-quantile of the sample and let “q75” denote

the 0.75-quantile of the sample. We can then define the interquartile range, denoted IQR by  $\text{IQR} := q_{75} - q_{25}$ . In the context of boxplots, an outlier  $X_i$  is any numerical value such that the following holds if either of the following holds:

$$X_i < q_{25} - 1.5 \times \text{IQR}, \quad \text{or} \\ X_i > q_{75} + 1.5 \times \text{IQR}.$$

Create a function called "num\_outliers" which computes the number of outliers within a sample (with missing values excluded).

Answers

```
num_outliers <- function(x){
  q25 <- quantile(x, 0.25, na.rm=TRUE)
  q75 <- quantile(x, 0.75, na.rm=TRUE)
  iq_range <- q75 - q25
  num <- sum( (x>q75+1.5*iq_range)|(x<q25-1.5*iq_range), na.rm=TRUE )
  return (num)
}
```

Test your "num\_outliers" function using the code below:

```
num_outliers( c(0, 40,60,185))
```

```
## [1] 1
```

(Q4) *Outliers by group*

Now combine your function num\_outliers() with the functions group\_by() and summarise() to compute the number of outliers for the three samples of hawk weights broken down by species. Your result should look as follows:

Answers

```
group_by(Hawks, Species) %>%
  summarise(num_outliers_weight = num_outliers(Weight))
```

```
## # A tibble: 3 x 2
##   Species num_outliers_weight
##   <fct>          <int>
## 1 CH              3
## 2 RT             13
## 3 SS              4
```

## 1.6 Covariance and correlation under linear transformations

(Q1) Compute the covariance and correlation between the Weight and Wing of the Hawks data. You can use the cov and cor functions.

Answers:

```
cov(Hawks$Weight, Hawks$Wing, use='complete.obs')
```

```
## [1] 41174.39
```

```
cor(Hawks$Weight, Hawks$Wing, use='complete.obs')
```

```
## [1] 0.9348575
```

(Q2) Suppose that we have a pair of variables:  $X$  with values  $X_1, \dots, X_n$  and  $Y$  with values  $Y_1, \dots, Y_n$ . Suppose that  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  have the sample covariance  $S$  and correlation  $R$ . Let  $a, b \in R$  be

real numbers and define a new variable  $\tilde{X}$  with  $\tilde{X}_1, \dots, \tilde{X}_n$  defined by  $\tilde{X}_i = aX_i + b$  for  $i = 1, 2, \dots, n$ . In addition, Let  $c, d \in \mathbb{R}$  be real numbers and define a new variable  $\tilde{Y}$  with  $\tilde{Y}_1, \dots, \tilde{Y}_n$  defined by  $\tilde{Y}_i = cY_i + d$ . What is the covariance between  $\tilde{X}_1, \dots, \tilde{X}_n$  and  $\tilde{Y}$  with  $\tilde{Y}_1, \dots, \tilde{Y}_n$  (as a function of  $S, a, b, c, d$ )? Assuming that  $a \neq 0$  and  $c \neq 0$ , what is the correlation between  $\tilde{X}_1, \dots, \tilde{X}_n$  and  $\tilde{Y}$  with  $\tilde{Y}_1, \dots, \tilde{Y}_n$ ?

Answers covariance:  $a \cdot b \cdot S$  correlation:

$$\text{sign}(a \cdots b) \cdot R$$

Let  $a = 2.4, b = 7.1, c = -1, d = 3$ , and let  $X$  be the hawk's weight and  $Y$  be the hawk's Wing. Verify your conclusion with R codes in a similar way to Section 1.3 (Q1).

Answers

```
cov(Hawks$Weight, Hawks$Wing, use='complete.obs')*2.4*(-1) -
  cov(Hawks$Weight*2.4+7.1, Hawks$Wing*(-1)+3, use='complete.obs')

## [1] 0
```

Answers

```
cor(Hawks$Weight, Hawks$Wing, use='complete.obs')*sign(2.4*(-1)) -
  cor(Hawks$Weight*2.4+7.1, Hawks$Wing*(-1)+3, use='complete.obs')

## [1] 1.110223e-16
```

## 2. Random experiments, events and sample spaces, and the set theory

In this exercise, we will learn about Random experiments, events and sample spaces and set theory that were introduced in Lecture 8.

In this section, you are not required to compute your results using R codes.

### 2.1 Random experiments, events and sample spaces

(Q1) Firstly, write down the definition of a random experiment, event and sample space.

Answer: A random experiment is a procedure (real or imagined) which:

1. has a well-defined set of possible outcomes;
2. could (at least in principle) be repeated arbitrarily many times.

An event is a set (i.e. a collection) of possible outcomes of an experiment A sample space is the set of all possible outcomes of interest for a random experiment

(Q2) Consider a random experiment of rolling a dice twice. Give an example of what is an event in this random experiment. Also, can you write down the sample space as a set? What is the total number of different events in this experiment? Is the empty set considered as an event?

Answer:

1.  $\{(1,2), (2,3)\}$  is an event. Here (1,2) means the number on the top of the first die is 1 and the number on the top of the second die is 2.
2. The sample space is  $\{(a,b) | a, b \in \{1, 2, 3, 4, 5, 6\}\}$
3. There are 36 different outcomes, so the total number of events is  $2^36$ .
4. The empty set is an event.



## 2.2 Set theory

Remember that a set is just a collection of objects. All that matters for the identity of a set is the objects it contains. In particular, the elements within the set are unordered, so for example the set  $\{1, 2, 3\}$  is exactly the same as the set  $\{3, 2, 1\}$ . In addition, since sets are just collections of objects, each object can only be either included or excluded and multiplicities do not change the nature of the set. In particular, the set  $\{1, 2, 2, 2, 3, 3\}$  is exactly the same as the set  $A = \{1, 2, 3\}$ . In general there is no concept of “position” within a set, unlike a vector or matrix.

**(Q1)** Set operations:

Let the sets  $A, B, C$  be defined by  $A := \{1, 2, 3\}$ ,  $B := \{2, 4, 6\}$ ,  $C := \{4, 5, 6\}$ .

1. What are the unions  $A \cup B$  and  $A \cup C$ ?
2. What are the intersections  $A \cap B$  and  $A \cap C$ ?
3. What are the complements  $A \setminus B$  and  $A \setminus C$ ?
4. Are  $A$  and  $B$  disjoint? Are  $A$  and  $C$  disjoint?
5. Are  $B$  and  $A \setminus B$  disjoint?
6. Write down a partition of  $\{1, 2, 3, 4, 5, 6\}$  consisting of two sets. Also, write down another partition of  $\{1, 2, 3, 4, 5, 6\}$  consisting of three sets.

Answers:

1.  $A \cup B = \{1, 2, 3, 4, 6\}$ .  $A \cup C = \{1, 2, 3, 4, 5, 6\}$ .
2.  $A \cap B = \{2\}$ .  $A \cap C = \emptyset$
3.  $A \setminus B = \{1, 3\}$ .  $A \setminus C = \{1, 2, 3\}$ .
4.  $A$  and  $B$  are not disjoint, since  $A \cap B = \{2\}$ .  $A$  and  $C$  are disjoint, since  $A \cap C = \emptyset$ .
5. Yes.
6. A partition consisting of two sets:  $\{1, 2, 3\}$  and  $\{4, 5, 6\}$ . A partition consisting of three sets:  $\{1, 2\}$ ,  $\{3, 4, 5\}$ ,  $\{6\}$ .

**(Q2)** Complements, subsets and De Morgan’s laws

Let  $\Omega$  be a sample space. Recall that for an event  $A \subseteq \Omega$  the complement  $A^c := \Omega \setminus A := \{w \in \Omega : w \notin A\}$ . Take a pair of events  $A \subseteq \Omega$  and  $B \subseteq \Omega$ .

1. Can you give an expression for  $(A^c)^c$  without using the notion of a complement?
2. What is  $\Omega^c$ ?
3. (Subsets) Show that if  $A \subseteq B$ , then  $B^c \subseteq A^c$ .
4. (De Morgan’s laws) Show that  $(A \cap B)^c = A^c \cup B^c$ . Let’s suppose we have a sequence of events  $A_1, A_2, \dots, A_K \subset \Omega$ . Can you write out an expression for  $(\cap_{k=1}^K A_k)^c$ ?
5. (De Morgan’s laws) Show that  $(A \cup B)^c = A^c \cap B^c$ .
6. Let’s suppose we have a sequence of events  $A_1, A_2, \dots, A_K \subset \Omega$ . Can you write out an expression for  $(\cup_{k=1}^K A_k)^c$ ?

Answers:

1.  $A$
2. The empty set.
3. Suppose that  $A \subseteq B$ . Now suppose  $x \in B^c$ . Then  $x \notin B$ . Hence,  $x \notin A$  since if it were the case that  $x \in A$  we would also have  $x \in B$ . Hence,  $x \in A^c$ . Since this holds for all elements of  $B^c$ , we must have  $B^c \subseteq A^c$ .
4. First, we show that  $x \in (A \cap B)^c$  implies  $x \in A^c \cup B^c$ . Suppose that  $x \in (A \cap B)^c$ . Then  $x \notin A \cap B$ . So either  $x \notin A$ , in which case  $x \in A^c$ , or  $x \notin B$ , in which case  $x \in B^c$  (or both holds). Hence, if  $x \in (A \cap B)^c$  we must have  $x \in A^c \cup B^c$ . Second, we show that  $x \in A^c \cup B^c$  implies  $x \in (A \cap B)^c$ . Suppose that  $x \in A^c \cup B^c$ . Then either  $x \in A^c$ , so  $x \notin A$ , so  $x \notin A \cap B$ , or  $x \in B^c$ , so  $x \notin B$  so  $x \notin A \cap B$ . Either way  $x \notin A \cap B$ , and hence  $x \in (A \cap B)^c$ .
5. Let’s apply  $(A \cap B)^c = A^c \cup B^c$  with  $A^c$  in place of  $A$  and  $B^c$  in place of  $B$ . This gives the result  $(A^c \cap B^c)^c = (A^c)^c \cup (B^c)^c$ . By taking complements, and applying the result  $(S^c)^c = S$  with  $A, B$ , in

place of  $S$  yields

$$A^c \cap B^c = ((A^c \cap B^c)^c)^c = ((A^c)^c \cup (B^c)^c)^c = (A \cup B)^c.$$

$$6. (\cup_{k=1}^K A_k)^c = \cap_{k=1}^K A_k^c$$

**(Q3)** Cardinality and the set of all subsets:

Suppose that  $\Omega = \{w_1, w_2, \dots, w_K\}$  contains  $K$  elements for some natural number  $K$ . Here  $\Omega$  has cardinality  $K$ .

Let  $E$  be a set of all subsets of  $\Omega$ , i.e.,  $E := \{A | A \subset \Omega\}$ . Give a formula for the cardinality of  $E$  in terms of  $K$ .

Answer: The cardinality of  $E$  is  $2^K$ .

**(Q4)** Disjointness and partitions.

Suppose we have a sample space  $\Omega$ , and events  $A_1, A_2, A_3, A_4$  are subsets of  $\Omega$ .

1. Can you think of a set which is disjoint from every other set? That is, find a set  $A \subseteq \Omega$  such that  $A \cap B = \emptyset$  for all  $B \subseteq \Omega$ .
2. Define events  $S_1 := A_1$ ,  $S_2 = A_2 \setminus A_1$ ,  $S_3 = A_3 \setminus (A_1 \cup A_2)$ ,  $S_4 = A_4 \setminus (A_1 \cup A_2 \cup A_3)$ . Show that  $S_1, S_2, S_3, S_4$  form a partition of  $A_1 \cup A_2 \cup A_3 \cup A_4$ .

Answers:

1. The empty set satisfies  $\emptyset \cap B = \emptyset$  for all  $B \subseteq \Omega$ .
2. We must show (1) that  $S_1 \cup S_2 \cup S_3 \cup S_4 = A_1 \cup A_2 \cup A_3 \cup A_4$  and (2) that  $S_1, S_2, S_3, S_4$  are disjoint.

(1). Take  $x \in S_1 \cup S_2 \cup S_3 \cup S_4$ . Then  $x \in S_i = A_i \setminus \cup_{j < i} A_j$  for some  $i \in \{1, 2, 3, 4\}$  and so  $x \in A_i$  and hence  $x \in A_1 \cup A_2 \cup A_3 \cup A_4$ . On the other hand, if  $x \in A_1 \cup A_2 \cup A_3 \cup A_4$ , we may choose  $i \in \{1, 2, 3, 4\}$  as small as possible with  $x \in A_i$ . It follows that  $x \in S_i = A_i \setminus \cup_{j < i} A_j$  and so  $x \in S_1 \cup S_2 \cup S_3 \cup S_4$ . Hence, the sets  $S_1 \cup S_2 \cup S_3 \cup S_4$  and  $A_1 \cup A_2 \cup A_3 \cup A_4$  have exactly the same elements and so are equal.

(2). To show  $S_1 \cup S_2 \cup S_3 \cup S_4$  are disjoint, let's show that there cannot be any  $x \in S_{i_0} \cap S_{i_1}$  with  $i_0 < i_1$ . Indeed, if  $x \in S_{i_1} = A_{i_1} \setminus (\cup_{j < i_1} A_j)$  so  $x \notin A_{i_0}$  and so  $x \notin S_{i_0}$ .

**(Q5)** Indicator function.

Suppose we have a sample space  $\Omega$ , and the event  $A$  is a subset of  $\Omega$ . Let  $\mathbf{1}_A$  be the indicator function of  $A$ .

1. Write down the indicator function  $\mathbf{1}_{A^c}$  of  $A^c$  (use  $\mathbf{1}_A$  in your formula).
2. Can you find a set  $B$  whose indicator function is  $\mathbf{1}_{A^c} + \mathbf{1}_A$ ?
3. Recall that  $\mathbf{1}_{A \cap B} = \mathbf{1}_A \cdot \mathbf{1}_B$  and  $\mathbf{1}_{A \cup B} = \max(\mathbf{1}_A, \mathbf{1}_B) = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A \cdot \mathbf{1}_B$  for any  $A \subseteq \Omega$  and  $B \subseteq \Omega$ . Combining this with the conclusion from Question (Q5) 1, Use indicator functions to prove  $(A \cap B)^c = A^c \cup B^c$  (De Morgan's laws).

Answer:

1.  $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$ .
2.  $\mathbf{1}_{A^c} + \mathbf{1}_A = 1$ , so this is the indicator function of  $\Omega = A \cup A^c$ .
3.  $\mathbf{1}_{A^c \cup B^c} = (1 - \mathbf{1}_A) + (1 - \mathbf{1}_B) - (1 - \mathbf{1}_A) \cdot (1 - \mathbf{1}_B) = 1 - \mathbf{1}_A \mathbf{1}_B = 1 - \mathbf{1}_{A \cap B} = \mathbf{1}_{(A \cap B)^c}$ . Since  $(A \cap B)^c$  and  $A^c \cup B^c$  have the same indicator function, they must be the same.

**(Q6)** Uncountable infinities (this is an optional extra).

This is a challenging optional extra. You may want to return to this question once you have completed all other questions.

Show that the set of numbers  $\Omega := [0, 1]$  is uncountably infinite.

Answer:

We use a proof due to Cantor known as the diagonalisation argument. We shall suppose that  $[0, 1]$  is countable and then deduce a contradiction. So suppose that  $[0, 1]$  is countable and that there is an

enumeration  $(a_n)_{n \in \mathbb{N}}$  of  $[0, 1]$ . For each  $n \in \mathbb{N}$ , let  $a_{n,j}$  be the corresponding decimal expansion, so each  $a_{n,j} \in \{0, 1, \dots, 9\}$  and  $a_n = 0.a_{n,1}a_{n,2}a_{n,3}a_{n,4} \dots$ . Now choose  $x \in [0, 1]$  with decimal expansion  $(x_j)_{j \in \mathbb{N}}$  by setting  $x_j \in \{1, 2, \dots, 9\} \setminus \{a_{j,j}\}$  for all  $j \in \mathbb{N}$ . It follows that  $x \neq a_n$  for all  $n \in \mathbb{N}$ , and hence  $(a_n)_{n \in \mathbb{N}}$  is not an enumeration of  $[0, 1]$ . This is a contradiction, hence there does not exist such an enumeration, so  $[0, 1]$  is uncountable.

### 3. Visualisation

Now let's move on to data visualisation. This experiment covers some of the concepts of data visualisation from Lecture 6.

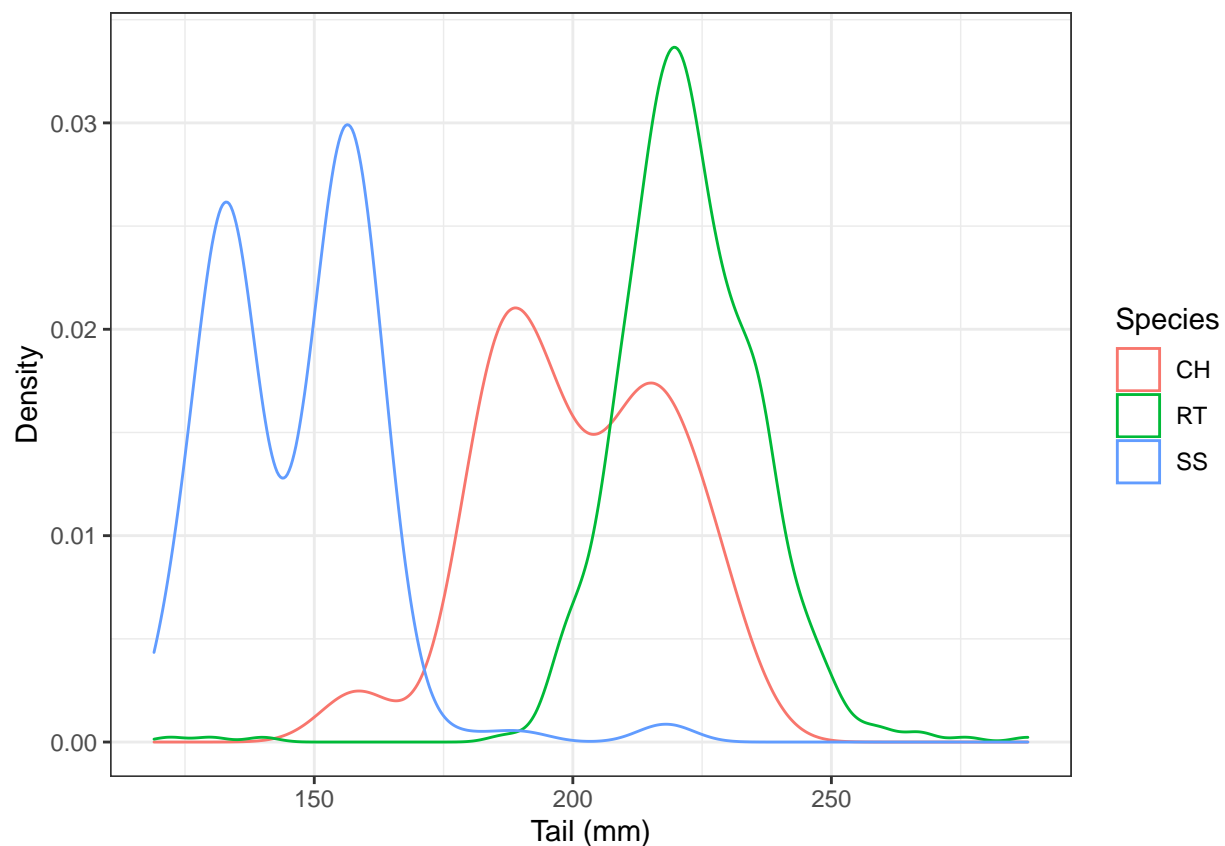
In Assignment 2, we have learned how to create univariate plots using ggplot2 histogram and density plot functions. In this assignment, we will explore bivariate and multivariate plots.

(Q1) Density plot:

Use the ggplot and geom\_density() functions to create the following density plot for the three species.

Answers

```
ggplot(Hawks, aes(x=Tail, color=Species)) + geom_density() +
  xlab('Tail (mm)') + ylab('Density') + theme_bw()
```

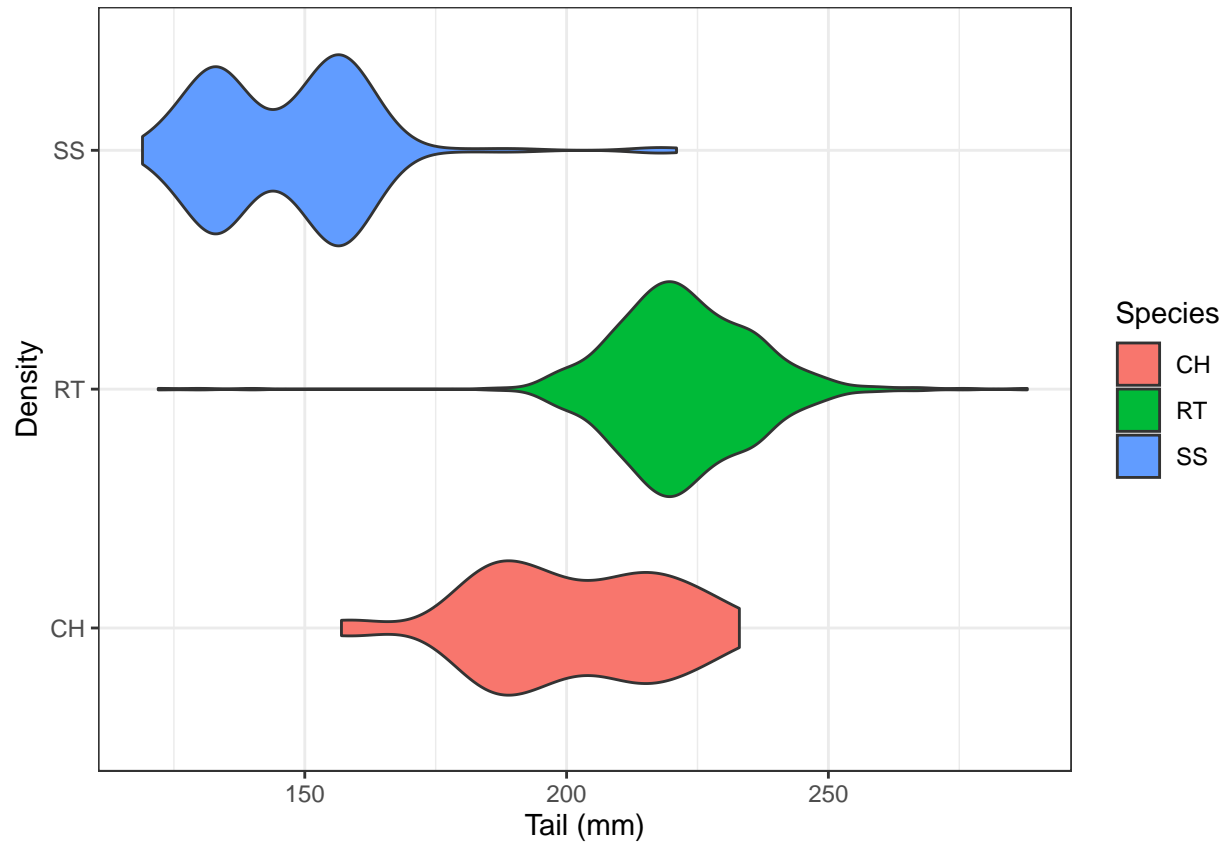


(Q2) Violin plot:

Use the ggplot and geom\_violin() functions to create the following violin plot for the three species.

Answers

```
ggplot(Hawks, aes(x=Tail, y=Species, fill=Species)) + geom_violin() +
  xlab('Tail (mm)') + ylab('Density') + theme_bw()
```



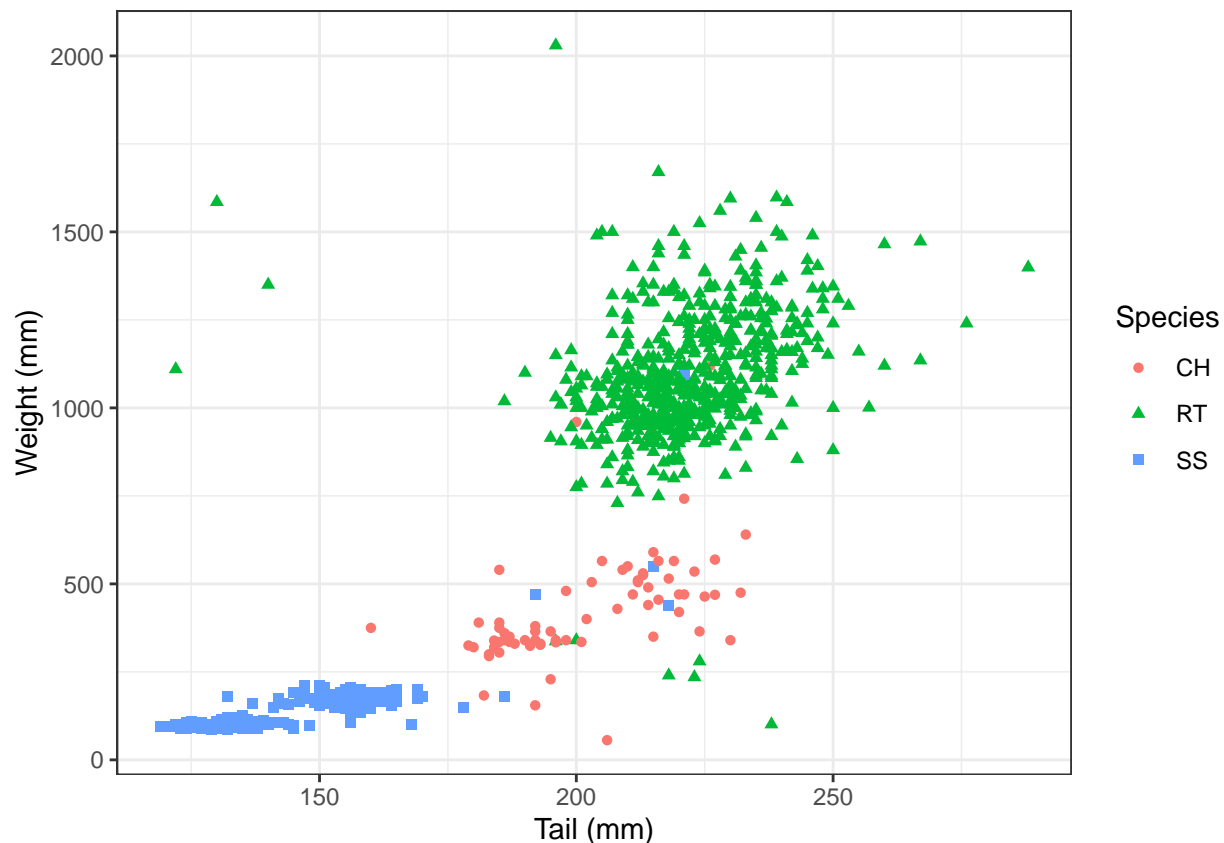
### Q(3) Scatter plot

Generate a plot similar to the following plot using the `ggplot()` and `geom_point()` functions.

1. How many aesthetics are present within the following plot?
2. What are the glyphs within this plot?
3. What are the visual cues being used within this plot?

Answers

```
ggplot(Hawks, aes(x=Tail, y=Weight, color=Species, shape=Species)) + geom_point() +
  xlab('Tail (mm)') + ylab('Weight (mm)') + theme_bw()
```



Answers:

1. There are four aesthetics: (1) The tail length is mapped to the horizontal position, (2) The weight is mapped to the vertical position, (3) The species is mapped to the colour and (4) The species is mapped to shape.
2. The glyphs are the small elements (in this particular case represented by the shapes) corresponding to individual cases.
3. The visual cues include horizontal and vertical position, shape, color.

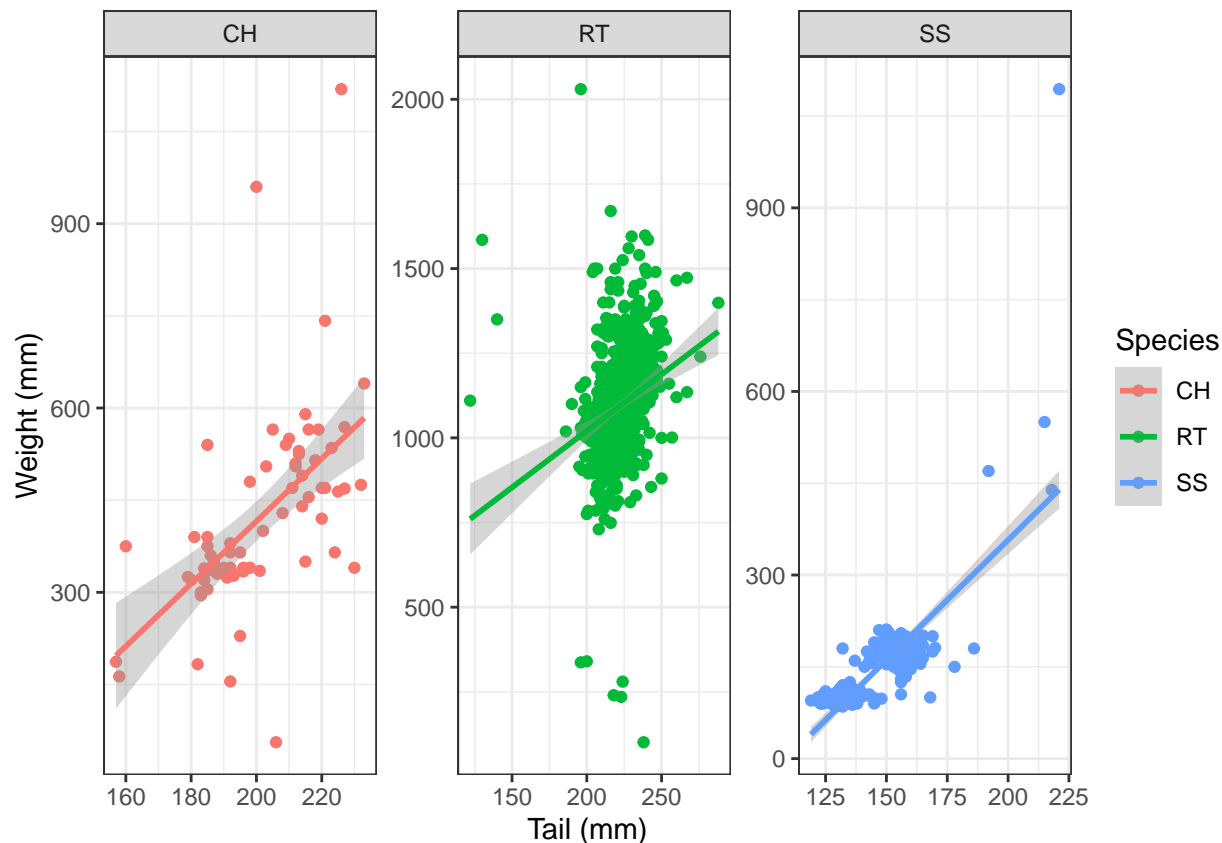
**Q(4)** Trend lines and facet wraps:

Generate the following plot using the `ggplot()`, `geom_point()`, `geom_smooth()` and `facet_wrap()` functions. Note that in the facet plot, the three panels use different scales.

1. What are the visual cues being used within this plot?
2. Based on the plot below, what can we say about the relationship between the weight of the hawks and their tail lengths?

Answers

```
ggplot(Hawks, aes(x=Tail, y=Weight, color=Species)) + geom_point() +
  facet_wrap(~Species, scales='free') + geom_smooth(method='lm') +
  xlab('Tail (mm)') + ylab('Weight (mm)') + theme_bw()
```



Answers:

1. The visual cues include horizontal and vertical position, shape, color and direction.
2. Based on this sample, longer tail lengths appear to be predictive of larger weights within each species.

#### Q(5) Adding annotations

First, compute the `Weight` and the `Tail` of the heaviest hawk in the dataset. You can use `filter()` and `select()` function to select proper data.

Answers

```
max_weight<-max(Hawks$Weight, na.rm=TRUE)
max_weight_row = head(Hawks %>% filter(Weight>=max_weight), 1)
max_weight_tail = max_weight_row$Tail
```

Second, reuse the code that you create from Q(3), adding an arrow and an annotation to indicate the heaviest hawk. Your result should look similar to this:

Answers

```
ggplot(Hawks, aes(x=Tail, y=Weight, color=Species)) + geom_point() +
  xlab('Tail (mm)') + ylab('Weight (mm)') + theme_bw() +
  geom_curve(x=max_weight_tail,xend=max_weight_tail,y=max_weight-200,yend=max_weight,
    arrow=arrow(length=unit(0.1,"cm")), color="black")+
  geom_text(x=max_weight_tail,y=max_weight-200,label='heaviest hawk',size=3,color="black")
```

