Self-organized criticality with Vicsek-like dynamics

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I. INTRODUCTION

A. Self-organized criticality

II. MODEL

Similar to the Vicsek model [], we define our model as particles interacting inside a 2-dimensional box of length L. Each particle at a time t is defined by its two positional coordinates along with the unit vector pointing along its direction of motion, i.e. $\left(\vec{r}(t), \vec{\theta}(t)\right)$. Starting with random initial conditions depicted in Fig. 1, we impose interaction dynamics between particles.

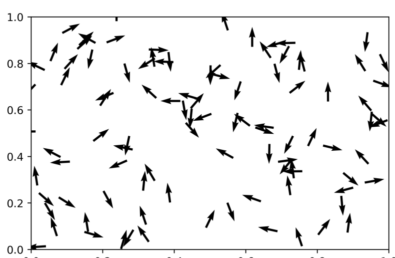

FIG. 1. Initial configuration of $N = 100$ particles in a box of size $L = 1$

The interaction is constrained to only act when two particles have a spatial distance less than, say r_c, between them. So the i-th particle, has a defined neighborhood $|\vec{r}-\vec{r}_i| < r_c$ only within which it will interact with others. Additionally, the interaction after each time step only changes the direction of the particle and not the absolute speed, thus the orientation vector $\vec{\theta}$ is sufficient to define its velocity configuration. The interaction rules for the particles are two-fold. The first is a minority interaction used to provide a mechanism for "drive" to the system, which is necessary for a model to display a self-organized critical state. The second is a polar alignment interaction exactly the same as in the Vicsek model.

1. *Minority interaction:*

The minority interaction acts on particles which satisfy two criteria. The first is a threshold used to ensure that the interaction does not affect a defector approaching a cluster but rather the members of the cluster itself, depicted in Fig. 2.

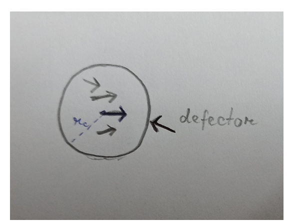

FIG. 2. A defector approaching a cluster of almost-aligned particles

For the i-th particle, this is ensured by demanding that the dot product of the average neighborhood orientation, $\vec{D}_i$ and particle orientation, $\vec{\theta}_i$ is more than ϵ (say).

$$\vec{D}_i(t).\vec{\theta}_i(t) > \epsilon$$

The absence of this criterion would result in the dynamics leading to checkerboard-like patterns, an artifact. The second criterion is used to ensure that the defector, i.e. the most disordered particle in the neighborhood (say $\vec{\theta}_i^{\max}(t)$), in fact, has a deviation angle from the average neighborhood orientation sufficiently large. The defector

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is given $\vec{\theta}_i^{\max}(t) = \vec{\theta}_j(t)$ which satisfies the constraint $\min\{\vec{D}_i(t).\vec{\theta}_j(t) : |\vec{r}_j - \vec{r}_i| < r_c\}$. Hence the second criterion can be written as,

$$\vec{D}_i(t).\vec{\theta}_i^{\max}(t) < \gamma$$

If a particle satisfies both conditions, we assign the orientation of $\vec{\theta}_i^{\max}(t)$ to it in the next time step, $t + \Delta t$, with an uncertainty due to noise ξ

$$\vec{\theta}_i(t + \Delta t) = \vec{\theta}_i^{\max}(t) + \xi$$

The noise ξ is chosen from a uniform random distribution between $(-\eta/2, \eta/2)$.

2. Polar alignment (Vicsek interaction):

If any of the two criteria above are not satisfied at time t, then the polar alignment interaction is applied to those particles. After the time step, Δt, each particle aligns in the direction given by the mean of its neighbors along with a noise ξ once again. For the i-th particle,

$$\vec{\theta}_i(t + \Delta t) = \langle \vec{\theta}_j(t) \rangle_{|\vec{r}_j(t) - \vec{r}_i(t)| < r_c} + \xi$$

The noise ξ once again is chosen from a uniform random distribution between $(-\eta/2, \eta/2)$.

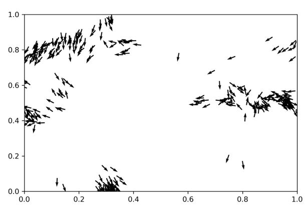

FIG. 3. Configuration at two different time-steps ($\Delta t =$) with the interactions defined.

3. Propagation step:

The positional update after each time step Δt with a fixed absolute speed (without loss of generality, say 1) will be.

$$\vec{r}_(t + \Delta t) = \vec{r}_i(t) + \vec{\theta}_i(t)\ \Delta t$$

III. OBSERVATIONS

A. Order parameter

The order parameter is defined as the vector sum of the particle orientations averaged over the total number of particles.

$$\phi(t) = \frac{1}{N} \left| \sum_i \vec{\theta}_i(t) \right|$$

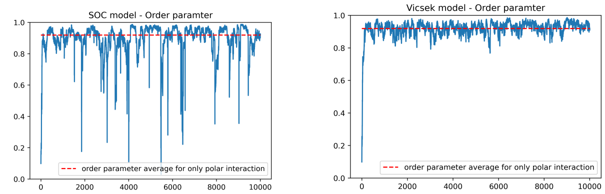

FIG. 4. a) (left) Order Parameter $\phi(t)$ for $L = 1$ with $\rho = 100$, $\gamma = -0.6$, $\epsilon = 0.6$. The red dashed line corresponds to the time averaged order parameter for the Vicsek model, i.e. only the polar interaction, with same parameter values. b) (right) Corresponding Vicsek order parameter.

It is observed that the steady-state behavior of the order parameter $\phi(t)$ is very different for our model when compared to the Vicsek model for the same parameter values (Fig. 4). In order to characterize this, we first define our system to be "ordered" at time t if $\phi(t) > \overline{\phi_v}$, where ϕ_v is the same order parameter but for the Vicsek model and $\overline{\phi_v}$ denotes time-average. Any configuration where $\phi(t) < \overline{\phi_v}$ is defined as "disordered", i.e. the breakdown of order due to avalanches induced by defectors. The return time of the order parameter is defined as the time required to return to order from the time disorder had set in. The distribution of the return time

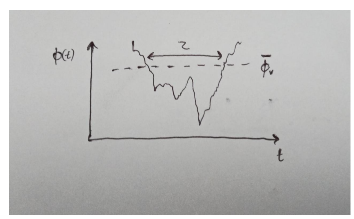

FIG. 5. The return time for the order parameter here is τ

is computed by averaging over time and ensembles and observed to approximately follow a power law as seen in Fig. 6.

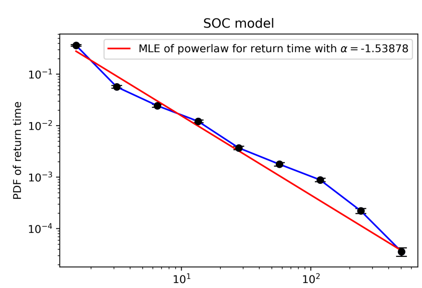

FIG. 6. Distribution of the return times of order parameter for $L = 1$, $\rho = 100$, $\gamma = -0.6$, $\epsilon = 0.6$. The red solid line is the maximum likelihood estimator for the power law fit $y = cx^\alpha$ with $\alpha = -1.538$

B. Spatial velocity correlation

The next observable we look at is the two-point spatial correlation function of velocity fluctuation formally defined as,

$$C(l) = \frac{\sum_{i,j}(\vec{v}_i - \vec{u}_{\text{avg}}).(\vec{v}_j - \vec{u}_{\text{avg}})\delta(l - r_{ij})}{\sum_{i,j}\delta(l - r_{ij})}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$ and $\vec{u}_{\text{avg}} = \frac{1}{N}\sum_k \vec{v}_k$. Since the absolute velocity of all particles remain unchanged, $\vec{v} = \vec{\theta}$. This also makes any normalization of the correlation by absolute velocity redundant.

However for our system, they have been computed by averaging the correlation in annular rings of $\Delta l = 0.05 = r_c$, where r_c is the neighbourhood radius for a particle. Assuming $C(l)$ to be stationary in time for an ergodic system, it is reasonable to average the correlation over time, instead of ensembles. The time-averaged velocity correlations is computed for a wide range of system sizes starting from $L = 0.4$ to $L = 2$ for a fixed density $\rho = 100$. Some typical correlation functions have been reported in fig. 7.

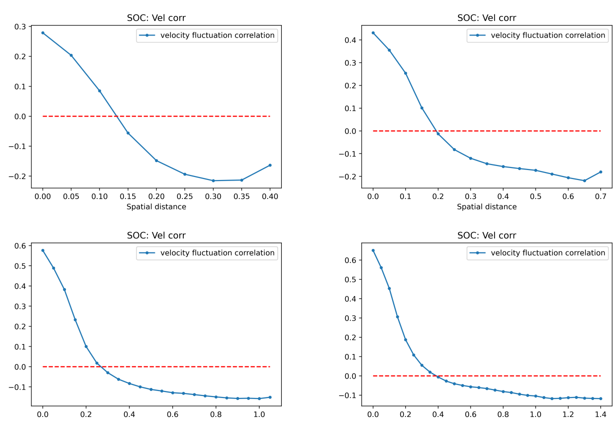

FIG. 7. Velocity fluctuations correlations as functions of distance for lattice sizes: a) $L = 0.6$, b) $L = 1.0$, c) $L = 1.5$, d) $L = 2.0$. Correlation length is defined as the distance at which the correlation function crosses over from positive correlated to negatively correlated region.

Firstly, we note that the correlation is not at $l = 0$ is not unity. This is because the correlations are averaged over Δl. Thus, there remains a contribution to the correlation at $l = 0$ from particles other than itself.

The other thing to note is that the correlation functions are not decaying exponentially, as it should for a system away from criticality. Rather, we see that the function crosses over from positive to negative values at a finite length, say ξ. Thus clearly, there is a region $l < \xi$, where the velocity fluctuations are positively correlated and a region $l > \xi$ where it is negatively correlated. Thus, ξ is playing the role of a correlation length in our system. Looking at the dependence of ξ on system size L, we find it increases linearly with L evident in Fig. 8.

Interestingly, this implies that our system is spatially scale-free. In general, the correlation function up to the leading term can be written as,

$$C(l) = \frac{1}{\xi^\gamma} f\left(\frac{l}{\xi}\right)$$

Since ξ scales linearly with L, we can write

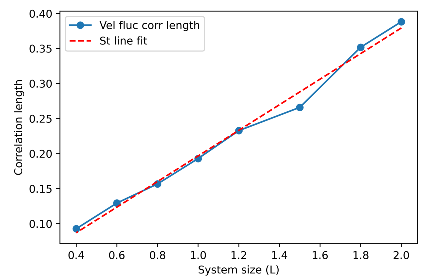

FIG. 8. Correlation length of velocity fluctuations scales linearly with system size L, implying that our system is spatially scale-free.

$$\xi(aL) = a\xi(L)$$

This leads to the scaling relation,

$$C(l, L) = a^\gamma C(al, aL)$$

Now we choose $a = 1/l$ which leads to,

$$C(l, L) = \frac{1}{l^\gamma} C(1, l/L) = \frac{1}{l^\gamma} g\left(\frac{l}{L}\right)$$

It is clear that when the system size is large, i.e. $L \to \infty$ the function $g\left(\frac{l}{L}\right) \to g(0)$ and we have,

$$C_\infty(l) = \frac{c_0}{l^\gamma}$$

where c_0 is some constant independent of system size. Consequently, the spatial correlation of our system must follow a power law, which is the signature of a scale-free system.

C. Susceptibility

The final observable is susceptibility. Let us start by looking at how susceptibility behaves for simple Ising spins. It is defined as the change in magnetization m, per unit change in field h and also given by the variance of magnetization m, which is the order parameter for Ising spins.

$$\chi = \frac{\partial m}{\partial h} = \langle m^2 \rangle - \langle m \rangle^2$$

Near criticality, it scales as $\chi \sim |T_c - T|^{-\gamma}$ It is also an extensive quantity here, i.e. it scales as $\chi \sim V^{\alpha}$ with $\alpha = 1$.

For self-organized critical systems on the other hand, we have an order parameter ϕ which is always non-zero. The generalized susceptibility χ is still defined as the variance in the order parameter,

$$\chi \sim \langle \phi^2 \rangle - \langle \phi \rangle^2$$

However, the order parameter generally takes only positive values in this case, Heuristically, for a positive quantity to have a finite mean with an infinite variance is an impossibility. Thus, the scaling of χ has to be hyperuniform, i.e. $\chi \sim V^{\alpha}$ where $\alpha < 1$. The above argument also holds for our system, since the definition of our global order parameter warrants that it only assumes positive values. The scaling of the susceptibility with system size is shown in fig. 9

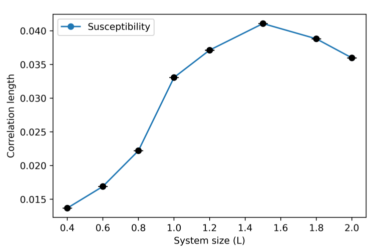

FIG. 9. Susceptibility as a function of system size L keeping density ρ constant. It is clear that susceptibility grows sublinearly which is a necessary condition for our system.

IV. CONCLUSION

Before we can claim that our model is indeed in a self-organized critical state, we need to go back to the necessary conditions for self-organized criticality []. They are as follows:

- Non-trivial scaling (finite size scaling; no dependence on control parameters).
- Spatio-temporal power law correlations.
- Apparent self tuning to the critical point (of a possibly identified, underlying continuous order phase transition).

We have already looked at the finite size scaling relation for susceptibility. The power law coefficients for the return time were also computed, varying system parameters and were seen to yield similar values, barring fluctuations.

Next, we looked at the temporal power-law of our system through the return times, as well the spatial scale-free nature of our system through velocity correlations.

Finally, we need to identify an underlying mechanism for the system to tune itself to its critical point and an associated second-order phase transition. The second-order phase transition for the Vicsek model has been studied in details in []. Additionally for our model, the minority interaction acts a mechanism to "drive" the avalanches by forcing particles to go against the majority direction if they satisfy certain thresholds. The polar alignment on the other hand, relaxes the system by making particles align in a single direction, which acts as a dissipation mechanism for our system. Thus, there is an underlying mechanism by which the system can self-tune itself to its critical point.

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