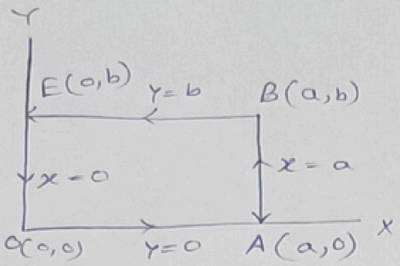
1. Evaluate $\oint_c (x^2+y^2)\,dx - 2xy\,dy$ where c in the rectangle in x-y plane bounded by $x=0$, $x=a$ & $y=0$, $y=b$

$\Rightarrow$



Here the curve 'C' consist of the straight lines OA, AB, BE & EO.

On OA, $y=0$, $dy=0$ & x-axis varies from 0 to a.

on AB, $x=a$, $dx=0$ & y-varies from 0 to b.

on BE, $y=b$, $dy=0$ & x-varies from a to 0.

on EO, $x=0$, $dx=0$ & y-varies from b to 0.

So,

$$\int_c (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{OA} x^2\,dx + \int_{AB} -2ay\,dy + \int_{BE} (x^2+b^2)\,dx + \int_{EO} 0.dy$$

$$= \int_0^a x^2\,dx - 2a\int_{AB}^{b} y\,dy + \int_a^0 (x^2+b^2)\,dx + 0$$

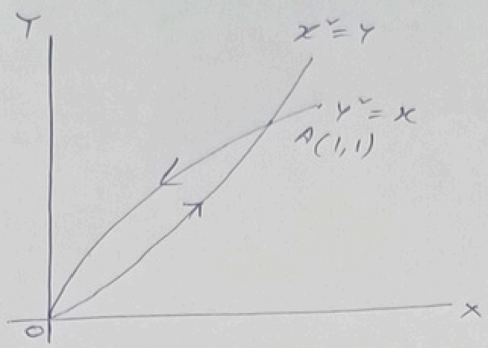
$$= \frac{1}{3}\left[x^3\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0$$

$$= \frac{1}{3}a^3 - ab^2 - \frac{1}{3}a^3 - ab^2$$

$$= -2ab^2 \text{ (Ans)}$$

2. Evalute $\oint_c (xy - x^2)dx + y\,dy$ where c is the closed curve of the region bounded by $y = x^2$ & $y^2 = x$.

$\Rightarrow$



The two curves $y = x^2$ & $y^2 = x$, cuts at origin O & A(1,1).

$$\therefore \int_c (xy - x^2)dx + y\,dy$$

$$= \int_{OA} (xy - x^2)dx + y\,dy + \int_{AO} (xy - x^2)\,dx + y\,dy$$

$$= \int_0^1 (x \cdot x^2 - x^2)dx + x^2 d(x^2) \quad (\because y = x^2)$$

$$+ \int_1^0 (y^2 \cdot y - y^4)\, d(y^2) + y\,dy \quad (\because x = y^2)$$

$$= \int_0^1 (x^3 - x^2 + 2x^3)dx + \int_1^0 (2y^4 - 2y^5 + y)dy$$

$$= \left[3\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 + \left[2\frac{x^5}{5} - 2 \cdot \frac{y^6}{6} + \frac{y^2}{2}\right]_1^0$$

$$= \left(\frac{3}{4} - \frac{1}{3}\right) + \left(\frac{2}{5} - \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{5}{12} - \frac{17}{30}$$

$$= -\frac{3}{20} \text{ (Ans)}$$

3. Evaluate $\int_0^{\pi/2}\int_0^{\pi} \sin(x+y)\,dx\,dy$

$\Rightarrow$ Here the region R is rectangle formed by the straight lines $x=0$, $x=\frac{\pi}{2}$ & $y=0$, $y=\pi$.

$$\therefore \int_{x=0}^{\pi/2}\int_{y=0}^{\pi} \sin(x+y)\,dx\,dy$$

$$= \int_{x=0}^{\pi/2}\left[\int_{y=0}^{\pi}\sin(x+y)\,dy\right]dx$$

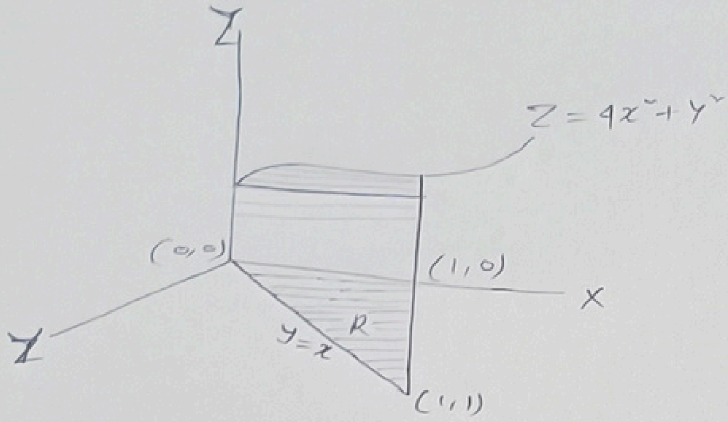
$$= \int_{x=0}^{\pi/2}\left[-\cos(x+y)\right]_{y=0}^{\pi} dx$$

$$= 2\int_0^{\pi/2}\cos x\,dx$$

$= 2$ (Ans)

4. Evalute $\iint \sqrt{4x^2+y^2}\, dx\, dy$ over the triangle formed by the Straight line $y=0$, $x=1$, $y=x$.

$\Rightarrow$



To find the volume of the region beneth $Z=4x^2+y^2$ & above the triangle with Verties $(0,0)$, $(1,0)$ & $(1,1)$ is given by,

$$\therefore \iint_R (4x^2+y^2)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (4x^2+y^2)\, dx\, dy$$

$$= \int_{x=0}^{1} \left[4x^2 y + \frac{y^3}{3}\right]_{y=0}^{x} dx$$

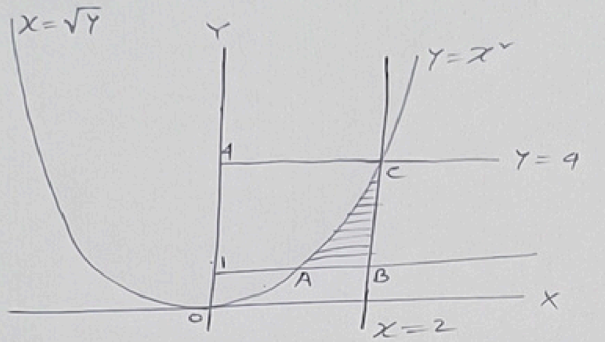
$$= \int_0^1 \left(4x^3 + \frac{x^3}{3}\right) dx$$

$$= \frac{13}{12} \left(x^4\right)_0^1$$

$$= \frac{13}{12} \text{ (Ans)}$$

5. Determine $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region bounded by $y=x^2$, $x=2$, $y=1$

$\Rightarrow$



The region 'R' is shown by shade in the fig. The boundary of 'R' can be decomposed into two curves, one BC represented by $x=2$ another AC represented by $x=\sqrt{y}$ defined on the interval $1 \le y \le 4$

So, in the region R, $1 \le y \le 4$ & for any y, $\sqrt{y} \le x \le 2$

$\therefore \iint_R (x^2+y^2)\,dx\,dy$

$$= \int_{y=1}^{4}\int_{x=\sqrt{y}}^{2} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=1}^{4}\left[\int_{x=\sqrt{y}}^{2} (x^2+y^2)\,dx\right]dy$$

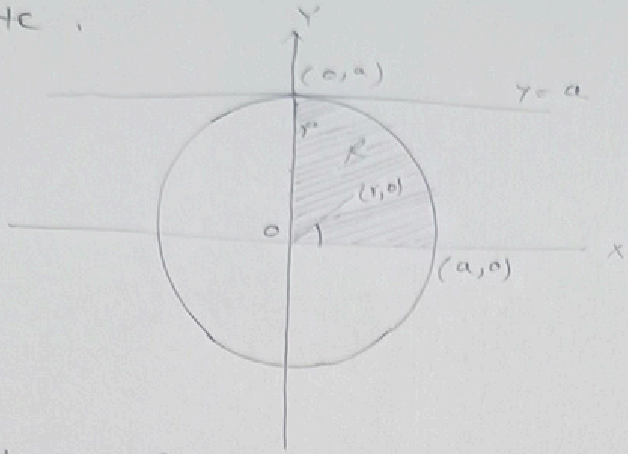
$$= \int_{y=1}^{4}\left[\frac{x^3}{3}+\frac{y^2}{1}x\right]_{x=\sqrt{y}}^{2}$$

$$= \int_{y=1}^{4}\left(\frac{8}{3}+2y^2-\frac{y\sqrt{y}}{3}-\frac{y^2}{1}\sqrt{y}\right)dy$$

$$= \frac{1006}{105} \text{ (Ans)}$$

6. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$ changing to Ploar co-ordinate.

$\Rightarrow$



In x-y plane of integration R is $0\le y\le a$, $0\le x\le\sqrt{a^2-y^2}$

i.e. $0\le y\le a$, $x^2\le a^2-y^2$ i.e. $0\le y\le a$, $x^2+y^2\le a^2$.

To change to ploar co-ordinate system we put

$x=r\cos\theta$, $y=r\sin\theta$

Now, the region 'R' is shown in the fig. by shade.

In that region for any point having Ploar co-ordinate (r,θ), we see $0\le r\le a$ ($\because$ radius of circle is a) & $0\le\theta\le\frac{\pi}{2}$ ($\because$ the point can lie in 1st quadrant of x-y plane). So in (r,θ) plane i.e. in Ploar-plane, the transformed region 'R' is $0\le\theta\le\frac{\pi}{2}$, $0\le r\le a$.

Now the Jacobian $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

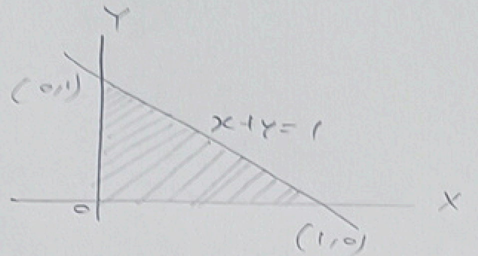
$= r\cos^2\theta + r\sin^2\theta = r.$

Now, $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} (r^2\cos^2\theta + r^2\sin^2\theta)\cdot \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$

$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2\cdot r\,dr\,d\theta$ ($\because r>0$) $= \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\right]_0^a = \frac{\pi a^4}{8}$ (Ans)

7. Evaluate $\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}} dy$ using transformation $u = x+y$, $uv = y$

$\Rightarrow$



Let, R be the region of integration in x-y plane. Then $0 \le x \le 1$ & $0 \le y \le 1-x$. i.e. $0 \le x \le 1$ & $x+y \le 1$. So, 'R' is the triangle as shaded in the fig.

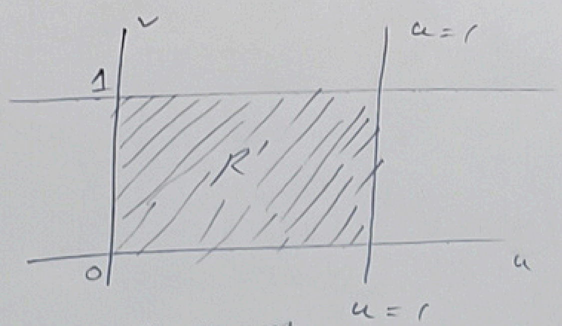
Now, the transformation is $x+y = u$, $y = uv$ Solving x, y. We get $x = u(1-v)$, $y = uv$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u$$

The region R will be transformed to R' in following.

(x,y) eq. for boundry of R'	Corresponding (u,v) eq. for the boundary of R'	Simplified (u,v) eq
$x+y=1$	$u(1-v)+uv=1$	$u=1$
$x=0$	$u(1-v)=0$	$u=0$, $v=1$
$y=0$	$uv=0$	$u=0$, $v=0$

Thus R' (in x-y) is transformed to R' (in u-v plane) bounded by $u=0$, $u=1$, $v=0$, $v=1$ (Shown in fig.)



u-v plane

$$\therefore \int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}}\, dx\, dy$$

$$= \iint_R e^{\frac{y}{x+y}}\, dx\, dy$$

$$= \iint_{R'} e^{\frac{uv}{u}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \iint_{R'} e^{v}\, u\, du\, dv$$

$$= \int_{u=0}^{1} \int_{v=0}^{v=1} e^{v}\, u\, du\, dv$$

$$= \frac{1}{2}(e-1) \quad \text{(Ans)}$$

8. Find $\int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz\, dy\, dx$

$\Rightarrow$ This integral is nothing but $\int_0^y \ldots$ is fn of y, $\int_0^x \ldots$ is fn. of x. So they would be integrated w.r.t y & w.r.t x respectively. Thus, x^3y^2z is to be integrated w.r.t z.

$$\therefore \int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz\, dy\, dz$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{x} \left[\int_{z=0}^{y} (x^3 y^2 z)\, dz \right] dy \right] dx$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{x} \frac{1}{2} x^3 y^4 \, dy \right] dx$$

$$= \int_{x=0}^{a} \left[\frac{1}{2} x^3 \frac{y^5}{5} \right]_{y=0}^{x} dx$$

$$= \frac{1}{10} \int_0^a x^3 . x^5 . dx$$

$$= \frac{1}{10} \int_0^a x^8 dx$$

$$= \frac{a^9}{90} \text{ (Ans)}$$

9. Find the max, min fo the fn.

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

Find also the saddle points.

$\Rightarrow$ Let, $f(x,y) = x^3 + y^3 - 3x - 12y + 20$, So. $f_{(x)}(x,y) = 3x^2 - 3$ and $f_y(x,y) = 3y^2 - 12$

consider the two equation, $3x^2 - 3 = 0$ & $3y^2 - 12 = 0$ that is $x^2 - 1 = 0$ & $y^2 - 4 = 0$. Solving these two equation we get $x = \pm 1$ & $y = \pm 2$ so the critical points are $(1,2), (-1,2), (1,-2)$ & $(-1,-2)$.

Now $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = 0$.

Then $H(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \{f_{xy}(x,y)\}^2$

$= 36xy$.

Now, $H(1,2) = 36 \times 1 \times 2 = 72 > 0$ & $f_{xy}(1,2) = 6 \times 1$
$= 6 > 0$

So, the fn. has min value at $(1,2)$. The min value $= f(1,2) = 1^3 + 2^3 - 3 \times 1 - 12 \times 2 + 20 = 2$.

Now, $H(-1,-2) = 72 > 0$ & $f_{xx}(-1,-2) = 6 \times -1 = -6 < 0$

So; $f(x,y)$ has maximum value at $(-1,-2)$. The max. value $= f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$
$= 38$.

$\therefore$ Clearly the saddle points ar $(-1,2)$ & $(1,-2)$.

10. Find the max & min of the given fn.

$$f(x,y) = 4x^2 + 4y^2 + x^3y + xy^3 - xy - 4.$$

$\Rightarrow$ Here, $f_x(x,y) = 8x + 3x^2y + y^3 - y$ &

$f_y(x,y) = -x + 8y + x^3 + 3xy^2$

$f_{xx}(x,y) = 8 + 6xy$, $f_{yy}(x,y) = 6xy + 8$, $f_{xy}(x,y) = 3x^2 + 3y^2 - 1$.

Consider the two eq. $f_x(x,y) = 0$ & $f_y(x,y) = 0$ that

$$8x + 3x^2y + y^3 - y = 0 \quad \text{——— (1)}$$

$$-x + 8y + x^3 + 3xy^2 = 0 \quad \text{——— (2)}$$

Adding these two we get

$$(x+y)\{(x+y)^2 + 7\} = 0$$

or, $x + y = 0$ $\quad \because (x+y)^2 + 7 = 0$

$y = -x$. ——— (3)

from (1) & (3) we get $\quad 9x - 4x^3 = 0$

$\Rightarrow x(4x^2 - 9) = 0 \Rightarrow x = 0, \frac{3}{2}, \frac{-3}{2}$

corresponding $y = 0, -\frac{3}{2}, \frac{3}{2}$.

$\therefore$ we have three critical points $(0,0)$, $\left(\frac{3}{2}, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

Now, $H(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (8 + 6xy)(6xy + 8) - (3x^2 + 3y^2 - 1)^2$

$= (6xy + 8)^2 - (3x^2 + 3y^2 - 1)^2$.

Now, $H(0,0) = 8^2 - (-1)^2 = 63 > 0$ & $f_{xx}(0,0) = 8 > 0$

So, $f(x,y)$ is minimum at $(0,0)$.

Again $H\left(\frac{3}{2}, -\frac{3}{2}\right) = \left(6 \times \frac{3}{2} \times -\frac{3}{2} + 8\right)^2 - \left(3 \times \frac{9}{4} + 3 \times \frac{9}{4} - 1\right)^2$

$$= -126 < 0.$$

So, $f(x,y)$ has neither maximum nor minimum at $\left(\frac{3}{2}, -\frac{3}{2}\right)$. This point is saddle point.

Now, $H\left(-\frac{3}{2}, \frac{3}{2}\right) = -126 < 0$, so $f(x,y)$ has no extrema at this point also.