Transport in Porous Media **53:** 117–122, 2003.

A New Model for Viscous Dissipation in Porous Media Across a Range of Permeability Values

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(Received: 14 March 2002; in final form: 16 October 2002)

Abstract. In this paper a unified mathematical theory for the viscous dissipation term in the governing Brinkman equation is derived. This term has, unlike other models, the correct asymptotic behaviour in both the fully Darcy and Newtonian fluid flow limits.

Key words: viscous dissipation, Brinkman equation, Darcy equation.

1. Introduction

The past few decades have seen an enormous expansion in research in porous media for both fluid flows, see Nield and Bejan (1999) and Ingham and Pop (1998, 2002), and convective heat transfer, see Pop and Ingham (2001). In addition, the modification of the Darcy equation, by way of Brinkman (1974a, b), to incorporate the whole range of permeability values has enabled the study of this equation to predict the correct limiting situations as the permeability $k \to 0$, that is, $\widetilde{\mu} \to 0$, and as $k \to \infty$, that is, $\widetilde{\mu} \to \mu_{\mathrm{f}}$, where $\widetilde{\mu}$ and μ_{f} are the effective viscosity and the viscosity of clear fluid, respectively. Whilst, Ingham *et al.* (1990) modified the energy equation when considering the Darcy equation in order to include the effects of dissipation, as yet there appears to be no satisfactory similar formulation of the energy equation when the flow is governed by the Brinkman equation. In a recent approach, Nield (2000, 2002) has concentrated on relating the viscous dissipation to the power of the drag force, resulting in the expression

$$\{(\mu_{\mathrm{f}}/k)\mathbf{V}\cdot\mathbf{V} - \widetilde{\mu}\mathbf{V}\cdot(\nabla^2\mathbf{V})\} \tag{1}$$

for the viscous dissipation. Although, this expression has the correct behaviour as $k \to 0$, $\widetilde{\mu} \to 0$, that is, the Darcy limit, namely the viscous dissipation is given by

$$(\mu_{\mathrm{f}}/k)\mathbf{V}\cdot\mathbf{V} \tag{2}$$

expression (1) does not appear to possess the correct limiting behaviour as $k \to \infty$ ($\widetilde{\mu} \to \mu_{\mathrm{f}}$), as does the Brinkman equation. Hence, it is with this in mind that we propose the following energy equation for situations where viscous dissipation

*Author for correspondence: e-mail: amt6dbi@amsta.leeds.ac.uk effects are important. We have attempted to demonstrate that if we consider the work done (W.D.) by the frictional forces when the Brinkman equation is the governing equation then we can obtain a model that possesses the correct asymptotic behaviours, together with a sensible physical constraint.

2. Analysis

2.1. MOTION GOVERNED BY THE NAVIER–STOKES (N–S) EQUATION

The work done (W.D.), in Cartesian coordinates, by frictional forces is given by

$$\begin{aligned}
\text{W.D.} &= \frac{\partial}{\partial X}\big\{(\sigma_X\mathbf{i}+\sigma_{XY}\mathbf{j}+\sigma_{XZ}\mathbf{k})\cdot(U\mathbf{i}+V\mathbf{j}+W\mathbf{k})\big\}+\\
&\quad+\frac{\partial}{\partial Y}\big\{(\sigma_{YX}\mathbf{i}+\sigma_{Y}\mathbf{j}+\sigma_{YZ}\mathbf{k})\cdot(U\mathbf{i}+V\mathbf{j}+W\mathbf{k})\big\}+\\
&\quad+\frac{\partial}{\partial Z}\big\{(\sigma_{ZX}\mathbf{i}+\sigma_{ZY}\mathbf{j}+\sigma_{Z}\mathbf{k})\cdot(U\mathbf{i}+V\mathbf{j}+W\mathbf{k})\big\}\\
&= \frac{\partial}{\partial X}\big\{(\sigma_X U+\sigma_{XY}V+\sigma_{XZ}W)\big\}+\frac{\partial}{\partial Y}\big\{(\sigma_{YX}U+\sigma_Y V+\\
&\quad+\sigma_{YZ}W)\big\}+\frac{\partial}{\partial Z}\big\{(\sigma_{ZX}U+\sigma_{ZY}V+\sigma_Z W)\big\}\\
&= U\left(\frac{\partial\sigma_X}{\partial X}+\frac{\partial\sigma_{XY}}{\partial Y}+\frac{\partial\sigma_{XZ}}{\partial Z}\right)+V\left(\frac{\partial\sigma_{XY}}{\partial X}+\frac{\partial\sigma_{Y}}{\partial Y}+\frac{\partial\sigma_{ZY}}{\partial Z}\right)+\\
&\quad+W\left(\frac{\partial\sigma_{XZ}}{\partial X}+\frac{\partial\sigma_{YZ}}{\partial Y}+\frac{\partial\sigma_{Z}}{\partial Z}\right)+\sigma_X\frac{\partial U}{\partial X}+\sigma_{XY}\frac{\partial V}{\partial X}+\sigma_{XZ}\frac{\partial W}{\partial X}+\\
&\quad+\sigma_{YX}\frac{\partial U}{\partial Y}+\sigma_Y\frac{\partial V}{\partial Y}+\sigma_{YZ}\frac{\partial W}{\partial Y}+\sigma_{ZX}\frac{\partial U}{\partial Z}+\sigma_{ZY}\frac{\partial V}{\partial Z}+\sigma_Z\frac{\partial W}{\partial Z}
\end{aligned}\tag{3}$$

where $\mathbf{V}=(U\mathbf{i}+V\mathbf{j}+W\mathbf{k})$ is the fluid velocity vector, $\mathbf{X}=(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})$ is the space vector and σ_X, σ_Y, σ_Z, σ_{XY}, σ_{XZ} and σ_{YZ} are the components of the stress tensor.

Using the equations of motion, namely

$$\begin{aligned}
\mathcal{P}\frac{\mathrm{D}U}{\mathrm{D}\mathcal{T}} &= \frac{\partial\sigma_X}{\partial X}+\frac{\partial\sigma_{YX}}{\partial Y}+\frac{\partial\sigma_{ZX}}{\partial Z}, \qquad \mathcal{P}\frac{\mathrm{D}V}{\mathrm{D}\mathcal{T}} = \frac{\partial\sigma_{XY}}{\partial X}+\frac{\partial\sigma_{Y}}{\partial Y}+\frac{\partial\sigma_{ZY}}{\partial Z}\\
\mathcal{P}\frac{\mathrm{D}W}{\mathrm{D}\mathcal{T}} &= \frac{\partial\sigma_{XZ}}{\partial X}+\frac{\partial\sigma_{YZ}}{\partial Y}+\frac{\partial\sigma_{Z}}{\partial Z}
\end{aligned}\tag{4}$$

then the work done becomes

$$\begin{aligned}
\text{W.D.} &= \mathcal{P}\frac{\mathrm{D}}{\mathrm{D}\mathcal{T}}\left(\frac{U^2+V^2+W^2}{2}\right)+\sigma_X\frac{\partial U}{\partial X}+\sigma_{XY}\frac{\partial V}{\partial X}+\sigma_{XZ}\frac{\partial W}{\partial X}+\\
&\quad+\sigma_{YX}\frac{\partial U}{\partial Y}+\sigma_Y\frac{\partial V}{\partial Y}+\sigma_{YZ}\frac{\partial W}{\partial Y}+\sigma_{XZ}\frac{\partial U}{\partial Z}+\sigma_{YZ}\frac{\partial V}{\partial Z}+\sigma_Z\frac{\partial W}{\partial Z}
\end{aligned}\tag{5}$$

where $\mathcal{P}$ is the fluid density and $\mathcal{T}$ is the dimensional time.

On introducing the constitutive equations

$$\sigma_X = -P + 2\mu_f \frac{\partial U}{\partial X}, \qquad \sigma_Y = -P + 2\mu_f \frac{\partial V}{\partial Y},$$
$$\sigma_Z = -P + 2\mu_f \frac{\partial W}{\partial Z}, \qquad \sigma_{XY} = \mu_f \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right),$$
$$\sigma_{XZ} = \mu_f \left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X} \right), \qquad \sigma_{YZ} = \mu_f \left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y} \right) \tag{6}$$

then the work done is given by

$$\begin{aligned} \text{W.D.} = {} & \frac{\mathrm{D}}{\mathrm{D}\mathcal{T}}(\text{Kinetic energy}) - P \left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} \right) + \\ & + \mu_f \left\{ 2 \left(\frac{\partial U}{\partial X} \right)^2 + 2 \left(\frac{\partial V}{\partial Y} \right)^2 + 2 \left(\frac{\partial W}{\partial Z} \right)^2 + \right. \\ & + \left. \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right)^2 + \left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y} \right)^2 \right\} \end{aligned} \tag{7}$$

where the terms represent the work converted into kinetic energy, work in compressing the fluid and work dissipated into heat, respectively, see Beckett (1980) and Beckett and Friend (1984).

2.2. MOTION GOVERNED BY THE DARCY EQUATION

The Darcy equation in a Cartesian vector form, which utilises the fact that $\sigma_{XY} = \sigma_{XZ} = \sigma_{YZ} = 0$, is given by

$$\frac{\mu_f}{k}\mathbf{V} = \frac{\partial(\sigma_X)}{\partial X}\mathbf{i} + \frac{\partial(\sigma_Y)}{\partial Y}\mathbf{j} + \frac{\partial(\sigma_Z)}{\partial Z}\mathbf{k} \tag{8}$$

or in component form

$$\frac{\mu_f}{k}U = \frac{\partial \sigma_X}{\partial X}, \quad \frac{\mu_f}{k}V = \frac{\partial \sigma_Y}{\partial Y} \quad \text{and} \quad \frac{\mu_f}{k}W = \frac{\partial \sigma_Z}{\partial Z} \tag{9}$$

Introducing the constitutive equations $\sigma_X = \sigma_Y = \sigma_Z = -P$ then

$$\frac{\mu_f}{k}U = -\frac{\partial P}{\partial X}, \quad \frac{\mu_f}{k}V = -\frac{\partial P}{\partial Y} \quad \text{and} \quad \frac{\mu_f}{k}W = -\frac{\partial P}{\partial Z} \tag{10}$$

and the work done by the frictional forces (since $\sigma_{XY} = \sigma_{XZ} = \sigma_{YZ} = 0$) is given by

$$\begin{aligned} \text{W.D.} &= \frac{\partial(\sigma_X U)}{\partial X} + \frac{\partial(\sigma_Y V)}{\partial Y} + \frac{\partial(\sigma_Z W)}{\partial Z} \\ &= -\frac{\partial(PU)}{\partial X} - \frac{\partial(PV)}{\partial Y} - \frac{\partial(PW)}{\partial Z} \end{aligned}$$

$$
\begin{aligned}
&= -P\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}\right)-U\frac{\partial P}{\partial X}-V\frac{\partial P}{\partial Y}-W\frac{\partial P}{\partial Z}\\
&= -P\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}\right)+\frac{\mu_{\mathrm{f}}}{k}(U^2+V^2+W^2) \qquad (11)
\end{aligned}
$$

where the terms represent the work in compressing the fluid and the work dissipated into heat, respectively, see Ingham *et al.* (1990).

2.3. MOTION GOVERNED BY THE BRINKMAN EQUATION

The Brinkman equation in the vector form is given by

$$
\begin{aligned}
\frac{\mu_{\mathrm{f}}}{k}\mathbf{V} &= \frac{\partial}{\partial X}(\sigma_X\mathbf{i}+\sigma_{YX}\mathbf{j}+\sigma_{ZX}\mathbf{k})+\frac{\partial}{\partial Y}(\sigma_{YX}\mathbf{i}+\sigma_Y\mathbf{j}+\sigma_{YZ}\mathbf{k})+\\
&\quad+\frac{\partial}{\partial Z}(\sigma_{ZX}\mathbf{i}+\sigma_{ZY}\mathbf{j}+\sigma_Z\mathbf{k}) \qquad (12)
\end{aligned}
$$

or in component form

$$
\begin{aligned}
\frac{\mu_{\mathrm{f}}}{k}U &= \frac{\partial\sigma_X}{\partial X}+\frac{\partial\sigma_{YX}}{\partial Y}+\frac{\partial\sigma_{ZX}}{\partial Z}, \qquad \frac{\mu_{\mathrm{f}}}{k}V=\frac{\partial\sigma_{XY}}{\partial X}+\frac{\partial\sigma_Y}{\partial Y}+\frac{\partial\sigma_{ZY}}{\partial Z}\\
\frac{\mu_{\mathrm{f}}}{k}W &= \frac{\partial\sigma_{ZX}}{\partial X}+\frac{\partial\sigma_{ZY}}{\partial Y}+\frac{\partial\sigma_Z}{\partial Z} \qquad (13)
\end{aligned}
$$

Introducing the constitutive equations

$$
\begin{aligned}
&\sigma_X=-P+2\widetilde{\mu}\frac{\partial U}{\partial X}, \qquad \sigma_Y=-P+2\widetilde{\mu}\frac{\partial V}{\partial Y}, \qquad \sigma_Z=-P+2\widetilde{\mu}\frac{\partial W}{\partial Z}\\
&\sigma_{XY}=\widetilde{\mu}\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right), \qquad \sigma_{XZ}=\widetilde{\mu}\left(\frac{\partial U}{\partial Z}+\frac{\partial W}{\partial X}\right),\\
&\sigma_{YZ}=\widetilde{\mu}\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right) \qquad (14)
\end{aligned}
$$

then the work done by the frictional forces is given by

$$
\begin{aligned}
\text{W.D.} &= \frac{\partial}{\partial X}(\sigma_X U+\sigma_{XY}V+\sigma_{XZ}W)+\frac{\partial}{\partial Y}(\sigma_{YX}U+\sigma_Y V+\sigma_{YZ}W)+\\
&\quad+\frac{\partial}{\partial Z}(\sigma_{ZX}U+\sigma_{ZY}V+\sigma_Z W)\\
&= U\left(\frac{\partial\sigma_X}{\partial X}+\frac{\partial\sigma_{YX}}{\partial Y}+\frac{\partial\sigma_{ZX}}{\partial Z}\right)+V\left(\frac{\partial\sigma_{XY}}{\partial X}+\frac{\partial\sigma_Y}{\partial Y}+\frac{\partial\sigma_{ZY}}{\partial Z}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+ W\left(\frac{\partial \sigma_{XZ}}{\partial X}+\frac{\partial \sigma_{YZ}}{\partial Y}+\frac{\partial \sigma_{Z}}{\partial Z}\right)+\sigma_X\frac{\partial U}{\partial X}+\sigma_{XY}\frac{\partial V}{\partial X}+\sigma_{XZ}\frac{\partial W}{\partial X}+\\
&+\sigma_{YX}\frac{\partial U}{\partial Y}+\sigma_Y\frac{\partial V}{\partial Y}+\sigma_{YZ}\frac{\partial W}{\partial Y}+\sigma_{ZX}\frac{\partial U}{\partial Z}+\sigma_{ZY}\frac{\partial V}{\partial Z}+\sigma_Z\frac{\partial W}{\partial Z}\\
= \; & -P\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}\right)+\frac{\mu_{\mathrm{f}}}{k}(U^2+V^2+W^2)+\\
&+\widetilde{\mu}\left\{2\left(\frac{\partial U}{\partial X}\right)^2+2\left(\frac{\partial V}{\partial Y}\right)^2+2\left(\frac{\partial W}{\partial Z}\right)^2+\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2+\right.\\
&+\left.\left(\frac{\partial U}{\partial Z}+\frac{\partial W}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)^2\right\} \qquad (15)
\end{aligned}
$$

where the first term represents the work in compressing the fluid, whilst the second and the third terms combine to produce work dissipated into heat.

Points to note from the expression

$$
\begin{aligned}
&\frac{\mu_{\mathrm{f}}}{k}(U^2+V^2+W^2)+\widetilde{\mu}\left\{2\left(\frac{\partial U}{\partial X}\right)^2+2\left(\frac{\partial V}{\partial Y}\right)^2+2\left(\frac{\partial W}{\partial Z}\right)^2+\right.\\
&+\left.\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2+\left(\frac{\partial U}{\partial Z}+\frac{\partial W}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)^2\right\} \qquad (16)
\end{aligned}
$$

- As $k \to \infty$, ($\widetilde{\mu} \to \mu_{\mathrm{f}}$), the viscous dissipation becomes

$$
\begin{aligned}
&\mu_{\mathrm{f}}\left\{2\left(\frac{\partial U}{\partial X}\right)^2+2\left(\frac{\partial V}{\partial Y}\right)^2+2\left(\frac{\partial W}{\partial Z}\right)^2+\right.\\
&+\left.\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2+\left(\frac{\partial U}{\partial Z}+\frac{\partial W}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)^2\right\} \qquad (17)
\end{aligned}
$$

  and this agrees with the result presented in Section 2.1, which is the standard result for viscous flows.
- As $k \to 0$, the viscous dissipation becomes

$$
\frac{\mu_{\mathrm{f}}}{k}(U^2+V^2+W^2) \qquad (18)
$$

  and this agrees with the result presented in Section 2.2, which corresponds with the standard result for flows governed by the Darcy model.
- The difference in the dissipation term that is presented here and that given by Nield (2000, 2002) is that our result possesses the correct asymptotic behaviour as $k \to \infty$ (clear fluid) and as $k \to 0$ (Darcy model). In addition, the dissipation is always positive, a condition that one would expect to occur physically and which may not be the situation when using the expression (1).

3. Conclusion

The present model for the viscous dissipation term in porous medium commences with the same physical form for the work done (W.D.) by the frictional forces, regardless of the constitutive equations. Further, it possesses the correct asymptotic behaviour for both $k \to 0$ and $k \to \infty$, and finally it produces an expression for the dissipation that is always positive. For an example of the use of this theory in a practical application, see Al-Hadhrami *et al.* (2002).

Nield, in a private communication, questions the validity of the Brinkman equation at very high values of the porosity. He argues that the Brinkman equation itself is breaking down when his expression for the viscous dissipation goes negative. Thus there is an urgent need to investigate the validity of the Brinkman equation at very high values of the porosity.

Acknowledgements

The authors would like to acknowledge the pertinent comments made by Professor Nield during the preparation of this paper. Further, the financial support provided by Petroleum Development Oman (PDO) for A.K. Al-Hadhrami is gratefully acknowledged and appreciated.

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