Volume Averaging Technique

The averaging theorems have been used in many multi-phase systems. The detailed averaging theorems and associated averaging operations can be found in many literatures (Ishii, 1995, Hassanizadeh and Gray, 1979a,b, Drew, 1983, and Ni and Beckermann, 1991). The main definitions and relations are highlighted as follows.

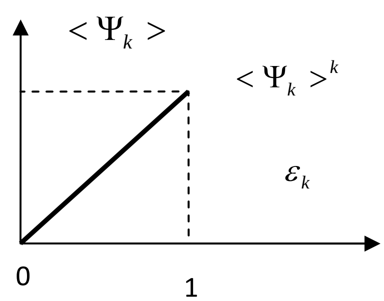

Figure The relationship between superficial and intrinsic quantities

The definition of the superficial volume average of a transport quantity Ψ in phase k is given as

$$<\Psi_k>\equiv \frac{1}{V_o}\int_{V_o} X_k \Psi_k dV$$

X_k is a medium exiting function, being equal to unity in medium k and zero otherwise. It functions as a filter. In the present system, for example, within the tissue, $X_k = X_t = 1$, while $X_{k\neq t} = X_p = X_b = 0$. For nanoparticles, $X_k = X_p = 1$, while $X_{k\neq p} = X_t = X_b = 1$. The superficial averaged quantity stands for the volume-based mean value of the transport quantity within the averaging volume.

One can also define intrinsic and superficial volume averages of a transport quantity Ψ in medium k as

$$<\Psi_k>^k \equiv \frac{1}{V_k}\int_{V_o} X_k \Psi_k dV \text{ , } <\Psi_k>\equiv \frac{1}{V_o}\int_{V_o} X_k \Psi_k dV \text{ , and } <\Psi_k>=\varepsilon_k <\Psi_k>^k$$

The intrinsic averaged quantity stands for the volume-based mean value of the transport quantity within the volume of medium k, while an average superficial quantity is proposal to the intrinsic one with a factor of volume fraction ε_k. The superficial volume averaged quantity is always smaller than the intrinsic counterpart. If the whole volume was completely occupied by medium k (i.e. ε_k =1), the superficial and intrinsic quantities are identical. The discrete expression for average superficial and intrinsic quantities can be expressed as

$$<\Psi_k>^k = \frac{1}{V_o}\sum_{i=1}^{N}(v_{p_i})(\Psi_p)_i$$

The volume fraction portion is relatively small. Once the volume averaged quality is obtain, one can give an expression for the deviation between the instant and averaged quantities, as

$$\hat{\Psi}_k \equiv (\Psi_k - <\Psi_k>^k) X_k$$

Based on definition, the average of the deviation is zero, i.e. $<\hat{\Psi}_k>=0$. This quantity gives the estimation of how the averaged value changes due to averaging processing. In the volume averaging theorems, one can also found the operation rules:

$$<\Psi_k \pm \Phi_k>=<\Psi_k> \pm <\Phi_k> , <\Psi_k \pm \Phi_k>^k = <\Psi_k>^k \pm <\Phi_k>^k ,$$

and $<c\Psi_k>^k = c<\Psi_k>^k$

where c is a constant. The average of the product of two quantities Ψ_k and Φ_k can be given as

$$< \Psi_k \Phi_k >^k = \frac{1}{V_k} \int_{V_o} X_k \Psi_k \Phi_k dV =< \Psi_k >^k < \Phi_k >^k + < \hat{\Psi}_k \hat{\Phi}_k >^k$$

Similarly, the dot production can be expressed as

$$< \Psi_k \cdot \Phi_k >^k =< \Psi_k >^k \cdot < \Phi_k >^k + < \hat{\Psi}_k \cdot \hat{\Phi}_k >^k$$

$$< \frac{\Psi_k}{\Phi_k} >^k =< \Psi_k >^k < \frac{1}{\Phi_k} >^k + < \hat{\Psi}_k \frac{1}{\hat{\Phi}_k} >^k \text{ or} < \Psi_k \Phi_k^{-1} >^k =< \Psi_k >^k < \Phi_k^{-1} >^k + < \hat{\Psi}_k \hat{\Phi}_k^{-1} >^k \quad (7)$$

The averaging theorems also give the average of a temporal derivative of a quantity as

$$< \frac{\partial \Psi_k}{\partial t} >= \frac{\partial < \Psi_k >}{\partial t} - \frac{1}{V_o} \int_{A_k} \Psi_k \vec{w}_k \cdot \vec{n}_k dA$$

Where $\vec{w}_k$ is the cell interfacial velocity, and $\vec{n}_k$ is the normal vector outward from the interface. The average of a spatial derivative of a quantity can be expressed as

$$< \nabla \Psi_k >= \nabla < \Psi_k > + \frac{1}{V_o} \int_{A_k} \Psi_k \vec{n}_k dA = \varepsilon_k \nabla (< \Psi_k >^k) + \frac{1}{V_o} \int_{A_k} \hat{\Psi}_k X_k \vec{n}_k dA$$

and

$$\frac{1}{V_o} \int_{A_i} X_k \vec{n}_k dA = -\nabla \varepsilon_k$$

This is an important relationship in the volume average theorem. It links the spatial derivative of volume fraction to the total interfacial area (the left-hand-side term).