AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Sittana Osman Afifi Mohamedelmubarak ch 2

Zero-Knowledge proofs: Implementation of the Graph Isomorphism Protocol Date: May 23, 2020

1 Graph isomorphism

[Jan: Please start each section with a short description of what will be in the section. For example, "In this section we will make and explain definitions of graphs and the main problem concerning this thesis, that is the graph isomorphism problem."]

Definition 1 (Graph [1])**.** *An undirected graph consists of a set of vertices (nodes) V and a set of edges E.*

[Jan: ..."where each edge consists of two distinct vertices"] *Two nodes u and v are said to be adjacent if there is an edge $(u, v) \in E$.*

[Jan: In general, I don't understand why you keep using \\ instead of starting new paragraph.]

We can describe graph using its adjacency matrix which is a square matrix $M_{n\times n}$, with $m_{ij} = 1$ if $(i, j) \in E$ and 0 otherwise.

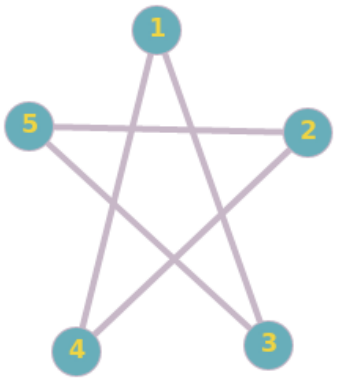
[Jan: Why is M written with a big letter and m_{ij} with a small letter?]

Definition 2. *Let $V(G)$, $E(G)$ denote the vertex set and edge set of a graph G respectively. Then, a pair of graphs (G_0, G_1) are **isomorphic** (denoted $G_0 \cong G_1$) if there exists a bijective map $\Pi : V(G_0) \longmapsto V(G_1)$such that $\forall x, y \in V(G_0), (x, y) \in E(G_0)$ if and only if $(\Pi(x)\Pi(y)) \in E(G_1)$. The permutation Π is called an isomorphism.[2]* [Jan: There should be a comma in $(\Pi(x), \Pi(y))$. Also you can delete citation in this case.]

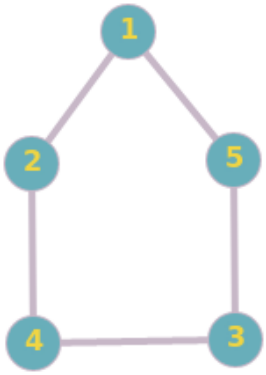
In other words: two graphs are said to be isomorphic if after we relabel vertices in one graph we get the other graph (with the same adjacency matrix).

Example 3. *Two isomorphic graphs with their corresponding adjacency matrices.* [Jan: This is not even a sentence. Please write something like "We show a simple example of two isomorphic graphs and their adjacency matrices".]

In Figure 1 G_1 is obtained, by relabeling the vertices of G_0 according to the following permutation: (1, 4, 5, 2, 3). This means that Node 3 in G_0 becomes Node 5 in G_1, Node 4 becomes Node 2. [Jan: You can also comment that adjacency matrix of G_1 is obtained by permuting rows and columns of adjacency matrix of G_0 according to the permutation Π.]



(a) G_0.



(b) G_1.

Figure 1: Two isomorphic graphs.

	1	**2**	**3**	**4**	**5**
1	0	0	1	1	0
2	0	0	0	1	1
3	1	0	0	0	1
4	1	1	0	0	0
5	0	1	1	0	0

Table 1: adjacency matrix of G_0

	1	**2**	**3**	**4**	**5**
1	0	1	0	0	1
2	1	0	0	1	0
3	0	0	0	1	1
4	0	1	1	0	0
5	1	0	1	0	0

Table 2: adjacency matrix of G_1

1.1 Graph Isomorphism based Zero-Knowledge Proofs

[Jan: Section title: "Zero-Knowledge Proof for Graph Isomorhpism]

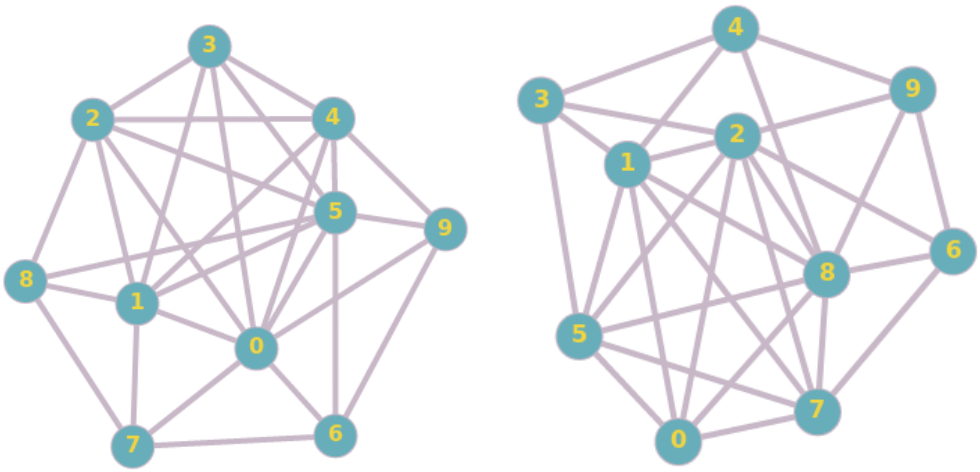
Suppose we have two isomorphic graphs G_0 and G_1 and $G_1 = \Pi(G_0)$, with limited messages between the prover (P) and verifier(V), P wants to prove to V he knows the secret Π without showing him what is Π exactly. [Jan: Try dividing the previous sentence into two and making them clearer.] In the previous example, we can see that it is easy to show if two graphs are isomorphic or not but this process isnt always simple; suppose we have two graphs each with 10 vertices and 28 edges, such as the graphs in Figure 2: [Jan: "For the graphs in Example 3, it is easy to see that they are isomorphic..."]
Then ZKP can provide a protocol that P can prove to V he knows the secret Π without revealing Π itself. [Jan: "A zero-knowledge proof is a protocol where...". In general, I suggest you try and get some help (your tutor?) with proofreading and the language.]
The protocol is done by applying a random permutation (φ) on G_0 by P, with:

$$H = \varphi(G_0)$$

and the honest prover has to be able to find a permutation such that he could transform H to either G_0 or G_1.(i.e to prove $H \cong G_0$ or $H \cong G_1$)
[Jan: Period is in the wrong place in the preceding sentence.]

1.2 Zero-Knowledge Protocol for Graph Isomorphism

We have two graphs known by both parties G_0 and G_1 such that they have n vertices, define S_n as a set of permutations of n elements.



(a) G_0. (b) G_1.

Figure 2: Two isomorphic graphs.

The protocol proceeds by the following: [2]
Input: pair of graphs (G_0, G_1)

1. **prover** chooses random permutation σ from S_n, and sends $H = \sigma(G_0)$.
2. **verifier** chooses ch randomly from $\{0, 1\}$ and sends it to the prover.
3. **prover** if $ch = 0$: then sends $\varphi = \sigma$ else sends $\varphi = \sigma \circ \Pi^{-1}$.
4. **verifier** output ACCEPT if $H = \varphi(G_{ch})$ else output is REJECT.

1.3 graph isomorphism (GI) decision problem

[Jan: This section should be before you introduce the ZK proof.] !
[Jan: Again, do not start with the definition. Explain in a few sentences what you are going to do and why.] !

Definition 4 (graph isomorphism (GI) decision problem [3])**.** *Input: pair of two finite graphs* (G_0, G_1).
Output: ACCEPT if and only if they are isomorphic.
Complexity can be measured in the number of vertices. [Jan: Better: "We are going to measure the computational complexity of this problem as a function of the number of vertices."] !

1.3.1 The complexity of GI [3]

[Jan: Please don't insert citations everywhere like that. Instead write a sentence like "The discussion in this section is based on *cite*". Then there is no need to keep making citations in the rest of the section.] If we have two graphs and we want to check whether they are isomorphic or not; the brute force algorithm is to apply all possible permutation in S_n but this run in $n!$ and its slower than any exponential algorithm.
There is a better algorithm but not in polynomial-time, because no polynomial-time algorithm yet is known to solve it (if it was in P, then there is no need to apply ZKP; the verifier can

check if graphs are isomorphic).
Moreover, GI also isn't considered to be NP-complete, This put GI in intermediate complexity problems (neither P nor NP-complete). In László Babais paper [4] according to **Corollary, 1.1.2** GI can be solved in quasipolynomial time and its run in $O(exp(log(n)^c))$ [Jan: Do not cite "Corollary 1.1.2". Also say that this is the best known algorithm. Also mention that this running time is not polynomial, but better than any exponential.] [Jan: Also use \exp, \log \Pr when writing functions.]

Theorem 5. *[3]*
The above protocol satisfies completeness, soundness $\frac{1}{2}$*, and zero-knowledge.*

[Jan: What above protocol? This is in a wrong place.] **Proof**
[Jan: Why aren't you using \begin{proof}?] **Completeness.** Completeness is quite straight forward, here we are dealing with honest prover $G_0 \cong G_1$ who knows Π, the verifier will check φ for both possible cases: [Jan: Please say one thing at a time. We assume that the graphs are isomorphic. We also assume that the prover is honest and knows Π. What are the two cases you discuss below? You need to say it: We are going to consider two cases, depending on the value of ch that was sent by the verifier.]

1. $(ch = 0)$: from step (1) P calculates $H = \sigma(G_0)$ and he should find a bijective map from H to G_0:
   it's clear that $\varphi = \sigma$ works because $\sigma(G_0) = \sigma(G_0)$
2. $(ch = 1)$: P has to find a map from H to G_1 or to show that $H \cong \varphi(G_1)$:
   [Jan: You are showing that $H = \varphi(G_1)$, not that they are isomorphic.] We know that:

$$G_1 = \Pi(G_0) \tag{1}$$

$$H = \sigma(G_0) \tag{2}$$

After we comibne equation 1 and equation 2 :

$$H = \sigma(\Pi^{-1}(G_1))$$
$$H = (\sigma \circ \Pi^{-1})(G_1)$$

So

$$\varphi = \sigma \circ \Pi^{-1}$$

The V has to check that $H \cong \varphi(G_1)$:

$$\varphi(G_1) = \sigma \circ \Pi^{-1}(G_1) = (\sigma \circ \Pi^{-1} \circ \Pi)(G_0) = H$$

As we see if the prover knows Π he can always return the correct permutation.
Soundness $\frac{1}{2}$**.** For any prover P^* such that $G_0 \ncong G_1$ interacting with an honest verifier V^*, then P^* must submit some φ to the verifier, P^* can do something like:

1. choose random φ.
2. pick random H and send it to V.

Since $G_0 \ncong G_1$ then H such is isomorphic to G_0 or G_1, so whatever prover does, after it sends H, H cannot be isomorphic to both G_0 and G_1.
[Jan: Again, say one thing at a time, slowly. First, you assume G_0 and G_1 are not isomorphic. Then, you take any prover P^*. Such a prover can do many things. For example, it can choose b and sent random

permutation of G_b. Or it can send random permutation of G_1. Or it can send an arbitrary graph H. But these are just examples and your argument works for any possible prover. You need to make that clear.]
So whatever happens, honest verifier always has at least $1/2$ chance to choose b that fails the prover, so:

$$Pr[P^* \text{ convinces } V \text{ that } G_0 \cong G_1] \leq 1/2$$

If we want to decrease the soundness to be negligible we have to repeat the verification procedure sufficiently many times, if we repeat it n times it will be soundness with 2^{-n} which is smaller than any polynomial-time algorithm. [Jan: Not algorithm, just polynomial. You can add that 2^{-n} is a negligible function.]

zero-knowledge. We want to show that there exists a polynomial-time simulator S such that if G_0 and G_1 are isomorphic then for an honest prover and for every verifier V^*:

$$\text{Distribution of transcript } (P, V^*) = \text{Distribution of transcript } S(G_0, G_1, V^*) \tag{3}$$

How S works:

1. $S_n \longleftarrow$ random permutation from S_n.
2. $b \longleftarrow$ random bit from $\{0, 1\}$..
3. $H = \sigma(G_b)$.
4. Send H to V^* to get ch.
5. If $ch = b$, output σ and return the transcript, else loop.

Now we want to show equation 3 is satisfied:
First: the transcript is $\{H, ch(H, V^*), \sigma\}$ where H is a result of applying a random permutation on G_0 (same as a result of applying a random permutation on G_1), ch was generated by V^* from H and σ is a random permutation from S_n. [Jan: I would write: H is a result of applying random permutation on G_b (same as applying on G_1) and σ satisifes $\sigma(G_b) = H$].
In the simulator, the distribution is generated in the same way, only there is always probability $\frac{1}{2}$ the simulator loop. But this probability ($\Pr[b \neq ch]$) is independent of H so it doesnt change the distribution of transcripts.
Then we want to show is a polynomial-time:
Let k be the number of loops, we want to show that S is polynomial-time by calculating the expected value of k:
When $k = 1 : E[k] = 1/2$ [Jan: No, the correct equation should be $\Pr[k = 1] = 1/2$]
When $k = 2 : E[k] = 1/4$
When $k = 3 : E[k] = 1/8$ and so on
But this is a geometric distribution:

$$P(X = x) = q^{x-1}p$$

where $q = 1 - p$ With $E[k] = \sum_{i=1} i * pr(k = i) = \frac{1}{p} = 2$
Since the expected value of k while k is the number of loops is 2, then S is polynomial-time. [Jan: "...S runs in expected polynomial time".]

References

[1] J. L. Gross and J. Yellen, *Handbook of graph theory*. CRC press, 2003.

[2] V. Goyal and J. Ackerman, "Introduction to Cryptography,Lecture 19: Zero-Knowledge Proofs I," 2018. URL: `https://www.cs.cmu.edu/~goyal/s18/15503/scribe_notes/lecture21.pdf`. Last visited on 2020/05/13.

[3] O. Goldreich, *Foundations of cryptography: volume 1, basic tools*. Cambridge university press, 2007.

[4] L. Babai, "Graph isomorphism in quasipolynomial time," in *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pp. 684–697, 2016.