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Chebyshev Polynomials and Spectral Method for Optimal Control Problem

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Abstract

This paper presents efficient algorithms which are based on applying the idea of spectral method using the Chebyshev polynomials: including Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind and shifted Chebyshev polynomials of the first kind. New properties of Chebyshev polynomials are derived to facilitate the computations throughout this work. In addition the convergence criteria for the proposed algorithms are derived. The use of the three algorithms has been demonstrated with example.

Keywords: spectral method chebyshev polynomials quadratic optimal control QOC.

متعددات حدود شيبيشف وطريقة الطيف لمسألة السيطرة المُثلى

الخلاصة

هذا البحث يقدم خوارزميات كفوءة والتي استندت على تطبيق فكرة طريقة الطيف بأستخدام متعددات حدود شيبيشف: والمتضمنة : متعددات حدود شيبيشف من النوع الأول، متعددات حدود شبشف من النوع الثاني، متعددات حدود شيبيشف المزاحة من النوع الأول . أشتقت بعض الخواص الجديدة لمتعددات حدود شيبيشف لتسهيل الحسابات.
أضافه إلى ذلك، اشتقت صيغة الاقتراب للخوارزميات المقترحة، وأستخدام الخوارزميات الثلاثة وضحت بمثال.

1. Introduction

Optimal control theory arises from the consideration of physical systems, which are required to achieve a definite objective as cheap as possible .The translation of the design objectives into a mathematical model gives rise to control problem.

Optimal control is an important branch of mathematics and the applications for it abound in engineering, science and economics [1], [2], [3], and [7].

Optimal control is important in a large number of applications, and successful implementations have been reported in the literature. In particular the well known quadratic optimal control QOC problems have found wide acceptance.

The work throughout this paper is concerned with the QOC problems and is associated with finite time of minimizing a performance index subject to the linear control dynamics.

The LQOC problem can be stated as follows:
Find the OC that minimizes the quadratic performance index

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$$J = \int_{t_0}^{t_f} (x^T Qx + u^T Ru)\,dt$$

subject to the linear system state equations

satisfying the initial conditions

where $A(t) \in \Re^n \times \Re^n$, $B(t) \in \Re^n \times \Re^m$, $x \in \Re^n$, $u \in \Re^m$, Q is an $n \times n$ positive semi definite matrix , $x^T Qx \geq 0$, and R is a $m \times m$ positive definite matrix,

$u^T Ru > 0$ unless $u(t) = 0$.

There are a great number of papers that present approximate and numerical methods for finding optimal controls [4], [6],[8],[11] and [13].

In this work, three kinds of Chebyshev Polynomials will be used with the aid of the **spectral** method to find the approximate solutions for the linear optimal control problem.

2. Chebyshev Polynomials and Their Properties

There are several kinds of Chebyshev Polynomials. In particular we shall introduce the first and second kind polynomials $T_n(x)$ and $U_n(x)$, as well as the shifted polynomials $T_n^*(x)$.

2.1 The First Kind Chebyshev Polynomials $T_n(x)$ [11]

The Chebyshev Polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n , defined by the relation $T_n(x) = \cos n\theta \;\; when \;\; x = \cos\theta$, $x \in [-1,1]$, $\theta \in [0,\pi]$

The important Properties of $T_n(t)$ are: $x\& = Ax + Bu$

- The fundamental recurrence relation of Chebyshev polynomial can be obtained as follows
 $x(t_0) = x_0$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n = 2,3,\ldots$$

where

$T_0(x) = 1$, $T_1(x) = x$

- The Chebyshev product formula is

$$T_m(x)T_n(x) = \frac{1}{2}\left(T_{m+n}(x) + T_{|m-n|}(x)\right)$$

- The Chebyshev integral is

$$\int T_n(x)\,dx = \begin{cases} \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{|n-1|}(x)}{n-1}\right], & n \neq 1 \\ \frac{1}{4}T_2(x), & n = 1 \end{cases}$$

- The Chebyshev derivative is

$$\frac{d}{dx}T_n(x) = 2n \sum_{\substack{r=0 \\ n-r\ odd}}^{n-1} {}' T_r(x)$$

The differentiation operationl matrix of Chebyshev polynomials of the first kind D_T can be given as follows

$$D_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 5 & 0 & 10 & 0 & 10 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 12 & 0 & 12 & 0 & 12 & 0 & 0 & \cdots & 0 \\ 7 & 0 & 14 & 0 & 14 & 0 & 14 & 0 & \cdots & 0 \\ 0 & 16 & 0 & 16 & 0 & 16 & 0 & 16 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m & 0 & 2m & 0 & 2m & 0 & 2m & 0 & \cdots & 0 \end{bmatrix}$$

In the previous matrix it is assumed that n odd .However, if n is even then the last row of D_T becomes $\begin{bmatrix} 0 & 2m & 0 & 2m & 0 & 2m & \cdots & 0 \end{bmatrix}$

2.2 The Second Kind Chebyshev Polynomials $U_n(x)$ [5]

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad when \quad x = \cos\theta$$

The ranges of x and θ are the same as for $T_n(x)$.

The important properties of $U_n(x)$ are:

- The polynomial $U_n(x)$ satisfies the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad n = 2,3,\ldots$$

with the initial conditions $U_0(x) = 1, \quad U_1(x) = 2x$

- New interesting formula concerning the product of Chebyshev polynomials of the second kind has been derived and given by the following lemma

Lamma (1):

The product of Chebyshev polynomials of the second kind $U_n U_m$ is given by

$$U_n(t)U_m(t) = \sum_{i=0}^{n} U_{m+n-2i}(t)$$

Proof:

The mathematical induction is used to prove this lemma. In order to establish the validity of this lemma, the following steps are needed:

i) prove that the lemma is true for $n = 0, n = 1$ *i.e*

$$U_0 U_m = \frac{\sin\theta}{\sin\theta} \cdot \frac{\sin(m+1)\theta}{\sin\theta} = \frac{\sin(m+1)\theta}{\sin\theta} = U_m$$

$$U_1 U_m = \frac{\sin 2\theta}{\sin\theta} \cdot \frac{\sin(m+1)\theta}{\sin\theta}$$

$$= 2 \cdot \frac{1}{2} \cdot \frac{1}{\sin\theta} \cdot [\sin(m+2)\theta + \sin m\theta]$$

$$= \frac{\sin(m+2)\theta}{\sin\theta} + \frac{\sin m\theta}{\sin\theta}$$

$$= U_{m+1} + U_{m-1}$$

and

ii) for fixed $n-1$, assume that lemma (1) is true.

$$U_{n-1} U_m = \sum_{i=0}^{n-1} U_{m+n-1-2i}$$

Then prove that lemma (1) is true for n, with the aid of the following formula

$$U_n = 2xU_{n-1} - U_{n-2}$$

We obtain

$$U_n U_m = (2xU_{n-1} - U_{n-2})U_m$$
$$= 2xU_{n-1}U_m - U_{n-2}U_m$$

Hence

$$U_n U_m = 2x \sum_{i=0}^{n-1} U_{m+n-1-2i} - \sum_{i=0}^{n-2} U_{m+n-2-2i}$$

since $x = \cos\theta$, this yields the following result

$$U_n U_m = 2\cos\theta \sum_{i=0}^{n-1} \frac{\sin(m+n-2i)}{\sin\theta} - \sum_{i=0}^{n-2} \frac{\sin(m+n-1-2i)}{\sin\theta}$$

By expanding the above formula, we get:

$$U_n U_m = \left(2\cos\theta \cdot \frac{\sin\,(m+n)}{\sin\theta} - \frac{\sin\,(m+n-1)}{\sin\theta} \right) + \left(2\cos\theta \cdot \frac{\sin\,(m+n-2)}{\sin\theta} - \frac{\sin\,(m+n-3)}{\sin\theta} \right) + \dots$$
$$+ \left(2\cos\theta \cdot \frac{\sin\,(m-n+1)}{\sin\theta} - \frac{\sin\,(m-n+2}{\sin\theta} \right)$$

Then, simplyfing the result to get the required result.

• The differentiation operational matrix of Chebyshev polynomials of the second kind D_u can be given by

$$D_U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 0 & 6 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 8 & 0 & 0 & \cdots & 0 \\ 2 & 0 & 6 & 0 & 10 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 8 & 0 & 12 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 0 & 6 & 0 & 10 & 0 & \cdots & 2n \end{bmatrix}$$

In the previous matrix it is assumed that n is odd.
However, if n is even then the last row of D_U becomes

$$\begin{bmatrix} 0 & 4 & 0 & 8 & 0 & 12 & 0 & \cdots & 2n \end{bmatrix}$$

The integral of $U_n(x)$ is

$$\int U_n(x)\,dx = \frac{1}{n+1} \cdot T_{n+1}(x) + constant$$

2.3 The First Kind Shifted Chebyshev Polynomial $T_n^*(x)$ [5]

The shifted chebyshev polynomials $T_n^*(t)$ are defined in the interval $x \in [0,1]$ as

$$T_n^*(x) = T_n(t) = T_n(2x-1)$$

These polynomials can be generated by noting $T_0^*(x) = 1$, $T_1^*(x) = 2x-1$

The important properties of $T_n^*(x)$ is $T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x)$

- The recurrence relation
- The product of two shifted Chebyshev polynomials is

$$T_n^*(t) T_m^*(t) = \frac{1}{2}\left(T_{n+m}^*(t) + T_{|n-m|}^*(t) \right)$$

- The differentiation operation matrix of shifted Chebyshev polynomials of the first kind D_T^* is

$$D_{T^*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 6 & 0 & 12 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 16 & 0 & 16 & 0 & 0 & \cdots & 0 & 0 \\ 10 & 0 & 20 & 0 & 20 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 4(m-2) & 0 & 4(m-2) & 0 & 4(m-2) & \cdots & 0 & 0 \\ 2(m-1) & 0 & 4(m-1) & 0 & 4(m-1) & 0 & \cdots & 4(m-1) & 0 \end{bmatrix}$$

The matrix D_{T^*} is an $(m \times m)$ matrix. Note that the matrix is square exactly because the derivative of a polynomial is a polynomial of lower degree.

The matrix has a row of zeros because the derivative of a constant is zero.

3. Spectral Method for OCP

In this method the solution is assumed to be a finite linear combination of some sets of analytic basis functions. However, as the number of basis functions increases we might be able to get more accurate solution to QOC problems. The most important practical issue regarding such method is the choice, type, of the basis functions $\{\phi_i\}$. Throughout the work, in this chapter, the basis functions that will be used are: Chebyshev polynomials of the first kind $\{T_i\}$, Chebyshev polynomials of the second kind $\{U_i\}$ and shifted Chebyshev polynomials of the first kind $\{T^*_i\}$.

The spectral method for a finite LQOCP is described as follows:

- Write the necessary conditions to determine the optimal solution of the finite LQOCP

$$\dot{x} = Ax - \frac{1}{2}BR^{-1}B^T\lambda \quad ,$$

$$\dot{\lambda} = -2Qx - A^T\lambda \quad ,$$

$$u = -\frac{1}{2}R^{-1}B^T\lambda$$

with the initial conditions $x(0) = x_0$ and the final conditions $\lambda(t_f) = 0$.

- Choose a set of state and adjoint variables and then approximate them by using a finite length series ϕ_i.

$$x_j(t) \approx x_j^N(t) = \sum_{i=0}^{N} a_{ij}\phi_i(t),$$

$$\lambda_j(t) \approx \lambda_j^N(t) = \sum_{i=0}^{N} b_{ij}\phi_i(t)$$

$; j = 1,2,\ldots,q$

The remaining $2(n-q)$ state and adjoint variables are obtained from the system state and the system adjoint equations.

- Form the $q(2N \times 2N)$ system of linear algebric equations of the unknown parameters a_{ij} and b_{ij}; $i = 1,2,\ldots,N$, $j = 1,2,\ldots,q$, from the unused state and adjoint equations as well as from the initial and final conditions. That is the $q(2N \times 2N)$ system of equations can be formed from the equations:

$$\dot{x}_j(t) = Ax_i - \frac{1}{2}BR^{-1}B^T\lambda_i$$

$$\dot{\lambda}_j(t) = -2Qx_i - A^T\lambda_i,$$

$$j = q+1, q+2,\ldots,n; \quad i = 1,2,\ldots,n.$$

with the conditions $x_j(0) = x_0$ and $\lambda_j(t_f) = 0$; $j = 1,2,\ldots,n$

The approximations for the state variables $x_j^N(t)$ and the adjoint variables $\lambda_j^N(t)$; $j = 1,2,\ldots,n$, can be written in a matrix form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{01} & a_{11} & a_{21} & \cdots & a_{N1} \\ a_{02} & a_{12} & a_{22} & \cdots & a_{N2} \\ \vdots & \vdots & & & \vdots \\ a_{0n} & a_{1n} & a_{2n} & \cdots & a_{nN} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} b_{01} & b_{11} & b_{21} & \cdots & b_{N1} \\ b_{02} & b_{12} & b_{22} & \cdots & b_{N2} \\ \vdots & \vdots & & & \vdots \\ b_{0n} & b_{1n} & b_{2n} & \cdots & b_{nN} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

The two matrices can be written in the form

$$x = \alpha\phi \text{ and } \lambda = \beta\phi \qquad \ldots (2)$$

Differentiating the systems with respect to t to obtain

$$\dot{x} = \alpha\ \dot{\phi}\ ,$$
$$\dot{\lambda} = \beta\ \dot{\phi} \qquad \ldots (3)$$

where the matrix D_ϕ is the differentiation operational matrix of the basis functions ϕ .

Now, the approximations (2) and their derivatives (3) are inserted into eqns. (1) and equate the coefficient of the bases functions ϕ_i to yield

$$\alpha D_\phi \phi = A\alpha\phi - \frac{1}{2} B R^{-1} B^T \beta\phi \quad ,$$
$$\beta D_\phi \phi = -2Q\alpha\phi - A^T \beta\phi \qquad \ldots(4)$$

Both the initial conditions and the final conditions can also be expressed using the basis functions ϕ at $t = 0$ and $t = t_f$ respectively to obtain the equations

$$\sum_{i=0}^{N} a_{ij}\phi_i(0) = x_0 \qquad \sum_{i=0}^{N} b_{ij}\phi_i(t_f) = 0$$
$$; \quad j = 1,2,\ldots,n \qquad \ldots (5)$$

The resulting system which we obtain from equations (4),(5) can be solved by using Gauss elimination procedure, with pivoting, to find the unknown parameters a_{ij} and b_{ij}; $i = 1,2,\ldots,n$; $j = 1,2,\ldots,q$.

- Approximate the performance index

$$J^* = \int_0^{t_f} (\phi^T \alpha^T Q \alpha\phi + \phi^T \gamma^T R \gamma\phi) dt$$

where J^* is the approximate value of J.

Let $\alpha^T Q \alpha = M$ and $\gamma^T R \gamma = S$, then

$$J^* = \int_0^{t_f} (\phi^T M \phi + \phi^T S \phi) dt$$

Now, the numerical solution has been obtained by using three types of basis functions ϕ_i ; $i = 0,1,\ldots,N$.$(T_i(t), U_i(t), T_i^*(t))$

4. The Convergence Test for the proposed Algorithms:[5]

In the spectral method, the state and adjoint variables are expanded interms of a set of orthogonal functions (basis set) or at least linearly independed set,

$$x_i(t) = \sum_{k=1}^{\infty} a_{ik}\phi_k(t)$$
$$\lambda_i(t) = \sum_{k=1}^{\infty} b_{ik}\phi_k(t)$$
$$i = 1,2,\ldots,n$$

It is not possible to perform computations on an infinite number of terms, therefore; we must truncate the above series. That is we take

$$x_{iN}(t) = \sum_{k=1}^{N} a_{ik}\phi_k(t) \qquad \text{and}$$
$$\lambda_{iN}(t) = \sum_{k=1}^{N} b_{ik}\phi_k(t)$$

so that

$$x_i(t) = x_{iN}(t) + \sum_{k=N+1}^{\infty} a_{ik}\phi_k(t) = x_{iN}(t) + r_i(t)$$

we must select coefficients such that the norm of the residual function $\|r(t)\|$ is less than some convergence criterion ε , where $r(t) = \max(r_1(t), r_2(t), \ldots, r_N(t))$.

Now we will return to the question of how large N must be later. There is a convergence test that must be used with spectral method. It is to do with the number of terms kept in the basis set *N*. The most useful test of convergence in terms of *N* comes from examining the L^2 norm of x_i and λ_i, $i=1,2,\ldots,n$ (the state and adjoint variables that is approximated), i.e.,

$$\left[\int_a^b \left(x_i(t)-x_{iN}(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon_i$$

and

$$\left[\int_a^b \left(\lambda_i(t)-\lambda_{iN}(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon_i ,$$

Let $\varepsilon=\max(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$, therefore

$$\left[\int_a^b \left(x(t)-x_N(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$ and

$$\left[\int_a^b \left(x(t)-x_N(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

for all *N* greater than some value N_0. Since we do not know $x(t)$ and $\lambda(t)$, we replace the presumably better approximation $x_{N+M}(t)$ and $\lambda_{N+M}(t)$, where $M\ge 1$

$$\left[\int_a^b \left(x_{N+M}(t)-x_N(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$ and

$$\left[\int_a^b \left(x_{N+M}(t)-x_N(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

4.1 The Convergence Test for Spectral Method Using $T_n(t)$ [9]

If Chebyshev polynomials of the first kind are used to approximate both the state and adjoint variables, we will get

$$\lambda_i(t)=\lambda_{iN}(t)+\sum_{k=N+1}^{\infty} b_{ik}\phi_k(t)=\lambda_{iN}(t)+r_i(t)$$

$$\left[\int_{-1}^{1}\left(\sum_{i=0}^{N+M} a_iT_i(t)-\sum_{i=0}^{N} a_iT_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\left[\int_{-1}^{1}\left(\sum_{i=N+1}^{N+M} a_iT_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon \qquad i=1,2,\ldots,n$$

$$\Rightarrow$$

$$\left[\int_{-1}^{1}\left(\sum_{i=N+1}^{N+M} a_iT_i(t)\right)\left(\sum_{i=N+1}^{N+M} a_iT_i(t)\right)dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_ia_j\int_{-1}^{1} T_i(t)\,T_j(t)\,dt < \varepsilon$$

Since

$$T_i(t)T_j(t)=\frac{1}{2}\left(T_{i+j}(t)+T_{|i-j|}(t)\right)$$

Therefore,

$$\sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_ia_j\int_{-1}^{1}\frac{1}{2}\left(T_{i+j}(t)+T_{|i-j|}(t)\right)dt < \varepsilon$$

4.2 The Convergence Test for Spectral Method Using $U_n(t)$

If Chebyshev polynomials of the second kind are used to approximate both the state and adjoint variables, we will get

$$\left[\int_{-1}^{1}\left(\sum_{i=0}^{N+M} a_iU_i(t)-\sum_{i=0}^{N} a_iU_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\left[\int_{-1}^{1}\left(\sum_{i=N+1}^{N+M} a_iU_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\left[\int_{-1}^{1}\left(\sum_{i=N+1}^{N+M} a_i U_i(t)\right)\left(\sum_{i=N+1}^{N+M} a_i U_i(t)\right)dt\right]^{\frac{1}{2}} < \varepsilon$$

Therefore

$$\sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_i a_j \int_{-1}^{1} U_i(t)\, U_j(t)\, dt < \varepsilon$$

Since

$$U_i(t) U_j(t) = \sum_{k=0}^{i} U_{i+j-2k}(t)$$

Therefore,

$$\sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_i a_j \int_{-1}^{1}\sum_{k=0}^{i} U_{i+j-2k}(t)\;\; dt < \varepsilon$$

4.3 The Convergence Test for Spectral Method Using $T^*_n(t)$

If shifted Chebyshev polynomials are used to approximate both the state and adjoint variables, we will get

$$\left[\int_{0}^{1}\left(\sum_{i=0}^{N+M} a_i T^*_i(t) - \sum_{i=0}^{N} a_i T^*_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\left[\int_{0}^{1}\left(\sum_{i=N+1}^{N+M} a_i T^*_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\left[\int_{0}^{1}\left(\sum_{i=N+1}^{N+M} a_i T^*_i(t)\right)\left(\sum_{i=N+1}^{N+M} a_i T^*_i(t)\right)dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow$$

$$\sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_i a_j \int_{0}^{1} T^*_i(t)\, T^*_j(t)\, dt < \varepsilon$$

Since

$$T^*_i(t) T^*_j(t) = \frac{1}{2}\left(T^*_{i+j}(t) + T^*_{|i-j|}(t)\right)$$

Therefore,

$$\sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_i a_j \int_{0}^{1}\frac{1}{2}\left(T^*_{i+j}(t) + T^*_{|i-j|}(t)\right) dt < \varepsilon$$

Test Example

Example (1):

Consider the finite time quadratic problem

Minimize

$$J = \int_{0}^{1}\left(x^2 + u^2\right)dt$$

subject to $\dot{x} = u \;,\; x(0) = 1$

The exact solution to this problem is given by

$x(t) = \dfrac{\cosh(1-t)}{\cosh 1}$ and

$u(t) = -\dfrac{\sinh(1-t)}{\cosh 1}$,

while the exact value of the performance index is $J = 0.761594156$ .when we using $T_n(t)$ and $U_n(t)$, the time interval $t \in [0,1]$ of the optimal control problem is transformed into the interval $\tau \in [-1,1]$ using the transformation $\tau = 2t - 1$

This transforms the optimal control problem in example (1) into:

Minimize

$$J = \frac{1}{2}\int_{-1}^{1}\left(x^2 + u^2\right)d\tau$$

subject to $\dot{x} = \frac{1}{2}u \quad , \quad x(-1) = 1$

In order to apply the spectral method, one first finds:

- The Hamittonian: $H = \frac{1}{2}\left(x^2 + u^2\right) + \frac{1}{2}\lambda u$
- The adjoint equation: $\dot{\lambda} = -x$
- The sufficient condition for optimality:

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad u + \frac{1}{2}\lambda = 0$$

Therefore $u = -\frac{1}{2}\lambda$

- The final system is: $\dot{x} = -\frac{1}{4}\lambda$, $\dot{\lambda} = -x$

with the boundary conditions: $x(-1) = 1$, $\lambda(1) = 0$

In order to apply the spectral method, one first finds:

- The Hamittonian: $H = x^2 + u^2 + \lambda u$

The adjoint equation:

- Thesufficient condition for optimality:

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad 2u + \lambda = 0$$

Therefore $u = -\frac{1}{2}\lambda$

- The final system is: $\dot{x} = -\frac{1}{2}\lambda$, $\dot{\lambda} = -2x$

with the boundary conditions: $x(0) = 1$, $\lambda(1) = 0$

5. Discussion

The spectral methods have some advantages, some of these advantages are:

- The obtained solution using spectral methods can be implemented easy.
- In a simple way, equal number of equations and unknown parameters can be obtained, that is, square set of equations, so that, Gauss Eliminations, with pivoting, can be used to find the unknown parameters.
- An accurate approximation, using the above technique, depends on:

(i) The number of basis functions $\phi_i(t)$, i.e., as the order of the basis function increases, the approximate performance value will converge to the optimal value when the following stopping criterion is satisfied:

$$\left| J^*_{i=N} - J^*_{i=N+1} \right| < \varepsilon$$

where ε is a small prescribed value.

(ii) The type of the basis functions.

References

[1] AL-Janabi A. S., Identification of Control Systems, Ph.D. Thesis, EÖtvÖs Lorand University, Budapest, Hungary, 1983.
[2] Barnett S., Introduction to Mathematical Control Theory, Oxford University Press, 1975.

[3] Beeler S. C. & Tran H. T., Feedback Control Methodologies For Nonlinear Systems, Department of Mathematics, North Carolina State University, 1999.

[4] Bonnans J. F. , The Shooting Algorithm For Optimal Control Problems: A review of Some Theoretical And Numerical Aspects, can be found at: http://www.-rocq.inria.fr/~bonnans, Version of January, 2002.

[5] Butcher,E.A. & Bueler, E. , Stability of Linear Time-Periodic Delay-Differential Equations Via Chebyshev Polynomials, Int. J., Numer. Meth. Engng, 2004.

[6] Dontchev A. L. & Hager W., Second - Order Runge Kutta Approximations in Control Constrained Optimal Control, Society For Industrial And Applied Mathematics, 2000.

[7] Geering, H., P., Optimal Control with Engineering Application, Springer-Verlag Berlin Heidelberg, 2007.

[8] Hala, R. G., An Approximate Solution of Some Continuous Time Linear-Quadratic Optimal Control Problem Via Generalized Laguerre Polynomials, Msc Thesis , University of Technology, 2007.

[9] Jabbar, A, E, "Spectral method for continuous optimal control problems with chebyshev polynomials", Msc Thesis, University of Technology, 2008.

[10] Marcellan, F. , Orthogonal Polynomials and Special Functions: Computation and Applications, Springer-Verlag Berlin Heidelberg, 2006.

[11] Naevdal E., Numerical Optimal Control in Continuous Time Made Easy, Department of Economics and Social Sciences, Agriculture University of Norway, 2001.

[12] Navasca C. L. , Local Solutions of The Dynamic Programming Equations And the Hamilton Jacobi Bellman , PDE, Ph.D Thesis, University of California Davis, 2002.

[13] Oskar V. S. & Bulirsch R. , Direct And Indirect Methods For Trajectory Optimization, Annals of Operations Research, pp. 357-373 1992.

Tables (1), (3) shows the approximate values for J^* by applying the algorithms with $T_N(t)$, $U_N(t)$ and $T_N^*(t)$ for different values of N.

Table (1)

Approximate Values of J^* With $T_N(t)$

N	Using $T_N(t)$	$\lvert J_{exact} - J_{app.}\rvert$
3	0.8168638732	0.0552446
4	0.7615879193	0.0000062
5	0.7615938058	0.0000003
6	0.7615941507	0.0000000
7	0.7615941507	0.0000000

Table (2)

Approximat e Values of J^* With $U_N(t)$

N	Using $U_N(t)$	$\lvert J_{exact} - J_{app.}\rvert$
3	0.7619934561	0.0003993
4	0.7616027555	0.0000086
5	0.7616261081	0.0000032
6	0.7615941686	0.0000000
7	0.7615941686	0.0000000

Table (3)

Approximat e Values of J^* With $T_N^*(t)$

N	Using $T_N^*(t)$	$\lvert J_{exact} - J_{app.}\rvert$
3	0.7624202441	0.0008261
4	0.7615879192	0.0000062
5	0.7615937061	0.0000004
6	0.7615941663	0.0000000
7	0.7615941663	0.0000000