# Fractional Chebyshev Collocation Method For Solving Linear Fractional-Order Delay-Differential Equations

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# FRACTIONAL CHEBYSHEV COLLOCATION METHOD FOR SOLVING LINEAR FRACTIONAL-ORDER DELAY-DIFFERENTIAL EQUATIONS

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## ABSTRACT

An efficient numerical method, the fractional Chebyshev collocation method, is proposed for obtaining the solution of a system of linear fractional order delay-differential equations (FDDEs). It is shown that the proposed method overcomes several limitations of current numerical methods for solving linear FDDEs. For instance, the proposed method can be used for linear incommensurate order fractional differential equations and FDDEs, has spectral convergence (unlike finite differences), and does not require a canonical form. To accomplish this, a fractional differentiation matrix is derived at the Chebyshev-Gauss-Lobatto collocation points by using the discrete orthogonal relationship of the Chebyshev polynomials. Then, using two proposed discretization operators for matrix functions results in an explicit form of solution for a system of linear FDDEs with discrete delays. The advantages of using the fractional Chebyshev collocation method are demonstrated in two numerical examples in which the proposed method is compared with the Adams-Bashforth-Moulton method.

## INTRODUCTION

Recent studies have shown that fractional-order differential equations (FDEs) are superior mathematical tools to describe dynamical systems with memory or hereditary property. The list of FDE applications has grown rapidly in fields such as the study of creep or relaxation in viscoelastoplastic materials, diffusion process models, plasma physics, control problems etc. [1–5]. On the other hand, delay differential equations (DDEs) occupy a place of central importance in all areas of science with a multitude of practical applications. For instance, DDEs are used in time-delayed systems such as high-speed machining, lateral vibration of wheels of trucks and motorcycles, and control systems such as human and robotic balancing [6,7]. Fractional-order delay-differential equations (FDDEs) result from including delays in FDEs and are able to model some natural and engineered dynamical systems in a more natural and accurate way [8–16].

The use of efficient and reliable numerical methods for solving differential equations is important for a wide range of applications, such as engineering problems including model development, control, and simulation. Indeed, simulation of FDEs is computationally expensive due to the superficial mixture of integer order integral and derivative operators in fractional operators. Efforts have been conducted develop of numerically stable methods for solving FDEs, which also indicates the importance of this topic. [17–21]. We recall that numerical methods for solving ordinary differential equations include finite difference methods and spectral methods with their excellent

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error properties [22,23]. These methods have been also extended to solve FDEs. The common finite difference methods for solving FDEs in the sense of Caputo (C) are Podlubny's method and Adams-Bashforth-Moulton method [20, 21]. Podlubny's method describes a framework for solving linear FDEs by using the Grünwald-Letnikov (GL) definition [21]. This method is easy to implement but suffers from poor convergence especially for long-time integration. The Adams-Bashforth-Moulton method demonstrates a better convergence and accuracy compared to the aforementioned method, but it still has linear convergence properties [17,18].

On the other hand, spectral methods are superior in the sense of having nearly exponential convergence with smaller computation time compared to finite difference methods. Spectral collocation methods have been recently adapted to solve integer-order differential equations. In linear differential equations, they are easier to implement and the solution can be obtained in an explicit form [24]. A spectral collocation method is based on defining a differentiation collocation matrix, as linear transformations, at some collocation points such as the Chebyshev-Gauss-Lobatto (CGL) points. In [24–26], different differentiation collocation matrices have been used to accomplish discretized differentiation of continuous functions. Furthermore, fractional differentiation collocation matrices have been also defined to obtain fractional derivatives of a function [19, 21, 27–31]. All the previous studies are mainly limited to FDEs without delays. There are few numerical methods for solving FDDEs based on using the GL definition [9, 32], the Adams-Bashforth-Morton method [33, 34], the fractional backward difference method [32], and the Hermite wavelet method [35]. However, all these methods are still very arduous, time-consuming, and exhibit linear convergence.

The present paper is devoted to developing a numerical method, called the fractional Chebyshev collocation (FCC) method, for solving a system of linear FDDEs. This method is an extension of the authors' recent proposed technique that was implemented in [10–16] for control and stabilization of fractional periodic time-delay systems. The FCC method has spectral convergence and smaller computation time compared to previously proposed methods for solving FDEs and FDDEs, and can be applied to a system with commensurate or incommensurate fractional derivative orders. It can be also applied to a system of linear commensurate order FDEs. Then, the theory is extended for a system of linear FDDEs with initial function $\phi(t)$, $-\tau \leq t \leq 0$. Finally, two numerical examples are provided to demonstrate the validity and the efficiency of the FCC method in solving a system of linear commensurate order FDEs and a system of linear FDDEs. Furthermore, a comparison is made between the solution of the proposed method and that of the Adams-Bashforth-Moulton method.

The rest of paper is organized as follows. An overview of fractional-order derivative definitions is given in Section II. In Section III, first, a fractional Chebyshev differentiation matrix is derived at the CGL points by using the discrete orthogonality relationship for the Chebyshev polynomials. Then, discretized state transition matrices for a system of linear commensurate order FDEs and a system of linear FDDEs are given. This is followed by two illustrative examples in Section IV. Finally, the main results are summarized in Section V.

## PRELIMINARIES

**Definition 0.1.** *The Cauchy formula for repeated integration reduces the n-fold primitive integral of a continuous and integrable function $f(x)$ in $[a,x]$ to a linear Volterra equation of the first kind as [36]*

$$J^n f(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n)\, \mathrm{d}\sigma_n \cdots \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1$$
$$= \frac{1}{(n-1)!} \int_a^x (x-\zeta)^{n-1} f(\zeta)\, \mathrm{d}\zeta, \quad x > a, \tag{1}$$

*where $a \in \mathbb{R}$, $\zeta \in \mathbb{R}$, and the linear Volterra integral equation is a convolution equation with kernel $\frac{x^{n-1}}{(n-1)!}$.*

**Definition 0.2.** *The left side Riemann-Liouville (RL) fractional integral of order $\alpha, \alpha \in \mathbb{R}^+$, for a function $f(x)$ is defined by a generalization of n in Eq. (1) to the fractional number $\alpha$ as*

$${}_a\mathcal{J}_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(\zeta)\,(x-\zeta)^{\alpha-1} d\zeta, \quad x > a, \tag{2}$$

*in which $\Gamma(.)$ denotes the Gamma function.*

**Definition 0.3.** *The left side RL derivative follows by using the fractional-order RL integral and an integer derivative operator $\partial_x^m = \frac{d^m}{dx^m}$ as*

$${}_a^{RL}\mathcal{D}_x^{\alpha} f(x) = \partial_x^m\, {}_a\mathcal{J}_x^{m-\alpha} f(x), \quad x > a, \tag{3}$$

*where $m \in \mathbb{N}$ is the smallest integer greater than or equal to $\alpha$, i.e. $m = \lceil \alpha \rceil$ in which $\lceil . \rceil$ is the ceiling function.*

**Definition 0.4.** *Caputo [37] proposed a modified version of RL fractional-order derivative by switching the fractional integral* *operator and integer order differentiation operator as*

$$ {}_{a}^{C}\mathcal{D}_{x}^{\alpha} f(x) = {}_{a}\mathcal{J}_{x}^{m-\alpha}\, \partial_{x}^{m} f(x), \quad x > a. \tag{4} $$

**Definition 0.5.** *The left side GL derivative is defined as [1]*

$$ {}_{a}^{GL}\mathcal{D}_{x}^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{k=\frac{x-a}{h}} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} (-1)^{k} f(x-kh). \tag{5} $$

Equation (5) is the most common method to numerically approximate a fractional derivative and is used in this paper to obtain numerical solutions for FDEs.

## FRACTIONAL CHEBYSHEV COLLOCATION METHOD

In this section, the discretization FCC method is described to solve a system of linear commensurate FDEs and a system of linear FDDEs. First, fractional Chebyshev differentiation matrix is derived that linearly map discretized values of a function $x(t)$ at some collocation points to values of its fractional derivative or integral at those points. In other words, let $\mathbf{x}_{d}^{(\alpha)}$ be the vector of discretized values of the fractional derivative with order of $\alpha$ at those points. The fractional Chebyshev differentiation matrix in the sense of Caputo, ${}_{t_0}\mathrm{D}_{t_N}^{\alpha}$, is a linear map that maps the discretized function at the CGL points $\mathbf{t}_{d} = [t_0, t_1, \cdots, t_N]^T$ onto the discretized fractional derivative of the function at those points

$$ {}_{t_0}\mathrm{D}_{t_N}^{\alpha} \mathbf{x}_{d} = \left[ {}_{t_0}^{C}\mathcal{D}_{t}^{\alpha} x(t_0)\ {}_{t_0}^{C}\mathcal{D}_{t}^{\alpha} x(t_1)\ \cdots\ {}_{t_0}^{C}\mathcal{D}_{t}^{\alpha} x(t_N) \right]^{T}. \tag{6} $$

Then, it is shown that how a matrix function can be discretized in an interval at the CGL points by using two discretization operators. Finally, in two theories the solution of a system of linear commensurate order FDEs and a system of linear FDDEs are given by a state transition matrix.

### Fractional Chebyshev Differentiation Matrix

A derivative of a function approximated by finite differences is obtained by interpolation of a local $m$th-degree polynomial on $m$ local points of an equispaced grid. This results in a maximum accuracy $O(h^m)$. On the other hand, the main idea of spectral methods is the use of a global $N$th-degree polynomial that employs all $N$ points inside the domain and achieves maximum accuracy. If finite difference methods with equispaced collocation points are used, then the error of the interpolation increases when constructing a high degree polynomial interpolant. It is shown that interpolation at the CGL points gives the minimum error out of all sets of polynomial bases, so the Chebyshev polynomials are ideal to be used in spectral methods [23].

Fig. 1 illustrates the CGL points given by extreme points of the Chebyshev polynomials. Moreover, the Chebyshev polynomials of the first kind $T_n(t)$ are defined as

$$ T_n(t) = \cos(n \arccos(t)), |t| \le 1 \tag{7} $$

Thus, the CGL points are obtained by $t_j = \cos(j\pi/N)$, $j = 0, 1, \cdots, N$ in $[-1, 1]$. They are non-equispaced points and distributed more densely towards the edges of the interval. They can be interpreted as the projections of equispaced points on the upper half of the unit circle. Consider

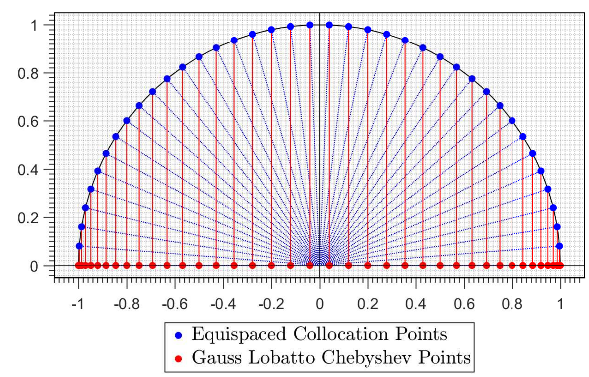


**FIGURE 1**. THE CHEBYSHEV-GAUSS-LOBATTO POINTS (RED POINTS) ARE DENSER TOWARDS THE EDGES OF THE INTERVAL COMPARED TO THE EQUISPACED POINTS (BLUE POINTS).

function $x(t)$ that is continuous on the real interval $[-1, 1]$, which can be approximated by using the Chebyshev polynomials as

$$ x(t) \approx \sum_{k=0}^{N} \hat{x}_k T_k(t), \tag{8} $$

where $\hat{x}_k \in \mathbb{R}$, $k = 0, 1, \cdots, N$, are called Chebyshev coefficients and can be obtained by the discrete orthogonality relationship for the Chebyshev polynomials as

$$\sum_{k=0}^{N} \frac{1}{c_k} T_i(t_k) T_j(t_k) = \frac{c_i}{2} \delta_{ij} N, \quad c_i = \begin{cases} 2, & i = 0 \text{ or } N, \\ 1, & i \neq 0 \text{ or } N, \end{cases} \tag{9}$$

in which $\delta_{ij}$ denotes the Kronecker delta. Using Eq. (9) and Eq. (8) yields

$$\hat{x}_k = \frac{2}{N c_k} \sum_{i=0}^{N} \frac{1}{c_i} T_k(t_i) x(t_i), \quad k = 0, 1, \ldots, N. \tag{10}$$

The Chebyshev coefficients can be obtained by a $N+1 \times N+1$ matrix $H$ defined as

$$H = \frac{1}{N} \begin{bmatrix} \frac{1}{2}(-1)^0 & 1 & \cdots & 1 & \frac{1}{2} \\ (-1)^1 & 2T_1(t_1) & \cdots & 2T_1(t_{N-1}) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{N-1} & 2T_{N-1}(t_1) & \cdots & 2T_{N-1}(t_{N-1}) & 1 \\ \frac{1}{2}(-1)^N & T_N(t_1) & \cdots & T_N(t_{N-1}) & \frac{1}{2} \end{bmatrix}. \tag{11}$$

Let $\hat{\mathbf{x}}$ and $\mathbf{x}_d$ are the Chebyshev coefficient vector and discretized values of $x(t)$ at the CGL points, respectively, then $\hat{\mathbf{x}} = H\mathbf{x}_d$. Thus, the approximation in Eq. (8) can be rewritten in vector form by substituting Eq. (11) in Eq. (8) as

$$x(t) \approx \mathbf{T}^T(t) H \mathbf{x}_d, \tag{12}$$

where $\mathbf{T}(t) = [T_0(t), \cdots, T_N(t)]^T$.

The fractional derivative operators are linear and hence the fractional differentiation collocation matrix in the sense of Caputo is given by

$${}_0\mathrm{D}_N^{\alpha} = \begin{bmatrix} {}_0^C\mathcal{D}_t^{\alpha} T_0(t)]_{t_0} & {}_0^C\mathcal{D}_t^{\alpha} T_1(t)]_{t_0} & \cdots & {}_0^C\mathcal{D}_t^{\alpha} T_N(t)]_{t_0} \\ {}_0^C\mathcal{D}_t^{\alpha} T_0(t)]_{t_1} & {}_0^C\mathcal{D}_t^{\alpha} T_1(t)]_{t_1} & \cdots & {}_0^C\mathcal{D}_t^{\alpha} T_N(t)]_{t_1} \\ \vdots & \vdots & \vdots & \vdots \\ {}_0^C\mathcal{D}_t^{\alpha} T_0(t)]_{t_N} & {}_0^C\mathcal{D}_t^{\alpha} T_1(t)]_{t_N} & \cdots & {}_0^C\mathcal{D}_t^{\alpha} T_N(t)]_{t_N} \end{bmatrix} H. \tag{13}$$

It is shown recently that the fractional Chebyshev differentiation matrix in Eq. (13) can be also obtained by using a modification on the well-known Chebyshev differentiation matrix [38,39]. The Chebyshev differentiation matrix ${}_{t_0}\mathrm{D}_{t_N}$ in $[-1,1]$ is defined as [22,23]

$$\begin{aligned} [{}_{t_0}\mathrm{D}_{t_N}]_{00} &= -[{}_{t_0}\mathrm{D}_{t_N}]_{NN} = \frac{2N^2+1}{6}, \\ [{}_{t_0}\mathrm{D}_{t_N}]_{jj} &= \frac{-t_j}{2\left(2-t_j^2\right)}, \quad j = 1, 2, \cdots, N-1, \\ [{}_{t_0}\mathrm{D}_{t_N}]_{ij} &= \frac{c_i}{c_j} \frac{(-1)^{i+j}}{\left(t_i - t_j\right)}, \quad i \neq j, \quad i, j = 1, 2, \cdots, N-1, \end{aligned} \tag{14}$$

where

$$c_i = \begin{cases} 2, & i = 0 \text{ or } N, \\ 1, & i \neq 0 \text{ or } N. \end{cases} \tag{15}$$

Let us define two types of operators ($\mathbf{P}$ and $\bar{\mathbf{P}}$) for discretization of any $n \times m$ matrix function $P(t)$ at collocation points $\mathbf{t}_d = [t_0, t_1, \cdots, t_{N-1}]^T$ as

$$\mathbb{P}(Q) = \begin{bmatrix} Q\,\mathrm{diag}\left(p_{11}(\mathbf{t}_d)\right) & \cdots & Q\,\mathrm{diag}\left(p_{1m}(\mathbf{t}_d)\right) \\ \vdots & \ddots & \vdots \\ Q\,\mathrm{diag}\left(p_{n1}(\mathbf{t}_d)\right) & \cdots & Q\,\mathrm{diag}\left(p_{nm}(\mathbf{t}_d)\right) \end{bmatrix}_{nN \times mN}, \tag{16}$$

where $Q$ is a $N \times N$ matrix, $\mathrm{diag}(v)$ returns a square diagonal matrix with the elements of vector $v$ on the main diagonal, and $p_{ij}$ are the elements of $P$. According to Eq. (16), $\mathbf{P}$ and $\bar{\mathbf{P}}$ are also defined as

$$\mathbf{P} = \mathbb{P}(I_N), \tag{17a}$$
$$\bar{\mathbf{P}} = \mathbb{P}(\bar{I}_N), \tag{17b}$$

where $I_N$ and $\bar{I}_N$ are defined as the $N \times N$ identity matrix and $N \times N$ identity matrix that its first row replaced by a zero row of the same size. For instance, the $(i,j)$th entry of $\bar{\mathbf{P}}$ is

$$(\bar{\mathbf{P}})_{ij} = \bar{I}_N \mathrm{diag}\left(p_{ij}(\mathbf{t}_d)\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & p_{ij}(t_1) & 0 & 0 & 0 \\ 0 & 0 & p_{ij}(t_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & p_{ij}(t_N) \end{pmatrix}. \tag{18}$$

## Linear Commensurate Order Fractional Differential Equations

Consider the following linear commensurate order FDE

$$\sum_{i=0}^{r} a_i\, y^{(\gamma_i)}(t) = \sum_{i=0}^{m} b_i\, u^{(\beta_i)}(t), \tag{19}$$

where $0 \le \gamma_i \le 1, i = 0,1,\cdots,r, 0 \le \beta_i \le 1, i = 0,1,\cdots,m, a_i$, $i = 0,1,\cdots,n$ and $b_i, i = 0,1,\cdots,m$ are constants. Its transfer function can be obtained by the Laplace transform of the both sides of Eq. (19) for zero initial conditions as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i\, s^{\beta_i}}{\sum_{i=0}^{r} a_i\, s^{\gamma_i}}. \tag{20}$$

The transfer function (20) is called *commensurate* if $\gamma_i \in \mathbb{Q}$, $i = 0,1,\cdots,r$, $\beta_i \in \mathbb{Q}, i = 0,1,\cdots,m$. Then, the transfer function (20) can be written in a commensurate form as

$$G_\alpha(s^\alpha) = \frac{Y(s^\alpha)}{U(s^\alpha)} = \frac{\sum_{i=0}^{m} b_i\, (s^\alpha)^{f_i}}{\sum_{i=0}^{r} a_i\, (s^\alpha)^{g_i}}, \tag{21}$$

where $f_i, g_i \in \mathbb{N}$, $i = 0,1,\cdots,r$, $\alpha = 1/q$ denotes the commensurate order in which $q \in \mathbb{N}$ is the least common denominator of $\alpha_i$ and $\beta_i$ and $\alpha$. If a fractional-order FDE is not commensurate order, then it is called an *incommensurate* order FDE. One can show that any incommensurate order FDE can be approximated by a commensurate order FDE if $\alpha$ is chosen small enough.

The canonical state space realization of the commensurate order FDE Eq. (21) is

$$\begin{aligned} {}_{0}^{C}\mathcal{D}_t^{(\boldsymbol{\alpha})} x &= A\, x + B\, u, \\ y &= C\, x + D\, u, \end{aligned} \tag{22}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $u \in \mathbb{R}^{m\times n}$, and

$${}_{0}^{C}\mathcal{D}_t^{(\boldsymbol{\alpha})} x(t) := \left[{}_{0}^{C}\mathcal{D}_t^{\alpha_1} x_1(t), {}_{0}^{C}\mathcal{D}_t^{\alpha_2} x_2(t), \cdots, {}_{0}^{C}\mathcal{D}_t^{\alpha_n} x_n(t)\right]^T, \tag{23}$$

in which $\boldsymbol{\alpha} = \{0 < \alpha_i \le 1, i = 1,2,\cdots,n\}$, i.e. they can be chosen irrational or incommensurate orders.

**Theorem 0.1.** *The solution of the commensurate order FDE (22) in $[0,\tau]$ is given by*

$$\mathbf{X}_{\tau_d} = {}_{0}^{C}\mathbf{T}_{\tau_d}^{\alpha} \mathbf{X}_{0_d}, \tag{24}$$

*where finite dimensional vector $\mathbf{X}_{\tau_d}$ represents the discretized $x(t)$ in $[0,\tau]$ at the CGL points and finite dimensional vector $\mathbf{X}_{0_d} = [\mathbf{X}_{0_d}^{(1)T}, \cdots, \mathbf{X}_{1_d}^{(n)T}]$ in which $\mathbf{X}_{1_d}^{(i)} = [0,\cdots,x_i(0)]^T$, $i = 1,\cdots,n$, and ${}_{0}^{C}\mathbf{T}_{\tau_d}^{\alpha}$ is called state transition matrix defined as*

$${}_{0}^{C}\mathbf{T}_{\tau_d}^{\alpha} = \left(\bar{\Delta}^{\bar{\alpha}} - \bar{\mathbf{A}}_d + I_n \otimes J\right)^{-1} \left(\bar{\mathbf{B}}_d + I_n \otimes J_d\right), \tag{25}$$

*where $\bar{\Delta}^{\bar{\alpha}} = I_n \otimes \bar{I}_N\, {}_0D_\tau^{\alpha}$, $J$ and $J_d$ are $N \times N$ zero matrices with their first rows replaced by $[1,0,0,\cdots,0,0]$ and $[0,0,\cdots,0,0,1]$, respectively, $\mathbf{A}_d$ and $\mathbf{B}_d$ are the $nN \times nN$ discretized matrices associated with $A$ and $B$ in $t \in [0,\tau]$, and $\otimes$ is the Kronecker operator.*

*Proof.* First, substituting the state transition matrix (25) into Eq. (22) results in

$$\left(\bar{\Delta}^{\bar{\alpha}} + I_n \otimes J\right)\mathbf{X}_{\tau_d} = \bar{\mathbf{A}}_d \mathbf{X}_{\tau_d} + \left(\bar{\mathbf{B}}_d + I_n \otimes J_d\right)\mathbf{X}_{0_d}, \tag{26}$$

which is the discretized form of Eq. (22) in $[0,\tau]$.

The solution is continuous at $t = 0$ because all the elements of the first row of $a_{ij}\bar{I}_N$, $b_{ij}\bar{I}_N$, and $\bar{I}_{N0}\mathrm{D}^{\alpha_i}, i = 1,\cdots,n$ are zero; and $I_n \otimes J$ and $I_n \otimes J_d$ are modified such that the first element of $\mathbf{X}_{\tau_d}^{(i)}$, $i = 1,2,\cdots,n$ is equal to the last element of $\mathbf{X}_{0_d}^{(i)}$, $i = 1,2,\cdots,n$. Thus, this completes the proof of Theorem 0.1.

## Linear Fractional-Order Delay-Differential Equations

Consider the following linear FDDEs with initial function $\phi(t)$, $-\tau \le t < 0$,

$$\begin{aligned} \dot{x}(t) = A(t)\; x(t) + A_\alpha(t)\; {}_{0}^{C}\mathcal{D}_t^{(\alpha_1,\alpha_2,\cdots,\alpha_n)} x(t) + \\ B(t)\; {}_{0}^{C}\mathcal{D}_t^{(\beta_1,\beta_2,\cdots,\beta_n)} x(t-\tau), \end{aligned} \tag{27}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the system, $A(t)$, $A_\alpha(t)$ and $B(t)$ are the $n \times n$ time function matrices of the coefficients of the current and delayed states, respectively, $\tau \in \mathbb{R}^+$ is the time delay, $\phi(t)$ is the initial vector function in the interval of $[-\tau, 0]$, and $0 \le \alpha_i, \beta_i < 1$, $i = 1,2,\cdots,n$, are the fractional derivative orders.

**Theorem 0.2.** *The solution of FDDE (27) in $[0,\tau]$ is given by*

$$\mathbf{X}_{\tau_d} = {}_0^C\mathbf{T}^{\alpha}_{\tau_d}\mathbf{X}_{0_d}, \tag{28}$$

*where finite dimensional vectors $\mathbf{Y}_0$ and $\mathbf{X}_{\tau_d}$ represent the discretized $x(t)$ in $[0,\tau]$ and $\phi(t)$ in $[-\tau,0]$ at the CGL collocation points, respectively, and ${}_0^C\mathbf{T}^{\alpha}_{1_d}$ is named state transition matrix defined as*

$${}_0^C\mathbf{T}^{\alpha}_{\tau_d} = \left(\bar{\Delta} - \bar{\mathbf{A}}_d - \bar{\mathbf{A}}_{\mathbf{ff}_d}\bar{\Delta}^{\bar{\alpha}} + I_n \otimes J\right)^{-1}\left(\bar{\mathbf{B}}_d\bar{\Delta}^{\bar{\beta}} + I_n \otimes J_d\right), \tag{29}$$

*where $\mathbf{A}_d$, $\mathbf{A}_{\mathbf{ff}_d}$ and $\mathbf{B}_d$ are the $nN \times nN$ discretized matrices associated with $A(t)$, $A_\alpha(t)$ and $B(t)$ in $t \in [0,\tau]$.*

*Proof.* First, substituting the state transition matrix (29) into Eq. (27) results in

$$\begin{aligned}\left(\bar{\Delta} + I_n \otimes J\right)\mathbf{X}_{\tau_d} = \left(\bar{\mathbf{A}}_d + \bar{\mathbf{A}}_{\mathbf{ff}_d}\bar{\Delta}^{\bar{\alpha}}\right)\mathbf{X}_{\tau_d} + \\ \left(\bar{\mathbf{B}}_d\bar{\Delta}^{\bar{\beta}} + I_n \otimes J_d\right)\mathbf{Y}_{0_d}.\end{aligned} \tag{30}$$

Let $\mathbf{Y}^{(i)}_{0_d}$ and $\mathbf{X}^{(i)}_{\tau_d}$ represent the discretized initial and state components $\phi_i(t)$ and $x_i(t)$ in any interval of length $\tau$, respectively. The first element of $\mathbf{X}^{(i)}_{\tau_d}$, $i = 1,2,\cdots,n$ is equal to the last element of $\mathbf{Y}^{(i)}_{0_d}$, $i = 1,2,\cdots,n$ because of the form of $I_n \otimes J$ and $I_n \otimes J_d$, and the fact that all the elements of the first row of $\bar{I}_N\mathrm{diag}\left(a_{ij}(\mathbf{t}_d)\right)$,$\bar{I}_N\mathrm{diag}\left(a_{\alpha ij}(\mathbf{t}_d)\right)$, $\bar{I}_N\mathrm{diag}\left(b_{ij}(\mathbf{t}_d)\right)$, $\bar{I}_{N0}\mathrm{D}^1$, $\bar{I}_{N0}\mathrm{D}^{\alpha_i}$, $i = 1,\cdots,n$, and $\bar{I}_{N0}\mathrm{D}^{\beta_i}$, $i = 1,\cdots,n$ are zero. On the other hand, Eq. (30) represents the discretized form of Eq. (27), and thus this completes the proof of Theorem 0.2.

## NUMERICAL SIMULATIONS

In this section, the advantages of the FCC method are shown in two examples. The first example demonstrates how the FCC method can be implemented for the well-known canonical form of linear commensurate order FDEs. In the second example as a system of linear FDDEs, it is shown that the FCC method handles incommensurate order FDEs in non-canonical form as well as FDDEs with delays and time-varying coefficients.

**Example 1.** *Consider the following commensurate order FDEs*

$${}_0^C\mathcal{D}_t^{(\alpha)}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} x \tag{31}$$

*with initial condition $x_0 = [10,0,-10]^T$. The Adams-Bashforth-Moulton method proposed in [18] can be used to verify the results obtained by the FCC method since linear FDE (31) has commensurate order.*

*Fig. 2 shows the numerical solutions of Eq. (31) obtained for different values of $\alpha$ using the Adams-Bashforth-Moulton method and the FCC method. It is shown that the numerical solution obtained using the FCC method are in agreement with those obtained using the Adams-Bashforth-Moulton method. Moreover, the number of the CGL collocation points is $N = 20$, and hence the dimensions of $\mathbf{A}_d$ defined by Eq. (17) is 60×60.*

**Example 2.** *Consider the following system of linear FDDEs*

$$\begin{aligned}&\ddot{x}(t) + (50 + 5\cos(\Omega t))\,x(t) + \\ &0.5\,x(t-\tau) + {}_0^C\mathcal{D}_t^{\alpha}x(t) = 0\end{aligned} \tag{32}$$

*with the initial function $x(t) = [1,1,0]^T$, $-\tau \le t < 0$. Let us choose $\Omega = 2$ rad/s and $\tau = T = \pi$ s.*

*The state space realization of Eq. (32) in the form of Eq. (27) is*

$$\begin{aligned}{}_0^C\mathcal{D}_t^{(1,1)}x(t) = \begin{bmatrix} 0 & 1 \\ -(50 + 5\cos(\Omega t)) & 0 \end{bmatrix} x(t) + \\ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} {}_0^C\mathcal{D}_t^{(0,\alpha)}x(t) + \begin{bmatrix} 0 & 0 \\ -0.5 & 0 \end{bmatrix} x(t-\tau).\end{aligned} \tag{33}$$

*It is noticed that the system of linear FDDEs Eq. (33) is incommensurate and has a single delay. It can be shown the solution of Eq. (32) asymptotically converges to zero for any initial function $\phi(t)$ [11]. Using 40 CGL collocation points results in discretizing $A(t)$ and $A_\alpha(t)$ as 80×80 matrices $\mathbf{A}_d$ and $\mathbf{A}_{\alpha_d}$ defined by Eq. (17). Fig. 3 shows the numerical solutions for Eq. (32) obtained for different values of $\alpha$ using the FCC method.*

## CONCLUSIONS

In this paper, a new method was proposed to obtain the numerical solution of a system of commensurate order FDEs or a system of linear fractional-order delay differential equations. This method is based on discretizing the solution at the CGL points and defining a state-transition matrix in the interval of the solution. Finally, the proposed method was employed on a system of commensurate order or incommensurate order FDDEs. In the first example, the results were verified by using the Adams-Bashforth method.

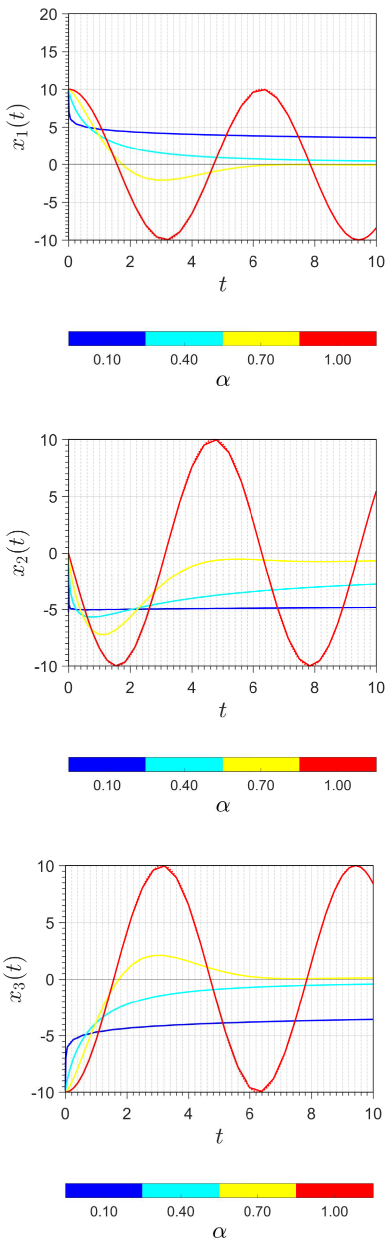


**FIGURE 2**. THE NUMERICAL SOLUTION OF EXAMPLE 1 USING THE FCC METHOD (SOLID LINES) AND ADAMS-BASHFORTH-MOULTON METHOD (DASHED LINES - INDISTINGUISHABLE FROM SOLID LINES).

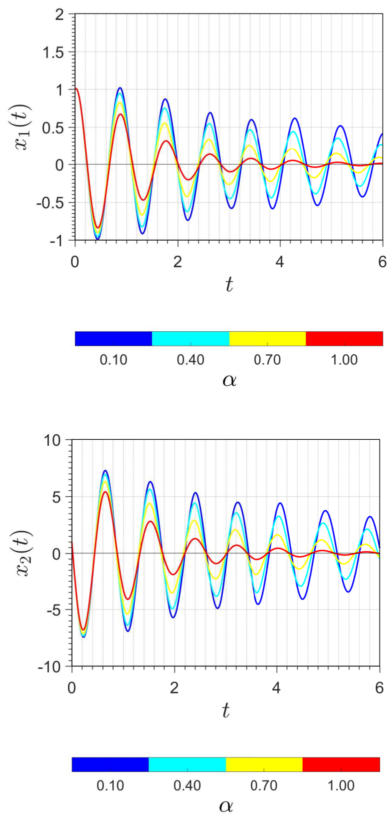


**FIGURE 3**. THE NUMERICAL SOLUTION OF (32) USING THE FCC METHOD FOR DIFFERENT VALUES OF $\alpha$.

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