OPTIMAL CONTROL APPLICATIONS & METHODS
Optim. Control Appl. Meth., **20**, 21–42 (1999)

COMPUTATION OF OPTIMAL CONTROL TRAJECTORIES USING CHEBYSHEV POLYNOMIALS: PARAMETERIZATION, AND QUADRATIC PROGRAMMING

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SUMMARY

An algorithm is proposed to solve the optimal control problem for linear and nonlinear systems with quadratic performance index. The method is based on parameterizing the state variables by Chebyshev series. The control variables are obtained from the system state equations as a function of the approximated state variables. In this method, there is no need to integrate the system state equations, and the performance index is evaluated by an algorithm which is also proposed in this paper. This converts the optimal control problem into a small size parameter optimization problem which is quadratic in the unknown parameters, therefore the optimal value of these parameters can be obtained by using quadratic programming results. Some numerical examples are presented to show the usefulness of the proposed algorithm. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: nonlinear optimal control; state parameterization; Chebyshev polynomials; quadratic programming

1. INTRODUCTION

It is well known that obtaining an exact analytical solution of general non-linear optimal control problem is out of reach. However, there are many methods which enable us to find either a numerical solution, such as gradient method, quasilinearization method, neighbouring extremal methods, etc., or approximate analytical solution in particular those which are based on the power series expansion.[1–3] One of the attractive numerical methods is the parameterization method.[4] This method transforms the optimal control problem into parameter optimization problem which is easier than the original optimal control problem.

The parameterization method is based on either parameterization of the control variables,[4,5] parameterization of both the state variables and the control variables[6,7] or parameterization of the state variables.[8–10] The use of the state parameterization was restricted to special cases, for example linear systems with equal number of state variables and control variables[9] or non-linear systems in phase-variable form.[10] Moreover, there was no details concerning which state variables to parameterize or how to treat problems in which the state equations are not in phase-variables form.

In this paper we clarify the state parameterization technique using Chebyshev polynomials and use it to solve general optimal control problems. On the other hand, Vlassenbroeck and Van

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CCC 0143–2087/99/010021–22$17.50

Received 20 November 1996
Revised 26 August 1998

Dooren[6] have also proposed to approximate the state variables and the control variables by Chebyshev series to transform the problem into non-linear programming problem. However, in this approach, the linear (non-linear) system state equations are replaced by a large number of linear (non-linear) equality constraints which arise from the new problem formulation. Therefore, the solution of the optimal control problem requires solution of a large number of linear (non-linear) algebraic equations.

In this paper, we propose a method to find the optimal open-loop solutions of both the linear quadratic and the non-linear optimal control problems by approximating the system state variables by Chebyshev series. The control variables are found from the state equations as a function of the approximated state variables. To handle the non-linear systems, we use the quasilinearization method. Therefore, after approximating the performance index, the optimal control problem becomes a small-size parameter optimization problem which is quadratic in the unknown parameters with a few linear equality constraints. The new problem can be solved by using the quadratic programming results. In the case of linear quadratic optimal control problem, the optimal solution is obtained in one step, and in the non-linear optimal control problem case we need to solve linear quadratic optimal control problems successively until an acceptable convergence error is achieved. We will consider the class of non-linear systems which are non-linear in state and linear in control.

2. LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

In this section, we propose an algorithm to find the optimal open-loop control of the linear quadratic optimal control problem, and in the next section, these results will be generalized to solve the non-linear optimal control problem.

Consider the time-invariant linear quadratic optimal control problem: find the optimal control $\mathbf{u}^*(t)$ that minimizes the quadratic performance index

$$J = \int_0^{t_f} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u})\, dt \tag{1}$$

subject to the system state equations and initial conditions given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $m \leqslant n$, $\mathbf{Q}$ and $\mathbf{R}$ are positive-semidefinite and positive-definite matrices, respectively.

The first step in the proposed method is to transform the time in the optimal control problem into the interval $\tau \in [-1, 1]$; therefore $\tau = -1$ will be the initial time for the new problem. This time transformation can be achieved by using

$$\tau = \frac{2t}{t_f} - 1 \tag{3}$$

This transforms (1) and (2) into

$$J = \frac{t_f}{2} \int_{-1}^{1} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u})\, d\tau \tag{4}$$

and

$$\frac{d\mathbf{x}}{d\tau} = \frac{t_f}{2} (\mathbf{A}\mathbf{x}(\tau) + \mathbf{B}\mathbf{u}(\tau)), \quad \mathbf{x}(-1) = \mathbf{x}_0 \tag{5}$$

To reformulate the problem (4)–(5) using the state parameterization via Chebyshev polynomials, we distinguish two cases:

1. The number of the state variables is equal to the number of the control variables i.e. $n = m$. In this case, each state variable is approximated by a finite length Chebyshev series

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau), \quad j = 1, 2, \ldots, \mathrm{n} \tag{6}$$

The derivative of this series[11] with respect to τ is given by

$$\dot{x}_j(\tau) = \frac{\acute{a}_0^{(j)}}{2} + \sum_{i=1}^{N-1} \acute{a}_i^{(j)} T_i(\tau), \quad j = 1, 2, \ldots, n \tag{7}$$

where

$$\acute{a}_{N-1} = 2Na_N \tag{8}$$

$$\acute{a}_{N-2} = 2(N-1)a_{N-1} \tag{9}$$

$$\acute{a}_{r-1} = \acute{a}_{r+1} + 2ra_r, \quad r = 1, 2, \ldots, N-2 \tag{10}$$

The control vector can be obtained as a function of the parameters of the approximated state variables as follows, assuming that $\mathbf{B}$ is non-singular,

$$\mathbf{u}(\tau) = \mathbf{B}^{-1}\left[\frac{2}{t_{\mathrm{f}}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} - \mathbf{A}\mathbf{x}(\tau)\right] \tag{11}$$

By substituting (6) and (7) into (11), the control variables can be expressed in series form as

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i(\tau), \quad l = 1, 2, \ldots, m = n \tag{12}$$

where $b_0^{(l)}, b_1^{(l)}, b_2^{(l)}, \ldots, b_N^{(l)}$ are functions of $a_0^{(j)}, a_1^{(j)}, a_2^{(j)}, \ldots, a_N^{(j)}$. This is because the right hand side of equation (11) is a function of $a_0^{(j)}, a_1^{(j)}, a_2^{(j)}, \ldots, a_N^{(j)}$ only.

2. The number of the control variables is less than the number of the state variables $m < n$. In this case, there is no need to approximate all the state variables. Because if all the state variables are approximated, then the state equations are replaced by a large number of equality constraints. Therefore, in this case, we choose and directly approximate a set of the state variables which will enable us to find the remaining state variables and the control variables as a function of this set. Assume that this set is $x_1, x_2, \ldots, x_q$ and $q < n$, then this set can be approximated by

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau), \quad j = 1, 2, \ldots, q \tag{13}$$

and the remaining $n - q$ state variables and the control variables are obtained from the system equations to obtain

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)}\ T_i(\tau), \quad j = q+1, q+2, \ldots, n \tag{14}$$

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i(\tau), \quad l = 1, 2, \ldots, m \tag{15}$$

where $a_0^{(j)}, a_1^{(j)}, \ldots, a_N^{(j)}$, $j = q+1, \ldots, n$ and $b_0^{(l)}, b_1^{(l)}, \ldots, b_N^{(l)}$, $l = 1, 2, \ldots, m$, are functions of the parameters $a_0^{(j)}, a_1^{(j)}, \ldots, a_N^{(j)}$, $j = 1, 2, \ldots, q$. The advantage of not approximating all the state variables is that the optimal control problem is reduced to a quadratic programming problem with few unknown parameters and few linear quality constraints. For the special case of a single-input–single-output systems expressed in phase-variable form, we need to approximate only one state variable and all other state variables and the control variable can be found as a function of the approximated state.

Remark 1

The set of q state variables to be selected and directly approximated is not unique. We can choose different sets, each of them can give us the remaining state variables and the control variables as in the following example:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = f(x_1, x_2, x_3) + u$$

In this example we can choose and directly approximate any one of the state variables, while all other states and control can be obtained as a function of this directly approximated state. In other words, there are three possible cases: The first case, if x_3 is directly approximated, then x_2 can be obtained by integrating the second equation, x_1 can be obtained by integrating the first equation and u can be found from the last equation. The second case, if x_2 is directly approximated, then x_1 can be obtained from the first equation by integration and x_3 can be obtained from the second equation by differentiation of x_2. The last case, if x_1 is approximated, then x_2 and x_3 can be obtained from the first and the second equations, respectively, by successive differentiation.

To limit the number of the state variables that can be directly approximated, we suggest to select the set of the state variables that enable us to express the remaining state variables and control variables as a function of this set by differentiation. Hence, in the previous example, x_1 should be directly approximated, while x_2, x_3 and u can be obtained as a function of x_1 by successive differentiation.

There are three reasons for our suggestion to employ the Chebyshev series differentiation and not the integration. The first reason is that in integration the length of the series will increase at each time we perform the integration, and this will complicate the computations. The second reason is that by differentiation we get more accurate results. Because in differentiation the unknown parameters are multiplied by an integer, therefore there is no truncation error. However, in integration, the unknown parameters are divided by an integer and hence there is a truncation error. The third reason is that the integration may lead to a very complicated approximation. For example, in $\dot{x}_1 = x_1 + x_2$, if x_2 is directly approximated, then x_1 will have a complicated form.

After performing the state parameterization as described above, we can express the approximated state variables and control variables in matrices as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{a_0^{(1)}}{2} & a_1^{(1)} & \cdots & a_{N-1}^{(1)} & a_N^{(1)} \\ \frac{a_0^{(2)}}{2} & a_1^{(2)} & \cdots & a_{N-1}^{(2)} & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_0^{(n)}}{2} & a_1^{(n)} & \cdots & a_{N-1}^{(n)} & a_N^{(n)} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \frac{b_0^{(1)}}{2} & b_1^{(1)} & \cdots & b_{N-1}^{(1)} & b_N^{(1)} \\ \frac{b_0^{(2)}}{2} & b_1^{(2)} & \cdots & b_{N-1}^{(2)} & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{b_0^{(m)}}{2} & b_1^{(m)} & \cdots & b_{N-1}^{(m)} & b_N^{(m)} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} \tag{17}$$

or in compact form

$$\mathbf{x}(\tau) = \alpha \mathbf{T}(\tau), \quad \mathbf{u}(\tau) = \beta \mathbf{T}(\tau) \tag{18}$$

where α, β are matrices of the unknown parameters and $\mathbf{T}(\tau)$ is a vector of Chebyshev polynomials.

The initial conditions can also be represented using Chebyshev series by

$$\frac{a_0^{(j)}}{2} - a_1^{(j)} + a_2^{(j)} - a_3^{(j)} + \cdots + (-1)^N a_N^{(j)} - x_j(-1) = 0, \quad j = 1, 2, \ldots, n \tag{19}$$

This equation will be treated as equality constraints.

The last step in the formulation of the new problem is to evaluate the performance index by using the previous Chebyshev series approximations of the state variables and the control variables. This can be done as follows:

By substituting (18) into (4) we get

$$\hat{J} = \min_{\mathbf{a}} \frac{t_f}{2} \int_{-1}^{1} (\mathbf{T}^T \alpha^T \mathbf{Q} \alpha \mathbf{T} + \mathbf{T}^T \beta^T \mathbf{R} \beta \mathbf{T}) \, d\tau \tag{20}$$

where $\hat{J}$ is the approximate value of J; $\mathbf{a}$ is $q(N+1) \times 1$ vector of all the unknown parameters of the directly approximated q state variables, $\mathbf{a} = [a_0^{(1)}, a_1^{(1)}, \ldots, a_N^{(1)}, a_0^{(2)}, \ldots, a_N^{(2)}, a_0^{(q)}, \cdots, a_N^{(q)}]$.

Let $\alpha^T \mathbf{Q} \alpha = \mathbf{M}$ and $\beta^T \mathbf{R} \beta = \mathbf{P}$, and notice that $\mathbf{M}$ and $\mathbf{P}$ are symmetric matrices. The first part of $\hat{J}$, namely $\mathbf{T}^T\mathbf{M}\mathbf{T}$ can be written as

$$\mathbf{T}^T \mathbf{M} \mathbf{T} = \sum_{i=1}^{N+1} T_{i-1} \sum_{j=1}^{N+1} m_{ij} T_{j-1} = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} T_{i-1} m_{ij} T_{j-1} \tag{21}$$

which can be expanded into

$$\begin{aligned}\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = m_{11}T_0T_0 + 2m_{12}T_0T_1 + 2m_{13}T_0T_2 + \cdots + 2m_{1N+1}T_0T_N \\ + m_{22}T_1T_1 + 2m_{23}T_1T_2 + \cdots + 2m_{2N+1}T_1T_N \\ + m_{33}T_2T_2 + \cdots + 2m_{3N+1}T_2T_N \\ \ddots \qquad \vdots \\ + m_{N+1N+1}T_NT_N\end{aligned} \tag{22}$$

The integration of all terms of (22) which contain T_iT_j, such that $i+j$ is odd, is zero. By considering the remaining terms, equation (22) reduces to

$$\begin{aligned}\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = m_{11}T_0T_0 + 0 + 2m_{13}T_0T_2 + 0 + 2m_{15}T_0T_4 + \cdots \\ + m_{22}T_1T_1 + 0 + 2m_{24}T_1T_3 + \cdots \\ + m_{33}T_2T_2 + 0 + 2m_{35}T_2T_4 + \cdots \\ \ddots \qquad \vdots \\ + m_{N+1N+1}T_NT_N\end{aligned} \tag{23}$$

Theorem 1

The integration of the term (23) can be given by

$$\int_{-1}^{1} \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}\,\mathrm{d}\tau = \frac{1}{2}\left\{-\frac{2}{K^2-1}\sum_{i=1}^{N+1-K} 2\acute{m}_{i,i+K} - 2\sum_{i=1}^{N+1-K}\frac{1}{(K-2+2i)^2-1}\,2\acute{m}_{i,i+K}\right\} \tag{24}$$

where $K = 0, 2, 4, \ldots, N$ (N even) or $N-1$ (N odd), and

$$\acute{m}_{i,i+K} = \begin{cases} m_{i,i+K}, & K \neq 0 \\ \dfrac{m_{ii}}{2}, & K = 0 \end{cases} \tag{25}$$

Proof. Equation (23) can be written as

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = \sum_{i=1} T_{i-1}m_{ii}T_{i-1} + 2\sum_{i=1}\sum_{j=1} T_{i-1}m_{i,i+2j}T_{i-1+2j} \tag{26}$$

$$= 2\sum_{j=0}\sum_{i=1} T_{i-1}\acute{m}_{i,i+2j}T_{i-1+2j} \tag{27}$$

where $\acute{m}_{i,i+2j}$ is as defined in the theorem.

To decide the upper limit of the summation, it is clear that the largest possible value for $i-1+2j$ is N, hence, the upper limit of i is $N+1-2j$. On the other hand, $2j$ cannot be greater than N, therefore, the upper limit for $2j$ is N if N is an even number, or $N-1$ if N is an odd. Substituting these limits into equation (27) and by using the product property of Chebyshev polynomials, we get

$$2\sum_{j=0}^{N/2,(N-1)/2}\ \sum_{i=1}^{N+1-2j} \frac{1}{2}\acute{m}_{i,i+2j}(T_{2i-2+2j} + T_{2j}) \tag{28}$$

which can be integrated, using the integration property of Chebyshev polynomials, to give

$$2\sum_{j=0}^{N/2,(N-1)/2}\sum_{i=1}^{N+1-2j}\frac{1}{2}\acute{m}_{i,i+2j}\left(\frac{-2}{(2i-2+2j)^2-1}+\frac{-2}{(2j)^2-1}\right) \tag{29}$$

Letting $2j=K$, the previous equation can be written as

$$2\sum_{i=1}^{N+1-K}\frac{1}{2}\acute{m}_{i,i+K}\left(\frac{-2}{(2i-2+K)^{\;2}-1}+\frac{-2}{K^2-1}\right) \tag{30}$$

where $K=0,2,4,\ldots,N$ (N even) or $N-1$ (N odd). Equation (30) is the required result. $\square$

Following the same procedure, the integration of the second part of the performance index can be computed

$$\int_{-1}^{1}\mathbf{T}^{\mathrm{T}}\mathbf{P}\mathbf{T}\,\mathrm{d}\tau = 2\sum_{i=1}^{N+1-K}\frac{1}{2}\acute{p}_{i,i+K}\left(\frac{-2}{(2i-2+K)^2-1}+\frac{-2}{K^2-1}\right) \tag{31}$$

where

$$\acute{p}_{i,i+K}=\begin{cases} p_{i,i+K}, & K\neq 0\\ \dfrac{p_{ii}}{2}, & K=0\end{cases} \tag{32}$$

and $K=0,2,4,\ldots,N$ (N even) or $N-1$ (N odd).

From (30) and (31), $\hat{J}$ can be obtained

$$\hat{J}=t_{\mathrm{f}}\sum_{i=1}^{N+1-K}\frac{1}{2}\left(\acute{m}_{i,i+K}+\acute{p}_{i,i+K}\right)\left(\frac{-2}{(2i-2+K)^{\;2}-1}+\frac{-2}{K^2-1}\right) \tag{33}$$

which can be expressed as

$$\hat{J}=\min_{\mathbf{a}}\tfrac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{H}\mathbf{a} \tag{34}$$

because $\acute{m}_{i,i+K}$ and $\acute{p}_{i,i+K}$ are quadratic functions of the unknown parameters $\mathbf{a}$. The matrix $\mathbf{H}$ can be obtained by finding the Hessian of $\hat{J}$,

$$\mathbf{H}=\frac{\partial^2\hat{J}}{\partial a_i^{(p)}\,\partial a_j^{(p)}} \tag{35}$$

where $i,j=0,1,\ldots,N$, and $p=1,2,\ldots,q$.

The optimal control problem is transformed into parameter optimization problem which is quadratic in the unknown parameters and the new problem can be stated as

$$\min_{\mathbf{a}}\tfrac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{H}\mathbf{a} \tag{36}$$

subject to the linear constrains

$$\mathbf{F}\mathbf{a}-\mathbf{b}=0 \tag{37}$$

where the linear constraints are due to the initial conditions (19) and in some cases may appear to represent some of the system state equations which are not satisfied yet.

The optimal value of the vector $\mathbf{a}$ can be obtained from the quadratic programming results,[5] given that $\mathbf{H}$, is a positive-definite matrix.

$$\mathbf{a}^* = \mathbf{H}^{-1}\mathbf{F}^{\mathrm{T}}(\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^{\mathrm{T}})^{-1}\mathbf{b} \tag{38}$$

Lemma 1

The matrix $\mathbf{H}$ is a positive-definite matrix.

Proof. Previously, we wrote $\mathbf{x} = \alpha\mathbf{T}$, hence $\mathbf{x}$ can be written in another way,

$$\mathbf{x} = \mathscr{T}\mathbf{a} \tag{39}$$

where $\mathbf{a}$ *is* a $q(N+1)\times 1$ vector, ($q(N+1)$ is the total number of the unknown parameters used in the approximation of all the states) and $\mathscr{T}$ is a $n\times q(N+1)$ matrix of Chebyshev polynomials. The matrix $\mathscr{T}$ can have two forms: The first form is obtained if all the state variables are directly approximated, while the second form is obtained if only q state variables are directly approximated. In both cases, the rows of $\mathscr{T}$ are linearly independent, and hence its rank is n for all $\tau\in[-1, 1]$.

Also, writing $\mathbf{u}$ as

$$\mathbf{u} = \mathscr{L}\mathbf{a} \tag{40}$$

where $\mathscr{L}$ is $m\times q(N+1)$ matrix of Chebyshev polynomials. The rank of the matrix $\mathscr{L}$ is m because all of its rows are linearly independent. Hence $\mathscr{T}^{\mathrm{T}}\mathbf{Q}\mathscr{T} + \mathscr{L}^{\mathrm{T}}\mathbf{R}\mathscr{L}$ is a positive definite, and

$$\mathbf{H} = \int_{-1}^{1} (\mathscr{T}^{\mathrm{T}}\mathbf{Q}\mathscr{T} + \mathscr{L}^{\mathrm{T}}\mathbf{R}\mathscr{L})\,\mathrm{d}\tau \tag{41}$$

is also a positive-definite matrix. □

The algorithm to solve the linear quadratic optimal control problem can be summarized as follows:

(1) Approximate the system state variables by Chebyshev series, after changing the time interval into $\tau\in[-1, 1]$. Usually we do not need to approximate all the state variables.
(2) Find the control variables and the state variables which are not directly approximated, as a function of the approximated states in step (1).
(3) Calculate the matrix $\mathbf{M}$ from $\alpha^{\mathrm{T}}\mathbf{Q}\alpha$, and the matrix $\mathbf{P}$ from $\beta^{\mathrm{T}}\mathbf{R}\beta$.
(4) Find an expression of $\hat{J}$ from (33).
(5) Determine the set of equality constraints, due to the initial conditions and due to state equations which are not yet satisfied, if any.
(6) Find the matrix $\mathbf{H}$, by calculating the Hessian of $\hat{J}$.
(7) Find the optimal parameters from equation (38), and substitute these parameters into (18) to find the optimal trajectories and control.

3. NON-LINEAR OPTIMAL CONTROL PROBLEM

In order to be able to apply the method proposed in the previous section to handle non-linear systems, we propose to use the quasilinearization technique,[12] in which the performance index is expanded up to the second order and the system state equations are expanded up to the first order around nominal trajectories. Therefore, the non-linear optimal control problem can be replaced by a sequence of time-varying linear quadratic optimal control problems.

Consider the non-linear optimal control problem which can be stated as: Find the control $\mathbf{u}^*(t)$ that minimizes the performance index

$$J = \int_0^{t_f} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}) \, dt \tag{42}$$

subject to the non-linear system dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{43}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $m \leqslant n$, $\mathbf{B}$ is a constant matrix, $\mathbf{Q}$ is $n \times n$ positive-semidefinite matrix and $\mathbf{R}$ is $m \times m$ positive-definite matrix.

To solve this problem using the algorithm proposed in the previous section, the first step is to apply the quasilinearization technique around nominal trajectories $\mathbf{x}^k(t)$ and nominal control $\mathbf{u}^k(t)$. This replaces the non-linear optimal control problem by a sequence of time-varying linear quadratic optimal control problems

$$J^{k+1} = \int_0^{t_f} \left((\mathbf{x}^{k+1})^T \mathbf{Q} \mathbf{x}^{k+1} + (\mathbf{u}^{k+1})^T \mathbf{R} \mathbf{u}^{k+1} \right) dt \tag{44}$$

$$\dot{\mathbf{x}}^{k+1} = \mathbf{A}(t)\,\mathbf{x}(t)^{k+1} + \mathbf{B}\mathbf{u}^{k+1}(t) + \mathbf{h}^k(t) \tag{45}$$

where

$$\mathbf{h}^k(t) = \mathbf{f}(\mathbf{x}^k(t)) - \mathbf{A}(t)\, \mathbf{x}^k(t) \tag{46}$$

The next step is to change the time interval into $\tau \in [-1, 1]$, to obtain the following problem: Minimize

$$J^{k+1} = \frac{t_f}{2} \int_{-1}^{1} \left((\mathbf{x}^{k+1})^{\,T} \mathbf{Q} \mathbf{x}^{k+1} + (\mathbf{u}^{k+1})^{\,T} \mathbf{R} \mathbf{u}^{k+1} \right) d\tau \tag{47}$$

subject to

$$\frac{d\mathbf{x}^{k+1}}{d\tau} = \frac{t_f}{2} (\mathbf{A}(\tau)\mathbf{x}(\tau)^{k+1} + \mathbf{B}\mathbf{u}^{k+1}(\tau) + \mathbf{h}^k(\tau)), \quad \mathbf{x}^{k+1}(-1) = \mathbf{x}_0 \tag{48}$$

The state parameterization is applied in the same way as in linear quadratic optimal control problem of the previous section. However, there are two differences: The first difference is that, in equation (48), $\mathbf{A}(\tau)$ and $\mathbf{h}^k(\tau)$ are time-varying quantities expressed as function of Chebyshev polynomials. In this case, there is a need for an algorithm to multiply Chebyshev series. This algorithm is given by the following lemma.

Lemma 2

Given two Chebyshev series

$$X = \sum_{i=0}^{n} c_i T_i \tag{49}$$

$$Y = \sum_{j=0}^{m} d_j T_j \tag{50}$$

The multiplication of these two Chebyshev series is a Chebyshev series of length $n + m$, given by

$$\sum_{k=0}^{n+m} z_k T_k \tag{51}$$

where

$$z_k = \frac{1}{2}\sum_{i=0}^{n}[c_i d_{k-i} + c_i d_{i-k} + c_i d_{i+k}] \tag{52}$$

$$= \frac{1}{2}\sum_{j=0}^{m}[d_j c_{k-j} + d_j c_{j+k} + d_j c_{j-k}] \tag{53}$$

Remark 2

For $k = 0$, the second or the third term of z_k will be replaced by 0 because of the repetition of the same term for $k = 0$.

Proof. The multiplication of (49) and (50) can be given by

$$\sum_{k=0}^{n+m} z_k T_k = \sum_{i=0}^{n}\sum_{j=0}^{m} c_i d_j T_i T_j \tag{54}$$

and by using Chebyshev polynomials multiplication property, (54) can be written as

$$\sum_{i=0}^{n}\sum_{j=0}^{m} c_i d_j T_i T_j = \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} c_i d_j (T_{i+j} + T_{|i-j|}) \tag{55}$$

or

$$\sum_{k=0}^{n+m} z_k T_k = \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} c_i d_j (T_{i+j} + T_{i-j} + T_{j-i}) \tag{56}$$

Remark 3

In (56), we redefine the absolute value of $|i - j|$. It should be clear for this equation that T_{-x} is zero, although the Chebyshev polynomial of a negative order is equal to Chebyshev polynomial of positive order, i.e. $T_{-x} = T_x$. Hence, one of the second and the third terms will be zero for any i and j. Also, if $i = j$, then we just need one term of the last two terms of equation (56).

Equating the coefficients of Chebyshev polynomials of equation (56), we get

$$z_k T_k = \frac{1}{2}\sum_{i=0}^{n}[c_i d_{k-i} + c_i d_{i-k} + c_i d_{i+k}] T_k \tag{57}$$

or

$$z_k T_k = \frac{1}{2}\sum_{j=0}^{m}[d_j c_{k-j} + d_j c_{j+k} + d_j c_{j-k}] T_k \tag{58}$$

which is the required result. □

The second difference is that the approximation of the state variables and the control variables have different forms and different lengths than that of linear quadratic optimal control problem.

This is due to the fact that (48) is time-varying system and due to the existence of the term $\mathbf{h}^k(\tau)$. Following the state parameterization procedure of the previous section, we can, generally, express the approximated state variables and the control variables as

$$x_j^{k+1}(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{\mathcal{N}} a_i^{(j)} T_i + v_j(\tau), \quad j = 1, 2, \ldots, n \tag{59}$$

$$u_l^{k+1}(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i + g_l(\tau), \quad l = 1, 2, \ldots, m \tag{60}$$

where $\mathcal{N}$ depends on N (the order of the Chebyshev series of the directly approximated states), and on $\mathbf{A}(\tau)$ of (48). $v_j(\tau)$, $g_l(\tau)$ are known functions of τ. These two terms appear because of $\mathbf{h}^k(\tau)$ in (48). Equations (59) and (60) can be written in matrix form as

$$\mathbf{x}^{k+1}(\tau) = \alpha\mathbf{T} + \mathbf{VT} \tag{61}$$

$$\mathbf{u}^{k+1}(\tau) = \beta\mathbf{T} + \mathbf{GT} \tag{62}$$

where α, β are matrices of unknown parameters and $\mathbf{V}$, $\mathbf{G}$ are constant matrices.

From equation (59), the initial conditions can also be approximated, using Chebyshev polynomials property at $\tau = -1$, as follows

$$\frac{a_0^{(j)}}{2} - a_1^{(j)} + a_2^{(j)} - a_3^{(j)} + \cdots + (-1)^{\mathcal{N}} a_{\mathcal{N}}^{(j)} + v_j(-1) - x_j^{k+1}(-1) = 0 \quad j = 1, 2, \cdots, n \tag{63}$$

To approximate the performance index (47), substituting (61) and (62) into (47), yields

$$\hat{J}^{k+1} = \min_{\mathbf{a}} \frac{t_f}{2} \int_{-1}^{1} (\mathbf{T}^{\mathrm{T}}(\alpha^{\mathrm{T}} + \mathbf{V}^{\mathrm{T}})\mathbf{Q}(\alpha + \mathbf{V})\mathbf{T} + \mathbf{T}^{\mathrm{T}}(\beta^{\mathrm{T}} + \mathbf{G}^{\mathrm{T}})\mathbf{R}(\beta + \mathbf{G})\mathbf{T})\,\mathrm{d}\tau \tag{64}$$

The integration of this equation can be done using the method described in the previous section. However, in this case, two new terms will appear after the integration: a constant term due to the integration of $\mathbf{T}^{\mathrm{T}}(\mathbf{V}^{\mathrm{T}}\mathbf{QV} + \mathbf{G}^{\mathrm{T}}\mathbf{RG})\mathbf{T}$, and a linear term of the unknown parameters due to the integration of $\mathbf{T}^{\mathrm{T}}(\alpha^{\mathrm{T}}\mathbf{QV} + \mathbf{V}^{\mathrm{T}}\mathbf{Q}\alpha + \beta^{\mathrm{T}}\mathbf{RG} + \mathbf{G}^{\mathrm{T}}\mathbf{R}\beta)$ $\mathbf{T}$. This leads to parameter optimization problem, which can be stated as

$$\min_{\mathbf{a}} \tfrac{1}{2}\,\mathbf{a}^{\mathrm{T}}\mathbf{Ha} + \mathbf{c}^{\mathrm{T}}\mathbf{a} + d \tag{65}$$

subject to the linear constrains

$$\mathbf{Fa} - \mathbf{b} = 0 \tag{66}$$

where d is a constant, $\mathbf{c}$ is a $q(N+1) \times 1$ vector, and $\mathbf{H}$ is a $q(N+1) \times q(N+1)$ positive-definite matrix as proved in the previous section. The linear constraints are due to the initial conditions (63), and in some cases may appear to represent some of the system equations. The optimal parameters $\mathbf{a}^*$ can be calculated from the quadratic programming results,[5]

$$\mathbf{a}^* = -\mathbf{H}^{-1}\mathbf{c} + \mathbf{H}^{-1}\mathbf{F}^{\mathrm{T}}(\mathbf{FH}^{-1}\mathbf{F}^{\mathrm{T}})^{-1}(\mathbf{FH}^{-1}\mathbf{c} + \mathbf{b}) \tag{67}$$

To solve the non-linear optimal control problem, we need to solve time-varying linear quadratic optimal control problems, (47) and (48), successively until an acceptable convergence error is achieved. The convergence error is defined by

$$\text{Convergence Error} = |\hat{J}_{\min}^{(i+1)} - \hat{J}_{\min}^{(i)}| \tag{68}$$

The procedure which is summarized at the end of the previous section is still applied for the non-linear case, with minor modifications, due to the form of the approximation of the state variables and the control variables in (61) and (62).

4. EXAMPLES

4.1. Example 1

Find $u^*(t)$ that minimizes

$$\int_0^1 (x_1^2 + x_2^2 + 0.005\,u^2)\,\mathrm{d}t \tag{69}$$

subject to

$$\dot{x}_1 = x_2, \quad x_1(0) = 0 \tag{70}$$

$$\dot{x}_2 = -x_2 + u, \quad x_2(0) = -1 \tag{71}$$

The first step in solving this problem by the proposed method is to transform the time interval into $\tau \in [-1, 1]$. This will lead to the following problem:

minimize

$$\frac{1}{2}\int_{-1}^{1} (x_1^2 + x_2^2 + 0.005u^2)\,\mathrm{d}\tau \tag{72}$$

subject to

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \tfrac{1}{2}x_2, \quad x_1(-1) = 0 \tag{73}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}\tau} = \tfrac{1}{2}(-x_2 + u), \quad x_2(-1) = -1 \tag{74}$$

Then by approximating $x_1(\tau)$ by fifth order Chebyshev series of unknown parameters, we get

$$x_1(\tau) = \frac{a_0}{2} + \sum_{i=1}^{5} a_i T_i \tag{75}$$

Using Chebyshev polynomials differentiation property, $\dot{x}_1(\tau)$ can be determined. By substituting $\dot{x}_1(\tau)$ into (73), $x_2(\tau)$ can be determined:

$$x_2(\tau) = (2a_1 + 6a_2 + 10a_5) + (8a_2 + 16a_4)\,T_1 + (12a_3 + 20a_5)\,T_2 + 16a_4T_3 + 20a_5T_4 \tag{76}$$

and by substituting $x_2(\tau)$, $\dot{x}_2(\tau)$ into (74), the control $u(\tau)$ can also be found:

$$u(\tau) = 2\dot{x}_2 + x_2 \tag{77}$$

By these approximations of $x_1(\tau)$, $x_2(\tau)$ and $u(\tau)$, the system state equations (73) and (74) are satisfied directly. This is a clear advantage of using the state parameterization.

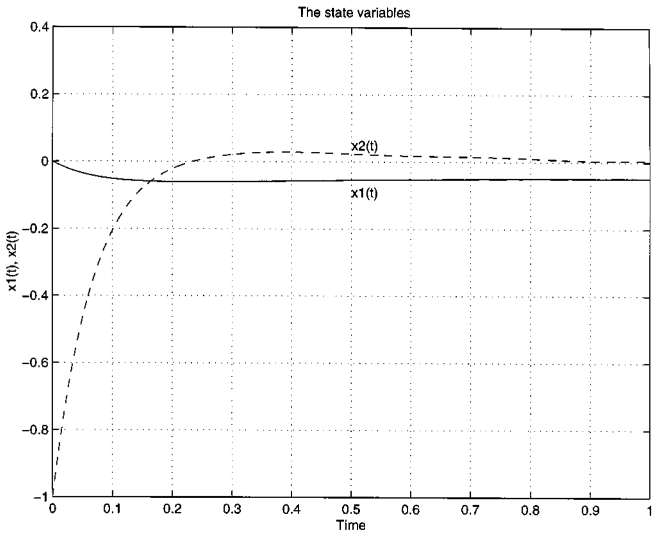

Figure 1. State variables $x_1(t)$ and $x_2(t)$ of Example 1

From equations (75) and (76), another two equations representing the initial conditions are obtained

$$\frac{a_0}{2} - a_1 + a_2 - a_3 + a_4 - a_5 = 0 \tag{78}$$

$$2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 + 1 = 0 \tag{79}$$

These two equations are considered as equality constraints.

Using (75)–(77), $\hat{J}$ can be determined. By solving the resulting quadratic programming problem, the optimal value is found to be 0.0759522.

The previous problem is also solved by approximating $x_1(\tau)$ by ninth-order Chebyshev series, and the optimal value is found to be 0.0693689 which is very close to both the exact value 0.06936094 and result obtained by Vlassenbroeck,[12] which is 0.069368, using ninth-order Chebyshev series. The method of Vlassenbroeck[13] requires the solution of quadratic programming problem of 30 unknown parameters and subject to 22 equality constraints. However, our method requires the solution of quadratic programming problem of 10 unknown parameters and subject to 2 equality constraints. The state trajectories and the optimal control are shown in Figures 1 and 2, respectively.

4.2. Example 2

Find $u^*(t)$ that minimizes

$$J = \int_0^{2.5} (x_1^2 + u^2)\,\mathrm{d}t \tag{80}$$

subject to

$$\dot{x}_1 = x_2, \quad x_1(0) = -5 \tag{81}$$

$$\dot{x}_2 = -x_1 + 1.4x_2 - 0.14x_2^3 + 4u, \quad x_2(0) = -5 \tag{82}$$

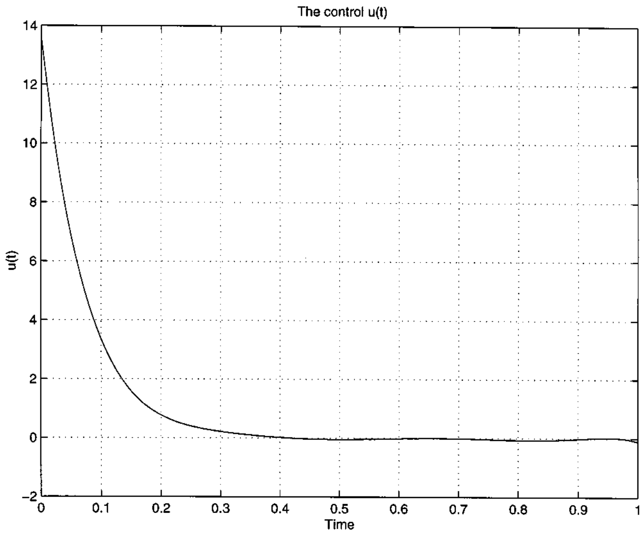

Figure 2. Control variable $u(t)$ of Example 1

Table I. Optimal value of 5 quasilinearization iterations of Example 2

QL step (i)	J_{min}	Convergence error
1	36.8601	—
2	29.4568	7.4033
3	29.4168	0.040
4	29.4092	0.0076
5	29.4081	0.0011

To solve this problem by the proposed algorithm, we apply the quasilinearization method around nominal trajectories $x_1^k(t)$ and $x_2^k(t)$. The expanded performance index is

$$J^{k+1} = \int_0^{2.5} ((x_1^{k+1})^2 + (u^{k+1})^2)\,\mathrm{d}t \tag{83}$$

and the linearized state equations are

$$\dot{x}_1^{k+1} = x_2^{k+1}, \quad x_1^{k+1}(0) = -5 \tag{84}$$

$$\dot{x}_2^{k+1} = -x_1^{k+1} + (1.4 - 0.42(x_2^k)^2)\, x_2^{k+1} + 4u^{k+1} + 0.28(x_2^k)^3, \quad x_2^{k+1}(0) = -5 \tag{85}$$

After changing the time into $\tau \in [-1, 1]$, $x_1(\tau)$ is approximated by ninth-order Chebyshev series, $x_2(\tau)$ is found from (84) while $u(\tau)$ is found from (85). The optimal value and the convergence error of five quasilinearization iterations, starting from zero nominal trajectories, are summarized in Table I.

Table II shows a comparison between the optimal value of the fifth quasilinearization iteration and the results obtained by other researchers. The optimal control for five quasilinearization iterations is shown in Figure 3.

Table II. Optimal value of Example 2

Source	$J_{\min}$
Nedeljkovic[14]	29.419
Sirisena[15]	29.451
This paper	29.4081

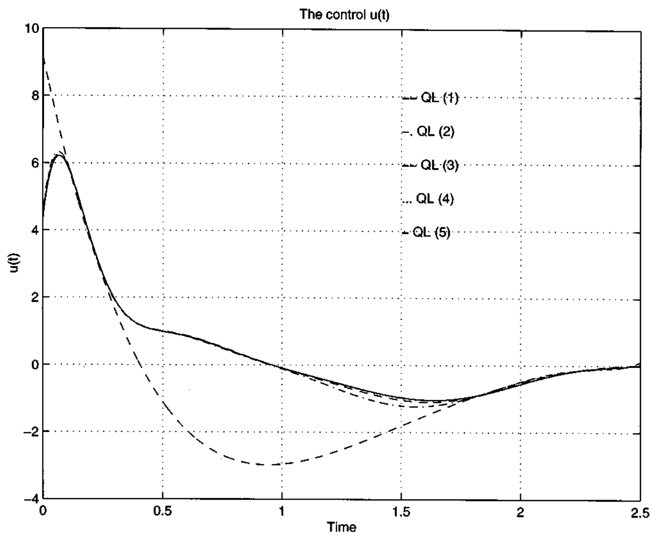

Figure 3. Control variable of Example 2

4.3. Example 3

As a practical application of the proposed algorithm, the problem which was treated by Garrard and Jordan,[16] using Lukes method,[2] is considered. The dynamic equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.877 & 0 & 1 \\ 0 & 0 & 1 \\ -4.208 & 0 & -0.396 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -x_1^2 x_3 - 0.088 x_1 x_3 - 0.019 x_2^2 + 0.47 x_1^2 + 3.846 x_1^3 \\ 0 \\ -0.47 x_1^2 - 3.564 x_1^3 \end{bmatrix} + \begin{bmatrix} -0.215 \\ 0 \\ -20.967 \end{bmatrix} u \quad (86)$$

The optimal control problem, which is considered by Garrard and Jordan, is to find the control u which minimizes the performance index

$$J = \frac{1}{2} \int_0^{\infty} (\mathbf{x}^{\mathrm{T}} \mathbf{Q} \mathbf{x} + r u^2)\, \mathrm{d}t \quad (87)$$

where $\mathbf{Q} = \mathrm{diag}\,[0.25, 0.25, 0.25]$ and $r = 1$.

In this paper, we consider the finite horizon version of this problem because the proposed algorithm cannot deal with infinite time. Arbitrarily, we select $t_{\mathrm{f}} = 10$ s.

Remark 4

Lee and Bien[17] proved that the infinite time optimal performance index can be approximated by finite time optimal performance index if the states $x^*(t_f)$ and $x^\dagger(t_f)$ are near the origin, where $x^*(t_f)$ is the optimal state of the infinite time problem at time $t = t_f$ and $x^\dagger(t_f)$ is the optimal state of the finite time problem at the end time.

Two cases of this problem are considered, the linearized system of (86) around the origin and the non-linear system (86).

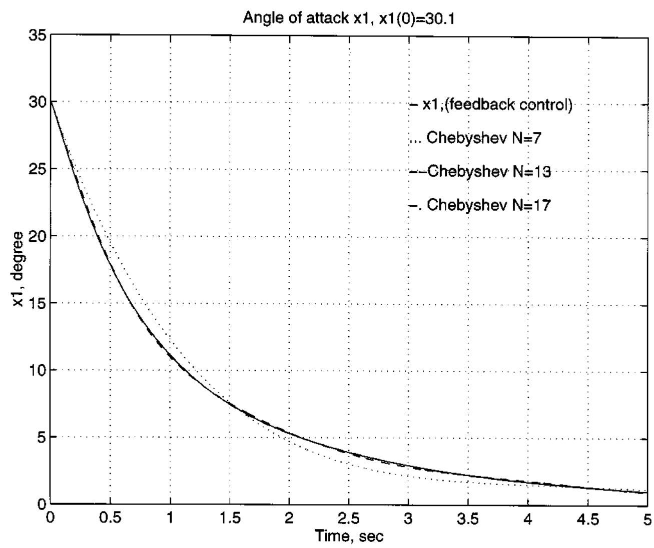

Figure 4. State variable $x_1(t)$

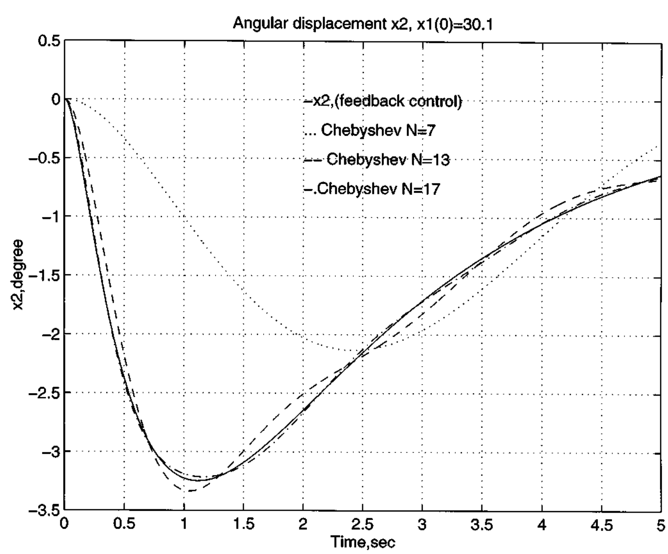

Figure 5. State variable $x_2(t)$

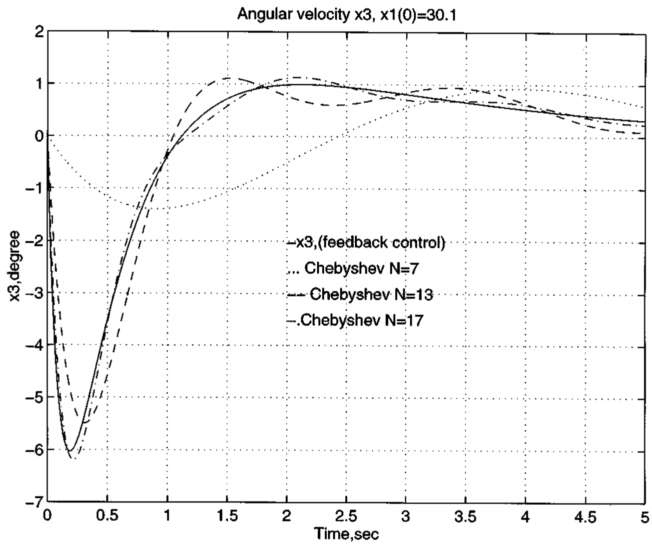

Figure 6. State variable $x_3(t)$

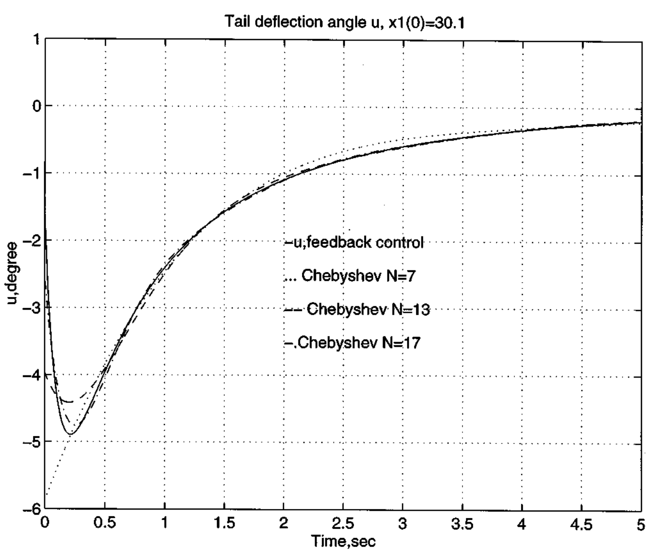

Figure 7. Control variable $u(t)$

In the linearized case and after changing the time interval into $\tau \in [-1, 1]$, $x_1(\tau)$ and $x_2(\tau)$ are approximated by Chebyshev series. The last state variable $x_3(\tau)$ and the control $u(\tau)$ are determined from the second and the first state equations, respectively.

For this problem the third state equation is not satisfied yet, therefore, this equation in addition to the initial conditions represents equality constraints.

Figures 4–7 show the state trajectories and the control variables for different lengths of Chebyshev series approximations along with the exact trajectories which are obtained by using

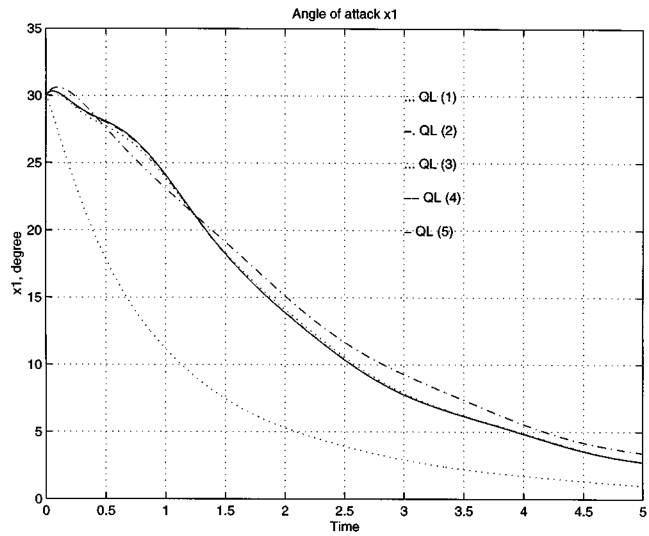

Figure 8. $x_1(t)$ for 5 quasilinearization iterations

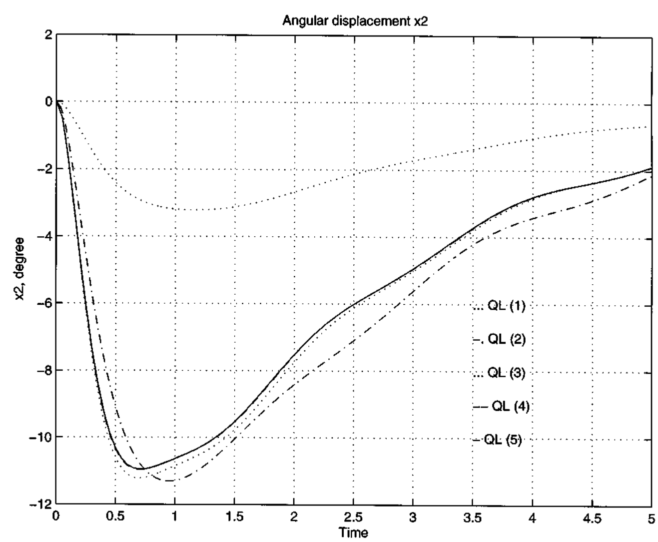

Figure 9. $x_2(t)$ for 5 quasilinearization iterations

the linear feedback control of the linearized system. The optimal trajectories are obtained for the initial conditions $x_1(0) = 30.1^\circ$, $x_2(0) = 0$ and $x_3(0) = 0$. From these figures, it is clear that the Chebyshev series approximation converges as the length of the series increased.

The optimal value of the linearized system of infinite horizon is 0.0222032. The optimal value of the finite horizon problem, using the Chebyshev series of order 17, is found to be 0.0222109 which gives an error of 7.7×10^{-6}. By using the estimate of Lee and Bien,[17] we can calculate the maximum error in the optimal value due to approximating the infinite time problem by finite time one. This maximum error is found to be of order 1.66×10^{-6}.

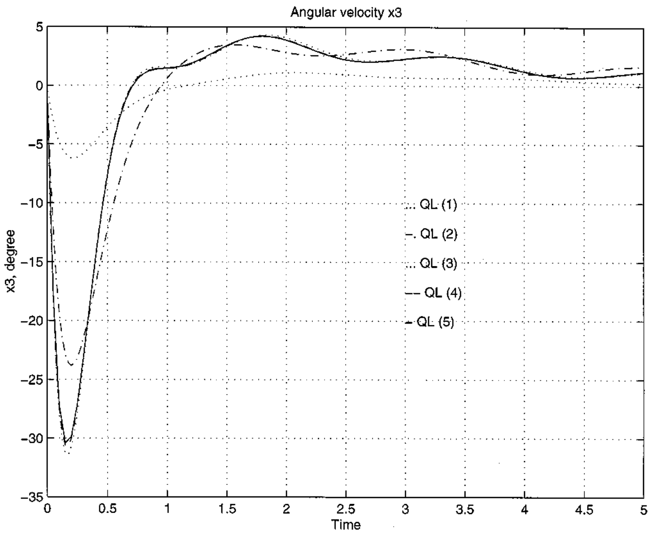

Figure 10. $x_3(t)$ for 5 quasilinearization iterations

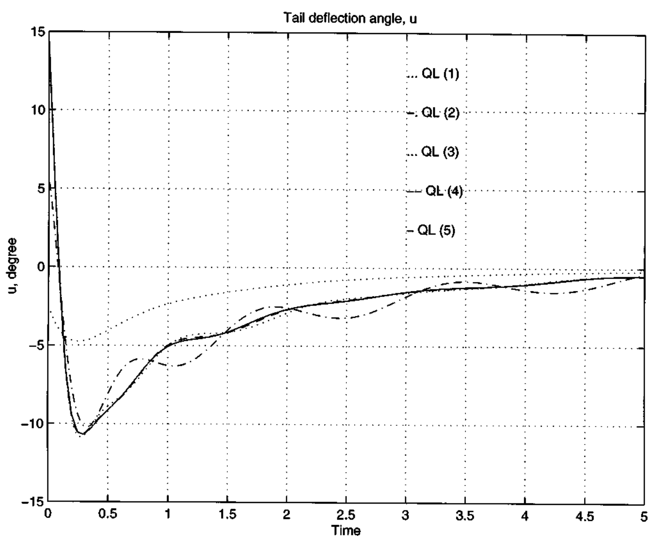

Figure 11. $u(t)$ for 5 quasilinearization iterations

For the non-linear case, the system equations (86) are expanded up to the first order around nominal trajectories $\mathbf{x}^k(t)$. Then after changing the time interval into $\tau \in [-1, 1]$, $x_1(\tau)$ and $x_2(\tau)$ are approximated by Chebyshev series of order 17, while $x_3(\tau)$ and $u(\tau)$ are determined from the second and the first equations respectively. In this case, also, the third state equation in addition to the initial conditions represents equality constraints.

This problem is solved for five quasilinearization iterations starting from the zero nominal trajectories. The state variables and the control variables are shown in Figures 8–11. Also Table III shows the optimal value and the convergence error for five quasilinearization iterations.

Table III. Optimal value of 5 quasilinearization iterations of Example 3

QL Step (i)	$J_{\min}$	Convergence error
1	0.0222109	—
2	0.0823456	0.0601
3	0.0831956	0.00085
4	0.0823616	0.000834
5	0.0823724	0.0000107

5. CONCLUSION

An effective algorithm is proposed to obtain the open-loop solutions for both the linear quadratic optimal control problem and the non-linear optimal control problem. The algorithm is based on parameterizing the state variables by using Chebyshev polynomials.

The main contributions of this paper are to clarify the state parameterization technique and apply it on general optimal control problems. Also in this paper we derived an explicit formula to approximate the performance index. Therefore there is no need to integrate either the performance index or the system state equations.

The main advantages of the proposed algorithm are: The difficult non-linear optimal control problem is converted into a sequence of quadratic programming problems with a few linear equality constraints. Another advantage of this approach is that the number of the unknown parameters is small. The third advantage is that the system state equations, in most cases, are satisfied directly.

At this stage, we do not have a mathematical convergence proof for the proposed algorithm. But from our experience in solving several problems using the proposed algorithm, we notice that as N increases the solution converge to the optimal value as it is clear in the examples.

APPENDIX

The Chebyshev polynomials of the first type are defined on the time interval $\tau \in [-1, 1]$, and are given by[11]

$$T_r(\tau) = \cos(r\theta), \quad \cos(\theta) = \tau, \quad -1 \leqslant \tau \leqslant 1 \tag{88}$$

For example, the first few Chebyshev polynomials are

$$\begin{aligned} T_0(\tau) &= 1 \\ T_1(\tau) &= \tau \\ T_2(\tau) &= 2\tau^2 - 1 \end{aligned} \tag{89}$$

The remaining Chebyshev polynomials can be obtained from the recurrence relation

$$T_{r+1}(\tau) = 2\tau T_r(\tau) - T_{r-1}(\tau), \quad r \geqslant 1 \tag{90}$$

The product of two Chebyshev polynomials is given by

$$T_r(\tau) T_i(\tau) = \tfrac{1}{2}(T_{r+i} + T_{|r-i|}) \tag{91}$$

and the integration of Chebyshev polynomials is given by

$$\int_{-1}^{1} T_n(\tau)\,\mathrm{d}\tau = \begin{cases} 0 & n \text{ odd} \\ \dfrac{-2}{n^2-1}, & n \text{ even.} \\ 2, & n=0 \end{cases} \tag{92}$$

The approximation of a state using a Chebyshev series of Nth order can be written as

$$x(\tau) = \frac{a_0}{2} + \sum_{i=0}^{N} a_i T_i(\tau) \tag{93}$$

as N approaches infinity the previous approximation approaches the exact value. The derivative of $x(\tau)$ with respect to τ is given by the series

$$\dot{x}(\tau) = \frac{b_0}{2} + \sum_{i=1}^{i=N-1} b_i T_i(\tau) \tag{94}$$

where

$$\begin{aligned} b_{N-1} &= 2Na_N \\ b_{N-2} &= 2(N-1)a_{N-1} \\ b_{r-1} &= b_{r+1} + 2ra_r, \quad r = 1, 2, \ldots, N-2 \end{aligned} \tag{95}$$

The initial and the final states can be represented, using Chebyshev series, by

$$x(1) = \frac{a_0}{2} + a_1 + a_2 + \cdots + a_N \tag{96}$$

$$x(-1) = \frac{a_0}{2} - a_1 + a_2 - \cdots + (-1)^N a_N \tag{97}$$

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