Proceedings of the American Control Conference
Chicago, Illinois • June 2000

Direct Trajectory Optimization by a Chebyshev Pseudospectral Method

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Abstract

A Chebyshev pseudospectral method is presented in this paper for directly solving a generic optimal control problem with state and control constraints. This method employs Nth degree Lagrange polynomial approximations for the state and control variables with the values of these variables at the Chebyshev-Gauss-Lobatto (CGL) points as the expansion coefficients. This process yields a nonlinear programming problem (NLP) with the state and control values at the CGL points as unknown NLP parameters. Numerical examples demonstrate this method yields more accurate results than those obtained from the traditional collocation methods.

1 Introduction

In recent years, direct solution methods have been used extensively in a variety of trajectory optimization problems [1, 10]. Their advantage over indirect methods, which rely on solving the necessary conditions derived from Pontryagin's minimum principle, is their wider radius of convergence to an optimal solution. In addition, since the necessary conditions do not have to be derived, the direct methods can be quickly used to solve a number of practical trajectory optimization problems.

Direct methods can be basically described as discretizing the optimal control problem and then solving the resulting nonlinear programming problem (NLP). The discretization is achieved by first dividing the time interval into a prescribed number of subintervals whose endpoints are called nodes [10]. The unknowns are the values of the controls and the states at these nodes, the state and control parameters. The cost function and the state equations can be expressed in terms of these parameters which effectively reduce the optimal control problem to an NLP that can be solved by a standard nonlinear programming code. The time histories of both the control and the state variables can be obtained by using an interpolation scheme. In most collocation schemes, linear or cubic splines are used as the interpolating polynomials [8, 13]. To impose the state differential equations, some form of integration scheme is used, among which the most popular is the Simpson-Hermite scheme [8]. Gauss-Lobatto quadrature rules such as trapezoidal, Simpson's or higher order rules with Jacobi polynomials as the interpolant are also used for collocation [3, 9].

Instead of using piecewise-continuous polynomials as the interpolant between prescribed subintervals, orthogonal polynomials such as Legendre and Chebyshev polynomials can be used for approximating the control and state variables [4]-[6], [12]. These polynomials are used extensively in *spectral methods* for solving fluid dynamics problems [2, 7], but their use in solving optimal control problems has created a new way of transforming these problems to NLP problems. One particular merit of the use of orthogonal polynomials is their close relationship to Gauss-type integration rules. This relationship can be used to derive simple rules for transforming the original optimal control problem to a system of algebraic equations. The efficiency and simplicity of these rules are best demonstrated in the spectral collocation (or pseudospectral) method that Elnagar *et al.* [4] and Fahroo and Ross [5, 6] have recently employed to solve a general class of optimal control problems. In their method, polynomial approximations of the state and control variables are considered where Lagrange polynomials are the trial functions and the unknown coefficients are the values of the state and control variables at the Legendre-Gauss-Lobatto (LGL) nodes. With this choice of collocation points and properties of the Lagrange polynomials, the state equations and the possible state and control constraints are readily transformed to algebraic equations. The state differential constraints are imposed by collocating the functions at the LGL nodes and using a differentiation matrix which is obtained by taking the analytic derivative of the interpolating polynomials and collocating them at the LGL points. In this sense, this method of imposing the state equations is in marked contrast to the numerical integration techniques that are used to approximate the differential equations in other collocation schemes [8]-[10].

Given the success of the Legendre pseudospectral method, we are encouraged to check the effectiveness of using Chebyshev polynomials and Chebyshev-Gauss-Lobatto (CGL) points in the discretization method. Chebyshev polynomials have been widely used in engineering applications, and aside from their popularity have the added computational advantage in that their corresponding quadrature weights and CGL points can be easily evaluated. On the other hand, the calculation of LGL points require the use of advanced numerical linear algebra techniques. In the two numerical examples presented in this paper, the results clearly show that Chebyshev polynomials are very effective in direct optimization techniques and offer superior results than the existing collocation methods.

2 Problem Formulation

Consider the following optimal control problem. Determine the control function $\mathbf{u}(\tau)$, and the corresponding state trajectory $\mathbf{x}(\tau)$, that minimize the Mayer cost function:

$$\mathcal{J}(\mathbf{x}, \tau_f) = \mathcal{M}[\mathbf{x}(\tau_f), \tau_f] \tag{1}$$

with $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$ subject to the state dynamics

$$\dot{\mathbf{x}}(\tau) = \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau), \qquad \tau \in [\tau_0, \tau_f] \tag{2}$$

and boundary conditions:

$$\boldsymbol{\psi}_0[\mathbf{x}(\tau_0), \tau_0] = \mathbf{0}, \quad \boldsymbol{\psi}_f[\mathbf{x}(\tau_f), \tau_f] = \mathbf{0}, \tag{3}$$

where $\boldsymbol{\psi}_0 \in R^p$ with $p \leq n$ and $\boldsymbol{\psi}_f \in R^q$ with $q \leq n$. Possible state and control inequality constraints are formulated as

$$\mathbf{g}[\mathbf{x}(\tau), \mathbf{u}(\tau)] \leq \mathbf{0}, \qquad \mathbf{g} \in R^r. \tag{4}$$

3 The Chebyshev Pseudospectral Method

The Chebyshev pseudospectral method is one special case of a more general class of spectral methods [7]. The basic formulation of these methods involve two essential steps: One is to choose a finite dimensional space (usually a polynomial space) from which an approximation to the solution of the differential equation is made. The other step is to choose a projection operator which imposes the differential equation in the finite dimensional space. One important feature of spectral methods which distinguishes it from finite-element or finite difference methods is that the underlying polynomial space is spanned by orthogonal polynomials which are infinitely differentiable global functions. Among examples of these orthogonal polynomials are Legendre and Chebyshev polynomials which are orthogonal on the interval $[-1, 1]$, with respect to an appropriate weight function ($w(x) = 1$ for Legendre polynomials, and $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ for Chebyshev polynomials of the first kind.) In the collocation methods, after choosing a polynomial approximation for a function, the requirement is to satisfy the differential equation exactly at some chosen nodes (or collocation nodes). In the orthogonal collocation methods, a given function is expanded in terms of orthogonal polynomials such that the expansion coefficients are exactly the values of the function at the collocation points. Also, since an arbitrary choice of collocation points can give very poor results in interpolation, different Gauss quadrature points are chosen to give the best accuracy in interpolation of a function. These two important aspects (choice of orthogonal polynomials as the trial functions and Gauss quadrature points) of orthogonal or spectral collocation methods separate them from the other collocation methods [8]-[10].

The basic idea behind the use of a Chebyshev pseudospectral method for solving the optimal control problem is to find global polynomial approximations for the state and control functions in terms of their values at the points. The time derivative of the approximate state vector, $\dot{\mathbf{x}}^N(\tau)$, is expressed in terms of the approximate state vector $\mathbf{x}^N(\tau)$ at the collocation points by the use of a differentiation matrix. In this manner, the optimal control problem is transformed to an NLP problem for the value of the states and the controls at the nodes. Since the problem presented in the previous section is formulated over the time interval $[\tau_0, \tau_f]$, and the CGL points lie in the interval $[-1, 1]$, we use the following transformation to express the problem in $t \in [t_0, t_N] = [-1, 1]$:

$$\tau = \frac{(\tau_f - \tau_0)t + (\tau_f + \tau_0)}{2}. \tag{5}$$

It follows that by using Equation (5) Equations (1-4) can be replaced by

$$\mathcal{J}(\mathbf{x}(\cdot), \tau_f) = \mathcal{M}[\mathbf{x}(1), \tau_f], \tag{6}$$

$$\dot{\mathbf{x}}(t) = (\frac{\tau_f - \tau_0}{2})\,[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)], \tag{7}$$

$$\boldsymbol{\psi}_0(\mathbf{x}(-1), \tau_0) = \mathbf{0}, \quad \boldsymbol{\psi}_f(\mathbf{x}(1), \tau_f) = \mathbf{0}, \tag{8}$$

$$\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \leq \mathbf{0}. \tag{9}$$

In most common Chebyshev collocation methods, the interpolation points in the interval $[-1, 1]$ are

$$t_l = \cos(\frac{\pi l}{N}), \qquad l = 0, \ldots N \tag{10}$$

which are the extrema of the Nth order Chebyshev polynomial $T_N(t)$. The ith order Chebyshev polynomial is expressed by

$$T_i(t) = \cos(i \cos^{-1} t), \qquad i = 0, \ldots, N$$

which yields $T_i(t_l) = \cos(\frac{\pi l i}{N})$. To keep the first CGL point at -1 and the last node at 1, we sort the nodes from (10) in our exposition and numerical implementation. For approximating the continuous equations, we seek polynomial approximations of the form

$$\mathbf{x}^N(t) = \sum_{l=0}^{N} \mathbf{x}(t_l)\phi_l(t), \tag{11}$$

$$\mathbf{u}^N(t) = \sum_{l=0}^{N} \mathbf{u}(t_l)\phi_l(t), \tag{12}$$

where, for $l = 0, 1, ..., N$

$$\phi_l(t) = \frac{(-1)^{l+1}}{N^2 c_l} \frac{(1-t^2)\dot{T}_N(t)}{t - t_l},$$

are the Lagrange polynomials of order N, with

$$c_l = \begin{cases} 2 & l = 0, N \\ 1 & 1 \leq l \leq N-1. \end{cases}$$

It can be shown that

$$\phi_l(t_k) = \delta_{lk} = \begin{cases} 1 & \text{if } l = k \\ 0 & \text{if } l \neq k. \end{cases}$$

From this property of ϕ_l it follows that

$$\mathbf{x}^N(t_l) = \mathbf{x}(t_l),\ \mathbf{u}^N(t_l) = \mathbf{u}(t_l). \tag{13}$$

To express the derivative $\dot{\mathbf{x}}^N(t)$ in terms of $\mathbf{x}^N(t)$ at the collocation points t_l, we differentiate (11) which results in a matrix multiplication of the following form:

$$\dot{\mathbf{x}}^N(t_l) = \sum_{k=0}^{N} D_{lk}\mathbf{x}(t_k), \tag{14}$$

where D_{lk} are entries of the $(N+1) \times (N+1)$ differentiation matrix $\mathbf{D}$

$$\mathbf{D} := [D_{lk}] := \begin{cases} \dfrac{c_l}{c_k} \dfrac{(-1)^{l+k}}{t_l - t_k} & l \neq k, \\ -\dfrac{t_l}{2(1-t_l^2)} & 1 \leq l = k \leq N-1, \\ \dfrac{2N^2+1}{6} & l = k = 0, \\ -\dfrac{2N^2+1}{6} & l = k = N. \end{cases} \tag{15}$$

To facilitate the NLP formulation, we simplify the notation using

$$\mathbf{a}_l := \mathbf{x}(t_l),\ \mathbf{b}_l := \mathbf{u}(t_l),$$

to rewrite (11)-(12) in the form:

$$\mathbf{x}^N(t) = \sum_{l=0}^{N} \mathbf{a}_l\phi_l(t), \quad \mathbf{u}^N(t) = \sum_{l=0}^{N} \mathbf{b}_l\phi_l(t). \tag{16}$$

For the derivative of the state vector $\mathbf{x}^N(t)$, collocated at the points t_l, we rewrite (14)

$$\mathbf{d}_l = \dot{\mathbf{x}}^N(t_l) = \sum_{k=0}^{N} D_{lk}\mathbf{a}_k. \tag{17}$$

Next, the cost function (6) is discretized: By evaluating the state functions in the cost function at the final CGL point, we have

$$\tilde{\mathcal{J}}(\mathbf{a}, \mathbf{b}, \tau_f) = \mathcal{M}(\mathbf{x}^N(1), \tau_f) = \mathcal{M}(\mathbf{a}_N, \tau_f). \tag{18}$$

The state equations and the initial and terminal state conditions are discretized by first substituting (14)-(16) in (7) and collocating at the CGL nodes, t_l. Using the notation for $\mathbf{a}$ and $\mathbf{b}$, the state equations are transformed into the following algebraic equations

$$(\frac{\tau_f - \tau_0}{2})\mathbf{f}(\mathbf{a}_k, \mathbf{b}_k) - \mathbf{d}_k = \mathbf{0}, \qquad k = 0, \ldots, N,$$

where $\mathbf{d}_k$ is as defined in (17), and the initial conditions are

$$\boldsymbol{\psi}_0(\mathbf{x}^N(-1), \tau_0) = \boldsymbol{\psi}_0(\mathbf{a}_0, \tau_0) = \mathbf{0}.$$

The terminal state conditions are

$$\boldsymbol{\psi}_f(\mathbf{x}^N(1), \tau_f) = \boldsymbol{\psi}_f(\mathbf{a}_N, \tau_f) = \mathbf{0}.$$

The control inequality constraints are approximated by

$$\mathbf{g}(\mathbf{x}^N(t_k), \mathbf{u}^N(t_k)) = \mathbf{g}(\mathbf{a}_k, \mathbf{b}_k) \leq \mathbf{0},\ k = 0, \ldots, N.$$

To summarize, the optimal control problem (6)-(9) is approximated by the following nonlinear programming problem: Find coefficients

$$\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_N),\ \mathbf{b} = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_N)$$

and possibly the final time τ_f to minimize

$$\tilde{\mathcal{J}}(\mathbf{a}, \mathbf{b}, \tau_f) = \mathcal{M}(\mathbf{a}_N, \tau_f) \tag{19}$$

subject to the following constraints for $k = 0, \ldots, N$:

$$(\frac{\tau_f - \tau_0}{2})\mathbf{f}(\mathbf{a}_k, \mathbf{b}_k, t_k) - \mathbf{d}_k = \mathbf{0}, \tag{20}$$

$$\mathbf{g}(\mathbf{a}_k, \mathbf{b}_k) \leq \mathbf{0}, \tag{21}$$

$$\boldsymbol{\psi}_0(\mathbf{a}_0, \tau_0) = \mathbf{0}, \tag{22}$$

$$\boldsymbol{\psi}_f(\mathbf{a}_N, \tau_f) = \mathbf{0}. \tag{23}$$

From the equations above, one can see the simplicity of the method which retains much of the structure of the continuous problem.

4 Numerical Examples

In this section, we present two numerical examples: one is the brachistochrone problem discussed in [3], and the other is the moon-landing problem [11]. The first example demonstrates the efficiency of this technique while the second example shows that the discontinuities are adequately captured by this approach.

4.1 Example 1: Brachistochrone

In this well-known problem, the control problem is formulated as finding the shape of a wire so that a bead sliding on the wire will reach a given horizontal displacement in minimum time. No frictional forces are considered and the gravity force is uniform. The problem is then to minimize τ_f subject to the equations

$$\dot{x} = \sqrt{2gy}\cos\theta, \tag{24}$$
$$\dot{y} = \sqrt{2gy}\sin\theta, \tag{25}$$

with boundary conditions

$$x(0) = y(0) = 0, \qquad x(\tau_f) = 0.5. \tag{26}$$

The control, angle θ, is the slope of the wire as a function of time. The analytic solutions to this problem are the equations of a cycloid

$$x(t) = (\frac{g\tau_f}{\pi})\left(t - \frac{\tau_f}{\pi}\sin\{\pi[1-(\frac{t}{\tau_f})]\}\right), \tag{27}$$

and

$$y(t) = \frac{2g\tau_f^2}{\pi^2}\cos^2\left[\frac{\pi}{2}\left(1-\frac{t}{\tau_f}\right)\right]. \tag{28}$$

The optimal control (angle) is given by the following expression:

$$\theta(t) = \frac{\pi}{2}\left[1-(\frac{t}{\tau_f})\right]. \tag{29}$$

For the value of $g = 1$, the minimum time is $\tau_f = 1.2533$.

For discretization of the problem, the Chebyshev pseudospectral method was used with NPSOL as the NLP solver. In Table 1, we show the minimum cost function obtained from our method with the Simpson collocation method and a fifth-degree Gauss-Lobatto method from [3], for $N = 11$. It is evident that even for a low number degree of discretization (and number of NLP variables), the Chebyshev method gives superior results than either of the other collocation methods.

In Figures 1 and 2, we show the time histories of the state and control variables against the analytic solutions for the same number $N = 11$. The graphs clearly demonstrate the accuracy of the results from the Chebyshev collocation method.

Method	J_i	$\lvert J_i - J_{ana}\rvert$	N_p
Analytic solution	1.2533		
Collocation (Simpson)	1.253005	0.000295	33
Collocation (5th-degree-GL)	1.253183	0.000117	83
Pseudospectral (CGL)	1.253309	9.099e-06	33

Table 1: Comparison of cost functions from different Collocation Methods for $N = 11$

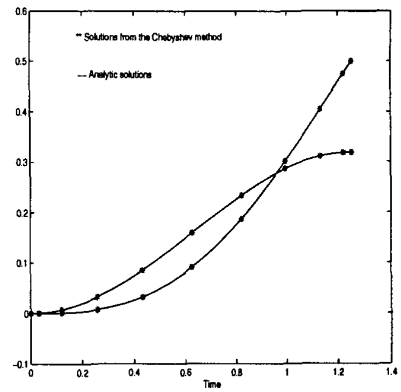

Figure 1: The time history of x and y

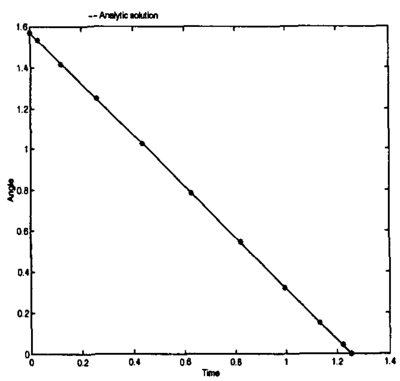

Figure 2: The time history of control θ

4.2 Example 2: Moon-Landing

The control problem is formulated as maximizing the final mass and hence minimizing

$$J = -m(\tau_f) \tag{30}$$

subject to the equations of motion

$$\frac{dh}{d\tau} = v, \tag{31}$$
$$\frac{dv}{d\tau} = -g + \frac{T}{m}, \tag{32}$$
$$\frac{dm}{d\tau} = -\frac{T}{I_{sp}g}, \tag{33}$$

where the state variables h, , v and m are altitude, speed and mass, respectively. The control is provided

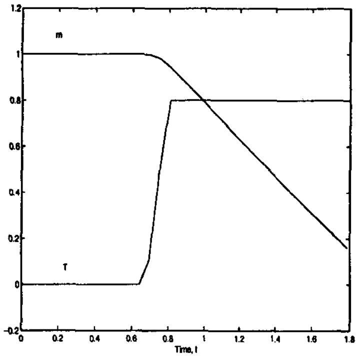

Figure 3: The time history of m and control T

by the thrust, T which is bounded by

$$0 \leq T \leq T_{max}.$$

The other parameters in the problem are g, the gravity of moon (or any planet without an atmosphere) and I_{sp}, the specific impulse of the propellent. Given any set of initial conditions h_0, v_0 and m_0, the normalized parameters for the problem were chosen as

$$\frac{T_{max}}{m_0 g} = 0.8, \quad \frac{I_{sp} g}{v_0} = 1, \quad \frac{h(0)}{h_0} = 1,$$

$$\frac{v(0)}{v_0} = -0.05, \quad \frac{m(0)}{m_0} = 1.$$

Therefore, we have the following *normalized* initial conditions:

$$h(0) = 1.0, \quad v(0) = -0.05, \quad m(0) = 1.0. \qquad (34)$$

For soft landing, we must have

$$h(\tau_f) = 0, \quad v(\tau_f) = 0. \qquad (35)$$

This problem has at most one switch in the control variable [11] and the results are displayed in Figure 3. It is clear that the method has adequately captured the optimal bang-bang structure of the control with one switch.

5 Conclusions

The simplicity and efficiency of the Chebyshev pseudospectral method allows one to perform rapid and accurate trajectory optimization. A low degree of discretization appears to be sufficient to generate good results. Thus, it is apparent that the technique has a high potential for use in optimal guidance algorithms that require corrective maneuver from the perturbed trajectory. In any case, reference optimal paths can be easily generated and the numerical examples indicate that the converged solutions are indeed optimal. Further tests and analysis are necessary to investigate the stability and accuracy of the method.

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