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Pseudospectral Chebyshev Optimal Control of Constrained Nonlinear Dynamical Systems

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Abstract. A pseudospectral method for generating optimal trajectories of linear and nonlinear constrained dynamic systems is proposed. The method consists of representing the solution of the optimal control problem by an mth degree interpolating polynomial, using Chebyshev nodes, and then discretizing the problem using a cell-averaging technique. The optimal control problem is thereby transformed into an algebraic nonlinear programming problem. Due to its dynamic nature, the proposed method avoids many of the numerical difficulties typically encountered in solving standard optimal control problems. Furthermore, for discontinuous optimal control problems, we develop and implement a Chebyshev smoothing procedure which extracts the piecewise smooth solution from the oscillatory solution near the points of discontinuities. Numerical examples are provided, which confirm the convergence of the proposed method. Moreover, a comparison is made with optimal solutions obtained by closed-form analysis and/or other numerical methods in the literature.

Keywords: optimal control, spectral numerical methods, Chebyshev polynomials, smoothing filters

1. Introduction

Optimal control theory is widely applied in aerospace, engineering, economics and other areas of science and has received considerable attention of researchers. Up to now enormous effort has been spent on the development of computational methods for generating solutions of optimal control problems (cf. [3, 4, 6, 9, 10, 14, 16, 17, 19, 23] and references therein). Although many computational methods have been developed and proposed, modification of the existing methods and development of new method should yet be explored to obtain accurate solutions successfully.

The approaches to numerical solutions of optimal control problems may be divided into two major classes: the indirect methods and the direct methods. The indirect methods are based on the Pontryagin maximum principle and require the numerical solution of boundary value problems that result from the necessary conditions of optimal control (cf. [5, 21]). For many practical optimization problems, these boundary value problems are quite difficult to solve. In fact, the manner in which Pontryagin maximum principle is used differs so significantly from one type of problem to another that no standard solution procedure can be devised. Therefore, one has to devise direct computational algorithms to solve optimal control problems.

Direct optimization methods transcribe the (infinite-dimensional) continuous problem to a finite-dimensional nonlinear programming problem (NLP) through some parametrization of the state and/or control vectors. In the direct methods initial guesses have to be provided only for physically intuitive quantities such as the states and possibly controls. However, continuous advances in NLP algorithms and software have made these the methods of choice in many applications (cf. [15, 20]).

In this paper, we present a direct approach that draws upon the power of nonlinear programming and a newly developed Chebyshev smoothing procedure to determine the optimal trajectories of high-order nonlinear (possibly discontinuous) dynamic systems. Central to the idea is the construction of, using Chebyshev nodes, the mth degree interpolating polynomial to approximate the state and the control vectors. The differential equations are collocated at the Chebyshev nodes and the integral involved in the definition of the performance index is discretized through a cell-averaging Chebyshev technique. The optimal control problem is thereby converted into a NLP to which existing well developed nonlinear programming algorithms may be applied. The advantages of recasting the optimal control problem as a NLP are:

(1) the proposed method eliminates the requirement of solving a (2PBVP);
(2) state and control inequality constraints are easier to handle;
(3) the definite integral, in the performance index, is calculated accurately once using the cell-averaging Chebyshev integration rule (see, [7, 12]);
(4) the pseudospectral Chebyshev approximation enjoys what is called spectral accuracy. By this we mean its truncation error decays as fast as the global smoothness of the underlying solution permits.

We remark that our approach is closest to that of [23]. However, the technique of [23], requires that the state and the control variables, the system dynamics, the constraints, and the performance index be expanded in Chebyshev series with unknown coefficients. The resulting system of nonlinear equations has to be solved, for these unknowns, by some kind of iterative method. The unknowns which evolve from the classical Chebyshev series expansion of the performance index have to be calculated at each step of the iteration. As a result, a rather large and complicated nonlinear system of equations have to be solved. The main drawback of polynomial approximation approaches presented in [9, 23], as well as in this paper, is that polynomial do not have local support. To get around this problem, [10] presents an interesting alternative method using a piecewise polynomial version of [9].

We note that the pointwise errors associated with the pseudospectral approximations, suffer from the limitation of being dependent on the smoothness of the function F over the whole domain, and not just on its local behavior in the neighborhood of the point of interest. This dependence of the local convergence rate on the global smoothness, which is reflected by (though not a consequence of) the error estimates (see Theorems 1 and 2, and the estimates (3.25) and (3.26)), is indeed inherent in both approximations. That is, the discontinuity of the function in one part of its domain, decelerates the convergence rate in the smooth part of it. Most notably is the case of pointwise smooth functions: not only that Gibbs phenomenon is recorded at points of discontinuity, but in addition, the spectral accuracy is lost at regions where the function is smooth.

In this paper we also show how pointwise values of the optimal trajectory $u(t), t \in [-1, 1]$, can be recovered from the information contained in its pseudospectral approximation, so that the accuracy solely depends on the local smoothness of optimal control u, that is, its smoothness in the neighborhood of the point of interest. If, in particular, the optimal solution u is infinitely smooth in that neighborhood, then the value $u(t)$ is approximated within infinite order of accuracy. Most notably, we recover pointwise values within spectral accuracy, despite the possible presence of discontinuities in the domain.

The paper is organized as follows. In Section 2, we state the problem statement. In Section 3, we present the details of the pseudospectral Chebyshev method, and state convergence and error estimates results. A Chebyshev smoothing procedure for extracting the piecewise smooth solution from the oscillatory data is presented in Section 4. Finally, we present numerical experiments in Section 5, which demonstrate the applicability and the superior accuracy of the pseudospectral Chebyshev method when applied to smooth problems. We also demonstrate the efficiency and the accuracy of the Chebyshev smoothing procedure for extracting, within spectral accuracy, the piecewise smooth optimal solution from the oscillatory pseudospectral Chebyshev solution near the point of discontinuity.

2. Problem statement

Find the control m-vector $U(\tau) = (U_0(\tau), U_1(\tau), \ldots, U_{m-1}(\tau))$, and the corresponding state n-vector $X(\tau) = (X_0(\tau), X_1(\tau), \ldots, X_{n-1}(\tau))$, $\tau \in [0, T]$, which minimize (or maximize) the functional

$$J = H(X(T), \dot{X}(T), T) + \int_0^T G(X(\tau), \dot{X}(\tau), U(\tau), \tau)\, d\tau, \tag{2.1}$$

subject to

$$F(X(\tau), \dot{X}(\tau), \ddot{X}(\tau), U(\tau), \tau) = 0, \quad \tau \in [0,\ T] \tag{2.2}$$

$$X(0) = X_0, \quad \dot{X}(0) = \dot{X}_0, \tag{2.3}$$

$$S(X(\tau), \dot{X}(\tau), U(\tau)) \leq 0, \quad \tau \in [0,\ T] \tag{2.4}$$

$$Z(X(T), \dot{X}(T), T) = 0. \tag{2.5}$$

The vector function F and the scaler functions H and G are generally nonlinear, and assumed to be continuously differentiable with respect to their arguments. T denotes the final time which may be free. It is assumed that the problem (2.1)–(2.5) has a unique solution. The time transformation $\tau = \frac{T}{2}(t+1)$ is introduced in order to use the pseudospectral Chebyshev interpolating polynomials defined on the interval $t \in [-1, 1]$. Using this transformation (2.1)–(2.5) are replaced by

$$J = h(x(1), \dot{x}(1), 1) + \int_{-1}^{1} g(x(t), \dot{x}(t), u(t), t, T)\, dt. \tag{2.6}$$

$$f(x(t), \dot{x}(t), \ddot{x}(t), u(t), t, T) = 0, \quad t \in [-1,\ 1] \tag{2.7}$$

$$x(-1) = x_{-1} = X_{-1}, \quad \dot{x}(-1) = \dot{x}_{-1} = \dot{X}_{-1}, \tag{2.8}$$

$$s(x(t), \dot{x}(t), u(t)) \leq 0, \quad t \in [-1, 1] \tag{2.9}$$

$$z(x(1), \dot{x}(1), 1) = 0. \tag{2.10}$$

3. The proposed method

In order to obtain spectral accuracy, the grids on which a physical problem is to be solved must also be obtained by spectrally accurate techniques. Thus, we let S_m denote the space of algebraic polynomials of degree $\leq m$, and let $T_k(t), k \geq 0, -1 \leq t \leq 1$, denote the orthogonal family of Chebyshev polynomials of the first kind in this space, with respect to the weight function $w(t) = (1-t^2)^{-1/2}$. We choose the grid (interpolation) points to be the extrema

$$t_j = \cos\left(\frac{j\pi}{m}\right), \quad j = 0, 1, \ldots, m, \tag{3.1}$$

of the mth order Chebyshev polynomial $T_m(t)$, $t \in [-1, 1]$. These grids, $t_m = -1 < t_{m-1} < \cdots < t_1 < t_0 = 1$, are also viewed as the zeros of $(1-t^2)\dot{T}_m(t)$, where $\dot{T}_m(t) = \frac{dT_m(t)}{dt}$.

In order to construct the interpolant of a function $F(t)$ at the point $t \in [-1, 1]$, we define the following Lagrange polynomials

$$\phi_k(t) = \frac{(-1)^{k+1}(1-t^2)\dot{T}_m(t)}{C_k m^2 (t - t_k)} = \frac{2}{mC_k} \sum_{j=0}^{m} \frac{T_j(t_k)T_j(t)}{C_j}, \quad (k = 0, 1, \ldots, m), \tag{3.2}$$

with $C_0 = C_m = 2$, $C_k = 1$ for $1 \leq k \leq m-1$. It is readily verified that

$$\phi_l(t_j) = \delta_{lj} = \begin{cases} 1 & \text{if } l = j \\ 0 & \text{if } l \neq j. \end{cases} \tag{3.3}$$

Associated with the $m+1$ Chebyshev nodes (grids) t_j, is a unique mth-degree interpolating polynomial $I_m F(t)$

$$I_m F(t) = \sum_{l=0}^{m} F(t_l)\phi_l(t), \tag{3.4}$$

such that $I_m F(t_j) = F(t_j)$, $j = 0, 1, \ldots, m$.

For each state variable $x_i(t)$ and control variable $u_n(t)$, we define the mth degree interpolating polynomials

$$x_i^m(t) = \sum_{l=0}^{m} a_{li}\phi_l(t), \quad (i = 0, 1, \ldots, N-1), \tag{3.5}$$

$$u_n^m(t) = \sum_{l=0}^{m} b_{li}\phi_l(t), \quad (n = 0, 1, \ldots, M-1), \tag{3.6}$$

where $I_m x_i(t) = x_i^m(t)$.

Let

$$\alpha = [\hat{a}_0\ \hat{a}_1\ \dots\ \hat{a}_m]^T = \begin{pmatrix} \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0(m)} \end{bmatrix}^T \\ \vdots \\ \begin{bmatrix} a_{(N-1)0} & a_{(N-1)1} & \dots & a_{(N-1)(m)} \end{bmatrix}^T \end{pmatrix}, \tag{3.7}$$

$$\beta = [\hat{b}_0\ \hat{b}_1\ \dots\ \hat{b}_m]^T = \begin{pmatrix} \begin{bmatrix} b_{00} & b_{01} & \dots & b_{0(m)} \end{bmatrix}^T \\ \vdots \\ \begin{bmatrix} b_{(M-1)0} & b_{(M-1)1} & \dots & b_{(M-1)(m)} \end{bmatrix}^T \end{pmatrix}, \tag{3.8}$$

and

$$\hat{\phi}_1(t) = \begin{pmatrix} \phi^T(t) & & 0 \\ & \ddots & \\ 0 & & \phi^T(t) \end{pmatrix}, \quad \hat{\phi}_2(t) = \begin{pmatrix} \phi^T(t) & & 0 \\ & \ddots & \\ 0 & & \phi^T(t) \end{pmatrix}, \tag{3.9}$$

where

$$\phi^T(t) = [\phi_0(t) \quad \phi_1(t) \quad \dots \quad \phi_m(t)]. \tag{3.10}$$

Note that $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ are $N \times N(m+1)$ and $M \times M(m+1)$ matrices, respectively.

Using (3.5)–(3.9), the state and the control vectors $x^m(t)$ and $u^m(t)$ can be expressed as

$$x^m(t) = \sum_{l=0}^{m} a_l \phi_l(t) = \hat{\phi}_1(t)\alpha, \tag{3.11}$$

$$u^m(t) = \sum_{l=0}^{m} b_l \phi_l(t) = \hat{\phi}_2(t)\beta, \tag{3.12}$$

where the unknown vectors a_l and b_l, $l = 0, 1, \dots, m$, are defined by

$$a_l = \begin{bmatrix} a_{0l} & a_{1l} & \dots & a_{(N-1)l} \end{bmatrix}^T, \quad b_l = \begin{bmatrix} b_{0l} & b_{1l} & \dots & b_{(M-1)l} \end{bmatrix}^T. \tag{3.13}$$

The relationship between $\dot{x}_i^m(t)$ and $x_i^m(t)$ at the Chebyshev nodes t_j, $j = 0, 1, \dots, m$, can be obtained by differentiating (3.5). The result is a matrix multiplication given in [12]

$$\dot{x}_i^m(t_k) = \sum_{l=0}^{m} D_{kl}^{(1)} a_{li}, \tag{3.14}$$

where $D^{(1)} = (D^{(1)}_{kl})$ is an $(m+1)\times(m+1)$ the pseudospectral Chebyshev first derivative matrix

$$D^{(1)} = \left(D^{(1)}_{kl}\right) = \begin{cases} \dfrac{C_k}{C_l}\dfrac{(-1)^{k+l}}{(t_k - t_l)} & k \neq l \\ \dfrac{2m^2+1}{6} & k = l = 0 \\ -\dfrac{2m^2+1}{6} & k = l = m \\ -\dfrac{t_l}{2\left(1-t_l^2\right)} & 1 \leq k = l \leq m-1. \end{cases} \tag{3.15}$$

Similarly, at the Chebyshev nodes t_j, the relationship between $\ddot{x}_i^m(t)$ and $x_i^m(t)$ is given in [12]

$$\ddot{x}_i^m(t_k) = \sum_{l=0}^{m} D^{(2)}_{kl} a_{li}, \quad i = 0, 1, \ldots, N-1, \tag{3.16}$$

where $D^{(2)} = (D^{(2)}_{kl})$ is an $(m+1)\times(m+1)$ the pseudospectral Chebyshev second derivative matrix

$$D^{(2)} = \left(D^{(2)}_{kl}\right) = \begin{cases} \dfrac{m^4-1}{15} & k = l;\, k = 0, m \\ -\dfrac{(m^2-1)\left(1-t_l^2\right)+3}{3\left(1-t_l^2\right)^2} & k = l;\, 1 \leq k \leq m-1 \\ \dfrac{2}{3}\dfrac{(-1)^l}{C_l}\dfrac{(2m^2+1)(1-t_l)-6}{(1-t_l)^2} & k = 0;\, 1 \leq k \leq m \\ \dfrac{2}{3}\dfrac{(-1)^{m+l}}{C_l}\dfrac{(2m^2+1)(1+t_l)-6}{(1+t_l)^2} & k = m;\, 0 \leq k \leq m-1 \\ \dfrac{(-1)^{k+l}}{C_l}\dfrac{t_k^2 + t_k t_l - 2}{\left(1-t_k^2\right)(t_k - t_l)^2} & 1 \leq k \leq m-1; \\ & \quad 0 \leq l \leq m-1;\ k \neq l. \end{cases} \tag{3.17}$$

In fact, the pseudospectral Chebyshev rth derivative matrix is $D^{(r)} = D^{(r)}_{jl} = \frac{d}{dt^r}\phi_l(t)|_{t=t_j}$.

In the discretization of the performance index J, we shall use the cell-averaging Chebyshev integration rule, stating that there exists an $m\times(m+1)$ matrix $(\bar{d}_{kl})$, $1 \leq k \leq m$, $0 \leq l \leq m$, such that for all $F \in C^r[-1,\ 1]$, $r > 0$, we have (see [7])

$$\begin{aligned} \int_{-1}^{1} F(s)\, ds = \sum_{k=1}^{m}\int_{t_k}^{t_{k-1}} F(s)\, ds &:= \sum_{k=1}^{m}\left[(t_{k-1} - t_k)\, \bar{F}_{k-\frac{1}{2}}\right], \\ &= \sum_{k=1}^{m}(t_{k-1} - t_k)\sum_{l=0}^{m}\bar{d}_{kl} F(t_l), \end{aligned} \tag{3.18}$$

where the cell-averages $\bar{F}_{\frac{1}{2}}, \bar{F}_{\frac{3}{2}}, \ldots, \bar{F}_{\frac{m-1}{2}}$ are related to $F_0 = F(t_0), \ldots, F_m = F(t_m)$ through the matrix $(\bar{d}_{kl})$. The entries of the cell averaging matrix $(\bar{d}_{kl})$, $1 \leq k \leq m, 0 \leq l \leq m$ are given by (see [7])

$$\bar{d}_{kl} = \bar{g}_l\big(t_{k-\frac{1}{2}}\big), \tag{3.19}$$

where

$$t_{k-\frac{1}{2}} = \cos\left[\frac{\big(k-\frac{1}{2}\big)\pi}{m}\right].$$

Finally, the cell averaging function $\bar{g}_l(t)$ is defined by

$$\bar{g}_l(t) := \frac{1}{m\, C_l}\left[1 + \sigma_1 T_1(t_l)U_1(t) + \sum_{k=2}^{m} \frac{T_k(t_l)(\sigma_k U_k(t) - \sigma_{k-2}U_{k-2}(t))}{C_k}\right], \tag{3.20}$$

where

$$\begin{aligned}
\sigma_\mu &= \frac{\sin\big((\mu+1)\frac{\pi}{2m}\big)}{(\mu+1)\sin\big(\frac{\pi}{2m}\big)},\\
U_\mu(t) &= \frac{1}{\mu+1}\dot{T}_{\mu+1}(t), \qquad \mu = 1, \ldots, m.
\end{aligned}$$

3.1. The polynomial interpolating operator

It is well known that polynomial interpolation based on Chebyshev nodes t_j is well behaved compared to that based on equally spaced points (consult [11, 13]). Clearly, I_m is a linear operator on $C = C[-1, 1]$, the Banach space of continuous, real-valued functions on $[-1, 1]$, with the property $I_m^2 = I_m$. This space is equipped with the uniform norm

$$\|F\|_\infty = \sup_{-1\leq t\leq 1} |F(t)|, \qquad F \in C. \tag{3.21}$$

Since I_m is a linear operator, with $I_m^2 = I_m$, then I_m is a projection operator, whose range is S_m, the set of all polynomials of degree $\leq m$. Furthermore, I_m is a bounded operator on C with

$$\|I_m\| = \sup_{-1\leq t\leq 1} \sum_{j=0}^{m} |\phi_j(t)|. \tag{3.22}$$

Since I_m is the interpolatory operator defined by (3.4), it follows that (consult [2, 13])

$$\lim_{m\to\infty} \int_{-1}^{1} |I_m F(t) - F(t)|^p (1-t^2)^{-1/2}\, dt = 0, \tag{3.23}$$

for every $F \in C$, and for every $p \in (0, \infty)$. In terms of the usual L_p norm (3.23) may be written as

$$\lim_{m\to\infty} \|I_m F(t) - F(t)\|_{p,w} = 0, \tag{3.24}$$

from which it follows that $\sup_m \|I_m F\|_{p,w} < \infty$.

For any positive integer m, let I_m be the orthogonal projection of L_2 onto the space spanned by $(\phi_0, \phi_1, \ldots, \phi_m)$. Then whenever $F \in H_w^l(-1, 1)$ for some $l \geq 1$, there is a constant M independent of F and m such that (see [8])

$$\|F - I_m F\|_{L_{2,w}(-1,1)} \leq M m^{-l} \|F\|_{H_w^l(-1,1)}. \tag{3.25}$$

The next theorem is a generalization of the last error estimate (see [8]):

Theorem 1. *For all* $F \in H_w^l(-1, 1), l \geq 0$, *there exist a constant* M *independent of* $F(t)$ *and* m *such that*

$$\|F - I_m F\|_{H_w^s(-1,1)} \leq M m^{2s-l} \|F\|_{H_w^l(-1,1)},$$

for all $0 \leq s \leq l$.

As a consequence, we have

$$\|F' - (I_m F)'\|_{L_{2,w}(-1,1)} \leq M m^{2-l} \|F\|_{H_w^l(-1,1)}. \tag{3.26}$$

The same estimate holds in the discrete $L_{2,w}$-norm at the collocation points t_j.

For the pseudospectral Chebyshev higher derivatives error estimates at t_j, we have

Theorem 2. *Let* $F(t)$ *have* σ *smooth derivatives for* $|t| \leq 1$. *Then there exists a constant* M *independent of* $F(t)$ *and* m *such that for all* $q < \frac{1}{2}\sigma$

$$\frac{\pi}{m} \sum_{K=0}^{q} \left[\sum_{j=0}^{m} \left(\frac{d^K h}{dt^K}(t_j) \right)^2 \right]^{1/2} \leq M m^{(2q-\sigma)} \|F\|_\sigma,$$

where $h(t) = F(t) - I_m F(t)$ and the norms $\|\cdot\|$, $\|\cdot\|_q$ are defined by

$$\|F\|^2 = \int_{-1}^{1} \frac{(F(t))^2}{\sqrt{1-t^2}}\, dt,$$

$$\|F\|_q^2 = \|F\|^2 + \left\| \frac{dF}{dt} \right\|^2 + \cdots + \left\| \frac{d^q F}{dt^q} \right\|^2.$$

Thus, if $F \in C^\infty$, then the rate of convergence of $I_m F$ to F is faster than any power of $\frac{1}{m}$. The next Theorem shows uniform convergence for the interpolating operator I_m (see [13]).

Theorem 3. *If t_j, $1 \le j \le m-1$ are the zeros of $\dot{T}_m(t)$ adjusted in the interval $(-1, 1)$, if $F(z)$ has no singularities except a finite number of poles, and if for some n, $\frac{F(z)}{z^n} \to 0$ as $|z| \to \infty$, then $I_m F(t) \to F(t)$ uniformly on $[-1, 1]$.*

Finally, since g in the functional (2.6) is continuously differentiable with respect to $x(t)$, $\dot{x}(t)$, and $u(t)$, then $g(x^m, \dot{x}^m, u^m, t) \to g(x, \dot{x}, u, t)$ as $m \to \infty$. From that it follows immediately that as $m \to \infty$, $J^m \to J$.

3.A. The system dynamics approximation. A formal substitution of (3.11), (3.12), (3.14) and (3.16) into the left-hand side of the (2.7) yields

$$f\left(\sum_{l=0}^{m} a_l \phi_l(t), \sum_{l=0}^{m} a_l \dot{\phi}_l(t), \sum_{l=0}^{m} a_l \ddot{\phi}_l(t), \sum_{l=0}^{m} b_l \phi_l(t), t, T\right). \tag{3.27}$$

Next, collocating at the Chebyshev nodes t_j, (3.27) becomes the system of algebraic equations:

$$A_j := f(a_j, d_j, c_j, b_j, t_j, T) = 0, \qquad j = 0, 1, \ldots, m. \tag{3.28}$$

In (3.28), the unknowns d_j and c_j are the jth component of $\hat{D}_1 \hat{a}$ and $\hat{D}_2 \hat{a}$, respectively, where

$$\hat{D}_1 = \begin{pmatrix} D^{(1)} & & 0 \\ & \ddots & \\ 0 & & D^{(1)} \end{pmatrix}, \quad \hat{D}_2 = \begin{pmatrix} D^{(2)} & & 0 \\ & \ddots & \\ 0 & & D^{(2)} \end{pmatrix}.$$

3.B. The performance index approximation. Substituting $x^m(t)$, $\dot{x}^m(t)$ and $u^m(t)$ in (2.6), the approximated performance index J^m has the form

$$\begin{aligned} J \approx J^m := {} & h(x^m(1), \dot{x}^m(1), T) \\ & + \int_{-1}^{1} g\left(\sum_{l=0}^{m} a_l \phi_l(t), \sum_{l=0}^{m} a_l \dot{\phi}_l(t), \sum_{l=0}^{m} b_l \phi_l(t), t, T\right). \end{aligned} \tag{3.29}$$

Setting $F(t) = g(\sum_{l=0}^{m} a_l \phi_l(t), \sum_{l=0}^{m} a_l \dot{\phi}_l(t), \sum_{l=0}^{m} b_l \phi_l(t), t, T)$ in (3.29) and using the cell-averaging Chebyshev integration rule (3.18), we obtain the following discretization of performance index J^m:

$$\begin{aligned} J^m &= h(x(1),\ \dot{x}(1),\ T) + \sum_{j=1}^{m} \int_{t_j}^{t_{j-1}} F(t)\, dt \\ &= h(x(1),\ \dot{x}(1),\ T) + \sum_{j=1}^{m} (t_{j-1} - t_j) \bar{F}_{j-1/2} \end{aligned}$$

$$= h(a_0,\ d_0,\ T) + \sum_{j=1}^{m}(t_{j-1} - t_j)\sum_{l=0}^{m}\bar{d}_{jl}\, g(a_l, d_l, b_l, t_l, T)$$
$$= h(a_0,\ d_0,\ T) + \sum_{l=0}^{m}\hat{d}_l\, g(a_l,\ d_l,\ b_l, t_l, T). \tag{3.30}$$

Note that $J^m = J^m(\alpha, \beta, T)$, where $\alpha,\ \beta$ are as in (3.7) and (3.8), and $\hat{d}_l = \sum_{j=1}^{m}(t_{j-1} - t_j)\,\bar{d}_{jl}$.

Finally, the equality and the inequality constraints (2.9) and (2.10) are incorporated in the above scheme as follows: Substituting $x^m(t)$, $u^m(t)$ and $\dot{x}^m(t)$ in (2.9) and collocating at the Chebyshev nodes t_j, yields the system of algebraic inequalities

$$B_j := s\left(\sum_{l=0}^{m} a_l\phi_l(t_j), \sum_{l=0}^{m} a_l\dot{\phi}_l(t_j), \sum_{l=0}^{m} b_l\phi_l(t_j)\right)$$
$$= s(a_j,\ d_j,\ b_j) \le 0, \quad j = 0,\ 1, \ldots,\ m, \tag{3.31}$$

and for the equality constraint, we get

$$C := z(a_0,\ d_0,\ 1) = 0, \tag{3.32}$$

where $d_j,\ j = 0,\ 1, \ldots,\ m$ are given in Section 3.A.

The optimal control problem has been converted into a nonlinear programming problem of the following general form: Given $a_m = x_{-1}$,

$$\begin{array}{ll} \text{Minimize} & J^N(\alpha,\ \beta,\ T), \\ \text{subject to} & A_l = 0,\, B_l \le 0, \quad l = 0, 1, \ldots, m, \quad \text{and} \\ & C\ = 0. \end{array}$$

Note that, the initial condition $d_m = \dot{x}^m(-1) = \sum_{l=0}^{m} D_{ml}^{(1)} a_l$ has been included in the formulation of A_m.

As the above theorems and error estimates promise us, the pseudospectral Chebyshev approximation of the NLP enjoys spectral accuracy, i.e., its truncation error decays as fast as the global smoothness of the underlying solution permits. However, it is essential to keep in mind that this superior accuracy cannot be realized in the presence of discontinuities, unless the pseudospectral Chebyshev solution is treated with an appropriate smoothing procedure in order to recover pointwise values within spectral accuracy.

4. Chebyshev smoothing procedure

In spectral theory, it is well known that spectral projections such as $I_m F$ provides highly accurate approximation of F, provided F itself is sufficiently smooth (see, Theorems 1 and 2 in Section 3). In fact, this projection enjoys spectral convergence rate. This superior accuracy is destroyed if F contains discontinuities; $I_m F$ produces spurious $O(1)$ Gibbs' oscillation which are *localized* in the neighborhoods of of the points of discontinuity, and moreover, their *global* accuracy is deteriorated to first order (see Example 4). To overcome this problem, we follow a similar treatment as in [1]. For simplicity of presentations and without loss of generality, we assume the function $F(t)$ has only one discontinuity at some unknown point t which we need to represent using a location index s; we therefore, denote this point by t_s. Note that in what follows, if s is an integer, then t_s is one of the preassigned Chebyshev nodes t_j, $0 \leq j \leq m$. On the other hand, if s is not an integer, then there exists a j, such that $t_s \in [t_j, t_{j-1}]$. See Example 4 for further illustrations.

Our basic interest is in the "inverse problems": Given the numerical results of the pseudospectral Chebyshev procedure, $F^m(t_j)$, which look oscillatory on the grids (collocation) points t_j—how do we extract out of this oscillatory solution the correct piecewise smooth solution (i.e., the location of the discontinuity, the magnitude of the jump and the structure of the smooth part of the solution)?

The approach we shall use to recover the "lost" information is as follows: Define the Heaviside function, $\Theta(t,\ t_s)$

$$\Theta(t,\ t_s) = \begin{cases} \theta_1 + \theta_2 & -1 \leq t \leq t_s \\ \theta_1 & t_s \leq t \leq 1, \end{cases} \tag{4.1}$$

where θ_1 is the state ahead of the discontinuity, and θ_2 is the magnitude of the jump discontinuity. The Chebyshev series approximation of the function $\Theta(t,\ t_s)$ has the form

$$\Theta_m(t,\ t_s) = \sum_{l=0}^{m} A_l(s) T_l(t), \tag{4.2}$$

where $T_l(t) = \cos[l \cos^{-1}(t)]$ is the Chebyshev polynomial of degree l, and the coefficients $A_l(s)$ are of course obtained using the orthogonality of Chebyshev polynomials:

$$\begin{aligned} A_0(s) &= \frac{\left(s + \frac{1}{2}\right)}{m} \\ A_l(s) &= \frac{\sin\left[\frac{\pi l}{m}(s + \frac{1}{2})\right]}{m \sin\left[\frac{\pi l}{2m}\right]}, \quad 1 \leq l \leq m-1 \\ A_m(s) &= \frac{\sin\left[s + \frac{1}{2}\right]}{2m}. \end{aligned} \tag{4.3}$$

Note that for a fixed integer s, simple calculations using $t_j = \cos(\frac{j\pi}{m})$ and (4.3) show that

$$\Theta(t_j,\ t_s) = \begin{cases} 1 & -1 \leq t_j \leq t_s \\ 0 & t_s \leq t_j \leq 1, \end{cases}$$

and therefore we have

$$\theta_1 + \theta_2 \Theta_m(t_j, t_s) = \Theta(t_j, t_s). \tag{4.4}$$

We now use the pseudospectral Chebyshev approximation $F^m(t)$ in order to remove the oscillations from the neighborhoods of the point of discontinuity t_s and accelerate the convergence rate. This can be done as follows: If we subtract from the discontinuous solution $F^m(t)$ the expansion (4.2) then $K^m(t) := F^m(t) - \Theta_m(t, t_s)$ is a continuous function because $F^m(t) - \Theta_m(t, t_s)$ does not have a jump anymore. It follows that $K^m(t)$ converges to $K(t)$ faster than $F^m(t)$ does to $F(t)$. We note that $F^m(t_j) - \Theta_m(t_j, t_s) = F^m(t_j) - (\theta_1 + \theta_2 \Theta_m(t_j, t_s))$.

The above observations lead to the following procedure that extracts from $F^m(t_j)$ which is oscillatory approximation to a (control or state) function $F(t)$ a piecewise smooth representation of $F(t)$. Given $F^m(t_j)$, we try to find an unknown smooth polynomial $\sum_{i=1}^{p} e_i T_i(t)$, $p < m$, and the Heaviside function $\Theta(t, t_s)$ with the unknown constants θ_1, θ_2 and an unknown location t_s, such that the L_2-norm

$$H := \sum_{j=0}^{m} \frac{1}{C_j} \left[F^m(t_j) - \theta_1 - \theta_2 \Theta_m(t_j, t_s) - \sum_{i=1}^{p<m} e_i T_i(t_j) \right]^2, \tag{4.5}$$

is minimized. Note that we have $p+3$ unknowns in (4.5): d_1, d_2, s and e_i $(1 \le i \le p \le m)$. Differentiating (4.5) with respect to the parameters d_1, d_2, s and e_i give

$$\frac{\partial H}{\partial \theta_1} = \sum_{j=0}^{m} \frac{1}{C_j} \left[F_j - \theta_1 - \theta_2 \Theta_j - \sum_{i=0}^{p} e_j T_i(t_j) \right] = 0, \tag{4.6}$$

where $F_j = F^m(t_j)$ and $\Theta_j = \Theta_m(t_j, t_s)$. Also

$$\frac{\partial H}{\partial \theta_2} = \sum_{j=0}^{m} \frac{1}{C_j} \left[F_j \Theta_j - \theta_1 \Theta_j - \theta_2 \Theta_j^2 - \sum_{i=0}^{p} e_j T_i(t_j) \Theta_j \right] = 0, \tag{4.7}$$

$$\frac{\partial H}{\partial s} = \sum_{j=0}^{m} \frac{1}{C_j} \left[\Theta'_j F_j - \Theta'_j \theta_1 - \Theta'_j \theta_2 \Theta_j - \Theta'_j \sum_{i=0}^{p} e_j T_i(t_j) \right] = 0, \tag{4.8}$$

where $\Theta'_j := \frac{\partial \Theta_m(t_j, t_s)}{\partial s} = \sum_{l=0}^{m} \frac{\partial A_l(s)}{\partial s} T_l(t)$.

Finally, $\frac{\partial H}{\partial e_\nu} = 0$, $\nu = 1, 2, \ldots, p$ gives

$$\sum_{j=0}^{m} \frac{1}{C_j} \left[F_j T_\nu(t_j) - \theta_1 T_\nu(t_j) - \theta_2 \Theta_j T_\nu(t_j) - T_\nu(t_j) \sum_{i=0}^{p} e_j T_i(t_j) \right] = 0. \tag{4.9}$$

Using the orthogonality relations for the Chebyshev polynomials we get the following system of $p+3$ nonlinear algebraic equations

$$e_\nu = \hat{F}_\nu - \theta_2 A_\nu, \tag{4.10}$$

$$\hat{F}_0 - \theta_2 A_0 - \theta_1 = 0, \tag{4.11}$$

$$\sum_{\nu=p+1}^{m} C_\nu A_\nu \hat{F}_\nu - \theta_2 \sum_{\nu=P+1}^{m} C_\nu A_\nu^2 = 0, \tag{4.12}$$

$$\sum_{\nu=p+1}^{m} C_\nu A'_\nu \hat{F}_\nu - \theta_2 \sum_{\nu=P+1}^{m} C_\nu A_\nu A'_\nu = 0, \tag{4.13}$$

where $\hat{F}_l = \frac{2}{mC_l} \sum_{j=0}^{m} \frac{1}{C_j} F(t_j) T_l(t_j)$, and $A'_l = \frac{\partial}{\partial s} A_l(s)$.

Next we combine the last two equations into a single nonlinear equation for the jump discontinuity location index, s:

$$\sum_{\nu=p+1}^{m} C_\nu A'_\nu \hat{F}_\nu \sum_{\nu=p+1}^{m} C_\nu A_\nu^2 - \sum_{\nu=P+1}^{m} C_\nu A_\nu \hat{F}_\nu \sum_{\nu=p+1}^{m} C_\nu A_\nu A'_\nu = 0. \tag{4.14}$$

The procedure for extracting the discontinuity location s, jump magnitude and smooth part of the solution from the data $F^m(t_j) = F(t_j)$ is as follows: The last equation is solved iteratively for s. Having found s, one immediately obtains from (4.3) all the $A'_l s$. Once θ_2 is found from (4.12), θ_1 can be evaluated from (4.11). Finally, from (4.10) we have the $e'_\nu s$.

5. Illustrative examples

In order to decide whether or not the computed solution is close enough to the optimal solution, we suggest, for computational purposes, practical and easy-to-use error estimations (see also, [14]):

Substituting the calculated $u^m(t)$ in (2.1) gives

$$f(x(t), \dot{x}(t), \ddot{x}(t), u^m(t), t, T) = 0, \quad -1 \le t \le 1. \tag{5.1}$$

Numerical integration of (5.1) is possible for a given initial or final conditions. Let $\hat{x}(t)$ be the solution obtained from numerical integration of (5.1), and define a practical easy-to-use error estimate for the the dynamical equations

$$\epsilon_{\rm dyn} := \|x^m - \hat{x}\|_\infty = \max_{-1\le t\le 1} |x^m(t) - \hat{x}(t)|. \tag{5.2}$$

Another important error estimate is the SAK

$$\mathrm{SAK} := \sum_{k=0}^{m} |A_k| = \sum_{k=0}^{m} |f(a_k, d_k, c_k, t_k, T) - b_k|. \tag{5.3}$$

For the performance index, we use the error estimate

$$|J^{r+1} - J^r|, \quad r = 1, 2, \ldots, m-1, \tag{5.4}$$

where $J^r := J(x^j(t_j), u^r(t_j), t_j, T)$.

An alternative empirical approach to verify the quality of the pseudospectral Chebyshev-based optimal control law is to check if it satisfies the necessary conditions for optimality which are derived by variational techniques

$$\left\| \frac{\partial H}{\partial u} \right\|_\infty = \max_{-1\le t\le 1} \left| \frac{\partial H}{\partial u}(t) \right| = 0,$$

where H is the Hamiltonian. In practice, this verification can be done by substituting the approximated optimal solution into an appropriate, standard, optimal algorithm and determining if the termination criterion of the selected algorithm can be satisfied. Thus, we can define the error on the necessary condition ENC $:= \|\frac{\partial H}{\partial u} - \frac{\partial H}{\partial u^m}\|_\infty$. Hence, based on the ENC, we may set and impose a precision parameter (PREC) on the unknowns α and β in order to stop the iterative procedure in nonlinear programming algorithms. Finally, we wish to note that a general purpose software package NLPQL [20] has been used to solve the test problems considered in this paper.

Example 1.

Case A. (Minimum time orbit transfer problem): Consider the problem of minimizing the transfer time of a constant low-thrust ion rocket between the orbits of Earth and Mars [3, 9, 10, 14, 23]. This involves the determination of the thrust angle history for which no exact solution is known. The transfer is governed by the following time-varying equations.

$$\dot{X}_1(\tau) = X_2(\tau), \quad 0 \le \tau \le T \quad \text{(free)}, \tag{5.5}$$

$$\dot{X}_2(\tau) = \frac{X_3^2(\tau)}{X_1(\tau)} - \frac{\gamma}{X_1^2(\tau)} + \frac{R_0 \sin U(\tau)}{m_0 + \dot{m}\tau}, \tag{5.6}$$

$$\dot{X}_3(\tau) = -\frac{X_2(\tau)X_3(\tau)}{X_1(\tau)} + \frac{R_0 \cos U(\tau)}{m_0 + \dot{m}\tau}, \tag{5.7}$$

where $\dot{X}_1(\tau)$ is the distance of the rocket from the Sun, $\dot{X}_2(\tau)$ is the radial velocity, and $\dot{X}_3(\tau)$ the tangential velocity. In (5.5)–(5.7), γ characterizes the gravitational attraction from the Sun, R_0 is the constant low-thrust magnitude, U is the control angle measured from the local horizonal, m_0, is the initial mass, and $\dot{m}$ is the constant propellant consumption rate. Using normalized values, we have $\gamma = 1$, $R_0 = 0.1405$, $m_0 = 1$ and $\dot{m} = -0.07487$. The time-unit is 58.18 days, according to the applied normalization. At the beginning of the maneuver the rocket is assumed to have velocity and radial position corresponding to the orbit of Earth, while at the end, the velocity and radial position correspond to the orbit of Mars. This leads to the following boundary conditions

$$\begin{aligned} X_1(0) &= 1, \quad X_2(0) = 0, \quad X_3(0) = 1, \\ X_1(T) &= 1.525, \quad X_2(T) = 0, \quad X_3(T) = 0.8098, \end{aligned} \tag{5.8}$$

Here T is the unknown final time to be minimized. Thus the performance index $J = T$.

Transforming the time interval $\tau \in [0,\ T]$ to $t \in [-1,\ 1]$ the system dynamics is replaced by

$$\dot{x}_1(t) = \frac{T}{2} x_2(t), \quad -1 \le t \le 1, \tag{5.9}$$

$$\dot{x}_2(t) = \frac{T}{2}\left(\frac{x_3^2(t)}{x_1(t)} - \frac{\gamma}{x_1^2(t)} + \frac{R_0 \sin u(t)}{m_0 + \dot{m}\frac{T}{2}(1+t)}\right), \tag{5.10}$$

$$\dot{x}_3(t) = \frac{T}{2}\left(-\frac{x_2(t)x_3(t)}{x_1(t)} + \frac{R_0 \cos u(t)}{m_0 + \dot{m}\frac{T}{2}(1+t)}\right), \tag{5.11}$$

with the boundary conditions

$$\begin{aligned} x_1(-1) &= 1, \quad x_2(-1) = 0, \quad x_3(-1) = 1, \\ x_1(1) &= 1.525, \quad x_2(1) = 0, \quad x_3(1) = 0.8098. \end{aligned} \tag{5.12}$$

Applying the proposed method to the system dynamics, we get for $k = 0, 1, \dots, m$

$$A_{0k} = \frac{2}{T}\left(\sum_{l=0}^{m} D_{kl}^{(1)} a_{1l}\right) - a_{2k} = 0, \tag{5.13}$$

$$A_{1k} = \frac{2}{T}\left(\sum_{l=0}^{m} D_{kl}^{(1)} a_{2l}\right) - \frac{a_{3k}^2}{a_{1k}} + \frac{\gamma}{a_{1k}^2} - \frac{R_0 \sin b_k}{m_0 + \dot{m}\left(\frac{T}{2}(1+t_k)\right)} = 0, \tag{5.14}$$

$$A_{2k} = \frac{2}{T}\left(\sum_{l=0}^{m} D_{kl}^{(1)} a_{3l}\right) + \frac{a_{2k}a_{3k}}{a_{1k}} - \frac{R_0 \cos b_k}{m_0 + \dot{m}\left(\frac{T}{2}(1+t_k)\right)} = 0. \tag{5.15}$$

The resulting NLP is: Given $a_{10} = 1.525,\ a_{1m} = 1,\ a_{20} = 0,\ a_{2m} = 0,\ a_{30} = 0.8098, a_{3m} = -1.$

$$\begin{aligned} &\text{Minimize} \quad J^m = T, \\ &\text{subject to} \quad A_{0k} = 0,\ A_{1k} = 0,\ A_{2k} = 0, \quad k = 0, 1, \dots, m. \end{aligned}$$

The nonlinear programming software package NLPQL developed in [20] is used to solve this NLP. The starting value for the final time T, and the starting values for a_{1r}, a_{2r}, a_{3r}, $r = 1, \dots, m-1$ and b_k, $k = 0, 1, \dots, m$, are chosen as follows: The final time is taken to

Table 1. Minimum time orbit transfer problem.

Methods	ϵ_{dyn}	SAIK	MEBC	PREC	ENC	J^m
Method of [23]						
$m = 7, K = 20$	0.58×10^{-2}	0.1×10^{-7}	$<10^{-13}$	—	—	3.33069
$m = 9, K = 30$	0.32×10^{-2}	0.24×10^{-7}	$<10^{-13}$	—	—	3.32263
$m = 11, K = 40$	0.11×10^{-2}	0.25×10^{-9}	$<10^{-13}$	—	—	3.31874
Method of [14]						
$m = 5$	4.1×10^{-4}	1.6×10^{-9}	0	10^{-6}	—	3.324016
$m = 7$	2.3×10^{-4}	0.87×10^{-10}	0	10^{-6}	—	3.319905
$m = 9$	2.0×10^{-4}	0.2×10^{-12}	0	10^{-6}	—	3.318742
Cell-averaging						
$m = 5$	0.21×10^{-4}	0.23×10^{-9}	0	10^{-4}	$<10^{-3}$	3.31860
$m = 7$	0.15×10^{-5}	0.71×10^{-10}	0	10^{-6}	$<10^{-5}$	3.318735
$m = 9$	0.32×10^{-5}	0.52×10^{-12}	0	10^{-8}	$<10^{-7}$	3.31873208

be $T = 3.00$ as in [14, 23]. If we assume that $x_1(t)$ behaves linearly between the specified initial and the terminal point, the starting values for a_{1l} can be determined. From (5.9) we drive the starting values for a_{2l}, and if we neglect the left-hand side, and the third term in (5.10), we obtain $a_3^2 = \frac{1}{a_1}$ from which starting values for a_{3l} can be derived. Finally, starting values for b_l can be determined by assuming that $u(t)$ varies linearly between 45 and 315°.

In Table 1 a comparison between the error estimate for the dynamical system ϵ_{dyn}, SAIK $= \sum_{k=0}^{m} |A_{ik}|$, the maximum error at the boundary conditions MEBC, the precision (PREC) imposed on α and β in order to stop the iterative procedure, the error on the necessary condition ENC $= \| \frac{\partial H}{\partial u} - \frac{\partial H}{\partial u^m} \|_\infty$ and the solutions of J obtained using the proposed method and the method of [14, 23].

From the results and the error estimate reported in Table 1, we conclude that the pseudospectral Chebyshev method gives the best results with minimal final time.

Case B. (Maximum radius orbit transfer problem): Given a constant low-thrust rocket operating for a given length of time T, we wish to find the thrust angle history to transfer this rocket from the orbit of Earth to the largest possible circular orbit. The system dynamics are described by (5.5)–(5.7), but the boundary conditions are now

$$X_1(0) = 1, \quad X_2(0) = 0, \quad X_3(0) = 1, \quad X_2(T) = 0, \quad X_3(T) = (X_1(T))^{-\frac{1}{2}}. \tag{5.16}$$

Here $X_1(T)$ is free and is to be maximized. Thus, the performance index is $J = X_1(T)$.

Table 2. Maximum radius orbit transfer problem.

Methods	E.T	$\epsilon_{\rm dyn}$	SAIK	MEBC	PREC	ENC	J^m
Method of [23]							
$m=7$	—	0.57×10^{-2}	—	$<10^{-13}$	—	—	1.521261
$m=9$	—	0.32×10^{-2}	—	$<10^{-13}$	—	—	1.524071
$m=11$	—	0.11×10^{-2}	—	$<10^{-13}$	—	—	1.525450
Cell-averaging							
$m=5$	3.82 s	0.19×10^{-4}	0.33×10^{-9}	0	10^{-4}	$<10^{-3}$	1.5251
$m=7$	4.22 s	0.20×10^{-5}	0.54×10^{-10}	0	10^{-6}	$<10^{-5}$	1.525398
$m=9$	4.90 s	0.31×10^{-5}	0.58×10^{-12}	0	10^{-8}	$<10^{-7}$	1.52545177

Applying a similar procedure as above, we obtain the constraints (5.13)–(5.15), and the NLP: Given $a_{1m}=1,\ a_{20}=0,\ a_{2m}=0,\ a_{30}=1,\ a_{3m}=(a_{10})^{-\frac{1}{2}}$,

$$\begin{aligned}&\text{Minimize} \quad J^m = a_{10}\\ &\text{subject to} \quad A_{0k}=0,\ A_{1k}=0 \text{ and } A_{2k}=0, \quad k=0,1,\ldots,m,\end{aligned}$$

where the constraints A_{ik}, $i=0,1,2$ are the same as (5.13)–(5.15).

In Table 2, the results from [23] are compared to our results. Note that our pseudospectral Chebyshev approximation J^5 is 0.25 percent larger than J^7 obtained in [23], and our boundary conditions are exactly satisfied.

Example 2. Consider the problem of transferring containers from a ship to a cargo truck (see [22]). The container crane is driven by a hoist motor and a trolley drive motor. The aim is to minimize the swing during and at the end of the transfer. After appropriate normalization, we summarise the problem as follows:

$$\text{Minimize} \quad J = 4.5\int_0^1 \left[X_3^2(\tau)+X_6^2(\tau)\right]d\tau, \tag{5.17}$$

subject to the dynamical equations

$$\begin{aligned}\dot{X}_1(\tau) &= 9X_4(\tau),\\ \dot{X}_2(\tau) &= 9X_5(\tau),\\ \dot{X}_3(\tau) &= 9X_6(\tau),\\ \dot{X}_4(\tau) &= 9[U_1(\tau)+X_3(\tau)],\\ \dot{X}_5(\tau) &= 9U_2(\tau),\\ \dot{X}_6(\tau) &= \frac{9(U_1(\tau)+27.0756X_3(\tau)+2X_5(\tau)X_6(\tau)}{X_2(\tau)}\end{aligned} \tag{5.18}$$

where

$$X(0) = [0, 22, 0, 0, -1, 0]^T,$$
$$X(1) = [10, 14, 0, 2.5, 0, 0]^T, \tag{5.19}$$

and

$$|U_1(\tau)| \leq 2.83374,$$
$$-0.80865 \leq U_2(\tau) \leq 0.71265, \quad \tau \in [0, 1] \tag{5.20}$$

with the continuous state inequality constraints

$$|X_4(\tau)| \leq 2.5,$$
$$|X_5(\tau)| \leq 1.0, \quad \tau \in [0, 1]. \tag{5.21}$$

Transforming the time interval $\tau \in [0, 1]$ to $t \in [-1, 1]$, then applying the proposed method to this nonlinear optimal control problem gives the following NLP: Given $(a_{10},\ a_{11},\ a_{12},\ a_{14},\ a_{15}) = (0, 22, 0, 0, -1, 0)$, and $(a_{m0},\ a_{m1},\ a_{m2},\ a_{m4},\ a_{m5}) = (10, 14, 0, 2.5, 0, 0)$,

$$\text{Minimize} \quad J^m = \frac{45}{20} \sum_{l=0}^{m} \hat{d}_l \left(a_{3l}^2 + a_{6l}^2\right)$$
$$\text{subject to} \quad A_{1l} = 0,\ A_{2l} = 0,\ A_{3l} = 0,\ A_{4l} = 0,\ A_{5l} = 0,\ A_{6l} = 0,$$

such that $b_{1l} \leq 2.83374$, $-b_{1l} \leq 2.83374$, $b_{2l} \leq 0.71265$, $-b_{2l} \leq 0.80865$, $a_{4l} \leq 2.5$, $-a_{4l} \leq 2.5$, $a_{5l} \leq 1.0$, $-a_{5l} \leq 1.0$, $l = 0, 1, \ldots, m$. In this NLP, the constraints A_{il}, $i = 1,\ 2, \ldots,\ 6$ are

$$A_{1l} = \sum_{l=0}^{m} D_{kl}^{(1)} a_{1l} - \frac{9}{2} a_{4l}, \quad A_{2l} = \sum_{l=0}^{m} D_{kl}^{(1)} a_{2l} - \frac{9}{2} a_{5l}$$
$$A_{3l} = \sum_{l=0}^{m} D_{kl}^{(1)} a_{3l} - \frac{9}{2} a_{6l}, \quad A_{4l} = \sum_{l=0}^{m} D_{kl}^{(1)} a_{4l} - \frac{9}{2} (b_{1l} + a_{3l}) \tag{5.22}$$
$$A_{5l} = \sum_{l=0}^{m} D_{kl}^{(1)} a_{5l} - \frac{9}{2} b_{2l}, \quad A_{6l} = \sum_{l=0}^{m} D_{kl}^{(1)} a_{6l} - \frac{9}{2} \frac{b_{1l} + 27.0756 a_{3l} + 2 a_{5l} a_{6l}}{a_{2l}},$$

The software package NLPQL developed in [20] is used to find the optimal solution for this NLP.

As can be seen from Table 3, the values of ϵ_{dyn} and SAK as m increases from $m = 5$ to $m = 11$ show that a good convergence rate has been achieved with zero MEBC. In addition, the error on the necessary conditions NEC of order $m = 11$, suggests that the optimal solution has been approximated within spectral accuracy. Note that the pseudospectral Chebyshev numerical results are superior to those reported in [22].

Table 3. Numerical results for Example 2.

Method	SAK	ϵ_{dyn}	ENC	PREC	E.T	J^m
Method of [22]						
$\epsilon = 10^{-3}, \gamma = 1 \times 10^{-4}$	—	—	—	—	—	0.5385×10^{-2}
$\epsilon = 10^{-4}, \gamma = 4 \times 10^{-5}$	—	—	—	—	—	0.5452×10^{-2}
$\epsilon = 10^{-5}, \gamma = 4 \times 10^{-6}$	—	—	—	—	—	0.5381×10^{-2}
$\epsilon = 10^{-6}, \gamma = 4 \times 10^{-7}$	—	—	—	—	—	0.5361×10^{-2}
Pseudospectral						
$m = 5$	$<10^{-5}$	$<10^{-3}$	10^{-3}	10^{-4}	1.48 s	0.5366×10^{-2}
$m = 7$	$<10^{-6}$	$<10^{-4}$	10^{-4}	10^{-6}	2.52 s	0.53614×10^{-2}
$m = 9$	$<10^{-6}$	$<10^{-4}$	10^{-4}	10^{-8}	3.0 s	$0.53610895 \times 10^{-2}$
$m = 11$	$<10^{-7}$	$<10^{-5}$	10^{-5}	10^{-10}	3.41 s	$0.5361102700 \times 10^{-2}$

Example 3. This example is adapted from [18].

$$\text{Minimize} \quad J = \frac{0.78}{2} \int_{-1}^{1} \left[x_1^2(t) + x_2^2(t) + 0.1u^2(t)\right] dt. \tag{5.23}$$

$$\text{subject to} \quad \dot{x}_1(t) = \frac{0.78}{2}\Bigg(-2[x_1(t) + 0.25] + [x_2(t) + 0.5] \exp\left[\frac{25x_1(t)}{x_1(t) + 2}\right] - [x_1(t) + 0.25]\, u(t)\Bigg), \tag{5.24}$$

$$\dot{x}_2(t) = \frac{0.78}{2}\left(0.5 - x_2(t) - [x_2(t) + 0.25] \exp\left[\frac{25x_1(t)}{x_1(t) + 2}\right]\right), \quad -1 \le t \le 1 \tag{5.25}$$

with the following initial conditions:

$$x_1(-1) = 0.05, \quad x_2(-1) = 0.00. \tag{5.26}$$

The state equations (5.23)–(5.25) of the system are highly nonlinear and coupled.

Applying the proposed method to the above optimal control problem gives the following NLP: Given $a_{10} = 0.05, \quad a_{20} = 0.00$,

$$\text{Minimize} \quad J^m = \frac{0.78}{2} \sum_{l=0}^{m} \hat{d}_l \left(a_{1l}^2 + a_{2l}^2 + 0.1b_l^2\right)$$

$$\text{subject to} \quad A_{1l} = 0 \text{ and } A_{2l} = 0, \quad l = 0, 1, \ldots, m,$$

Table 4. Numerical results for Example 3.

Method	MEBC	SAK	ϵ_{dyn}	ENC	PREC	E.T	J^m
Method of [18]	—	—	—	—	—	—	0.02662
Cell-averaging							
$m = 5$	0.00	$<10^{-5}$	$<10^{-3}$	10^{-4}	10^{-6}	2.2 s	0.026623
$m = 7$	0.00	$<10^{-6}$	$<10^{-4}$	10^{-4}	10^{-7}	2.54 s	0.0266214
$m = 9$	0.00	$<10^{-6}$	$<10^{-4}$	10^{-5}	10^{-8}	2.89 s	0.026621417
$m = 11$	0.00	$<10^{-7}$	$<10^{-5}$	10^{-6}	10^{-9}	3.10 s	0.0266214171

where A_{1l} and A_{2l} are given by

$$\begin{aligned} A_{1l} &= \sum_{l=0}^{m} D_{kl}^{(1)} a_{1l} - \frac{0.78}{2}\bigg(-2[a_{1l}+0.25] + [a_{2l}+0.5]\exp\left[\frac{25a_{1l}}{a_{1l}+2}\right] \\ &\qquad - [a_{1l}+0.25]b_l \bigg), \qquad (5.27) \\ A_{2l} &= \sum_{l=0}^{m} D_{kl}^{(1)} a_{2l} - \frac{0.78}{2}\left(0.5 - a_{2l} - [a_{2l}+0.25]\exp\left[\frac{25a_{1l}}{a_{1l}+2}\right]\right). \end{aligned}$$

This numerical evidence supports the assumption that the pseudospectral Chebyshev method is spectrally accurate and is cabable of solvig highly nonlinear coupled dynamical equations (see Table 4). Moreover, these numerical results show that the pseudospectral Chebyshev numerical results are superior to those reported in [18].

Example 4. This example considers a bang-bang control problem, adapted from [18].

$$\text{Minimize} \quad J = \frac{1}{2}\int_0^5 [X(\tau)^2 + \dot{X}(\tau)^2]\, d\tau, \qquad (5.28)$$

$$\text{subject to} \quad \ddot{X}(\tau) - \dot{X}(\tau) + X(\tau) = U(\tau), \quad \tau \in [0,\ 5], \qquad (5.29)$$

$$X(0) = 0.231, \quad \dot{X}(0) = 1.126, \qquad (5.30)$$

$$-0.8 \le U(\tau) \le 0.8. \qquad (5.31)$$

This example is chose to illustrate our smoothing procedure developed in Section 4. It serves as an appropriate test case, for it admits a finite jump which originates at $t_s \in (0,\ 5)$. Applying a similar procedure as above, we obtain the NLP: Given $a_m = 0.231$, $c_m = 1.126$,

$$\text{Minimize} \quad J^m = \frac{1}{4}\sum_{l=0}^{m} \hat{d}_l\left(a_l^2 + c_l^2\right)$$

$$\text{subject to} \quad A_l = 0,\ b_l \le 0.8 \text{ and } b_l \ge -0.8, \quad l = 0,\ 1, \ldots,\ m,$$

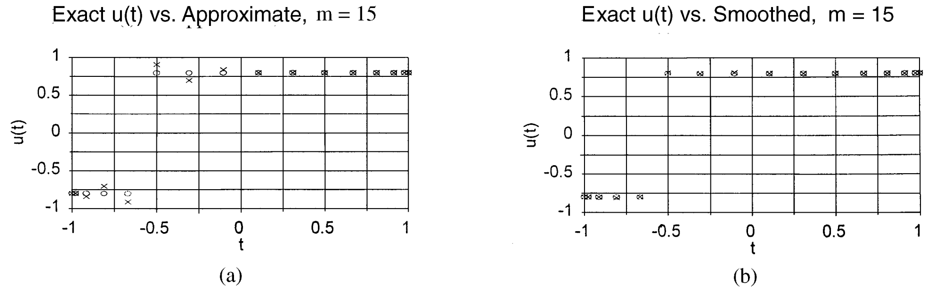

Figure 1.

where A_l are given by

$$A_l = \sum_{l=0}^{m} D_{kl}^{(2)} a_l - \sum_{l=0}^{m} D_{kl}^{(1)} a_l + a_l - b_l. \tag{5.32}$$

The nonlinear programming algorithm developed in [20] is used to solve the resulting NLP. All computations and the error estimates have been computed with very high precision on a Sun-SPARC-II workstation.

Figures 1(a) and (b) display the computed raw (without Chebyshev smoothing procedure) control variable $u^{15}(t_k)$, the treated pseudospectral Chebyshev optimal solution $u_*^{15}(t_k)$ and the exact control variable $u(t_k)$. Note that the raw results, $u^{15}(t_k)$, reported in figure 1(a) seem to indicate roughly the location of the point of jump discontinuity between the grid points t_8 and t_{11} with magnitude $\approx$1.82. In fact, the correct location of the jump discontinuity is at $t_s = t_{10} = -0.5$ with magnitude $\theta_2 = 1.6$ and with $\theta_1 = 0.8$.

The first result we present, figure 1(a), shows the necessity of the Chebyshev smoothing procedure. Indeed, the raw pseudospectral Chebyshev solution, or the raw results $u^{15}(t_k)$, see figure 1(a), is inconsistent with the exact optimal control $u(t)$ near the point of discontinuity, hence it fails to converge to the exact solution. As a result, the standard pseudospectral Chebyshev approximation J^{15} is not even close to the exact value J.

Applying the Chebyshev smoothing procedure, with $(\text{PREC} = 10^{-5})$, to the raw results $u^{15}(t_k)$ gives the treated pseudospectral Chebyshev optimal solution $u_*^{15}(t_k)$ which is recorded in figure 1(b). The location index s of the jump discontinuity is found to be $9.99896 \approx 10$ as we expected. Once the jump discontinuity location index s has been found, Eqs. (4.3) give all A_l', $l = 0, \ldots, 15$, and Eq. (4.12) gives the magnitude of the jump θ_2. Thus, once s and θ_2 are calculated, one can see that $u(t)$ has a jump discontinuity at $t_s = -0.4998 \approx t_{10} = -0.5$, the magnitude of the jump $\theta_2 \approx 1.5781$ with error 2.2×10^{-2}, and the state ahead $\theta_1 = 0.8064$ with error 6.4×10^{-3}.

From the results reported in Table 5, we see that treated performance index J_*^{15} is indeed much more accurate than the standard value J^{15}. The dramatic improvement in the convergence rate is evident. Moreover, These numerical results indicate the strong L_r-convergence $(r < \infty)$ for the treated pseudospectral Chebyshev method, in contrast to the

Table 5. Numerical results for Example 4.

m	E.T	$e^m_{\max}$	$\hat{e}^m_{\max}$	MEBC	ENC	ENC_*	$\lvert J^m - J\rvert$	PREC	$\lvert J^m_* - J\rvert$
15	3.41 s	$<10^{-5}$	$<10^{-4}$	0.0	4.3×10^{-1}	$<10^{-4}$	1.18×10^{-1}	10^{-4}	$<10^{-4}$
20	3.88 s	$<10^{-7}$	$<10^{-5}$	0.0	6.8×10^{-2}	$<10^{-5}$	3.2×10^{-2}	10^{-6}	$<10^{-5}$

standard pseudospectral Chebyshev method. Clearly, figure 1(b) shows a dramatic improvement in the convergence rate as well as the accuracy. Indeed, figure 1(b) also shows that the smoothing procedure recovers the smooth parts of the exact solution within spectral accuracy. However, the resolution of the treated pseudospectral Chebyshev solutions still suffers from smearing of spurious Gibbs' oscillations with very small magnitude. In fact, these oscillations would not be discerned by eye as we proceed to higher order approximations.

In Table 5, we report the maximum errors $e^m_{\max} := \max\lvert x^m_*(t_j) - x(t_j)\rvert,\ j = 0, 1, \ldots, m$ and $\hat{e}^m_{\max} := \max\lvert u^m_*(t_j) - u(t_j)\rvert$, the maximum error at the boundary conditions (MEBC), the errors on the necessary conditions ENC for the plain (untreated) optimal control, the errors on the necessary conditions ENC_* for the treated optimal control, the precision (PREC) imposed on the location index s, in order to stop the iterative procedure, $\lvert J^m - J\rvert$ and $\lvert J^m_* - J\rvert$ for $m = 15, 20$. We should mention that pseudospectral Chebyshev approximations of the performance index J of order $m = 15, 20$ were found to be $J^{15}_* = 5.86337$ with (PERC $= 10^{-5}$) and $J^{20}_* = 5.8633720$ with (PREC $= 10^{-7}$). Comparing our results with those of [18], we see that that our J^{15}_* is at least 25 percent smaller.

Remarks. Based on the above results, one can draw the following conclusion:

(i) the convergence rate of the treated pseudospectral Chebyshev solution is faster than any finite value, in agreement with the piecewise C^∞-regularity of the nonsmooth solution;
(ii) spectral convergence rate can be observed already for quite low values of the parameter m.

Conclusions

In this paper, the pseudospectral Chebyshev method has been used to generate the optimal solution of nonlinear constrained dynamical systems. With the availability of this methodology, it will now become possible to investigate the pseudospectral solution of general nonlinear optimal control problems which may describe many nonlinear Physical and Engineering problems. We conclude by noting that the above Chebyshev smoothing procedure of the pseudospectral Chebyshev solution plays a necessary key role in realizing the spectral accuracy of the pseudospectral Chebyshev method in smooth regions of the underlying solution.

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