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Chebyshev differentiation matrices for efficient computation of the eigenvalues of fourth-order Sturm–Liouville problems

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ABSTRACT

In this paper, an efficient technique based on the Chebyshev spectral collocation method for computing the eigenvalues of fourth-order Sturm–Liouville boundary value problems is proposed. The excellent performance of this scheme is illustrated through four examples. Numerical results and comparison with other methods are presented.

1. Introduction

The Sturm–Liouville problems arise throughout applied mathematics, classical and quantum mechanics. Most of physical phenomena, can describe by PDEs in several dimensions. This leads to a Sturm–Liouville problem when the equations are separable. The Sturm–Liouville boundary value problems for ODEs play an important role in both the theory and applications of physical, biological and chemical phenomena.

The general form of the second-order Sturm–Liouville equation is an ODE for $y(x)$ of the form

$$-(q_1(x)y'(x))' - q_2(x)y(x) = \lambda w_1(x)y(x), \quad x \in (a,b), \tag{1.1}$$

subject to some two point specified conditions at the boundary $x \in \{a, b\}$ on y and/or y' [1,2]. The second-order Sturm–Liouville problems are more frequently available in the literature. Although for second-order problems there are many software's like SLEIGN [3], SLEIGN2 [4], SLEGDGE [5] and MATSLISE [6] available, not much work has done for fourth-order problems. The only code available in this regard is the Sturm–Liouville Eigenvalues Using Theta matrices (SLEUTH) [7]. Attili and Lesnic [8] used the Adomian decomposition method (ADM), while Syam and Siyyam [9] developed a numerical technique based on variational iteration method (VIM) and Chanane [10] proposed the Extended Sampling Method (ESM) for computing the eigenvalues of fourth-order Sturm–Liouville boundary value problems. Recently, Yücel and Boubaker [11] applied the differential quadrature method (DQM) and the Boubaker Polynomials Expansion Scheme (BPES) to compute the eigenvalues of some regular fourth-order Sturm–Liouville problems.

In this paper, we propose a new technique based on Chebyshev differentiation matrix for computing eigenvalues of the following fourth-order Sturm–Liouville problem:

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$$(p_1(x)y''(x))'' = (s(x)y'(x))' + (\lambda w(x) - q(x))y(x), \quad x \in (a,b), \tag{1.2}$$

subject to some four point specified conditions at the boundary $x \in \{a,b\}$ on $y, y', \ p_1(x)y''$ and/or $(p_1(x)y'')' - s(x)y'$. Problem (1.2) is regular if (a,b) is finite and $p_1(x),\ q(x),\ s(x)$ and $w(x)$ are piecewise continuous functions and $p_1(x),\ w(x)$ are positive; otherwise it is singular. More information on the mathematical theory of fourth-order Sturm–Liouville problems may be found in Refs. [7,12,13]. In applications, there are three types of boundary conditions commonly used with Eq. (1.2). In this paper we will using the following two types of boundary conditions:

$$y = 0, \quad y' = 0, \tag{1.3}$$

at endpoints, namely the clamped conditions, or the hinged conditions

$$y = 0, \quad y'' = 0. \tag{1.4}$$

Moreover, a combination of these two types of boundary conditions is used.

Greenberg [12,13], showed that the problem (1.2) has an infinite sequence of eigenvalues $\{\lambda_k\}_{k\geqslant 1}$ which are bounded from below. These eigenvalues can be ordered as an increasing sequence, i.e.,

$$\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots,$$

where $\lim_{k\to\infty} \lambda_k = \infty$ and each eigenvalue has multiplicity at most 2.

Here, we propose a novel technique based on Chebyshev spectral differentiation matrices for solving problem (1.2) with boundary conditions (1.3) or (1.4). Spectral methods are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns. The three most widely used spectral versions are the Galerkin, collocation, and tau methods. Spectral collocation method have become increasingly popular for solving differential equations and also very useful in providing highly accurate solutions to differential equations [14,15]. In Refs. [16,17] the Chebyshev collocation method is proposed to approximate eigenvalues of second-order Sturm–Liouville problems, while mapped barycentric Chebyshev differentiation matrix method is given in Ref. [18] to solve second-order Sturm–Liouville problems. Phillips and Malek [19] presented pseudospectral collocation schemes to solve linear fourth order differential equations in one and two dimensions. Sari and Butcher [20] investigated the effects of damaged boundaries on natural frequencies and critical loads of beams and columns of variable cross section with conservative and non-conservative loads using Chebyshev polynomials. Comparison of perturbation and spectral collocation for the free vibration analysis of Kirchhoff plates with both partially and completely damaged boundaries is given in Ref. [21]. Sari [22] has developed a numerical technique based on Chebyshev collocation method for the free vibration analysis of isotropic rectangular and annular Mindlin plates with damaged boundaries. In Ref. [23], a collocation expansion at the Chebyshev collocation points, is used for the stability analysis of linear, time-periodic delay-differential equations and of a single degree-of-freedom model for the milling process in particular. Khasawneh et al. [24] investigated the stability of periodic delay systems with non-smooth coefficients using a multi-interval Chebyshev collocation approach. A spectral element approach to study the stability and equilibria solutions of delay differential equations is given in Ref. [25]. Accuracy, resolution and stability of Chebyshev collocation method has been widely studied in the literature (see for example [26,27]). The Chebyshev differentiation interpolation matrix studied systematically by Gottlieb et. al. [28], Solomonoff and Turkel [29], and Peyret [30]. In the year 2000, Trefethen [31] give a MATLAB code to solve fourth-order differential equations equipped with only the clamped boundary conditions. Weideman and Reddy [32] a MATALAB differentiation matrix suite which includes a function for computing fourth-order derivatives based on pseudospectral method. On the best knowledge of the authors there is no Chebyshev differentiation matrices solution to a general class of regular fourth-order Sturm–Liouville problems. The current method here is able to deal with general kind of the boundary conditions, since the boundary conditions are considered directly inside the generalized eigenvalue problem. Here, it is shown that the proposed technique has ability to solve regular fourth-order Sturm–Liouville problems, efficiently. Moreover, this method finds the high index eigenvalues with less computational costs (see Tables 1, 2 and 4).

The paper is organized as follows. The Chebyshev spectral collocation method, its convergence and stability is summarized in Section 2. Four numerical examples are discussed in Section 3 and some conclusions are drawn in Section 4.

2. Chebyshev spectral collocation

First, let us rewrite Eq. (1.2) in the following form

$$y^{(4)} + 2P_1(x)y''' + P_2(x)y'' - S(x)y' + Q(x)y = \lambda W(x)y, \tag{2.1}$$

where $P_1(x) = \frac{p_1'(x)}{p_1(x)}$, $P_2(x) = \frac{p_1''(x) - s(x)}{p_1(x)}$, $S(x) = \frac{s'(x)}{p_1(x)}$, $Q(x) = \frac{q(x)}{p_1(x)}$ and $W(x) = \frac{w(x)}{p_1(x)}$ are defined on the interval of $-1 < x < 1$ and λ is a parameter independent of x.

The Chebyshev collocation points are unevenly spaced points in the domain $[-1,1]$ corresponding to the extremum points of the Chebyshev polynomial of the first kind of degree N, [14,15,20,24,31,32].

Consider the basis functions ϕ_j that are polynomials of degree $\leqslant N$ satisfying $\phi_j(x_k) = \delta_{j,k}$ for the Chebyshev nodes which we can define as the projections of equispaced points on the upper half of the unit circle as

Table 1
The first forty-one eigenvalues and the related absolute error for Example 1.

k	λ_{Exact}	$N = 40$ λ_k	$N = 50$ λ_k	$N = 60$ λ_k	$N = 70$ λ_k	$N = 80$ λ_k	$N = 90$ λ_k	$N = 100$ λ_k	Absolute error
1	1	1	1	1	1	1	1	1	0
2	16	16	16	16	16	16	16	16	0
3	81	81	81	81	81	81	81	81	0
4	256	256	256	256	256	256	256	256	0
5	625	625	625	625	625	625	625	625	0
6	1296	1296	1296	1296	1296	1296	1296	1296	0
7	2401	2401	2401	2401	2401	2401	2401	2401	0
8	4096	4096	4096	4096	4096	4096	4096	4096	0
9	6561	6561	6561	6561	6561	6561	6561	6561	0
10	10000	10000	10000	10000	10000	10000	10000	10000	0
11	14641	14641	14641	14641	14641	14641	14641	14641	0
12	20736		20736	20736	20736	20736	20736	20736	0
13	28561		28561	28561	28561	28561	28561	28561	0
14	38416		38416	38416	38416	38416	38416	38416	0
15	50625		50625	50625	50625	50625	50625	50625	0
16	65536		65536	65536	65536	65536	65536	65536	0
17	83521			83521	83521	83521	83521	83521	0
18	104976			104976	104976	104976	104976	104976	0
19	130321			130321	130321	130321	130321	130321	0
20	160000			160000	160000	160000	160000	160000	0
21	194481			194481	194481	194481	194481	194481	0
22	234256				234256	234256	234256	234256	0
23	279841				279841	279841	279841	279841	0
24	331776				331776	331776	331776	331776	0
25	390625				390625	390625	390625	390625	0
26	456976				456976	456976	456976	456976	0
27	531441					531441	531441	531441	0
28	614656					614656	614656	614656	0
29	707281					707281	707281	707281	0
30	810000					810000	810000	810000	0
31	923521					923521	923521	923521	0
32	1048576						1048576	1048576	0
33	1185921						1185921	1185921	0
34	1336336						1336336	1336336	0
35	1500625						1500625	1500625	0
36	1679616						1679616	1679616	0
37	1874161							1874161	0
38	2085136							2085136	0
39	2313441							2313441	0
40	2560000							2560000	0
41	2825761							2825761	0

Table 2
Some eigenvalues and the related absolute error for Example 1 using $N = 250$.

k	λ_k	Absolute error	k	λ_k	Absolute error
100	100000000	0	110	146410000	0
101	104060401	0	111	151807041	0
102	108243216	0	112	157351936	0
103	112550881	0	113	163047361	0
104	116985856	0	114	168896016	0
105	121550625	0	115	174900625	0
106	126247696	0	116	181063936	0
107	131079601	0	117	187388721	0
108	136048896	0	118	193877776	0
109	141158161	0	119	200533921	0

$$x_k = cos\left(\frac{k\pi}{N}\right), \quad k = 0, \ldots, N, \tag{2.2}$$

where the number of collocation points used is $N + 1$. A spectral differentiation matrix for the Chebyshev collocation points is obtained by interpolating a polynomial through the collocation points, i.e. the polynomial

$$p(\mathbf{x}) = \sum_{j=0}^{N} \phi_j y_j, \tag{2.3}$$

interpolates the points (x_j, y_j), such that

$$p(\mathbf{x}) = \mathbf{y}, \tag{2.4}$$

where $\mathbf{x} = (x_0, x_1, \dots, x_N)^T$ and $\mathbf{y} = (y_0, y_1, \dots, y_N)^T$.

The dth derivative of the interpolating polynomial at the nodes is given by [31,32]

$$p^{(d)}(\mathbf{x}) = D^{(d)}\mathbf{y}, \tag{2.5}$$

where the i,jth element of the differentiation matrices $D^{(d)}$ is $\phi_j^{(d)}(x_i)$. For each $N \geqslant 1$, let the rows and columns of the $(N+1) \times (N+1)$ Chebyshev differentiation matrix D_N be indexed from 0 to N. Then the entries of the matrix are [31]

$$\begin{aligned} &(D_N)_{00} = \frac{2N^2+1}{6}, \quad (D_N)_{NN} = -\frac{2N^2+1}{6}, \\ &(D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}, \quad j = 1, \dots, N-1, \\ &(D_N)_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - xj)}, \quad i \neq j, \quad j = 1, \dots, N-1, \end{aligned} \tag{2.6}$$

where

$$c_i = \begin{cases} 2, & i = 0 \text{ or } N, \\ 0, & \text{otherwise}. \end{cases}$$

Evaluating Eq. (2.1) at each interior node $x_k,\ k = 0, 1, \dots, N$, leads to

$$y^{(4)}(x_k) + 2P_1(x_k)y'''(x_k) + P_2(x_k)y''(x_k) - S(x_k)y'(x_k) + Q(x_k)y(x_k) = \lambda W(x_k)y(x_k). \tag{2.7}$$

The interpolating polynomial $p(\mathbf{x})$ is to satisfy Eq. (2.1) at each interior node. Therefore, we have the following collocation equations

$$p^{(4)}(x_k) + 2\hat{P}_1(x_k)p'''(x_k) + \hat{P}_2(x_k)p''(x_k) - \widehat{S}(x_k)p'(x_k) + \widehat{Q}(x_k)p(x_k) = \lambda\widehat{W}(x_k)p(x_k), \tag{2.8}$$

where $\hat{P}_1 = diag(P_1),\ \hat{P}_2 = diag(P_2)$, $\widehat{S} = diag(S),\ \widehat{Q} = diag(Q)$ and $\widehat{W} = diag(W)$. Here, we see that the

$$p^{(4)}(x) = \sum_{j=1}^{N-2} \phi_j^{(4)}(x_k)y_j. \tag{2.9}$$

For the boundary conditions (1.3), we have

$$p(-1) = p'(-1) = 0, \quad p(1) = p'(1) = 0 \tag{2.10}$$

and for the hinged boundary conditions (1.4), we get

$$p(-1) = p''(-1) = 0, \quad p(1) = p''(1) = 0, \tag{2.11}$$

where $p''(-1) = \sum_{j=1}^{N} \phi_j''(-1)y_j$ and $p''(1) = \sum_{j=1}^{N} \phi_j''(1)y_j$. Rewriting Eq. (2.8) in the differentiation matrix form, we get

$$\left(D^{(4)} + 2\hat{P}_1 D^{(3)} + \hat{P}_2 D^{(2)} - \widehat{S}D^{(1)} + (\widehat{Q} - \lambda\widehat{W})\right)\mathbf{y} = 0. \tag{2.12}$$

The boundary conditions in (2.10) and (2.11) are modeled in the following forms respectively

$$I_{-1}\mathbf{y} = D_{-1}^{(1)}\mathbf{y} = 0, \quad I_1\mathbf{y} = D_1^{(1)}\mathbf{y} = 0 \tag{2.13}$$

and

$$I_{-1}\mathbf{y} = D_{-1}^{(2)}\mathbf{y} = 0, \quad I_1\mathbf{y} = D_1^{(2)}\mathbf{y} = 0. \tag{2.14}$$

Inserting four boundary conditions (2.13) or (2.14) into (1.2), yields generalized eigenvalue problems

$$\begin{pmatrix} D^{(4)} + 2\hat{P}_1 D^{(3)} + \widehat{P_2}D^{(2)} - \widehat{S}D^{(1)} + \widehat{Q} \\ I_{-1} \\ D_{-1} \\ I_1 \\ D_1 \end{pmatrix} \mathbf{y} = \lambda \begin{pmatrix} \widehat{W}I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix} \mathbf{y}, \tag{2.15}$$

or

$$\begin{pmatrix} D^{(4)} + 2\hat{P}_1 D^{(3)} + \hat{P}_2 D^{(2)} - \widehat{S} D^{(1)} + \widehat{Q} \\ I_{-1} \\ D^{(2)}_{-1} \\ I_1 \\ D^{(2)}_1 \end{pmatrix} \mathbf{y} = \lambda \begin{pmatrix} \widehat{W} I \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \mathbf{0}^T \end{pmatrix} \mathbf{y}, \tag{2.16}$$

respectively, where indices -1 and 1 represented boundary conditions in endpoints -1 and 1, respectively. Now, the approximate eigenvalues of Sturm–Liouville problem (1.2) can be obtained by solving the generalized eigenvalue problem (2.15) or (2.16). This shows the effect of boundary conditions in the model.

2.1. Convergence and stability

The Chebyshev polynomials of the first kind, $\phi_j(x)$, are the eigenfunctions of the singular Sturm–Liouville problems

$$((1-x^2)^{1/2}\phi_j'(x))' + \frac{j^2}{(1-x^2)^{1/2}}\phi_j(x) = 0, \quad x \in (-1,1), \tag{2.17}$$

according to Eq. (1.1), $q_1(x) = (1-x^2)^{1/2}, \; q_2(x) = 0$ and $w_1(x) = \frac{1}{(1-x^2)^{1/2}}$. If ϕ_j is normalized so that $\phi_j(1) = 1$, then

$$\phi_j(x)) = cos(k\theta), \quad \theta = cos^{-1}x. \tag{2.18}$$

Thus, the Chebyshev polynomials are cosine functions after a change of independent variable. This property is the origin of their widespread popularity in the numerical approximation of non-periodic boundary value problems [15]. We denote by $\mathbb{P}_N$ the space of Chebyshev polynomials ϕ_j of degree $\leqslant N$. These polynomials are orthogonal over the interval $(-1,1)$ with respect to a weight function $w_1(x)$, i.e.

$$\int_{-1}^{1} \phi_j(x)\phi_k(x)w_1(x)dx = 0, \quad if\ j \neq k. \tag{2.19}$$

It is will known that such a system is complete in the space $L^2_{w_1}(-1,1)$. This space is the space of functions $f(x)$ on $(-1,1)$ such that

$$\int_{-1}^{1} |f(x)|^2 w_1(x)dx < \infty. \tag{2.20}$$

$L^2_{w_1}(-1,1)$ is a Hilbert space with inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)w_1(x)dx \tag{2.21}$$

and norm

$$\|f\|_{L^2_{w_1}(-1,1)} = \left(\int_{-1}^{1} |f(x)|^2 w_1(x)dx\right)^{1/2}. \tag{2.22}$$

Now, if the $y(x) \in L^2_{w_1}(-1,1)$ is an eigenfunction of Sturm–Liouville problem, then the formal series of it in terms of the ϕ_j is

$$y(x) = \sum_{j=0}^{\infty} \phi_j \hat{y}_j, \tag{2.23}$$

where the expansion coefficients $\hat{y}_j$ are determined as

$$\hat{y}_j = \frac{\int_{-1}^{1} y(x)\phi_j(x)w_1(x)dx}{\|\phi_j\|^2_{L^2_{w_1}(-1,1)}}. \tag{2.24}$$

An approximate solution $\tilde{y}(x)$ is sought in the truncated Chebyshev series form

$$P_N\tilde{y} = \sum_{j=0}^{N} \hat{y}_j \phi_j. \tag{2.25}$$

Due to (2.19), P_N is the orthogonal projection of y upon $\mathbb{P}_N$ in the inner product (2.21), that is,

$$(P_N\tilde{y}, \psi)_{w_1} = (y,\psi)_{w_1}, \quad \forall \psi \in \mathbb{P}_N. \tag{2.26}$$

The completeness of the ϕ_j is equivalent to the property that,

$$\|y - P_N\tilde{y}\|_{w_1} \to 0, \quad as\ N \to \infty, \quad \forall y \in L^2_{w_1}(-1,1). \tag{2.27}$$

It is shown that the Chebyshev differentiation matrix method can be viewed as the *hp* version of the finite element method [15,31,18]. Following this process we may anticipate an error estimation of the form $O(k^p e^{-cN})$, for the kth eigenvalue.

3. Numerical results

In this Section, to confirm the efficiency and accuracy of the proposed technique that outlined in the previous section, four numerical examples of fourth-order Sturm–Liouville problems are presented. Here, we made use of the MATLAB 7 package to calculate the eigenvalues of generalized eigenvalue problem (2.15) and (2.16). The computational codes were conducted on an Intel (R) Core (TM) Duo CPU with power 2.10 GHz CPU, equipped with 2 GB of Ram.

Example 1. Consider the following fourth-order Sturm–Liouville problem

$$\begin{cases} y^{(4)}(x) = \lambda y(x), & x \in (0,\pi), \\ y(0) = y''(0) = 0, & y(\pi) = y''(\pi) = 0. \end{cases} \tag{3.1}$$

With the change of variables $x = \frac{\pi X}{2} + \frac{\pi}{2}$, the Sturm–Liouville problem (3.1)transforms to

$$\begin{cases} \frac{16}{\pi^4} Y^{(4)} = \lambda Y, & X \in (-1,1), \\ Y(-1) = Y''(-1) = 0, & Y(1) = Y''(1) = 0. \end{cases} \tag{3.2}$$

As the interpolating polynomial have to satisfy the differential equation at each interior node, in this example we obtain the following collocation equation

$$\frac{16}{\pi^4} p^{(4)}(x_k) - \lambda p(x_k) = 0. \tag{3.3}$$

The differentiation matrix relation is given by

$$\left(\frac{16}{\pi^4} D^{(4)} - \lambda I\right)\mathbf{y} = 0 \tag{3.4}$$

and the boundary conditions are

$$I_{-1}\mathbf{y} = \frac{4}{\pi^2} D^{(2)}_{-1}\mathbf{y} = 0, \quad I_1\mathbf{y} = \frac{4}{\pi^2} D^{(2)}_1 \mathbf{y} = 0. \tag{3.5}$$

Combining Eqs. (3.4) and (3.5), we get the following generalized eigenvalue problem

$$\begin{pmatrix} \frac{16}{\pi^4} D^{(4)} \\ I_{-1} \\ \frac{4}{\pi^2} D^{(2)}_{-1} \\ I_1 \\ \frac{4}{\pi^2} D^{(2)}_1 \end{pmatrix} \mathbf{y} = \lambda \begin{pmatrix} I \\ 0^T \\ 0^T \\ 0^T \\ 0^T \end{pmatrix} \mathbf{y}. \tag{3.6}$$

The first forty-one eigenvalues and the related absolute error are given in Table 1. Note that, the exact eigenvalues of problem (3.1) are $\lambda_k = k^4$, for $k = 1, 2, 3, \ldots$. Numerical results show that exact eigenvalues and fast convergence are achieved for $N = 40, 50, 60, 70, 80, 90$ and $N = 100$. By using $N = 250$, we can computing the exact values of first 119 eigenvalues of problem (3.1), the numerical results given in Table 2.

Example 2. Consider the following fourth-order Sturm–Liouville problem related to mechanical non-linear systems identification [8,9,11]

$$\begin{cases} y^{(4)}(x) - 0.02x^2 y''(x) - 0.04xy'(x) + (0.0001x^4 - 0.02)y(x) = \lambda y(x), & x \in (0,5), \\ y(0) = y''(0) = 0, y(5) = y''(5) = 0. \end{cases} \tag{3.7}$$

The change of variables $x = \frac{5X}{2} + \frac{5}{2}$ transforms the Sturm–Liouville problem (3.7) to

$$\begin{cases} \frac{16}{625} Y^{(4)} - SY'' - RY' + QY = \lambda Y, & X \in (-1,1), \\ Y(-1) = Y''(-1) = 0, & Y(1) = Y''(1) = 0, \end{cases} \tag{3.8}$$

where $S = 0.125(\frac{5X}{2}+\frac{5}{2})^2$, $R = 0.1(\frac{5X}{2}+\frac{5}{2})$ and $Q = (0.0001(\frac{5X}{2}+\frac{5}{2})^4 - 0.02)$. In this case the collocation equation is

$$\frac{16}{625}p^{(4)}(x_k) - \widehat{S}p''(x_k) - \widehat{R}p'(x_k) + \widehat{Q}p(x_k) = \lambda p(x_k), \tag{3.9}$$

where $\widehat{S} = diag(S(x_k)),\ \widehat{R} = diag(R(x_k))$ and $\widehat{Q} = diag(Q(x_k))$. The differentiation matrix relation can be written as

$$\left(\frac{16}{625}D^{(4)} - \widehat{S}D^{(2)} - \widehat{R}D^{(1)} + (\widehat{Q}I - \lambda I)\right)\mathbf{y} = 0 \tag{3.10}$$

and the boundary conditions are

$$I_{-1}\mathbf{y} = \frac{4}{25}D_{-1}^{(2)}\mathbf{y} = 0, \quad I_1\mathbf{y} = \frac{4}{25}D_1^{(2)}\mathbf{y} = 0. \tag{3.11}$$

Combination of Eqs. (3.10) and (3.11), leads to the following generalized eigenvalue problem

$$\begin{pmatrix} \frac{16}{625}D^{(4)} - \widehat{S}D^{(2)} - \widehat{R}D^{(1)} + \widehat{Q}I \\ I_{-1} \\ \frac{4}{25}D_{-1}^{(2)} \\ I_1 \\ \frac{4}{25}D_1^{(2)} \end{pmatrix}\mathbf{y} = \lambda\begin{pmatrix} I \\ 0^T \\ 0^T \\ 0^T \\ 0^T \end{pmatrix}\mathbf{y}. \tag{3.12}$$

Table 3 shows the first six computed eigenvalues of the problem (3.7), compared with Adomian decomposition method (ADM) [8], variational iteration method (VIM) [9] and polynomial-based differential quadrature (PDQ) [11]. The results of current work are in excellent agreement with the results of Refs. [8,9,11].

Example 3. In this example consider the same Sturm–Liouville problem given in Example 2 which is equipped with clamped boundary conditions [8,10,11]

$$\begin{cases} y^{(4)}(x) - 0.02x^2y''(x) - 0.04xy'(x) + (0.0001x^4 - 0.02)y(x) = \lambda y(x), & x \in (0,5), \\ y(0) = y'(0) = 0, & y(5) = y'(5) = 0. \end{cases} \tag{3.13}$$

We use the same procedure as we did in Example 2, except the boundary conditions which are modeled by the equations

$$I_{-1}\mathbf{y} = \frac{2}{5}D_{-1}^{(1)}\mathbf{y} = 0, \quad I_1\mathbf{y} = \frac{2}{5}D_1^{(1)}\mathbf{y} = 0. \tag{3.14}$$

Table 4 lists the first six computed eigenvalues using $N = 30$. It also shows the results using Adomian decomposition method (ADM) [8], Extended Sampling Method (ESM) [10] and polynomial-based differential quadrature (PDQ) [11]. Moreover, this problem is solved by the authors using the method described in the book by Trefethen [31]. As it is shown in the Table 4, there is an excellent agreement between the results of the current work and the results of Refs. [8,10,11,31]. The CPU time for the current work is compared with the program 40, page 151 using $N = 30$, [31]. As it is shown in Table 4, the CPU time for the proposed method in this work has less cost than the one introduced in Ref. [31]. We should note here that only the first four computed eigenvalues were reported in Ref. [10].

Example 4. In this example we solve the following fourth-order Sturm–Liouville problem which has combinations of two types of boundary conditions (1.3) and (1.4)

$$\begin{cases} y^{(4)}(x) = \lambda y(x), & x \in (0,1), \\ y(0) = y'(0) = 0, & y(1) = y''(1) = 0. \end{cases} \tag{3.15}$$

Table 3
Comparison of the eigenvalues of Example 2 for different methods $N = 30$.

k	λ_k			
	Current work	ADM [8]	VIM [9]	PDQ [11]
1	0.2150508643160	0.21505086436971596	0.21505086436971492	0.21505086437
2	2.7548099336169	2.754809934682985	2.754809934682884	2.75480993468
3	13.215351540581	13.215351540558824	13.215351540558812	13.2153515406
4	40.950819758144	40.95081975913761	40.95081975913755	40.9508197591
5	99.0534780383535	99.05347813813881	99.05347813813880	99.0534780633
6	204.355735479344	204.35449348957832	20435449348957833	204.355732256

Table 4
Comparison of the eigenvalues and CPU time (second) for Example 3.

k	λ_k				
	Current work	ADM [8]	ESM [10]	PDQ [11]	Trefethen [31]
1	0.8669025023919642026	0.8669025023997106	0.866902502399465	0.866902502602292	0.86690250242610
2	6.3576864481438599697	6.357686448145815	6.357686448174460	6.357686448439836	6.35768644812656
3	23.992746850326330588	23.992746850281375	23.99274697506674	23.99274686509660	23.9927468502824
4	64.978667594841567734	64.97866759571622	64.97863591597007	64.97866761311830	64.9786675950131
5	144.28062688384346757	144.28062803844648	–	144.2806272956158	144.280626925786
6	280.60096699712966029	280.58602048195377	–	280.6009637443962	280.600963283848
CPU time	0.138014	–	–	–	0.226369

This problem has been considered by several authors [11,8,10]. The change of variables $x = \frac{X}{2} + \frac{1}{2}$ transforms the Sturm–Liouville problem (3.15) to

$$\begin{cases} 16Y^{(4)} = \lambda Y, & X \in (-1,1), \\ Y(-1) = Y'(-1) = 0, & Y(1) = Y''(1) = 0. \end{cases} \tag{3.16}$$

The collocation equation for this problem can be written as

$$16p^{(4)}(x_k) - \lambda p(x_k) = 0. \tag{3.17}$$

The differentiation matrix relation is

$$(16D^{(4)} - \lambda I)\mathbf{y} = 0 \tag{3.18}$$

and the boundary conditions are

$$I_{-1}\mathbf{y} = 2D^{(1)}_{-1}\mathbf{y} = 0, \quad I_1\mathbf{y} = 4D^{(2)}_1\mathbf{y} = 0. \tag{3.19}$$

Combining Eqs. (3.18) and (3.19), we obtain the following generalized eigenvalue problem

$$\begin{pmatrix} 16D^{(4)} \\ I_{-1} \\ 2D^{(1)}_{-1} \\ I_1 \\ 4D^{(2)}_1 \end{pmatrix} \mathbf{y} = \lambda \begin{pmatrix} I \\ 0^T \\ 0^T \\ 0^T \\ 0^T \end{pmatrix} \mathbf{y}. \tag{3.20}$$

Solving Eq. (3.20), gives the eigenvalues of the problem (3.15). The exact eigenvalues of Eq. (3.15) can be obtained by solving [11]

$$\tanh(\sqrt{(\lambda)}) - \tan(\sqrt{(\lambda)}) = 0.$$

Yücel and Boubaker [11] proposed PDQ and the Fourier expansion-based differential quadrature (FDQ) methods to solve it. Table 5 shows comparison between the current work, exact solution and relative errors of the problem results in Table 5 show the high performance of the current work.

4. Conclusion

This paper presents a new technique based on Chebyshev spectral collocation method for directly computing eigenvalues of a general class of regular fourth-order Sturm–Liouville problems. Through the Example 1 which has the exact solution, one

Table 5
The first six eigenvalues and relative error for Example 4 with $N = 30$.

k	λ_k		Relative error		
	Current work	Exact	Current work	PDQ [11]	FDQ [11]
1	237.72106704645204900	237.72106753	2.03e−009	7.59e−009	3.24e−009
2	2496.4874358794290856	2496.48743786	7.93e−010	4.45e−008	4.83e−008
3	10867.582219516587429	10867.58221698	2.33e−010	1.71e−008	2.21e−008
4	31780.096453434191062	31780.09645408	2.03e−011	2.36e−008	1.72e−008
5	74000.849343595109531	74000.849349156	7.51e−011	2.99e−008	1.86e−008
6	148634.47725248429924	148634.47728577	2.24e−010	4.77e−008	6.41e−008

can see that the proposed technique gives the exact eigenvalues required for any index $\lambda_k,\ k = 1,2,3\ldots$ by selecting appropriate Chebyshev collocation points *N*. Here, in Examples 1–4, we attempted to cover the clamped, hinged and the combination of those boundary conditions that appear in conjunction with regular fourth-order Sturm–Liouville problems. In a contrast with other techniques, when a Chebyshev differentiation matrices for the solution of regular fourth-order Sturm–Liouville problems is used a highly accurate solution is yielded for the desired problem. Future work will concern developing the technique for solving regular and singular higher-order Sturm–Liouville problems.

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