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A Chebyshev Approximation for Solving Optimal Control Problems

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Abstract—This paper presents a numerical solution for solving optimal control problems, and the controlled Duffing oscillator. A new Chebyshev spectral procedure is introduced. Control variables and state variables are approximated by Chebyshev series. Then the system dynamics is transformed into systems of algebraic equations. The optimal control problem is reduced to a constrained optimization problem. Results and comparisons are given at the end of the paper.

Keywords—Chebyshev approximation; Optimal control problem; Ordinary and partial differential equations.

1. INTRODUCTION

Bellman's dynamic programming [1] and Pontryagin's maximum principle method [2] represent the most known methods for solving optimal control problems. In this paper, an alternative algorithm for solving such problems is presented. This approach is based on the expansion of the control variable in Chebyshev series with unknown coefficients. In the system dynamics, the state variables can be obtained by transforming the boundary value problem for ordinary and partial differential equations to integral formulae. Using El-Gendi's method [3], Chebyshev spectral approximations for these integrals [4] can be obtained. This is accomplished by starting with a Chebyshev spectral approximation for the highest order derivative and generating approximations to the lower order derivatives through successive integration. Therefore, the differential and integral expressions which arise for the system dynamics and the performance index, the initial or boundary conditions, or even for general multipoint boundary conditions are converted into algebraic equations with unknown coefficients. In this way, the optimal control problem is replaced by a parameter optimization problem, which consists of the minimization or maximization of the performance index, subject to algebraic constraints. Then, the constrained extremum problem can be replaced by an unconstrained extremum problem by applying the method of Lagrange [5]

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or the penalty function technique [6]. The same computational technique can be extended to solve the controlled Duffing oscillator.

2. MATHEMATICAL FORMULATION

The behaviour of a dynamic system can be represented by the following set of ordinary differential equations:

$$\frac{dx_i}{d\tau} = f_i(x_1, x_2, \dots, x_n,\ u_1, u_2, \dots, u_r,\ \tau), \qquad i = 1, 2, \dots, N,$$

or in the vector form

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{f}(\mathbf{X}, \mathbf{U}, \tau), \qquad 0 \le \tau \le T, \tag{2.1}$$

with initial condition

$$\mathbf{X}(0) = \mathbf{X}_0, \tag{2.2}$$

where $\mathbf{X}$ and $\mathbf{U}$ are vector functions of τ, $(x_1, x_2, \dots, x_n)$ are the state variables, and $(u_1, u_2, \dots, u_r)$ are the control variables.

The problem of optimal control is then to find the control u_i, $i = 1, \dots, N$, transferring the system (2.1) from the position $x_i = x_i(\tau_0)$ to the position $x_i = x_i(\tau_f)$ within the time $(\tau_f - \tau_0)$, and yielding the optimum of performance index I, given by [7]

$$I = h[\mathbf{X}(T), T] + \int_0^T g(\mathbf{X}, \mathbf{U}, \tau, T)\, d\tau. \tag{2.3}$$

The vector function $\mathbf{f}$ and the scalar functions h and g are generally nonlinear, and are assumed to be continuously differentiable with respect to their arguments. Without loss of generality, we will assume that $n = r = 1$. The time transformation

$$\tau = \frac{T}{2}(1 + t) \tag{2.4}$$

is introduced in order to use Chebyshev polynomials of the first kind, defined on the interval $[-1, 1]$. It follows that equations (2.1)–(2.3) are replaced by:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t), \qquad -1 \le t \le 1, \tag{2.5}$$

$$\mathbf{x}(-1) = \mathbf{x}_0, \tag{2.6}$$

$$I = H[\mathbf{x}(1), T) + \int_{-1}^{1} G(\mathbf{x}, \mathbf{u}, t, T)\, dt. \tag{2.7}$$

2.1. Approximation of the System Dynamics

To solve equation (2.5), we put [4]

$$\frac{d\mathbf{x}(t)}{dt} = \phi(t). \tag{2.8}$$

From the initial condition (2.6), and by integrating equation (2.8), we get

$$\mathbf{x}(t) = \int_{-1}^{1} \phi(t)\, dt + \mathbf{x}_0. \tag{2.9}$$

Here, we can give Chebyshev spectral approximations as follows:

$$\mathbf{x}_i = \mathbf{x}(t_i) = \sum_{j=0}^{N} b_{ij}\, \phi(t_j) + \mathbf{x}_0, \qquad i = 1, \dots, N, \tag{2.10}$$

where $t_i = -\cos\frac{i\pi}{N}$ are the Chebyshev points and b_{ij} are the elements of the matrix $\mathbf{B}$ as given in [3].

By expanding the control variables in a Chebyshev series of order m, we have [8]

$$u_m(t) = {\sum_{i=0}^{m}}' c_i \, T_i(t); \tag{2.11}$$

here $T_i(t)$ is the i^{th} Chebyshev polynomial. A summation symbol with the prime denotes a sum with first term halved. Hence, the system of equations (2.5) can be approximated as follows: substituting from equations (2.8), (2.10), and (2.11) into equation (2.5), we have

$$\phi(t_i) = F\left(\sum_{j=0}^{N} b_{ij}\,\phi(t_j) + x_0,\ {\sum_{k=0}^{m}}' c_k \, T_k(t_i),\ t_i\right), \qquad i = 1, \dots, N, \tag{2.12}$$

which can be written in the form:

$$\mathbf{F}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0, \tag{2.13}$$

where $\boldsymbol{\alpha} \equiv (\phi(t_0), \phi(t_1), \dots, \phi(t_N))$, $\boldsymbol{\beta} \equiv (c_0, c_1, \dots, c_m)$.

2.2. Approximation of the Performance Index

The performance index (2.7) can be approximated as follows: using the El-Hawary technique [4], and substituting from (2.10) and (2.11) into (2.7), we have

$$J = H[\mathbf{x}(T), T] + \sum_{j=0}^{N} b_{Nj}\, G\left(\mathbf{x}(t_j), \mathbf{u}(t_j), T\right) \tag{2.14}$$

$$= H\left(\sum_{j=0}^{N} b_{Nj}\,\phi(t_N),\ T\right) + \sum_{j=0}^{N} b_{Nj}\, G\left(\sum_{s=0} b_{js}\,\phi(t_s) + x_0,\ {\sum_{r=0}^{m}}' c_r\, T_r(t_j),\ T\right) = J(\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{2.15}$$

Generally, J is nonlinear in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$.

The optimal control problem has been reduced to a parameter optimization problem. The problem now is to find the minimum value of $J = J(\boldsymbol{\alpha}, \boldsymbol{\beta})$ given by (2.15), subject to the equality constraints (2.13), i.e.,

$$\begin{array}{ll} \text{Minimize} & J = J(\boldsymbol{\alpha}, \boldsymbol{\beta}), \\ \text{subject to} & \mathbf{F}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0. \end{array}$$

Many techniques are available in such case, such as Lagrange multipliers, penalty function, etc. We prefer the penalty function approach with partial quadratic interpolation method which is called Penalty Partial Quadratic Interpolation (PPQI) [9].

Urabe [10,11] has described a method to determine very accurately the numerical solution of nonlinear ordinary differential equations, and has shown how to study the existence and uniqueness problem of an exact solution near the calculated Chebyshev approximation, and how to estimate the error of the approximation. However, we believe it will be very hard to apply Urabe's results to the optimal control problem. Jacques and Renè [7] suggest, for engineering purposes, a somewhat different view of the error estimation problem [12]. We therefore use either

$$\left|J(\alpha_{N+1}, \beta_{N+1}) - J(\alpha_N, \beta_N)\right| < \varepsilon_1, \qquad \text{or}$$

$$\left(\sum_{i=0}^{N} \mathbf{F}_i^2\right)^{1/2} < \varepsilon_2,$$

or both, to decide whether the computed solution is close enough to the optimal solution.

3. NUMERICAL EXAMPLES

To illustrate the approximations (2.13) and (2.15), we consider the following problems.

3.1. Problem Treated by Feldbaum

The object is to find the optimal control $U(\tau)$ which minimizes

$$I = \frac{1}{2}\int_0^1 (X^2 + U^2)\,d\tau, \tag{3.1}$$

when

$$\frac{dX}{d\tau} = -X + U, \qquad 0 \le \tau \le 1, \qquad \text{and} \tag{3.2}$$
$$X(0) = 1$$

are satisfied. Transforming τ to the t-interval $[-1, 1]$, the exact solution of this linear-quadratic problem can be found in [7]. The problem is then redefined as

$$\text{Minimize} \qquad I = \frac{1}{4}\int_{-1}^1 (x^2 + u^2)\,dt, \tag{3.3}$$

$$\text{subject to} \qquad 2\frac{dx}{dt} = -x + u, \qquad -1 \le t \le 1, \tag{3.4}$$

$$\text{and} \qquad x(-1) = x_0 = 1.$$

From (2.10) and (2.11), we have

$$x(t_i) = \sum_{j=0}^{N} b_{ij}\,\phi(t_j) + 1,$$
$$u_m(t_i) = {\sum_{k=0}^{m}}' c_k\,T_k(t_i), \qquad i = 1, \dots, N.$$

Then, equation (3.4) gives us

$$g_i(\alpha, \beta) = 2\phi(t_i) + \sum_{j=0}^{N} b_{ij}\,\phi(t_j) + 1 - {\sum_{k=0}^{m}}' c_k\,T_k(t_i) = 0, \qquad i = 1, \dots, N, \tag{3.5}$$

i.e., a system of linear equations in $(N + m + 2)$ unknowns

$$\alpha \equiv (\phi(t_0), \phi(t_1), \dots, \phi(t_N)), \qquad \text{and}$$
$$\beta \equiv (c_0, c_1, \dots, c_m).$$

The approximate performance index to be minimized is

$$J = \frac{1}{4}\sum_{j=0}^{N} b_{Nj}\left[\left(\sum_{s=0}^{N} b_{js}\,\phi(t_s) + 1\right)^2 + \left({\sum_{k=0}^{m}}' c_k\,T_k(t_j)\right)^2\right]. \tag{3.6}$$

The parameter optimization problem (3.6), subject to (3.5), can be solved by using PPQI.

A comparison between the results for the exact solution and for the values $m = 3$ and $N = 3$ shows that the error in the performance index is of the order of 10^{-3}, while for the values $m = 5$ and $N = 11$, an agreement of about 9 decimal figures is obtained in the performance index. The results for different values of m and N are listed in Table 1. The results gradually tend to the exact results as we systematically proceed to higher order approximations. The exact solution for J is $J^* = 0.192909298$ as given in [7].

The present method for $N = 11$ and $m = 5$ is a very accurate approximation of the exact solution. The largest deviation in the coefficients is smaller than 10^{-9}.

Table 1. The Feldbaum problem.

m	$N=5$	$N=7$	$N=9$	$N=11$
3	0.192907464	0.192909305	0.192909306	0.192909306
5	0.192881804	0.192909292	0.192909299	0.192909298
7	—	0.192906918	0.192909299	0.192909298
9	—	—	0.192909306	0.192909298

3.2. Minimum Time Orbit Transfer Problem

One of the best known trajectory optimization examples is the problem of minimizing the transfer time of a constant low-thrust ion rocket between the orbits of Earth and Mars. This involves the determination of the thrust angle history, for which no exact solution is known [7]. The performance index of the problem can be stated as follows:

$$\text{Minimize} \quad I = T, \tag{3.7}$$

subject to the following time-varying equations [13]:

$$\begin{aligned} \frac{dX_1}{d\tau} &= X_2, \\ \frac{dX_2}{d\tau} &= \frac{X_3^2}{X_1} - \frac{\gamma}{X_1^2} + \frac{R_0 \sin u}{m_0 + m\tau}, \\ \frac{dX_3}{d\tau} &= -\frac{X_2 X_3}{X_1} + \frac{R_0 \cos u}{m_0 + m\tau}, \qquad 0 \le \tau \le T, \end{aligned} \tag{3.8}$$

with the boundary conditions

$$\begin{aligned} X_1(0) &= 1.0, & X_2(0) &= 0.0, & X_3(0) &= 1.0, \\ X_1(T) &= 1.525, & X_2(T) &= 0.0, & X_3(T) &= 0.8098, \end{aligned} \tag{3.9}$$

where T is the unknown final time to be minimized, γ characterizes the gravitational attraction from the sun, R_0 is the constant low-thrust magnitude, U is the control angle measured from the local horizontal, m_0 is the initial mass, and m is the constant propellant consumption rate. Using normalized values [7], we have $\gamma = 1$, $R_0 = 0.1405$, $m_0 = 1$, and $m = -0.07487$. After transforming the domain $t \in [0,1]$ to $t \in [-1,1]$, the state variables x_1, x_2, x_3, the control variable u, and the right-hand side of the differential equation are approximated by our technique; so we have:

$$\text{Minimize} \quad J = T, \tag{3.10}$$

subject to

$$\begin{aligned} \frac{2}{T}\frac{dx_1}{dt} &= x_2, \\ \frac{2}{T}\frac{dx_2}{dt} &= \frac{x_3^2}{x_1} - \frac{\gamma}{x_1^2} + \frac{R_0 \sin u}{m_0 + m\left[\frac{T}{2}(1+t)\right]}, \\ \frac{2}{T}\frac{dx_3}{dt} &= -\frac{x_2 x_3}{x_1} + \frac{R_0 \cos u}{m_0 + m\left[\frac{T}{2}(1+t)\right]}, \qquad -1 \le t \le 1, \end{aligned} \tag{3.11}$$

with the boundary conditions

$$\begin{aligned} x_1(-1) &= 1.0, & x_2(-1) &= 0.0, & x_3(-1) &= 1.0, \\ x_1(1) &= 1.525, & x_2(1) &= 0.0, & x_3(1) &= 0.8098. \end{aligned} \tag{3.12}$$

Let the following approximations for the state variables be

$$\frac{dx_1(t)}{dt} = \phi(t), \qquad \frac{dx_2(t)}{dt} = \psi(t), \qquad \frac{dx_3(t)}{dt} = \theta(t). \tag{3.13}$$

From equation (3.12) and by integrating equation (3.13), we get

$$x_1(t) = \int_{-1}^{t} \phi(t)\,dt + 1, \qquad x_2(t) = \int_{-1}^{t} \psi(t)\,dt, \qquad x_3(t) = \int_{-1}^{t} \theta(t)\,dt + 1,$$

$$\begin{aligned} x_1(t_i) &= \sum_{j=0}^{N} b_{ij}\,\phi_j + 1, \\ x_2(t_i) &= \sum_{j=0}^{N} b_{ij}\,\psi_j, \\ x_3(t_i) &= \sum_{j=0}^{N} b_{ij}\,\theta_j + 1, \qquad i = 1, \dots, N, \end{aligned} \tag{3.14}$$

where $t_i = -\cos\frac{i\pi}{N}$ are the Chebyshev points and b_{ij} are the elements of the matrix $\mathbf{B}$ as given in [3]. The boundary conditions give $C_1 = 1.0$, $C_2 = 0.0$, $C_3 = 1.0$, and using El-Gendi's technique, we get

$$\begin{aligned} x_1(1) &= \sum_{j=0}^{N} b_{Nj}\,\phi_j + 1 = 1.525, \\ x_2(1) &= \sum_{j=0}^{N} b_{Nj}\,\psi_j = 0.0, \\ x_3(1) &= \sum_{j=0}^{N} b_{Nj}\,\theta_j + 1 = 0.8098. \end{aligned} \tag{3.15}$$

By expanding the control variables in Chebyshev series of order m,

$$u_m(t) = {\sum_{k=0}^{m}}{}' \, c_k\, T_k(t), \tag{3.16}$$

and substituting from (3.13), (3.14), and (3.16) into the system dynamics (3.11), we have the following approximations:

$$\begin{aligned} g_{1s}(\phi_i, \psi_i, \theta_i, c_k, T) &= 0, \qquad s = 1, \dots, N, \\ g_{2p}(\phi_i, \psi_i, \theta_i, c_k, T) &= 0, \qquad p = 1, \dots, N, \\ g_{3q}(\phi_i, \psi_i, \theta_i, c_k, T) &= 0, \qquad q = 1, \dots, N, \end{aligned} \tag{3.17}$$

where $i = 1, \dots, N$, $k = 1, \dots, m$.

Equations (3.15) and (3.17) give $(3N + 6)$ equations in $(3N + m + 5)$ unknowns. Hence, the optimal control problem can be stated as follows.

Find $\phi_i, \psi_i, \theta_i, c_k, T$, so that the performance index $J = T$ is to be minimized, subject to the constraint equations in (3.17), with the boundary conditions (3.15).

The state and control variables obtained for $m = 11$ and $N = 9$ are listed in Table 2. For this case, we have the final time $T = 3.31171$. The boundary conditions (3.9) are accurately satisfied (see Table 2), and the estimated errors ε_1 and ε_2 are then of order 10^{-7}.

This problem has also been successfully solved by many authors [7]. In Table 3, there are comparisons between these methods and the present method.

Table 2. Converged functions for Example 3.2; $N = 9$, $m = 11$.

t	$X_1(t)$	$X_2(t)$	$X_3(t)$	$U(T)$
−1.000	1.00000	0.00000	1.00000	6.69204
−0.940	1.00029	0.00715	1.01246	6.75185
−0.766	1.00768	0.04608	1.04012	0.60110
−0.500	1.04584	0.13657	1.05170	−5.45747
−0.174	1.16243	0.28958	0.96983	1.63750
0.174	1.34165	0.29342	0.82897	4.55413
0.500	1.46420	0.15606	0.76869	−1.20047
0.766	1.51248	0.06813	0.77532	−1.09215
0.940	1.52435	0.01427	0.79789	−0.93693
1.000	1.52500	0.00000	0.80980	18.09990

Table 3. Results for the minimum-time orbit transfer problem.

Methods	T	Max. Error Boundary Conditions
Moyer, Pinkham [14]		
– gradient		
first	3.317	0.1%
second	3.317	0.05%
– gener. Newton-Raphson	3.3207	—
Falb, de Jong [15]	3.3193	—
Hontoir, Cruz [13]	3.3194	—
Taylor, Constantinides [12]	3.3819	0
Jacques, Renè [7]		
– $m = 7$	3.33069	$< 10^{-13}$
– $m = 9$	3.32263	$< 10^{-13}$
– $m = 11$	3.31874	$< 10^{-13}$
Present method	3.31171	0

4. THE CONTROLLED LINEAR OSCILLATOR

We will consider the optimal control of a linear oscillator governed by the differential equation

$$\ddot{x} + \omega^2 x = u, \tag{4.1}$$

in which a dot $(\cdot)$ means differentiation with respect to τ, where $-T \le \tau \le 0$ and T is specified. Equation (4.1) is equivalent to the dynamic state equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega^2 x_1 + u, \tag{4.2}$$

with the boundary conditions

$$\begin{aligned} x_1(-T) &= x_{10}, \quad x_2(-T) = x_{20}, \qquad \text{and} \\ x_1(0) &= 0, \qquad\quad x_2(0) = 0. \end{aligned} \tag{4.3}$$

One wishes to control the state of this plant such that the performance index

$$I = \frac{1}{2} \int_{-1}^{0} u^2 \, d\tau \tag{4.4}$$

is minimized over all admissible control functions $u(\tau)$.

Pontryagin's maximum principle method [2] applied to this optimal control problem yields the following exact analytical solution representation [7]:

$$\begin{aligned} x_1 &= \frac{1}{2\omega^2}\left[A\,\omega\tau\,\sin\omega\tau + B\,(\sin\omega\tau - \omega\tau\,\cos\omega\tau)\right], \\ x_2 &= \frac{1}{2\omega}\left[A\,(\sin\omega\tau + \omega\tau\,\cos\omega\tau) + B\,\omega\tau\sin\omega\tau\right], \\ u(\tau) &= A\,\cos\omega\tau + B\,\sin\omega\tau, \\ I &= \frac{1}{8\omega}\left[2\omega T\,(A^2+B^2) + (A^2-B^2)\,\sin 2\omega T - 4A\,B\,\sin^2\omega T\right], \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} A &= \frac{2\omega\left[x_{10}\,\omega^2\,T\,\sin\omega T - x_{20}\,(\omega T\cos\omega T - \sin\omega T)\right]}{\omega^2\,T^2 - \sin^2\omega T} \\ B &= \frac{2\omega^2\left[x_{20}\,T\,\sin\omega T + x_{10}\,(\sin\omega T + \omega T\,\cos\omega T\right]}{\omega^2\,T^2 - \sin^2\omega T}. \end{aligned} \tag{4.6}$$

5. SOLUTION OF THE PROBLEM AND ITS RESULTS

The optimal control problem described in (4.1), (4.3), and (4.4) can be restated as follows:

$$\text{Minimize} \qquad I = \frac{T}{4}\int_{-1}^{1} u^2\,dt, \tag{5.1}$$

$$\text{subject to} \qquad \ddot{x} = \frac{1}{4}\,T^2\,(-\omega^2\,x + u), \tag{5.2}$$

with

$$x(-1) = x_{-1}, \qquad \dot{x}(-1) = \dot{x}_{-1}, \qquad x(1) = 0, \qquad \dot{x}(1) = 0. \tag{5.3}$$

To solve equation (5.2), we put

$$\frac{d^2x(t)}{dt^2} = \phi(t), \qquad \text{then} \tag{5.4}$$

$$\frac{dx(t)}{dt} = \int_{-1}^{t} \phi(t')\,dt' + C_1, \tag{5.5}$$

$$x(t) = \int_{-1}^{t}\int_{-1}^{t''} \phi(t')\,dt'\,dt'' + C_1\,t + C_2. \tag{5.6}$$

From the boundary condition (5.3), we get

$$\dot{x}(-1) = \dot{x}_{-1} = C_1, \qquad x(-1) = -C_1 + C_2 = x_{-1},$$

hence

$$C_2 = x_{-1} + \dot{x}_{-1}.$$

Now we can give the following approximations:

$$x_i = x(t_i) = \sum_{j=0}^{N} b_{ij}^{(2)}\,\phi(t_j) + \dot{x}_{-1}(t_j+1) + x_{-1}, \tag{5.7}$$

$$\dot{x}(t_i) = \sum_{j=0}^{N} b_{ij}\,\phi(t_j) + \dot{x}_{-1}, \qquad \text{for } i = 1,\dots,N, \tag{5.8}$$

where $b_{ij}^{(2)} = (t_i - t_j)\,b_{ij}$, $t_i = -\cos\frac{i\pi}{N}$, and b_{ij} are the elements of the matrix $\mathbf{B}$ as given in [3].

The boundary conditions (5.3) are approximated by:

$$
\begin{aligned}
x(1) &= \int_{-1}^{1}\int_{-1}^{t} \phi(t')\,dt'\,dt + 2\dot{x}_{-1} + x_{-1} = 0,\\
x(1) &= \sum_{j=0}^{N} b_{Nj}^{(2)}\,\phi(t_j) + 2\dot{x}_{-1} + x_{-1} = 0,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
\dot{x}(1) &= \int_{-1}^{1} \phi(t)\,dt + \dot{x}_{-1} = 0,\\
\dot{x}(1) &= \sum_{j=0}^{N} b_{Nj}\,\phi(t_j) + \dot{x}_{-1} = 0.
\end{aligned}
\tag{5.10}
$$

As above, let the approximation of the control variable be given by

$$
u_m(t) = {\sum_{i=0}^{m}}{}' c_i\,T_i(t). \tag{5.11}
$$

Hence, the system dynamics (5.2) can be approximated as follows. Substituting from equations (5.4), (5.7), and (5.11) into equation (5.2), we get

$$
\phi(t_i) - \frac{1}{4}T^2\left(-\omega^2\left(\sum_{j=0}^{N} b_{ij}^{(2)}\,\phi(t_j) + \dot{x}_{-1}(t_j+1) + x_{-1}\right) + {\sum_{k=0}^{m}}{}' c_k\,T_k(t)\right) = 0, \qquad i = 1,\dots,N, \tag{5.12}
$$

which can be written in the form:

$$
\mathbf{F}_i(\boldsymbol{\alpha},\boldsymbol{\beta}) = 0, \qquad i = 1,\dots,N, \tag{5.13}
$$

where

$$
\boldsymbol{\alpha} \equiv (\phi(t_0),\phi(t_1),\dots,\phi(t_N)), \qquad \boldsymbol{\beta} \equiv (c_0,c_1,\dots,c_m),
$$

which is then a system of $(N+3)$ linear equations in $(N+m+2)$ unknowns $\boldsymbol{\alpha},\boldsymbol{\beta}$.

The performance index (5.1) can be approximated as follows. Substituting from (5.11) into (5.1) and using [3], we get

$$
J = \frac{T}{4}\sum_{j=0}^{N} b_{Nj}\left({\sum_{r=0}^{m}}{}' c_r\,T_r(t_j)\right)^2 = J(\boldsymbol{\beta}). \tag{5.14}
$$

Generally, J is nonlinear in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$.

The optimal control problem has been reduced to a parameter optimization problem. The problem now is to find the minimum value of $J = J(\beta)$ given by (5.14) subject to the equality constraints (5.13), i.e.,

$$
\begin{aligned}
&\text{Minimize} && J = J(\boldsymbol{\beta}),\\
&\text{subject to} && \mathbf{F}(\boldsymbol{\alpha},\boldsymbol{\beta}) = 0.
\end{aligned}
$$

By using the Penalty Partial Quadratic Interpolation (PPQI) [9], we get the results of the state and control variables with $N = 15$ and $m = 7$ listed in Table 4. The optimal value of the cost functional J for different values of m and N is given in Table 5.

The estimated errors on the boundary conditions are 0.2794E–03 and 0.2627E–08, respectively. The exact solution as given in [7] is $J = 0.184858542$.

Table 4. $w = 1$, $T = 2$, $x_{-1} = 0.5$, and $\dot{x}_{-1} = -0.5$.

t_i	$x(t_i)$	$\dot{x}(t_i)$	$U(t_i)$
−1.000	$0.5000E+00$	$-0.5000E+00$	$0.4892E+00$
−0.978	$0.4891E+00$	$-0.5001E+00$	$0.4946E+00$
−0.914	$0.4568E+00$	$-0.4982E+00$	$0.5090E+00$
−0.809	$0.4052E+00$	$-0.4890E+00$	$0.5278E+00$
−0.669	$0.3382E+00$	$-0.4658E+00$	$0.5440E+00$
−0.500	$0.2628E+00$	$-0.4238E+00$	$0.5493E+00$
−0.309	$0.1875E+00$	$-0.3626E+00$	$0.5365E+00$
−0.105	$0.1210E+00$	$-0.2874E+00$	$0.5012E+00$
0.105	$0.6921E-01$	$-0.2078E+00$	$0.4436E+00$
0.309	$0.3431E-01$	$-0.1348E+00$	$0.3685E+00$
0.500	$0.1430E-01$	$-0.7677E-01$	$0.2844E+00$
0.669	$-0.4846E-02$	$-0.3713E-01$	$0.2012E+00$
0.809	$0.1350E-02$	$-0.1448E-01$	$0.1280E+00$
0.914	$0.4283E-03$	$-0.4122E-02$	$0.7153E-01$
0.978	$0.2861E-03$	$-0.6585E-03$	$0.3620E-01$
1.000	$0.2794E-03$	$-0.2627E-08$	$0.2421E-01$

Table 5. The optimal cost functional J.

m	$N = 9$	$N = 11$	$N = 15$
7	0.184851218	0.184851229	0.184851242
9	—	0.184851228	0.184851242
11	—	—	0.184851242

6. THE CONTROLLED DUFFING OSCILLATOR

Let us now investigate the optimal control of the Duffing oscillator, described by the nonlinear differential equation

$$\ddot{x} + \omega^2 x + \varepsilon x^3 = u, \tag{6.1}$$

subject to the same boundary conditions as before and taking the same performance index expression. Of course, the exact solution in this case is not known. The approach system dynamics, boundary conditions, and performance index take the same expressions, at least formally, as equations (5.13), (5.14), (5.9), and (5.10).

Table 6 lists the optimal value of the cost functional J for different values of N with $m = 7$.

Table 6. The optimal cost functional J^* in case of $m = 7$.

$N = 9$	$N = 11$	$N = 13$	$N = 15$
0.187433708	0.187433709	0.187433709	0.187433791

The approximate solution ($m = 11$) for the performance index as given in [5] is 0.187444856.

7. CONCLUSIONS

Tables 1–3 give a comparison between the proposed technique and other methods. The results show that the suggested method is quite reliable. Comparing our method with the one of Jacques and Renè (which is better than other methods) [7], we notice that we have 33 equations and 43 unknowns in our method, against 39 equations and 49 unknowns in the Jacques and Renè method, which gives some superiority to our method; particularly, our minimum time $J^* = 3.31171$ is better than that of Jacques and Renè ($J^* = 3.31874$). The major advantage of this method is that we can deal directly with the highest-order derivatives in the differential equation without transforming it to a system of first order, and that will reduce the number of the unknowns. This fact has been shown in applying the suggested technique on the controlled Duffing oscillator. Finally, we notice that our technique is much easier than the numerical integration of the nonlinear TPBVP derived from Pontryagin's maximum principle method.

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