Method of particular solutions.

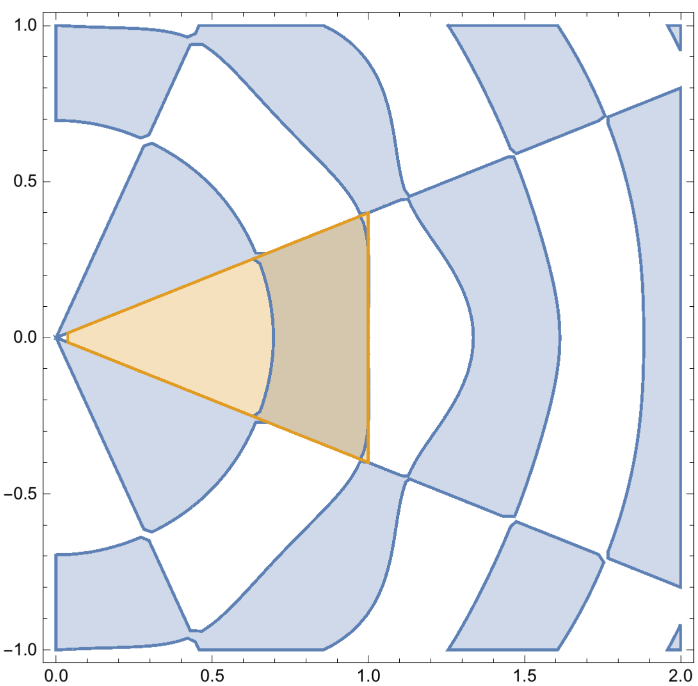
Fitting the sector eigenfunctions to a triangle.

In[18]:=
```
l = 5 / 2;
M = 1;
K = M;
q = π / (2 ArcTan[1 / l]);
u[r_, θ_] :=
  Table[10^(q i - q) BesselJ[q (2 i - 1), x r] Sin[q (2 i - 1) (θ - π / q / 2)], {i, 1, K + 1}];
α[i_] = (i + 0.4) π / q / 2 / (M + 1);
mat = Table[u[1 / Cos[α[i]], α[i]], {i, 0, M}];
u0 = Map[# == 0 &, mat.Table[c[i], {i, 0, K}]];
c[0] = 1;
Sequence @@ Table[{c[i], RandomReal[] / 10}, {i, 1, K}];
rt = FindRoot[u0, {x, BesselJZero[q, 2]}, Evaluate[%]]
approx = x^2 / (l)^2 /. %
u[r, θ].Table[c[i], {i, 0, K}] /. rt;
% /. r → Sqrt[x^2 + y^2] /. θ → ArcTan[y / x];
RegionPlot[
 {% > 0, Sqrt[x^2 + y^2] Cos[ArcTan[y / x]] ≤ 1 && -π / q / 2 < ArcTan[y / x] < π / q / 2},
 {x, 0, 2}, {y, -1, 1}]
fun = u[r, θ].Table[c[i], {i, 0, K}] /. rt
% /. θ → π / q / 2 // Simplify
```

Out[28]= {x → 11.1287, c[1] → -0.0000178389}

Out[29]= 19.8158

Out[32]=



Out[33]= $$\text{BesselJ}\left[\frac{\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]},\ 11.1287\,r\right]\ \text{Sin}\left[\frac{\pi\left(\theta-\text{ArcTan}\left[\frac{2}{5}\right]\right)}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right]-0.23963\,\text{BesselJ}\left[\frac{3\,\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]},\ 11.1287\,r\right]\ \text{Sin}\left[\frac{3\,\pi\left(\theta-\text{ArcTan}\left[\frac{2}{5}\right]\right)}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right]$$

Out[34]= 0.

We just need to rationalize the numerical values in the above function to get a symbolic eigenfunction approximation.

Finding approximation error and bounds for the true eigenvalue of the triangle.

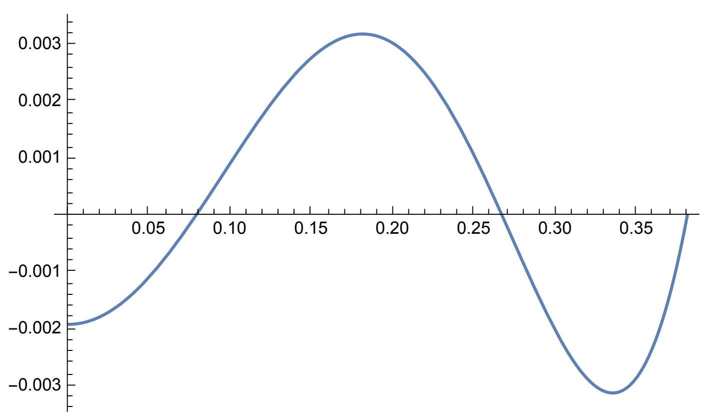
```
funr = fun /. rt // Rationalize[#, 0.001] &
fun2 = funr /. r → r / l // Simplify
funθ = % /. r → l / Cos[θ]
Plot[%, {θ, 0, π / q / 2}]
NMaximize[{Abs[%%], 0 ≤ θ ≤ π / q / 2}, θ, Method → "RandomSearch"][[1]]
NIntegrate[fun2^2 r, {θ, -π / q / 2, π / q / 2}, {r, 0, l}]^(1 / 2)
Sqrt[l] %% / %
{approx / (1 + %), approx / (1 - %)}
```

Out[43]= $\text{BesselJ}\left[\frac{\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}, \frac{345\,r}{31}\right] \text{Sin}\left[\frac{\pi\left(\theta - \text{ArcTan}\left[\frac{2}{5}\right]\right)}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right] - \frac{6}{25}\text{BesselJ}\left[\frac{3\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}, \frac{345\,r}{31}\right] \text{Sin}\left[\frac{3\pi\left(\theta - \text{ArcTan}\left[\frac{2}{5}\right]\right)}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right]$

Out[44]= $-\text{BesselJ}\left[\frac{\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}, \frac{138\,r}{31}\right] \text{Cos}\left[\frac{\pi\,\theta}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right] - \frac{6}{25}\text{BesselJ}\left[\frac{3\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}, \frac{138\,r}{31}\right] \text{Cos}\left[\frac{3\pi\,\theta}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right]$

Out[45]= $-\text{BesselJ}\left[\frac{\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}, \frac{345\,\text{Sec}[\theta]}{31}\right] \text{Cos}\left[\frac{\pi\,\theta}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right] - \frac{6}{25}\text{BesselJ}\left[\frac{3\pi}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}, \frac{345\,\text{Sec}[\theta]}{31}\right] \text{Cos}\left[\frac{3\pi\,\theta}{2\,\text{ArcTan}\left[\frac{2}{5}\right]}\right]$

Out[46]=



Out[47]= 0.00319679

Out[48]= 0.253096

Out[49]= 0.019971

Out[50]= {19.4278, 20.2196}

The last pair of numbers gives bounds the true eigenvalue. Note that we are using only 2 eigenfunctions of the sector.