Decomposition of Graphs: Topological Sort

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Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Implement the topological sort algorithm.
- Prove that a DAG can be linearly ordered.

Outline

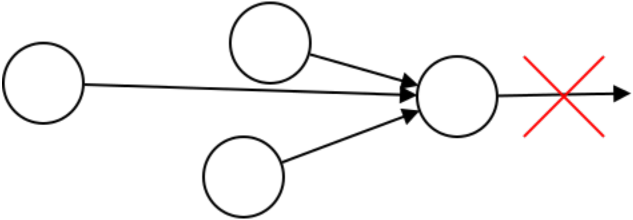
1. Idea
2. Algorithms
3. Correctness

Last Time

- Directed graphs.
- Linearly order vertices.
- Requires DAG.

Last Vertex

Consider the last vertex in the ordering. It cannot have any edges pointing out of it.



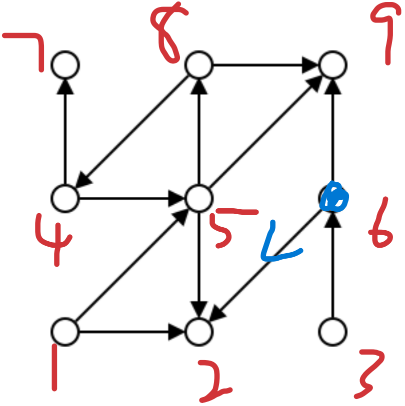
Sources and Sinks

Definition

A source is a vertex with no incoming edges.
A sink is a vertex with no outgoing edges.

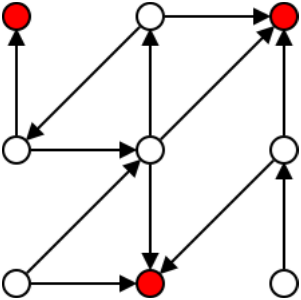
Problem

How many sinks does the graph below have?



Solution

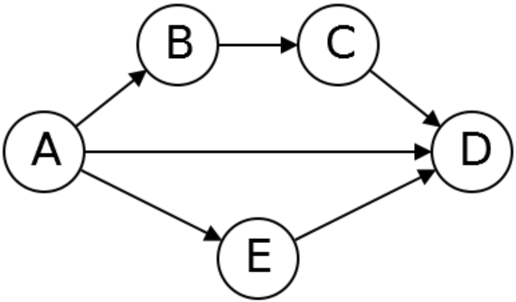
3.



Idea

- Find sink.
- Put at end of order.
- Remove from graph.
- Repeat.

Example



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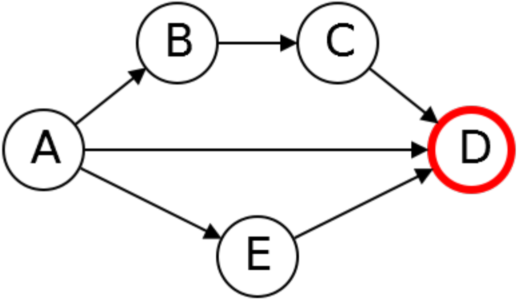

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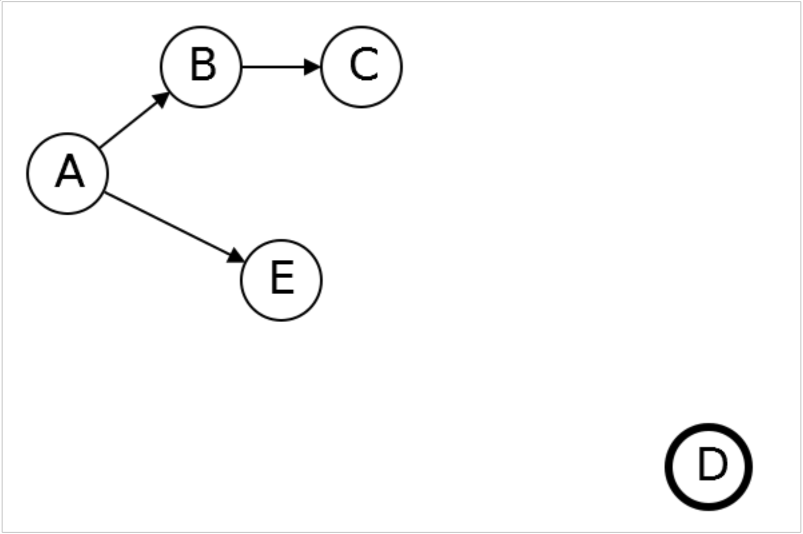

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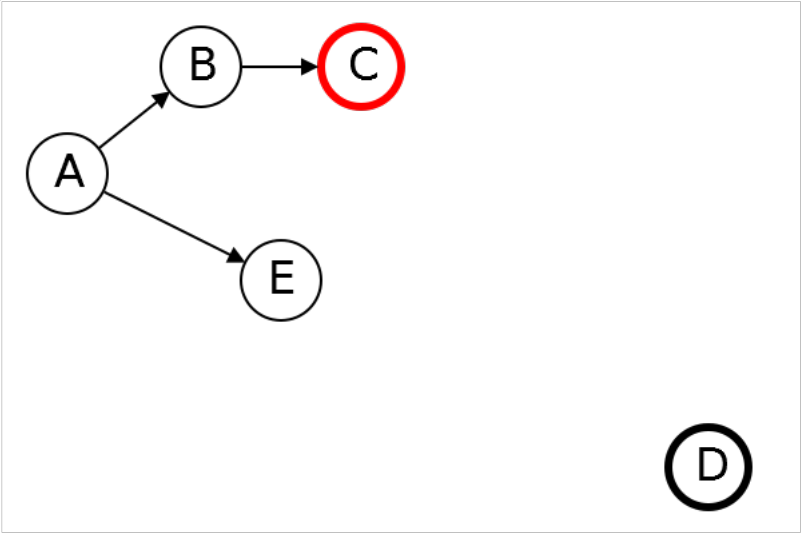

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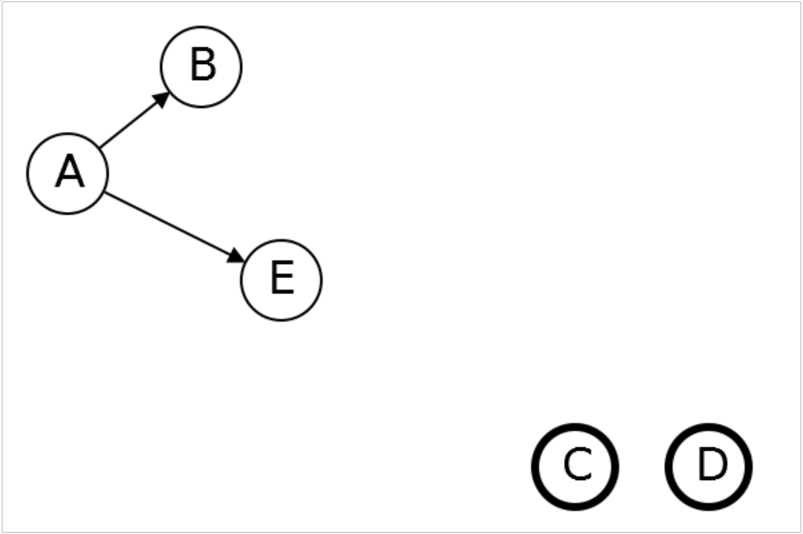

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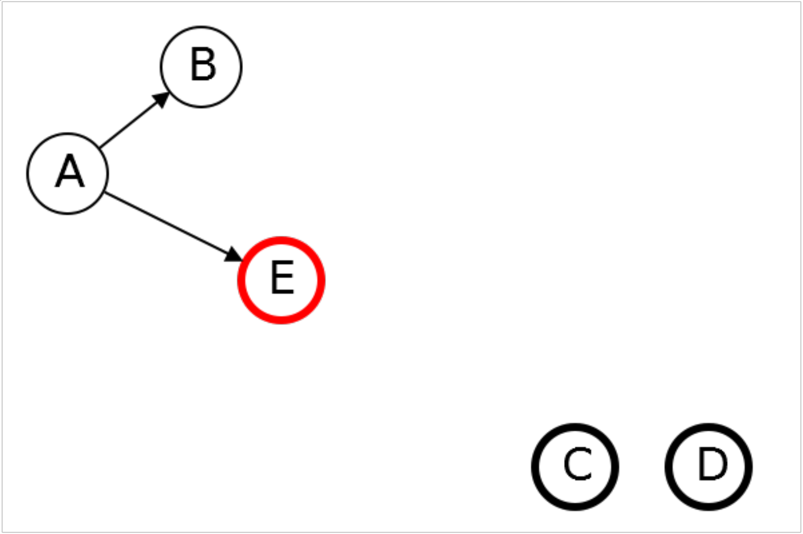

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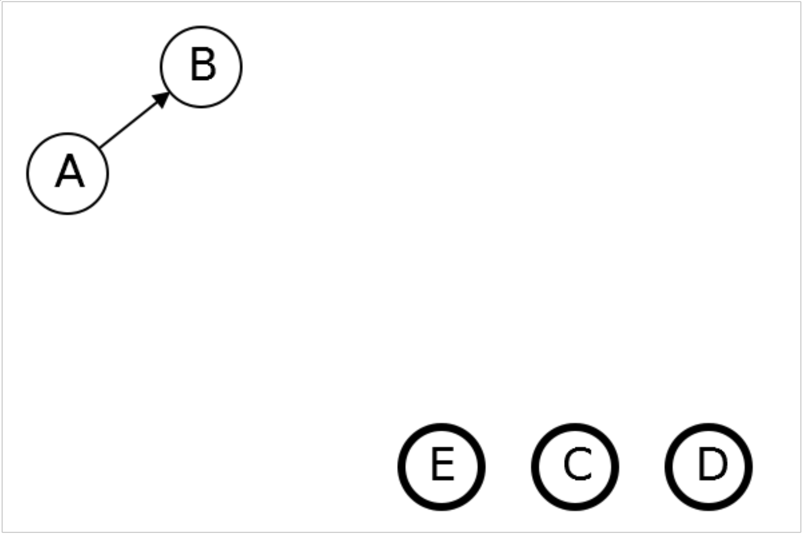

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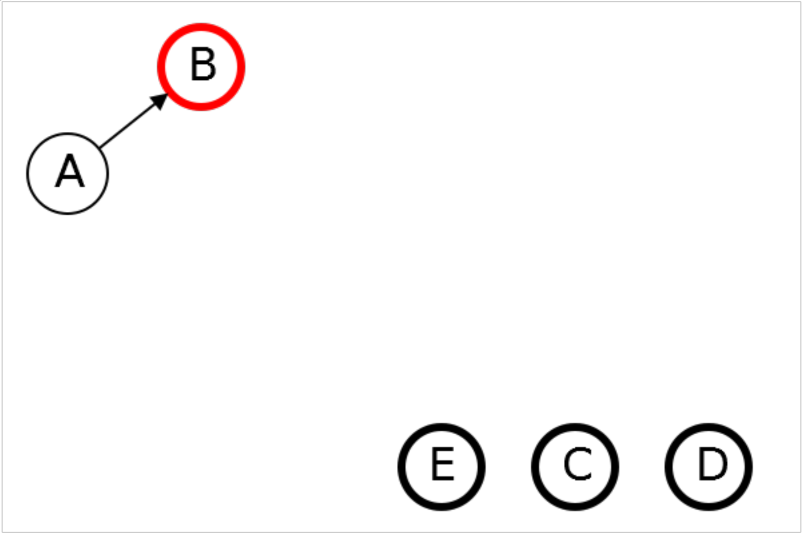

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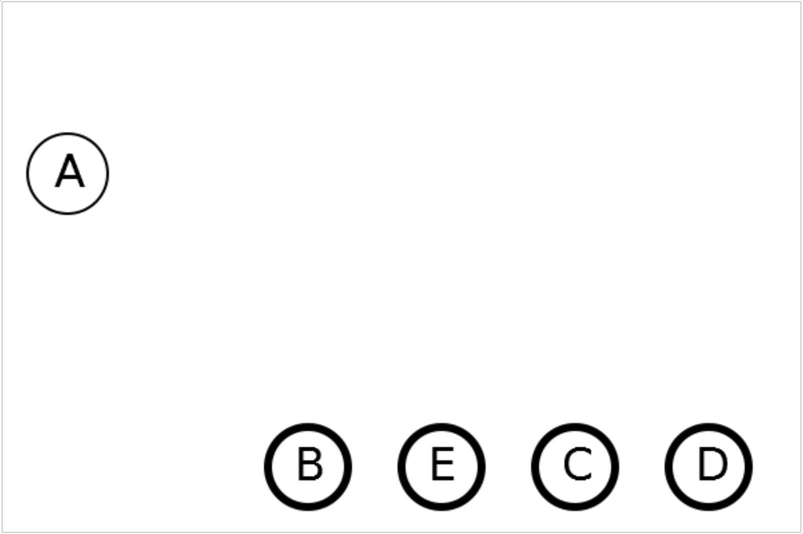

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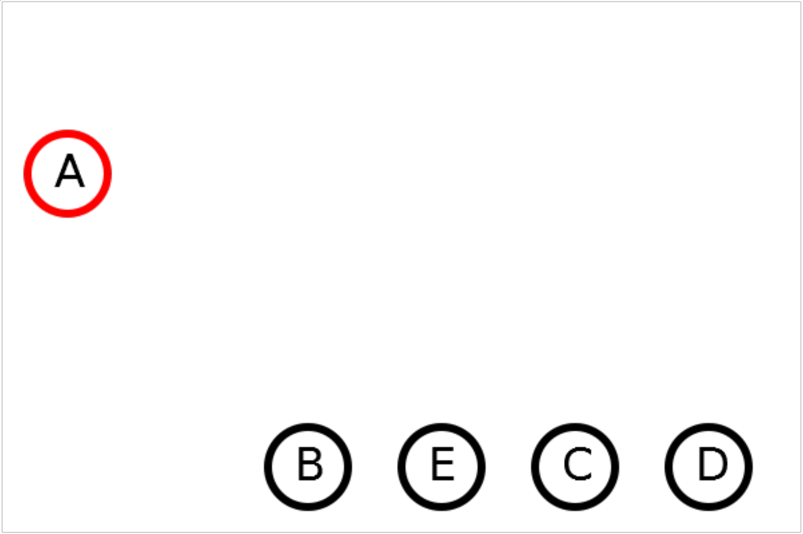

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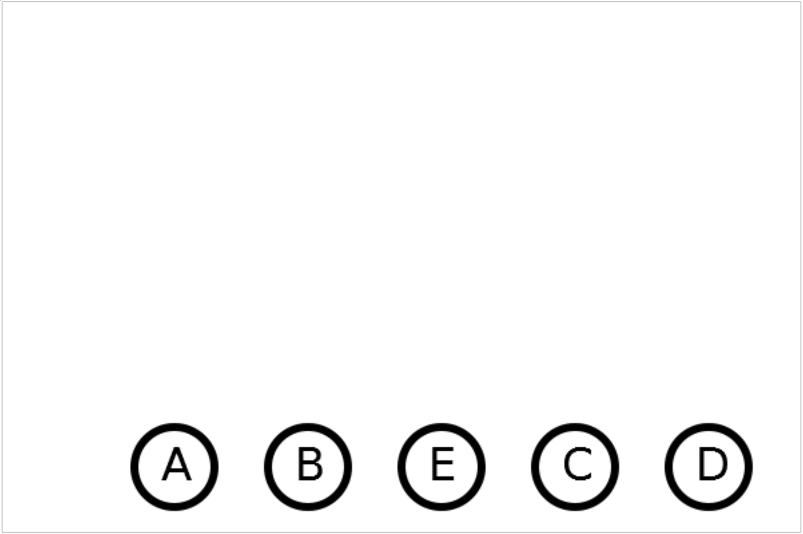


Finding Sink

Question: How do we know that there is a sink?

Follow Path

Follow path as far as possible
$v_1 \to v_2 \to \ldots \to v_n$. Eventually either:

- Cannot extend (found sink).
- Repeat a vertex (have a cycle).

Outline

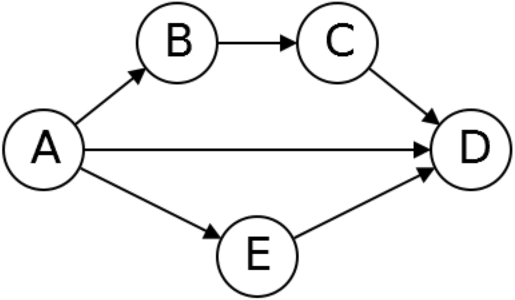
1. Idea
2. Algorithms
3. Correctness

First Try

LinearOrder(G)

```
while G non-empty:
   Follow a path until cannot extend
   Find sink v
   Put v at end of order
   Remove v from G
```

Example



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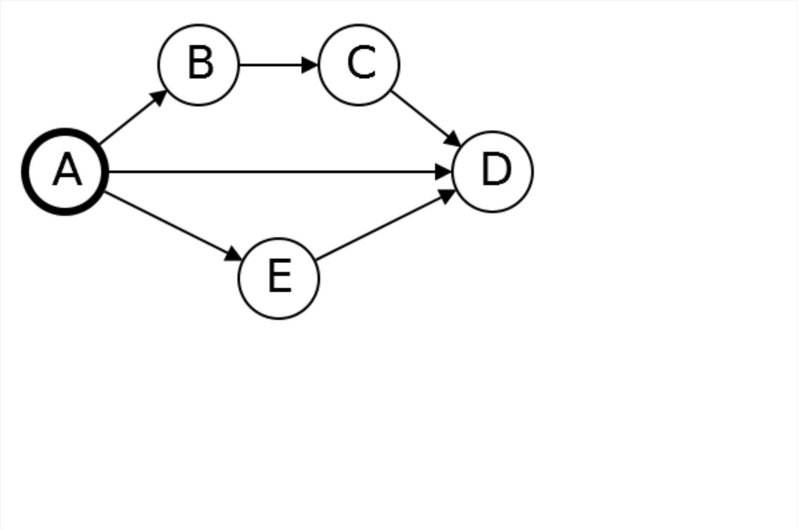

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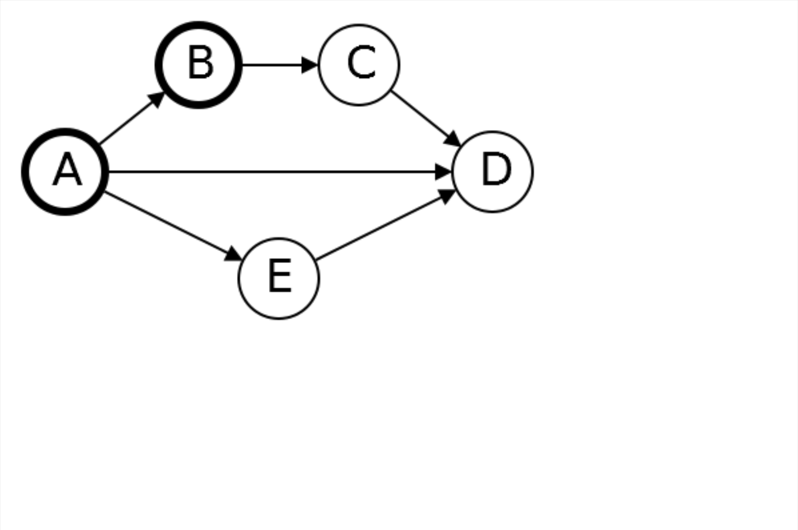

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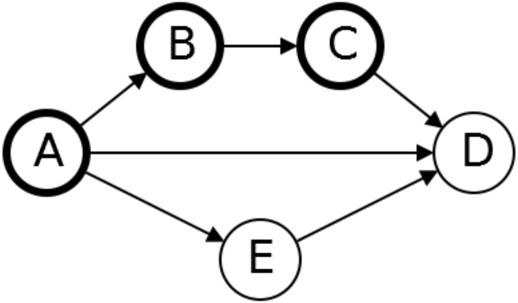

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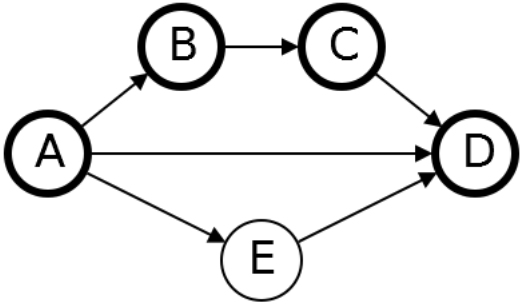

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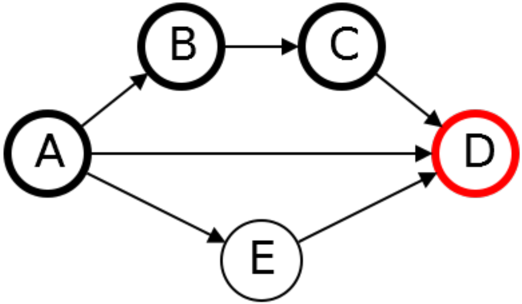

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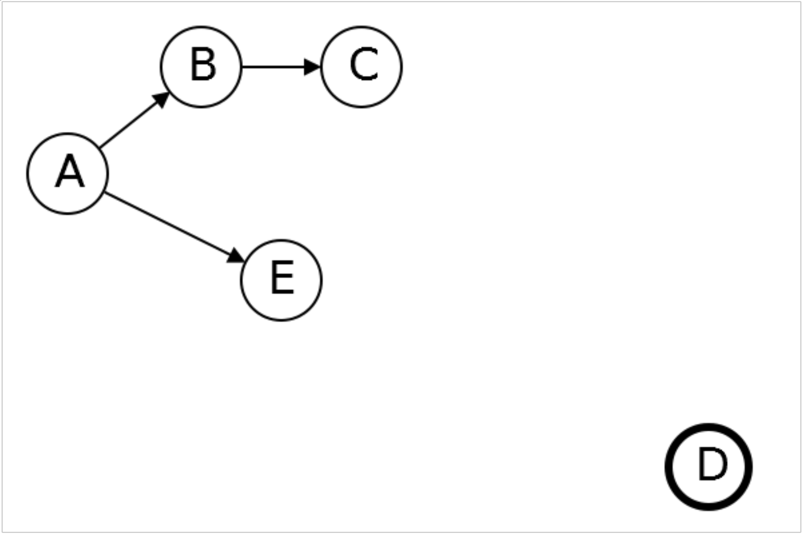

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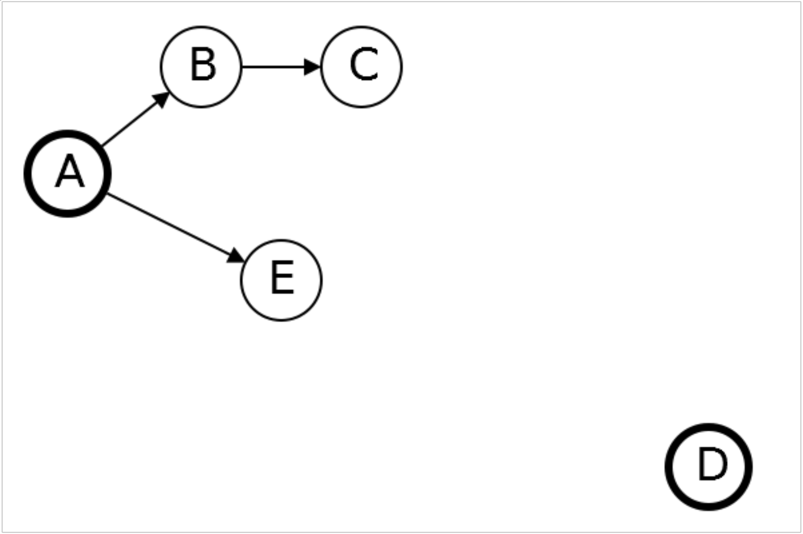

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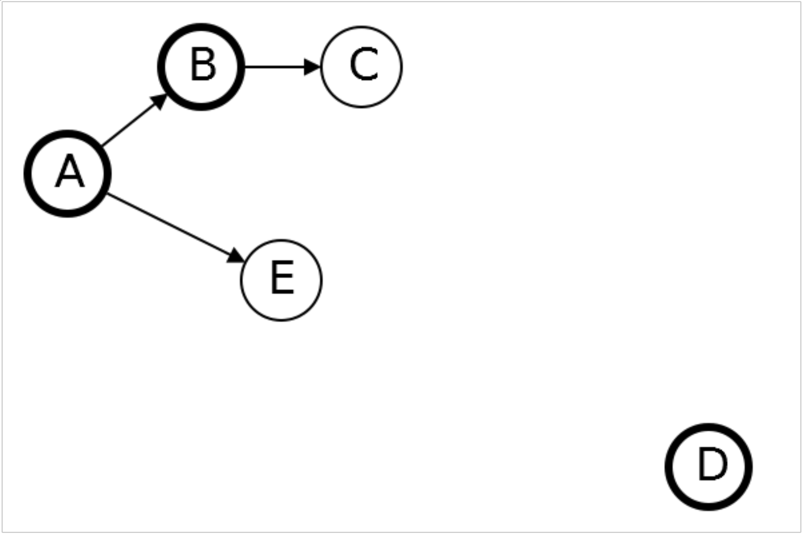

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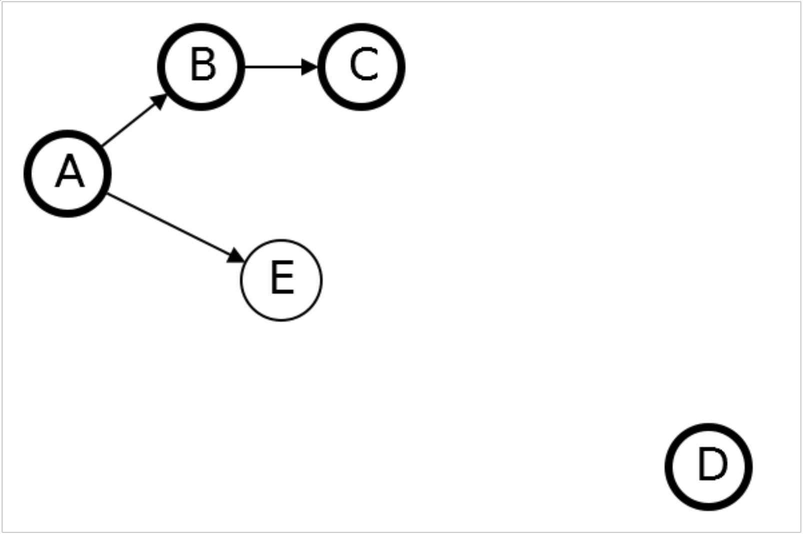

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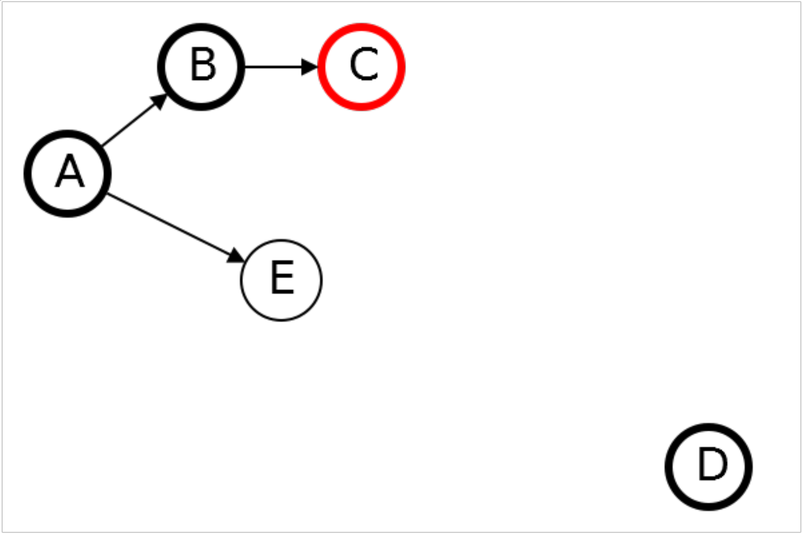

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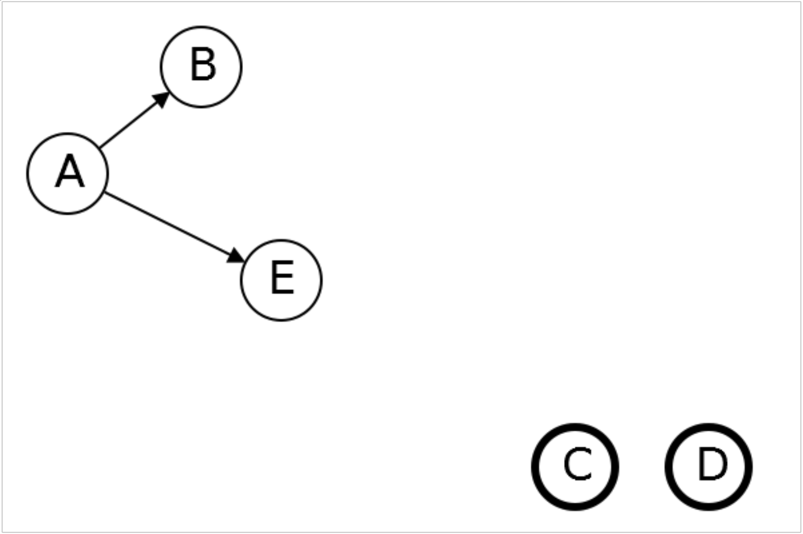

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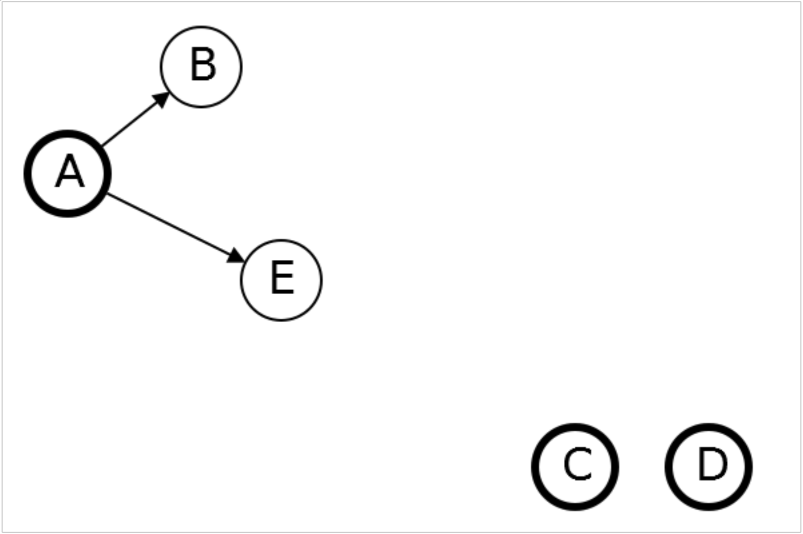

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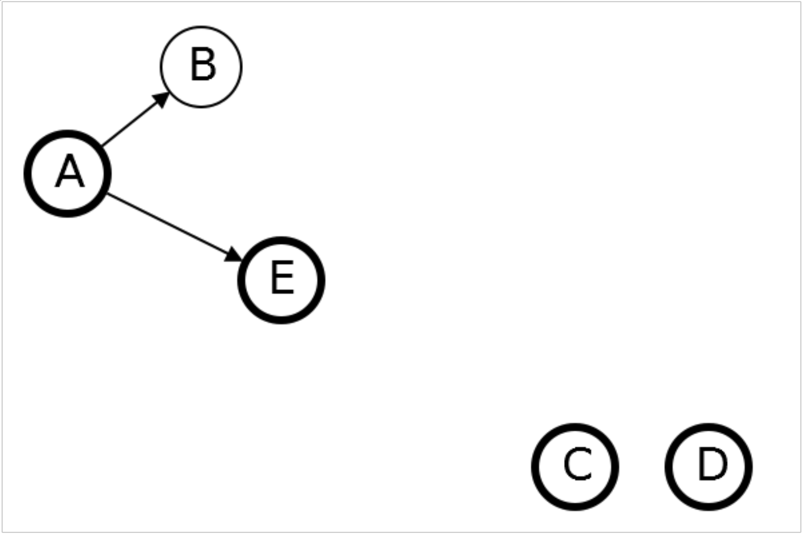

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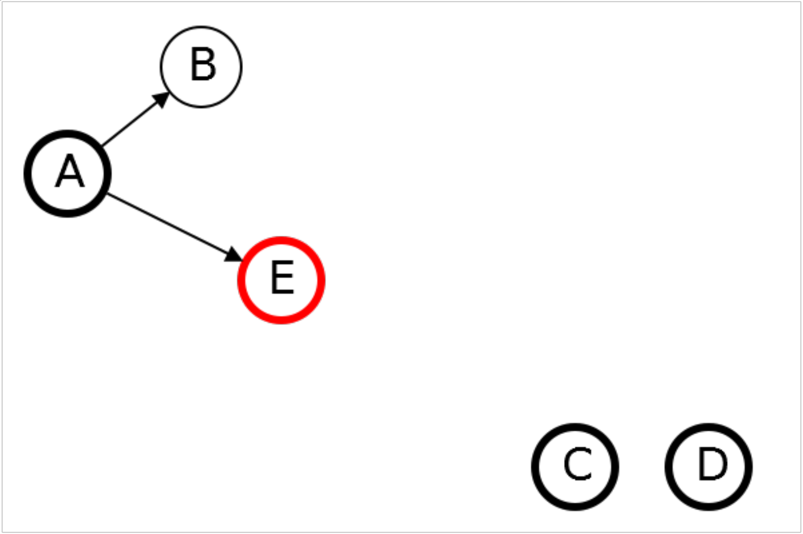

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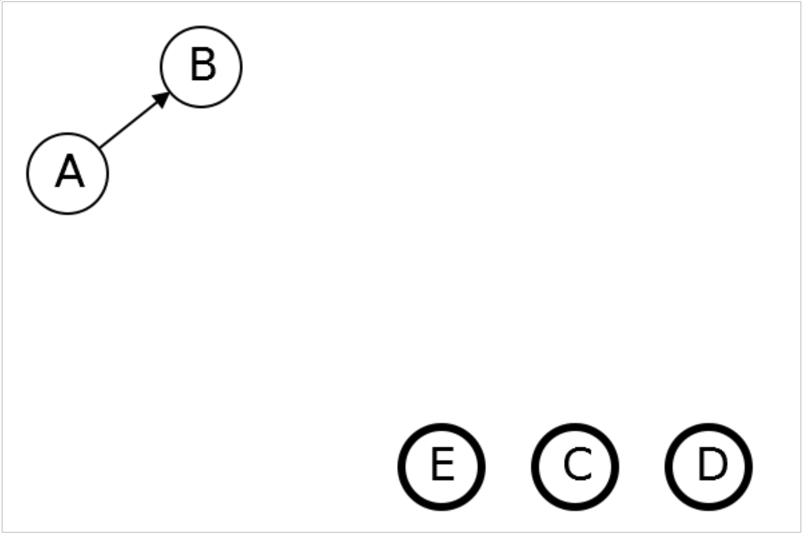

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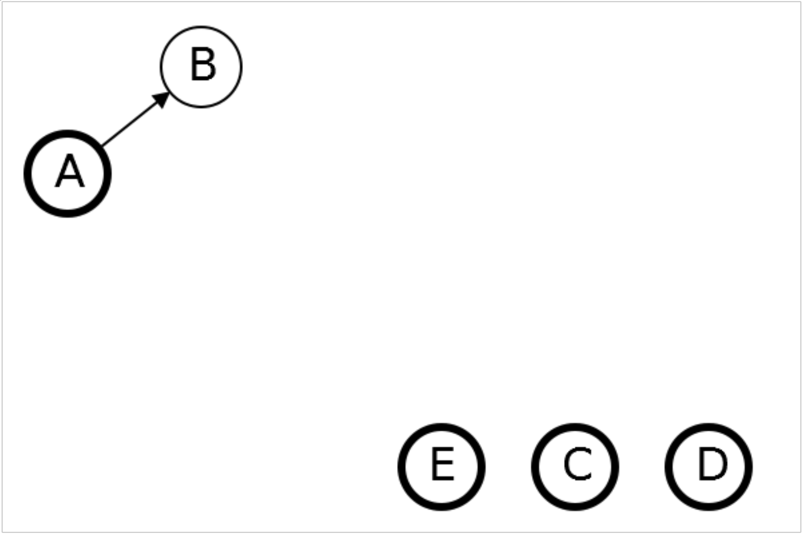

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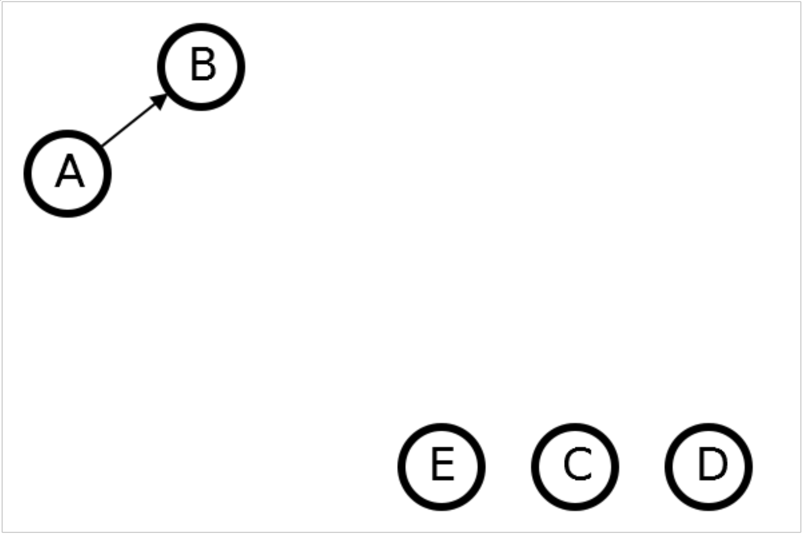

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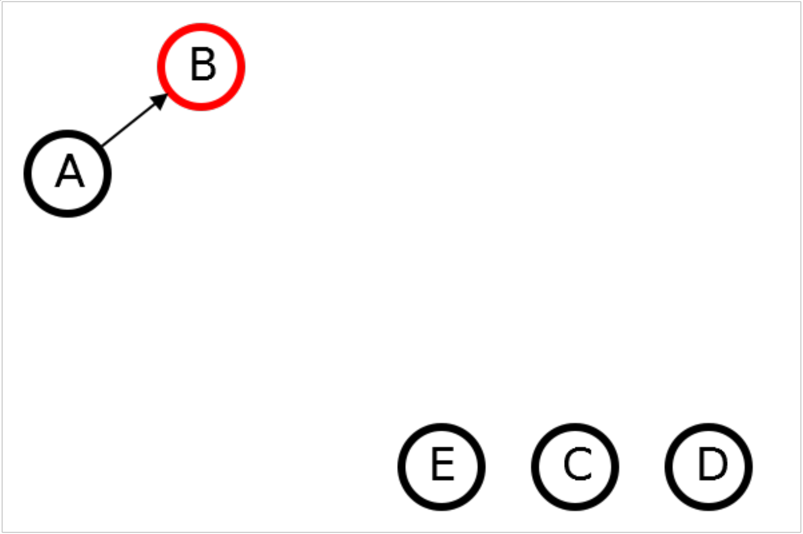

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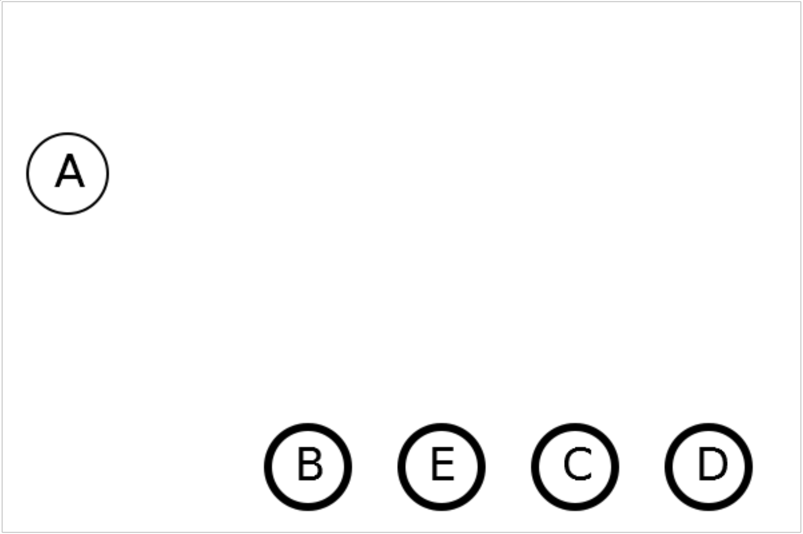

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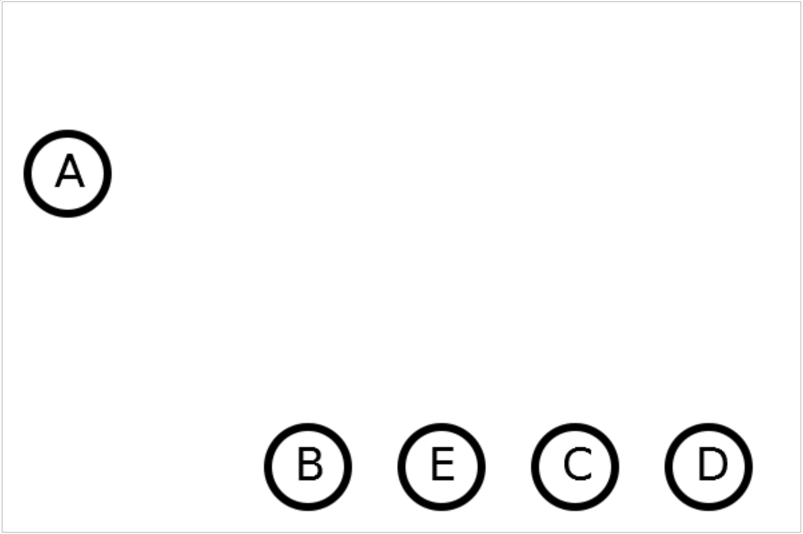

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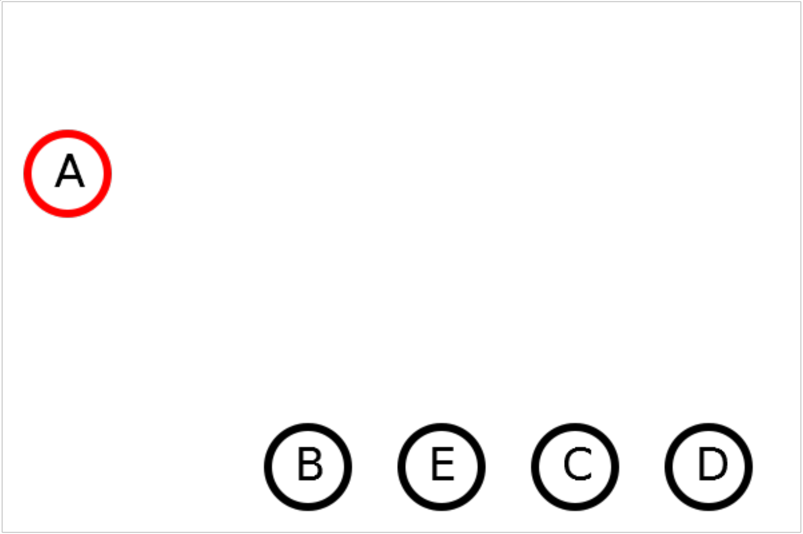

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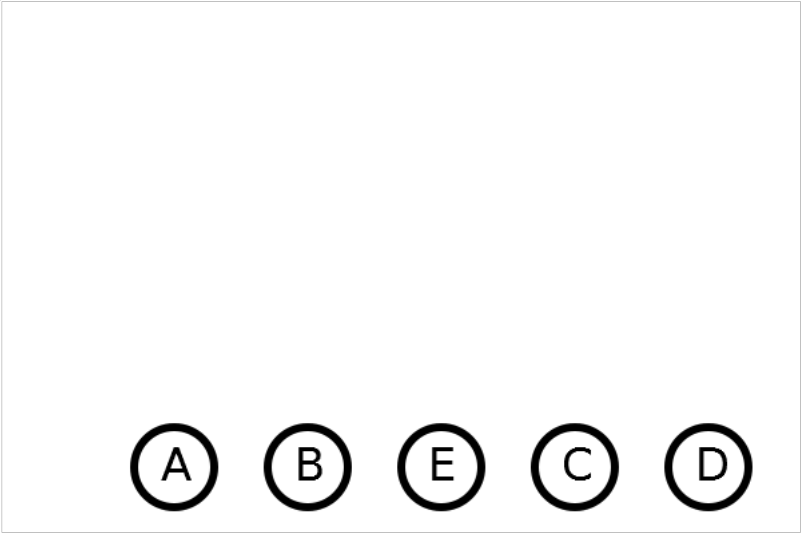


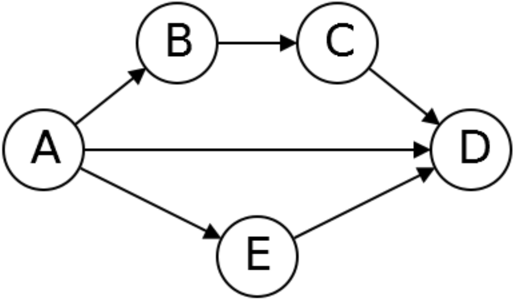
Runtime

- $O(|V|)$ paths.
- Each takes $O(|V|)$ time.
- Runtime $O(|V|^2)$.

Speed Up

- Retrace same path every time.
- Instead only back up as far as necessary.

Example



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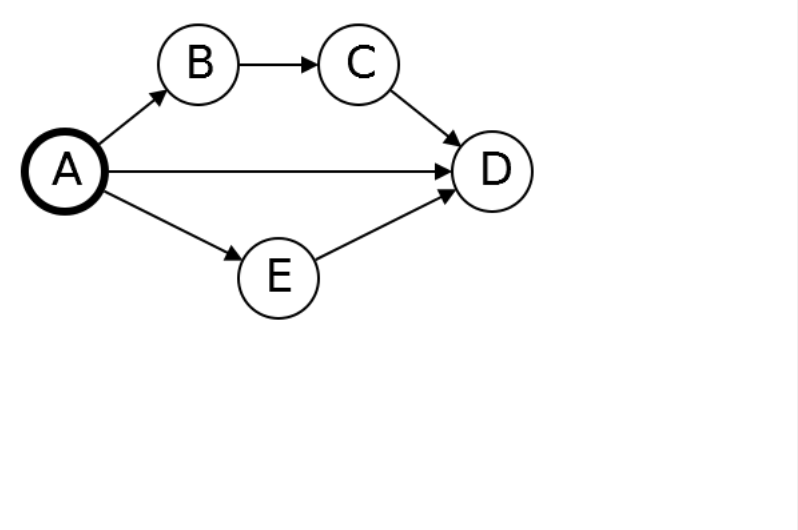

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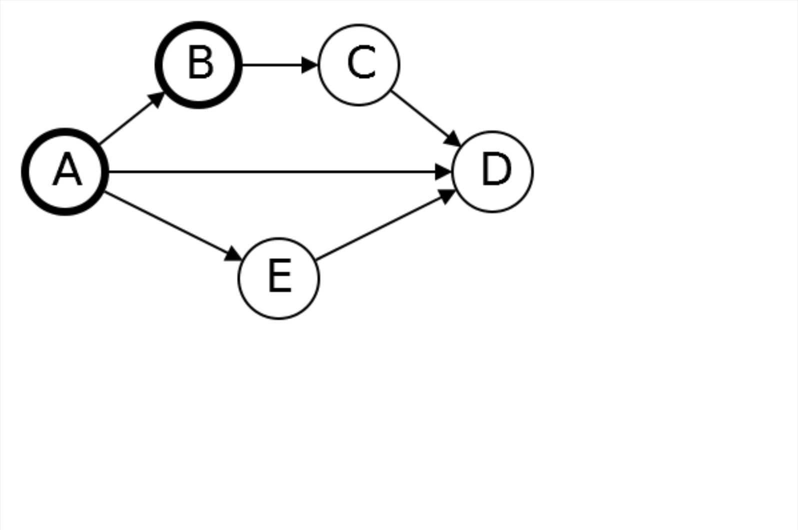

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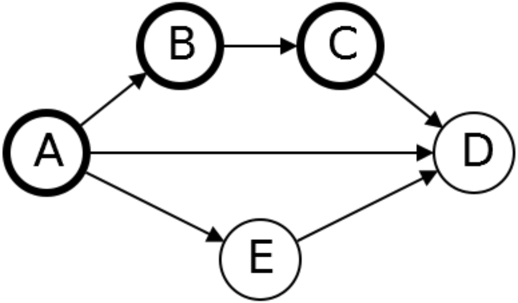

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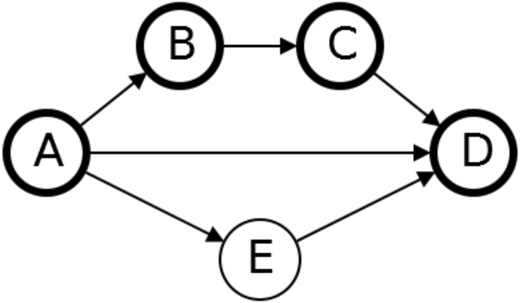

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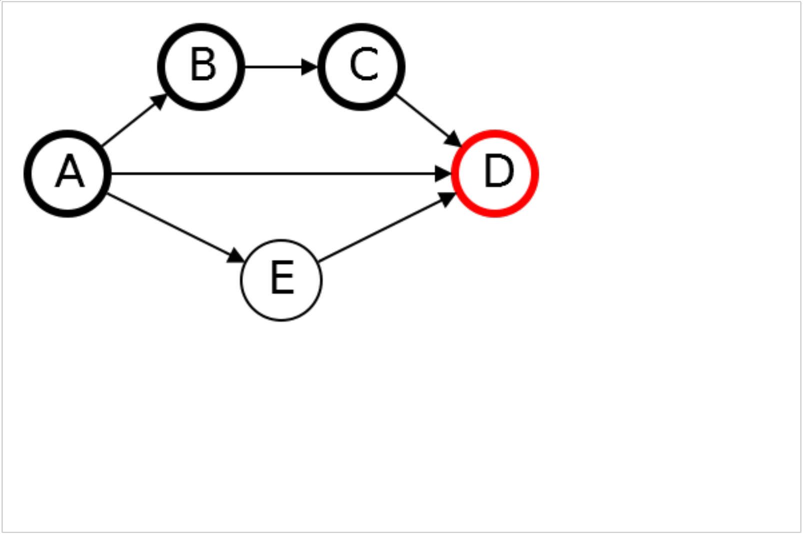

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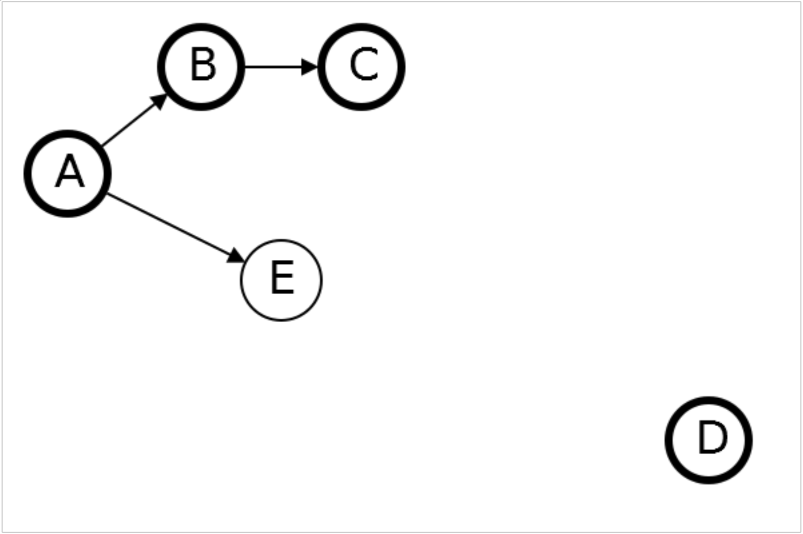

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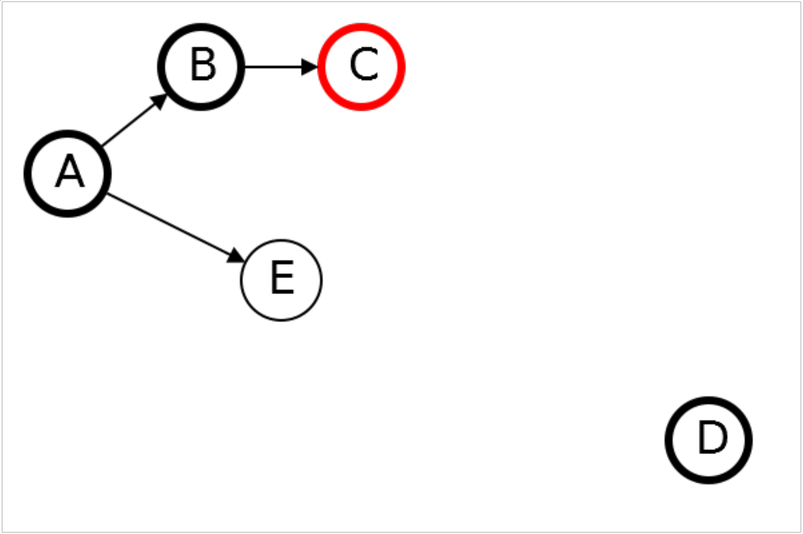

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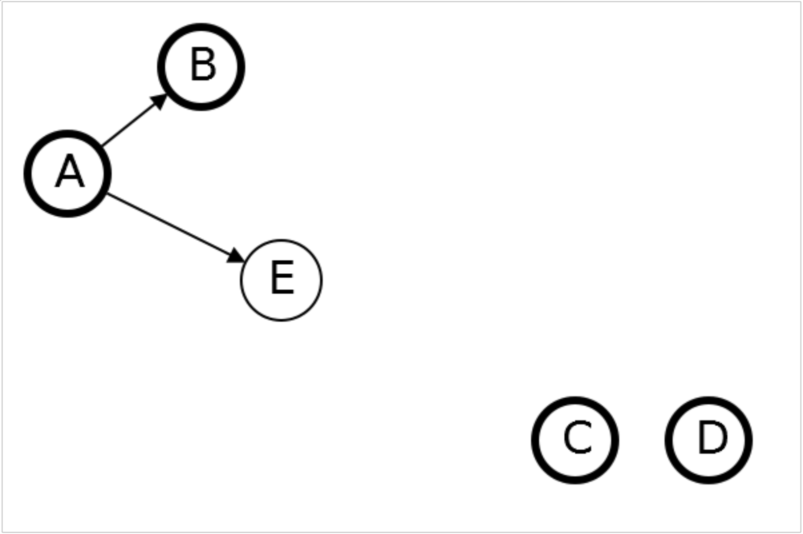

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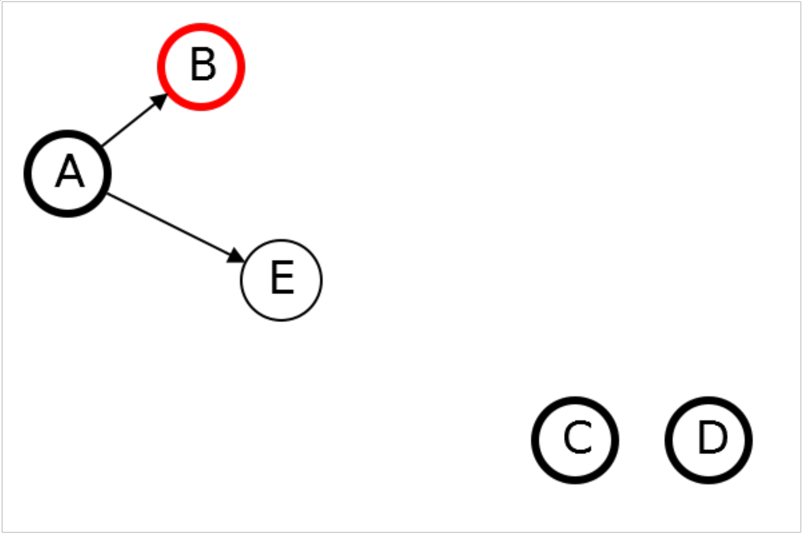

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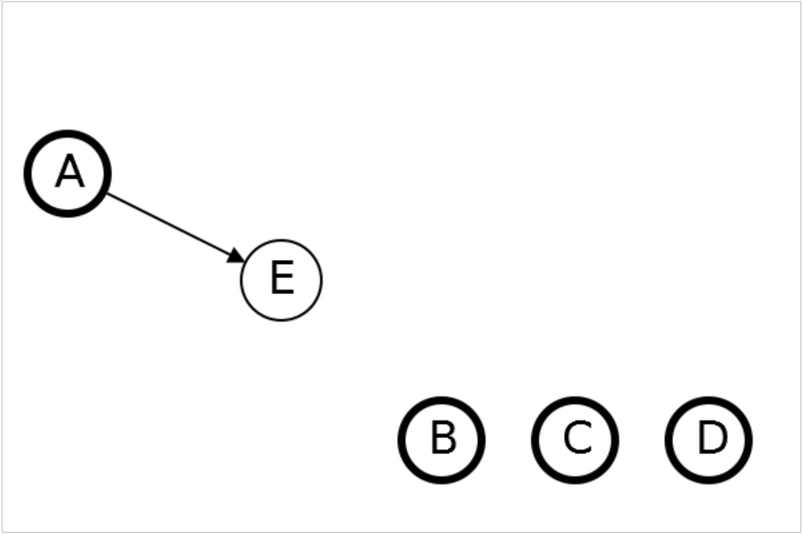

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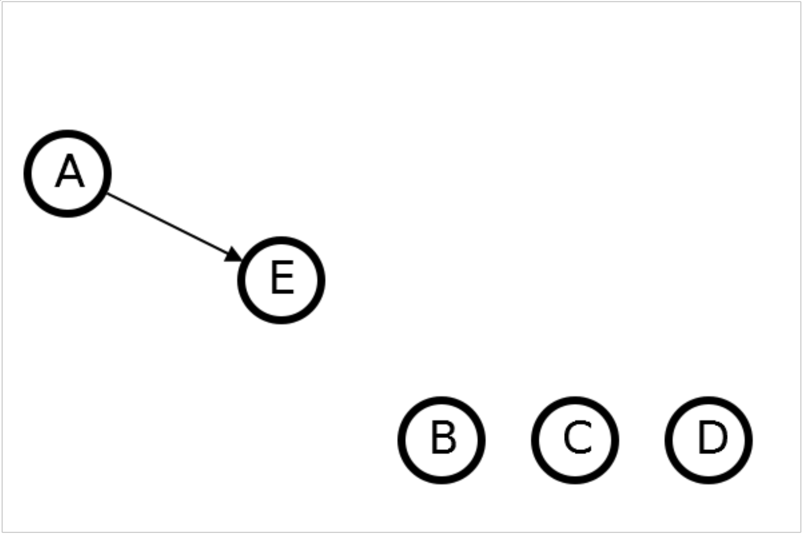

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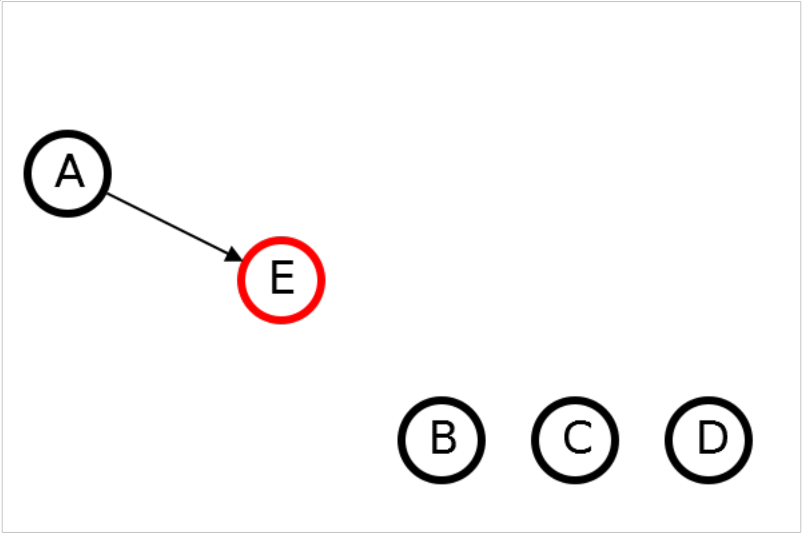

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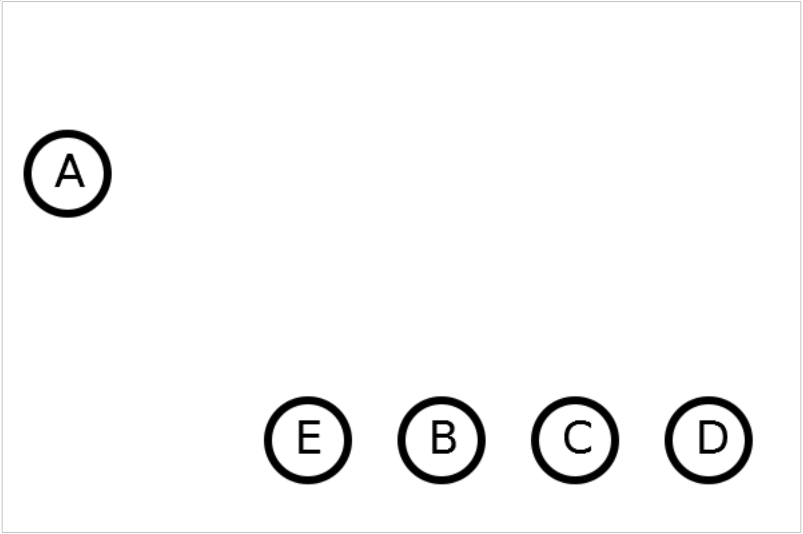

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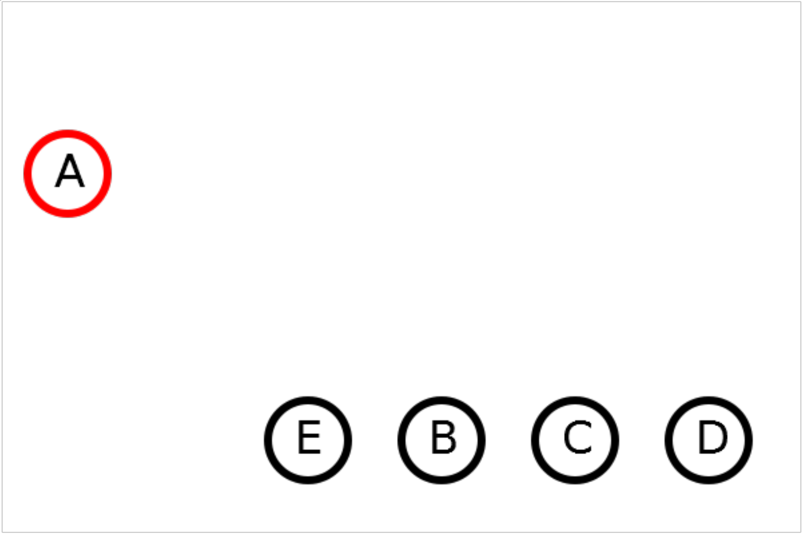

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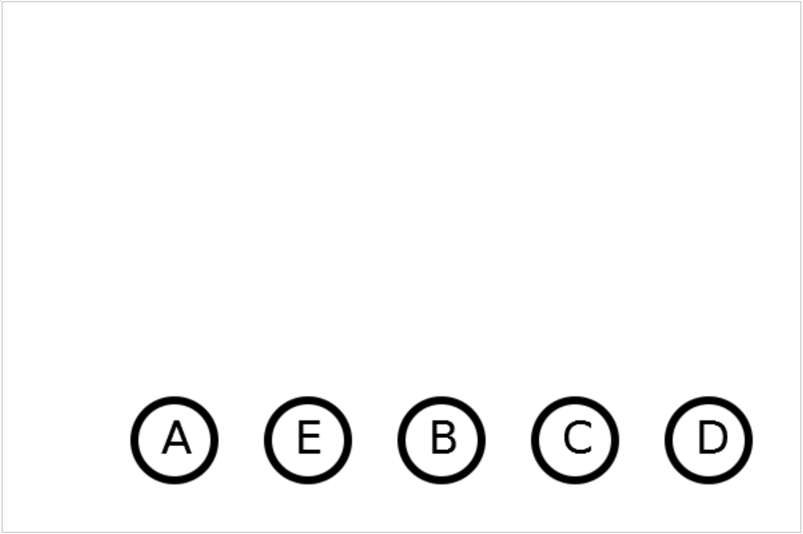

Observation

This is just DFS!

Observation

This is just DFS!

We are sorting vertices based in postorder!

Better Algorithm

TopologicalSort(G)

```
DFS(G)
sort vertices by reverse post-order
```

Outline

1. Idea
2. Algorithms
3. Correctness

Theorem

Theorem

If G is a DAG, with an edge u to v, $\texttt{post}(u) > \texttt{post}(v)$.

Proof

Consider cases

- Explore v before exploring u.
- Explore v while exploring u.
- Explore v after exploring u (cannot happen since there is an edge).

Case I

Explore v before exploring u.

- Cannot reach u from v (DAG)
- Must finish exploring v before find u
- $\mathtt{post}(u) > \mathtt{post}(v)$.

Case II

Explore v while exploring u.
Must finish exploring v before can finish exploring u. Therefore $\texttt{post}(u) > \texttt{post}(v)$.

Next Time

Connectivity in directed graphs.