CS 70 Discussion 8B

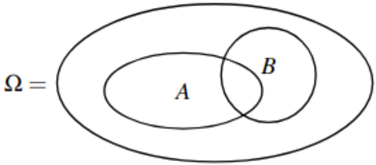
October 25, 2024

Conditional Probability

Problem: How do we represent the probability of an event B, *given* that event A happens?
Notation: $\mathbb{P}[B|A] = \mathbb{P}[B$ happens given A happened$]$. We can say:

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}$$

$$\mathbb{P}[A \cap B] = \mathbb{P}[B|A]\mathbb{P}[A] = \mathbb{P}[A|B]\mathbb{P}[B]$$



Bayes' Rule

I want to use $\mathbb{P}[A|B]$ to calculate $\mathbb{P}[B|A]$:

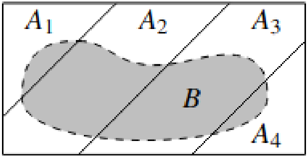
$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A|B]\mathbb{P}[B]}{\mathbb{P}[A]}$$

Application: I want to calculate the accuracy of a COVID test. I know $\mathbb{P}[\text{test is positive}|\text{a person has COVID}]$ from an experiment. But, the efficacy of the test is determined by $\mathbb{P}[\text{a person has COVID}|\text{test is positive}]$ (we want to predict whether someone has COVID from the test, not predict the output of the test after already knowing someone has COVID!).

Total Probability Rule

Problem: I want to calculate the probability of the event B, but I can only calculate $\mathbb{P}[B \cap A_i]$ for a list of events $A_1, A_2, \ldots, A_n$ where for any $i \neq j$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{n} A_i = \Omega$.
Formula:

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \cap A_i] = \sum_{i=1}^{n} \mathbb{P}[B|A_i]\mathbb{P}[A_i]$$



Independence

Motivation: We need a formal definition of what it means for two events to be *independent*, meaning the occurrence of one event doesn't effect the probability of the other event occurring.
Definition: Events $A_1, A_2, ..., A_n$ are **mutually independent** iff:

$$\mathbb{P}\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} \mathbb{P}[A_i]$$

$A_1, A_2, \ldots, A_n$ are **pairwise independent** iff:

$$(\forall i \neq j)(\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_i]\mathbb{P}[A_j])$$

Mutual Independence $\implies$ Pairwise Independence, but Pairwise Independence $\not\Longrightarrow$ Mutual Independence