CS 70 Discussion 6A

October 8, 2024

Counting

Problem: We want to count the number of scenarios/configurations that satisfy a particular condition.
Common Variants:

- Count the number of ways to pick k objects from a set of n objects *with replacement* and where *order matters*.
- Count the number of ways to pick k objects from a set of n objects *without replacement* and where *order matters*.
- Count the number of ways to pick k objects from a set of n objects *without replacement* and where *order doesn't matter*.
- Count the number of ways to pick k objects from a set of n objects *with replacement* and where *order doesn't matter*.

With Replacement + Order Matters

We are essentially creating a sequence of length k where each element is a value in $\{1, 2, ..., n\}$. The idea is as follows:

1. We have n options for each element in the sequence.
2. We have n possible sequences of length $k = 1$.
3. We have n^2 possible sequences of length $k = 2$.
4. $\vdots$
5. We have n^k possible sequences of length k.

So, $\boxed{n^k}$ is the answer.

Without Replacement + Order Matters

We are still making a sequence of length k, but we now can't reuse values! Therefore:

1. We have n options for the first element in the sequence.
2. For our second element, we cannot use the same value as we used in the first element, so we have $n - 1$ options here.
3. We have n possible sequences of length $k = 1$.
4. We have $n(n - 1)$ possible sequences of length $k = 2$.
5. We have $n(n - 1)(n - 2)$ possible sequences of length $k = 3$.
6. $\vdots$
7. We have $\boxed{\frac{n!}{(n-k)!} = \prod_{i=n-k+1}^{n} i}$ sequences of length k.

Note: $n! = 1 \times 2 \times 3 \times ... \times n$

Without Replacement + Order Doesn't Matter

How many different *sets* (order doesn't matter for sets, and no duplicate elements can be in sets) of k elements can we find? Notation (n *choose* k):

$$\boxed{\binom{n}{k}} = \frac{n!}{k!(n-k)!} = \# \text{ of sets of } k \text{ values from } n \text{ values}$$

The reasoning is that we divide the order matters case by $k!$ to not consider all different permutations of every sequence of k unique elements (i.e. $\{1, 2, 3\} \equiv \{3, 1, 2\} \equiv \dots$).

With Replacement + Order Doesn't Matter

A sequence where order doesn't matter is uniquely defined by the number of elements of each value that are present in the sequence.
Key Observation: Both of these values are equivalent:

- Number of ways to create a sequence of length k where order doesn't matter and values in $\{1, 2, ..., n\}$ can be reused
- Number of ways to distribute k identical balls among n bins

Balls-and-Bins: We have $k + n - 1$ positions. Insert $n - 1$ dividers at different positions. Now, we have divided our k indices into n groups. Here is an example for $n = 4$ and $k = 4$:

$$\star | \star \star | \star | \equiv 1, 2, 2, 3$$

This solves our problem:

$$\boxed{\binom{k+n-1}{n-1}} = \# \text{ of ways to split } k \text{ identical balls among } n \text{ bins}$$