CS 70 Discussion 9B

November 1, 2024

Random Variables

Problem: We need a way to compute statistics about a sample space (ex. if Ω is the set of all sequences of n coin flips, and we want to count the average number of heads over all outcomes)
Solution: A random variable is some function $X : \Omega \to \mathbb{R}$. X is *not an event*, but $X = k$ is an event:

$$(X = k) \equiv \{\omega \in \Omega | X(\omega) = k\}$$

Random Variables (Cont.)

Here are some example probability notations for random variables X and Y:

- $\mathbb{P}[X = k] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = k\}]$
- $\mathbb{P}[a \leq X \leq b] = \mathbb{P}[\{\omega \in \Omega : a \leq X(\omega) \leq b\}]$
- $\mathbb{P}[X + Y \geq k] = \mathbb{P}[\{\omega \in \Omega : X(\omega) + Y(\omega) \geq k\}]$

Distributions

Problem: How do we define the "spread" / "shape" of the probabilities of a random variable taking certain values?
Solution: A distribution is some assignment of probabilities (via $\mathbb{P}$) on an arbitrary sample space (a distribution does not need a sample space to define it, and it is solely defined by probability-values). For a discrete distribution:

$$\sum_{k \in \mathbb{R}} \mathbb{P}[X = k] = 1$$

Bernoulli Distribution

Situation: We flip a coin with probability p, do we get a heads?
Definition: We say $X \sim \text{Bernoulli}(p)$, and:

$$\mathbb{P}[X = k] = \begin{cases} p & \text{if } k = 1 \\ 1 - p & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$$

Binomial Distribution

Situation: We flip n coins each with an independent probability p of landing heads. How many heads do we get in total?
Definition: We say $X \sim \text{Binomial}(n, p)$, where:

$$\mathbb{P}[X = k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, \dots, n\} \\ 0 & \text{else} \end{cases}$$

Some facts:

- $X = \sum_{i=1}^{n} X_i$ if $(\forall i \in \{1, 2, \dots, n\})(X_i \sim \text{Bernoulli}(p))$ and $X_1, X_2, \dots, X_n$ are mutually independent
- $X + Y \sim \text{Binomial}(m + n, p)$ if $Y \sim \text{Binomial}(m, p)$ and X and Y are independent

Geometric Distribution

Situation: We flip a coin with heads probability p and we stop when we land our first heads. How many times do we flip the coin?
Definition: We say $X \sim \text{Geometric}(p)$, where:

$$\mathbb{P}[X = k] = \begin{cases} p(1-p)^{k-1} & \text{if } k \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

Some facts:

- $\mathbb{P}[X > k] = (1-p)^k$
- **Memoryless**: For $n \geq m$, $\mathbb{P}[X > n | X > m] = (1-p)^{n-m}$, meaning that if I have already failed to land heads m times, it doesn't mean I will have a heads anytime soon (each flip is independent).

Poisson Distribution

Situation: We have memoryless arrivals in some finite time interval. In particular, we have an average of λ arrivals within our chosen time slice. How many arrivals do we get in our interval?
Definition: We say $X \sim \text{Poisson}(\lambda)$, where:

$$\mathbb{P}[X = k] = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & \text{if } k \in \{0, 1, \dots\} \\ 0 & \text{else} \end{cases}$$

Some facts:

- If X represents arrivals in times $[a, b]$, Y represents arrivals in times $[c, d]$, and $b < c$, then X and Y are independent.
- $X + Y \sim \text{Poisson}(\lambda + \mu)$ if $Y \sim \text{Poisson}(\mu)$ and X and Y are independent