# CS 70 Discussion 6B

October 11, 2024

# Counting

Some facts that always hold:

- Symmetry: $\binom{n}{k} = \binom{n}{n-k}$
- Summation: $2^n = \sum_{i=0}^{n} \binom{n}{i}$
- Recursive Definition: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
- Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

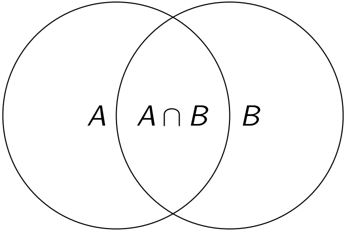
You can use Pascal's triangle to get binomial coefficients quickly. The row represents $n$, the column represents $k$, and $\binom{n}{k}$ is the value. The rows and columns are 0-based indexed (the first row and column are 0, not 1):

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

# Principle of Inclusion-Exclusion

**Problem**: I want to count the number of elements in the set $A_1 \cup A_2 \cup \ldots \cup A_n$.

**Solution**: For the $n = 2$ case:




We have $|A \cup B| = |A| + |B| - |A \cap B|$. But, general formula is:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{s=1}^{n} (-1)^{s-1} \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ |S|=s}} \left|\bigcap_{i \in S} A_i\right|$$

# Combinatorial Proofs

These are essentially informal proofs where you prove that two sets are the same size by setting-up a "story" where counting the elements in one set is functionally equivalent to counting the elements in the other set. Example for proving $\binom{n}{k} = \binom{n}{n-k}$:

- Left-Hand Side (LHS): We have $n$ people. We want to pick $k$ people out of these $n$ people to be representatives. Thus $\binom{n}{k}$.
- Right-Hand Side (RHS): We have $n$ people. We pick $n-k$ people to *not* be representatives, and we make the rest of the $k$ people representatives. Thus $\binom{n}{n-k}$.

Both these situations finds all possible groups of $k$ representatives.

# Countability

We define three main types of sets:

- **Finite-Sized, Countable Set**: A set with a finite number of elements
  - Examples: any set with a finite size (ex. $\{0.342, 5554, \frac{1}{4}\}$)
- **Infinite-Sized, Countable Set**: A set with an infinite number of elements, but you can *order* all of the elements (i.e. you can create an infinite-length list of all elements in the set and each element has an index)
  - Examples: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$
- **Infinite-Sized, Uncountable Set**: A set where there is no way to order the elements of the set.
  - Examples: $\mathbb{R}, \mathbb{C}$

**Cardinality**/Size Comparison: |Finite-Sized, Countable Set| $<$ |Infinite-Sized, Countable Set| $<$ |Infinite-Sized, Uncountable Set|

# Countability (Cont.)

**Problem**: How to prove a set $S$ is countable?
**Solution**: Here are some options:

- Prove that there is some function $f : S \rightarrow \mathbb{N}$ that is one-to-one (injective), which proves $|S| \leq |\mathbb{N}|$.
- Prove that there is some bijective function $f : S \rightarrow \mathbb{N}$ (one-to-one and onto), which proves $|S| = |\mathbb{N}|$ ($S$ is an infinite-sized, countable set).
- Prove that there are some one-to-one functions $f : S \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow S$, which proves $|S| = |\mathbb{N}|$ (this is another way to prove a bijection between $S$ and $\mathbb{N}$).
- Prove that $S \subseteq T$, where you already know $T$ is countable.

# Countability (Cont.)

**Problem**: How to prove a set $S$ is uncountable?
**Solution**: Here are some options:

- Prove that $S \supseteq T$, where $T$ is some uncountably infinite set.
- Use a **Cantor's Diagonalization** proof.

# Cantor's Diagonalization

Proof technique:

1. Assume that set $S$ is countable.
2. This means that all elements can be ordered. Example for $S = [0, 1]$:

$$
\begin{gathered}
0.0234562374533564\ldots \\
0.6547345765765462\ldots \\
\vdots
\end{gathered}
$$

3. Create a new element $x \in S$ by changing the diagonal elements of our table. Example: for $S = [0, 1]$, we can turn all non-5 digits into 5s and all 5-digits into 2s along the diagonal:

$$x = 0.525525...$$

4. $x \in S$, but it cannot be in our table. Contradiction!