CS 70 Discussion 11B

November 15, 2024

Markov's Inequality

Problem: If I can't calculate the exact probability that a random variable X takes some range of values, how can I easily upper-bound the probability?
Solution: For any *non-negative* random variable X and constant $c > 0$, **Markov's Inequality** always holds:

$$\mathbb{P}[X \geq c] \leq \frac{\mathbb{E}[X]}{c}$$

Chebyshev's Inequality

Problem: I want a stronger, two-sided bound of a probability (i.e. I want to upper-bound the probability that X deviates from its mean $\mathbb{E}[X]$ by at least some value).
Solution: For any random variable X with $\mathbb{E}[X] < \infty$ and constant $c > 0$:

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq c] \leq \frac{\mathrm{Var}(X)}{c^2}$$

Confidence Intervals

Problem: I have some distribution with variance $\leq \sigma^2$ and unknown expectation μ. I can take n **independent, identically-distributed (i.i.d.)** samples $X_1, X_2, \ldots, X_n$ of this distribution. How many times do I need to sample from the distribution to estimate the true value of μ?

Definition: We define an **unbiased estimator** $\hat{\mu}$ (i.e. $\mathbb{E}[\hat{\mu}] = \mu$):

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

We want n large enough such that for some **error margin** $\epsilon > 0$ and **confidence level** $\delta \in [0, 1)$:

$$\mathbb{P}[\hat{\mu} \in [\mu - \epsilon, \mu + \epsilon]] \geq \delta$$

The following lower-bound on n gives us the desired probability:

$$n \geq \frac{\sigma^2}{\epsilon^2(1 - \delta)}$$

Weak Law of Large Numbers (WLLN)

For a random variable $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ where the $X_1, X_2, \ldots, X_n$ are i.i.d. from a distribution with expectation $\mu < \infty$:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{i.p.}} \mu$$

This means that as we sample more and more times, the variance of the possible outcomes we get approaches 0 (i.e. $\mathrm{Var}(\hat{\mu}) \to 0$), which explains why sampling more times let's us be more confident in the accuracy of our estimate.

Sidenote: There is also a Strong Law of Large Numbers (SLLN) which shows a stronger type of convergence, but for this class, you can just treat SLLN and WLLN as being the same.

Recap: Discrete Random Variables

Here is a table of statistics of common discrete distributions:

Name	PMF	Expectation	Variance
Bernoulli	$\begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases}$	p	$p(1-p)$
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Geometric	$p(1-p)^{k-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}$	λ	λ

Note: I would recommend putting formulas in the notes on your cheat sheet! You can use these values on exams without proof.