CS 70 Discussion 14A

December 4, 2024

Markov Chains

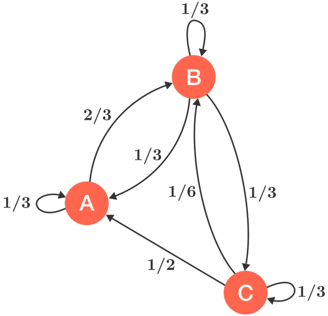
A Markov chain is a sequence of random variables $X_0, X_1, \ldots$ that follows the **Markov property**:

$$\begin{aligned}\mathbb{P}[X_t = x_t | X_0 = x_0, \ldots, X_{t-1} = x_{t-1}] &= \mathbb{P}[X_t = x_t | X_{t-1} = x_{t-1}] \\ &= P_{x_{t-1}, x_t}\end{aligned}$$

This means that the **state** that we are at at a certain time is only dependent on the state we were at at the previous timestep. We refer to P_{x_{t-1}, x_t} as a **transition probability** and P as the **transition matrix** (P is a square matrix with the number of rows and number of columns equaling the number of states in our chain). Each row of P also naturally sums to 1.

Markov Chains (Cont.)

Given the Markov property, you can view a Markov chain as a graph with each node representing a state, and each edge representing the probability of moving from one state to another in a timestep:



Initial Distribution

Problem: While we have the matrix P to represent our transition probabilities, how do we represent our distribution of states we *start* at?
Solution: We define π_0 as the **initial distribution**, where $\mathbb{P}[X_0 = x] = \pi_0(x)$. One thing to note is that π_t is common notation for the distribution of states we will be at in timestep t (i.e. $\mathbb{P}[X_t = x] = \pi_t(x)$). π is a *row vector*, and the following property holds for all times $t > 0$:

$$\pi_t = \pi_{t-1}P = \pi_0 P^t$$

Note: We are performing matrix multiplication to get P^t.

Hitting Probability

Problem: Let's say I start at a state s. I want to calculate the probability that I hit state w before I hit state ℓ. How do I calculate this probability?

Solution: We define $\alpha(x)$ as the probability of getting to w before ℓ if we are currently at state x. We now set-up a system of equations for your Markov chain as follows:

$$\alpha(w) = 1$$

$$\alpha(\ell) = 0$$

$$(\forall x \neq w, x \neq \ell)\left(\alpha(x) = \sum_{y=1}^{n} \alpha(y) P_{x,y}\right)$$

Our desired probability is $\alpha(s)$.

Hitting Time

Problem: Let's say I start at state s. I want to calculate the expected number of time steps to reach state w. How do I calculate this expectation?
Solution: We define $\beta(x)$ as the expected time to reach state w starting from x. Now, we can set-up the following system of equations for your Markov Chain:

$$\beta(w) = 0$$

$$(\forall x \neq w) \left(\beta(x) = 1 + \sum_{y=1}^{n} \beta(y) P_{x,y} \right)$$

Our desired expectation is $\beta(s)$.