UNIT-4

GRAPH THEORY

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Contents

Basic Concepts of Graphs, Matrix Representation of Graphs: Adjacency Matrix, Incidence Matrix, Isomorphic Graphs, Paths and Circuits, Euler and Hamilton Graphs, Planar Graphs and Euler's Formula.

Graphs

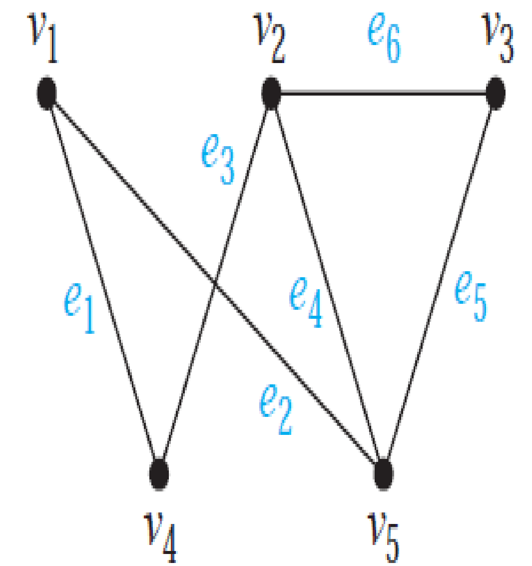
- Discrete structures consisting of vertices and edges that connect these vertices
- Depending on the type and number of edges that can connect a pair of vertices, there are many kinds of different graphs.
- Can be used to model a variety of areas
- Modelling road maps, assignment of jobs to employees of an organization, links between websites, modelling computer networks, social networks, outcomes of a round-robin tournament etc.,

Basic Definitions

Graph:

A graph G consists of two sets:
(1)A set V=V(G) whose elements are called vertices or nodes or points of G
(2)A set E=E(G) of unordered pairs of distinct vertices called edges of G.
A graph is denoted by G(V,E).

- A vertex is denoted by a dot or a small circle
- An edge is denoted by curve or a line



Adjacent:

Two vertices are said to be adjacent if the two vertices are connected by an edge. The vertices are called endpoints of the edge.

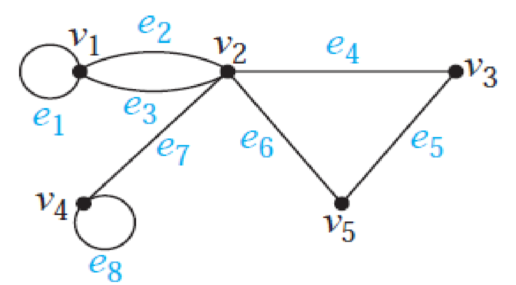
Incident:

The edge is said to incident on each of its endpoints.

Multiple or Parallel Edges:

If two vertices in a graph G are connected by more than one edge, then such edges are called multiple or parallel edges.

Ex: e2,e3 are multiple edges.



Loop:

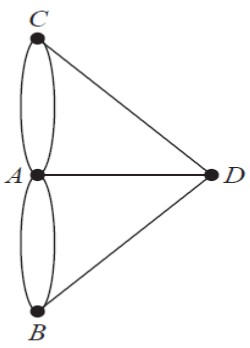
Any edge starting and ending at the same vertex is called a loop.

Ex:e1,e8 are loops.

Types of Graphs

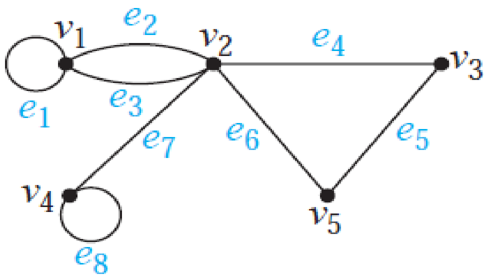
Multigraph:

A graph containing multiple edges is called Multigraph.



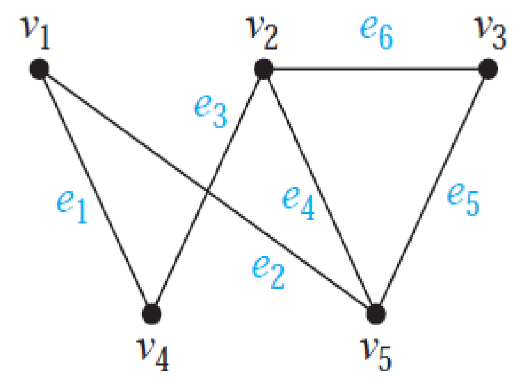
Pseudograph:

A graph containing multiple edges and/or loops is called Pseudo graph.



Simple Graph:

A graph without multiple edges and loops is called a simple graph.

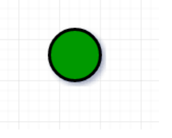


Finite Graph:

A graph is said to be finite if it has finite number of vertices and finite number of edges.

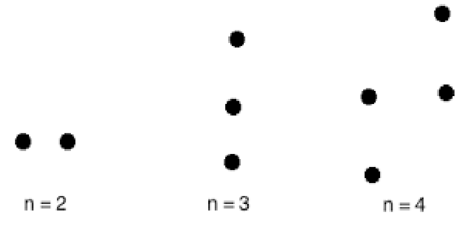
Trivial graph:

A graph with single vertex and no edges is called trivial graph.



Null graph:

A graph with finite number of vertices and no edges is called a Null graph.



Types of Vertices

Degree of a vertex:

In a graph, the degree of a vertex is defined as the number of edges that are incident on vertex v, denoted by deg(v).

Based on the degree of the vertex ,the vertices can be classified into

1.Odd vertex, if the degree of the vertex is an odd number

2.Even vertex, if the degree of the vertex is an even number

3.Isolated vertex, if the degree of the vertex is 0.

4.Pendant vertex, if the degree of the vertex is 1.

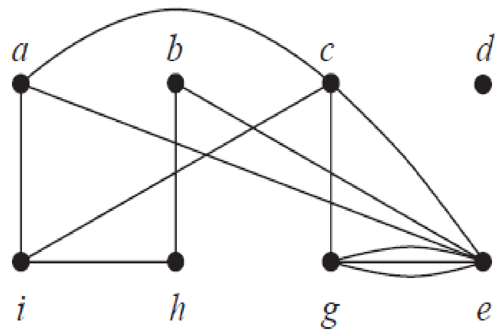
Note: The degree of the vertex having a loop is 2.

Ex:

i. The graph is a multigraph, since the vertices g,e have multiple edges.

ii. Degrees of the vertices:

Deg(a)=3
Deg(b)=2
Deg(c)=4
Deg(d)=0
Deg(e)=6
Deg(g)=4
Deg(h)=2
Deg(i)=3



iii.The vertices a, i are odd vertices
The vertices b, c, e, g,h are even vertices
d is an isolated vertex.

Some Useful Theorems

Theorem1(Handshaking Theorem):

If G=(V,E) is an undirected graph with m edges then $\sum_i \deg(v_i) = 2m$

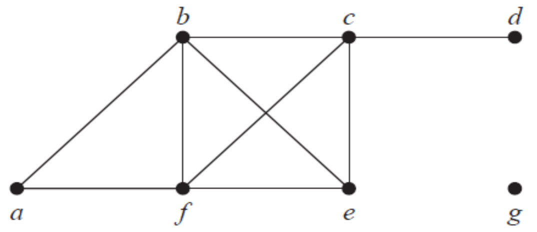
i.e., the sum of the degrees of all the vertices of an undirected graph is twice

the number of edges of the graph and hence a even number.

Theorem2:

The number of vertices of odd degree in an undirected graph is even.

Verify Handshaking Theorem for the following graph



Sol:No.of vertices=7

No.of edges=m=9

Deg(a)=2,deg(b)=4,deg(c)=4,deg(d)=1,deg(e)=3,deg(f)=4,deg(g)=0

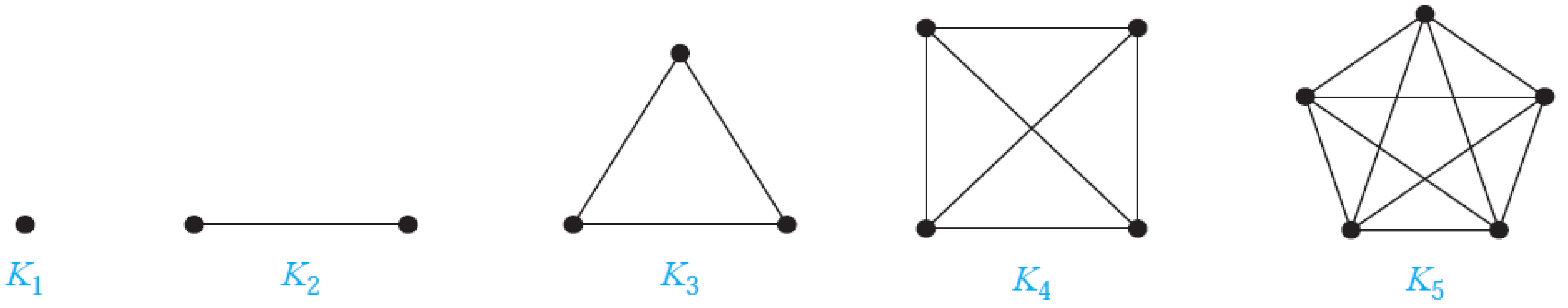
By Handshaking theorem,

$\sum_i \deg(v_i) = 2+4+4+1+3+4+0 = 18 = 2m$.Theorem is verified.

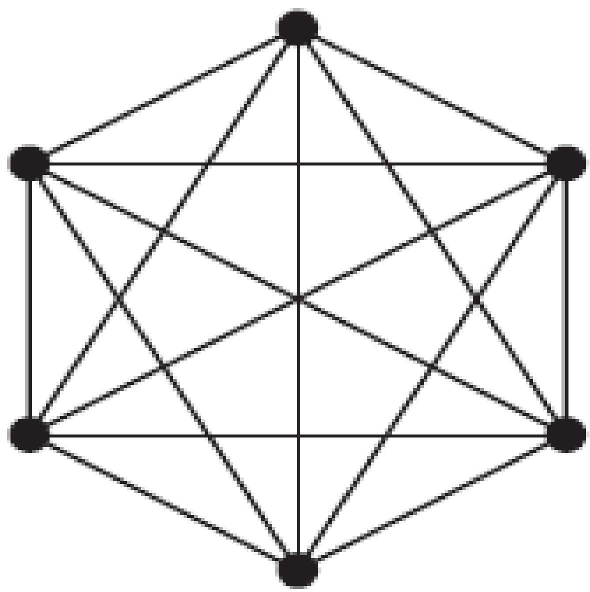
Some Special Simple Graphs

Complete Graph

A simple graph in which there is exactly one edge between each pair of distinct vertices is called a Complete Graph ,denoted by K_n.



Draw the Complete Graph K_6.



K_6

Note:

1.The number of edges in K_n is $\frac{n(n-1)}{2}$

2.The maximum number of edges in a simple graph with

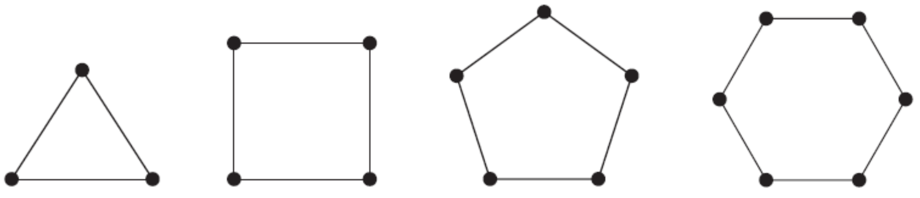
n vertices is $\frac{n(n-1)}{2}$

Regular Graph

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

If every vertex in a regular graph has degree n, then the graph is called n-regular.

Ex:2-regular graphs



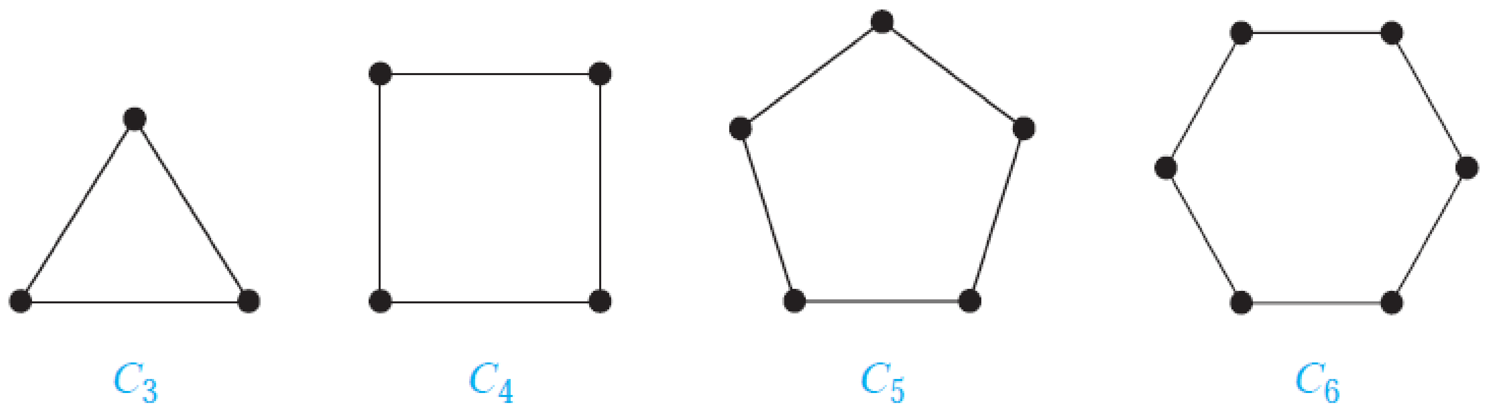
Some Special Simple Graphs

Cycles

A cycle $C_n, n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$ and edges

$\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}$.

A Cycle is a 2-regular graph



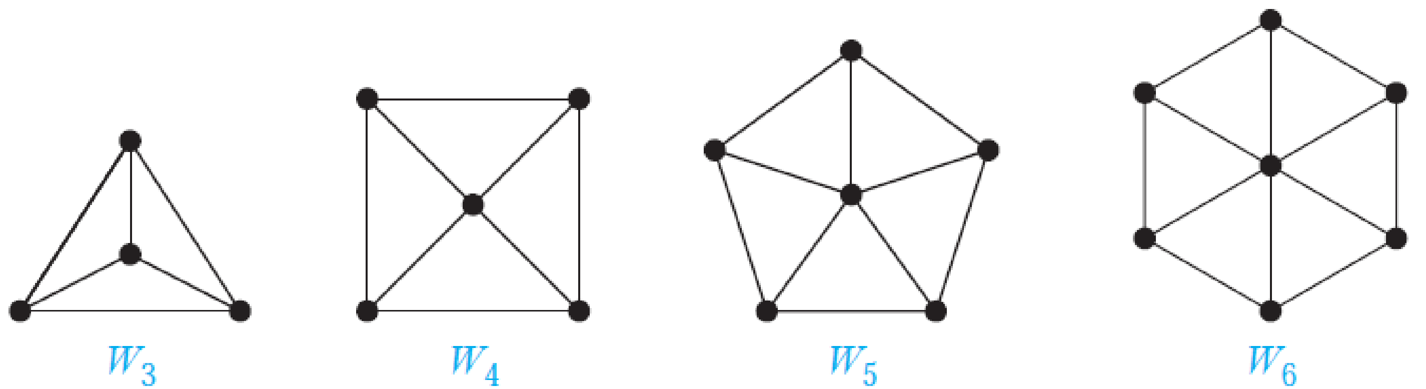
Wheels

A Wheel Wn is obtained by adding a additional vertex to a cycle

Cn,n≥3 and connect this new vertex to each of the other vertices

by new edges.

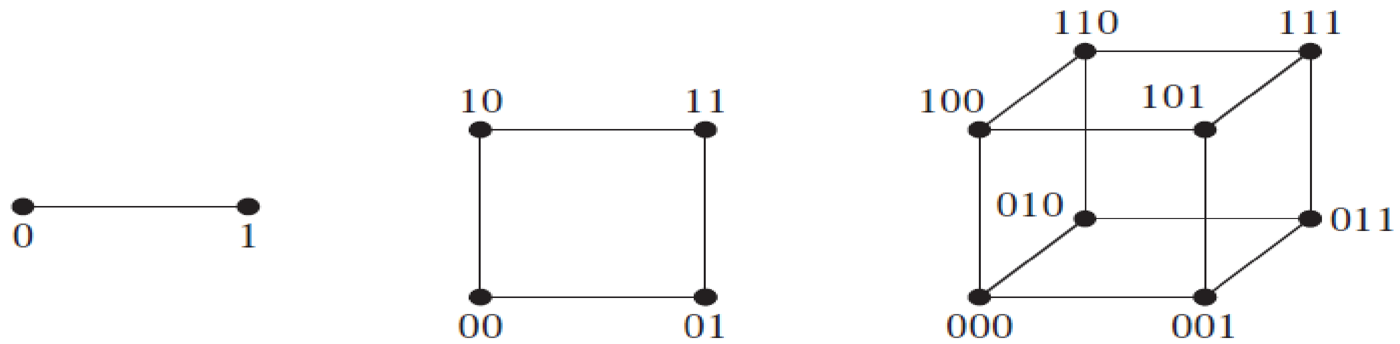
A Wheel is a 3-regular graph



Some Special Simple Graphs

n-Cubes

An n-cube is a graph that has vertices representing the 2^n bit strings of length n.Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position,denoted by Q_n.



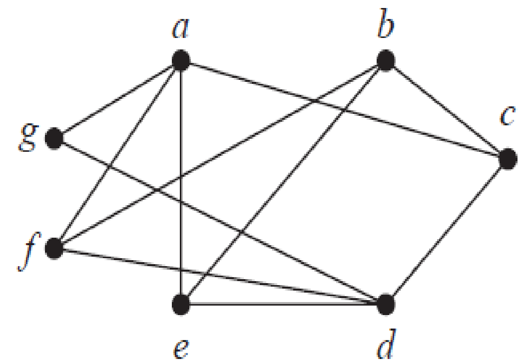
Q_1 Q_2 Q_3

Some Special Simple Graphs

Bipartite Graphs:

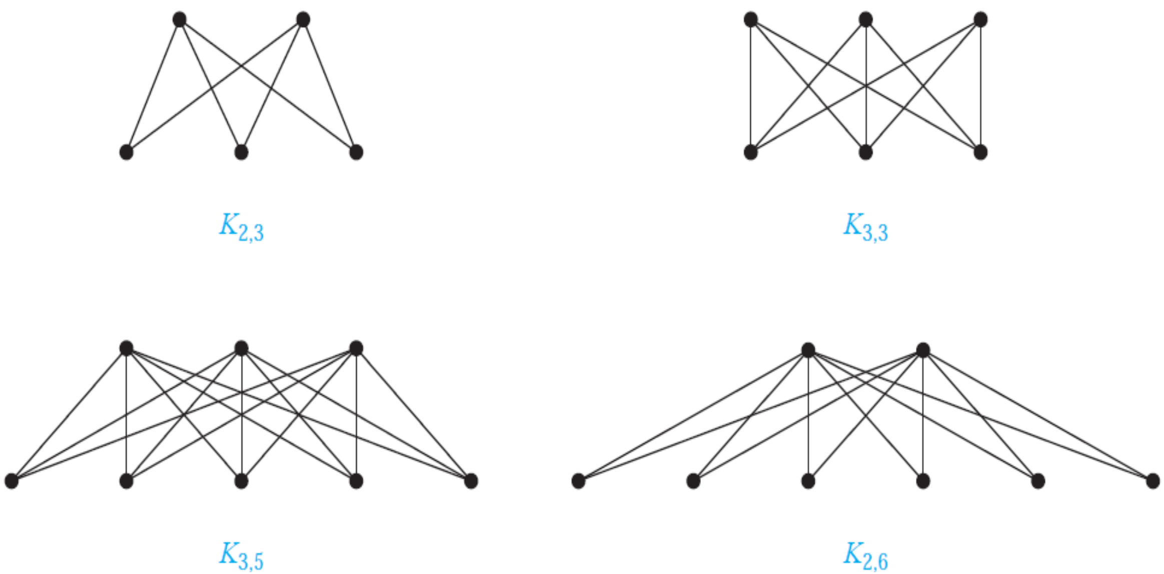
If the vertex set of a simple graph G=(V,E) can be partitioned into two subsets V1 and V2 such that every edge of G connects a vertex in V1 and a vertex in V2(so that no edge in G connects either two vertices in V1 or two vertices in V2),then G is called a bipartite graph

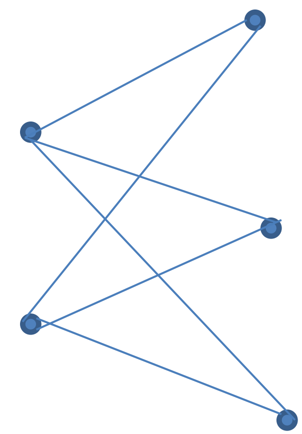
Ex:The vertex set can be divided into two disjoint sets {a,b,d} and {c,e,f,g} such that there is an edge from one set to another and not inside the sets.



Complete Bipartite Graphs:

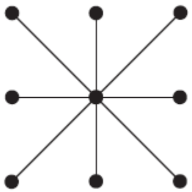
If each vertex of V_1 is connected with each vertex of V_2 by an edge, then G is called a completely bipartite graph denoted by $K_{m,n}$ where m is no. of vertices in V_1 set and n is no. of vertices in v_2 set.





Applications of Special Graphs

1. A local area network can be represented as $K_{1,n}$

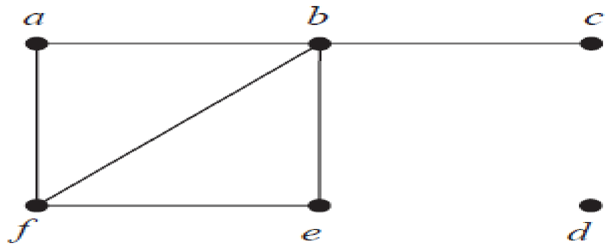


2.

Practice Problems

1. In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

1.



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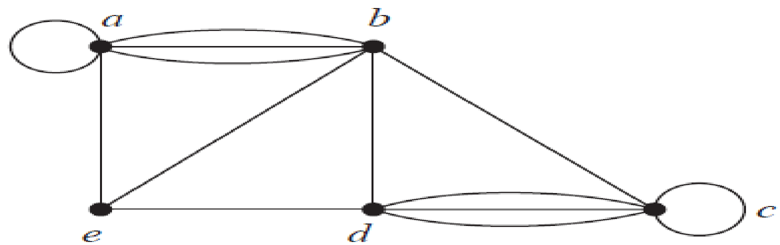

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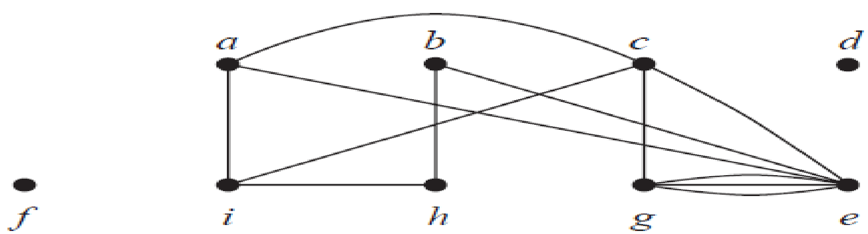


Also check the handshaking theorem in each case

Practice Problems

2. Draw these graphs.

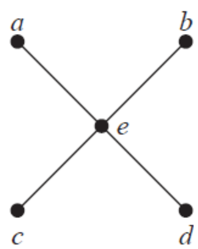
a) K_7 **b)** $K_{1,8}$ **c)** $K_{4,4}$
d) C_7 **e)** W_7

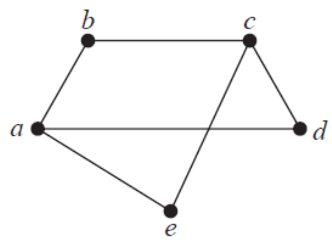
3. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

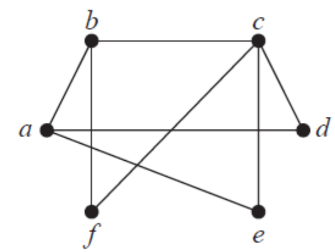
   a) Model the possible marriages on the island using a bipartite graph.

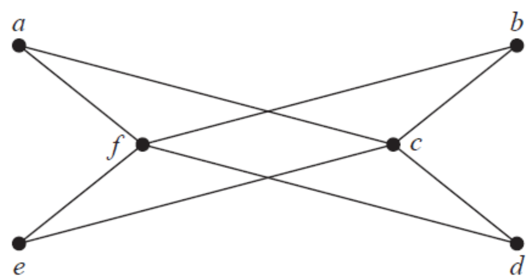
Practice Problems

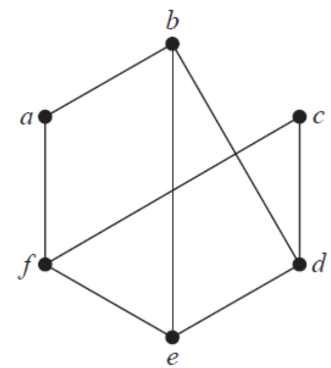
4.Check whether the following graphs are bipartite?

a) 

b) 

c) 

d) 

e) 

a) V={a,b,c,d,e} v1={a,b,c,d }, v2={ e}

Basic Definitions

Directed Graph or digraph:

A digraph G consists of two sets:

(1)A set V=V(G) whose elements are called vertices or nodes or points of G

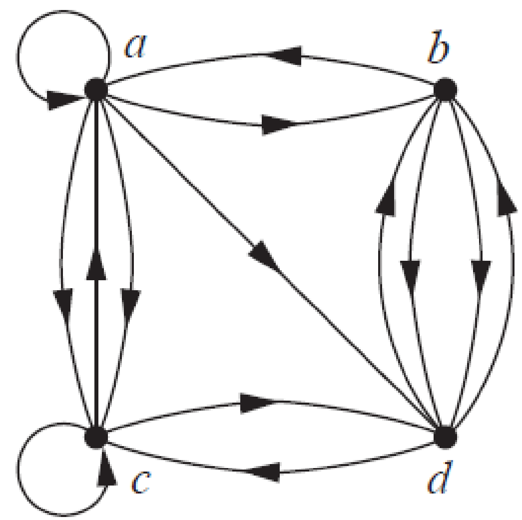
(2)A set E=E(G) of ordered pairs of vertices called edges or directed edges or arcs of G.

- A vertex is denoted by a dot or a small circle
- An edge is denoted by an arrow indicating the direction
- If a directed edge e =(u,v) starts at ‘u’ and ends at ‘v’ then u is called origin or initial point of the edge ‘e’ and v is called destination or terminal point of e
- u is adjacent to v and v is adjacent from u

Parallel Edges:

Two directed edges are said to be parallel if they both begin at vertex u and end at vertex v.

Ex: Edges between the vertices (a,c) and (b,d) are parallel.
Edges between the vertices (a,b) and(c,d) are not parallel,
Since they don’t have the same direction



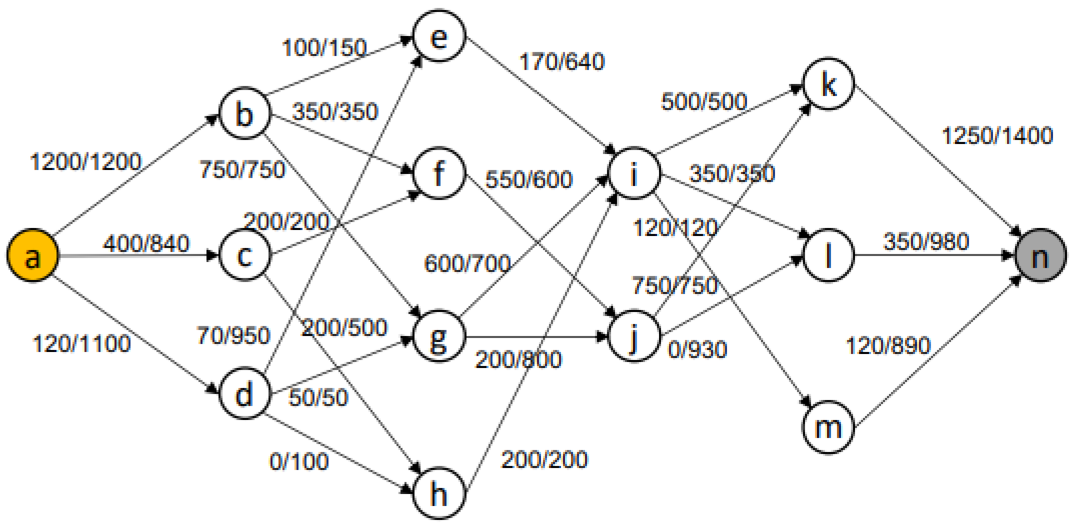
Loop:

Any directed edge starting and ending at the same vertex is called a loop.

Ex: The directed edges at the vertices a and c are loops.

Labelled Directed Graph:

If the edges and/or vertices of a directed graph are labelled with some type of data then ‘G’ is called a labelled directed graph.



Types of Vertices in digraphs

Indegree of a vertex:

The indegree of a vertex v is defined as the number of edges ending at the vertex v or entering into the vertex v, denoted by indeg(v).

Outdegree of a vertex:

The outdegree of a vertex v is defined as the number of edges beginning at the vertex v or leaving the vertex v, denoted by outdeg(v).

Source:

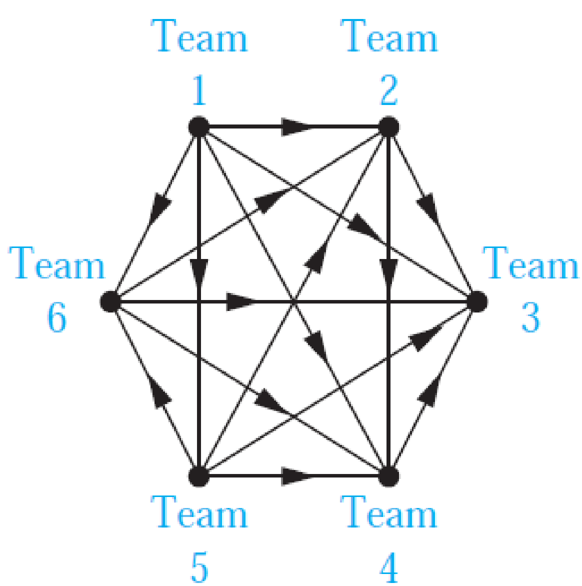
A vertex with zero indegree is called a source.

Sink:

A vertex with zero outdegree is called a sink.

Various Graph Models

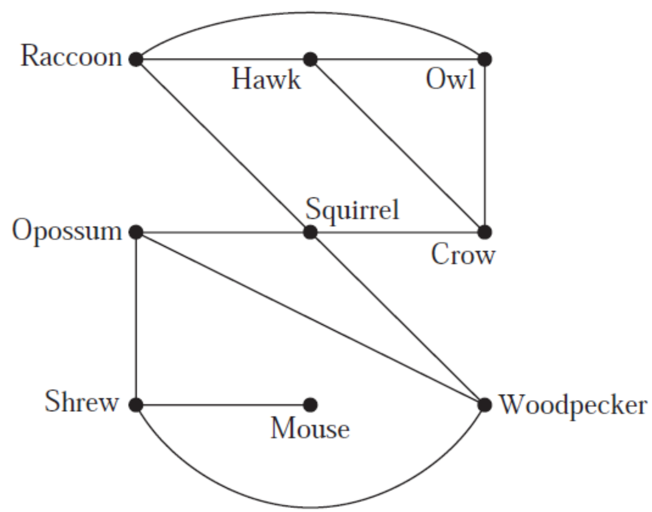
Outcomes of a round-robin tournament



Various Graph Models

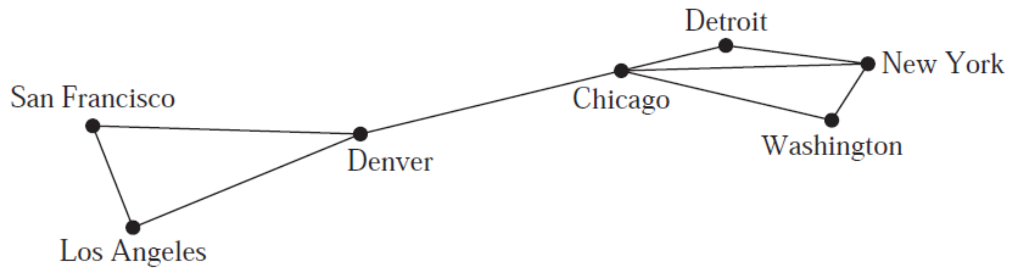
Niche Overlap Graph

The competition between the species in ecosystem can be modelled using these graphs.

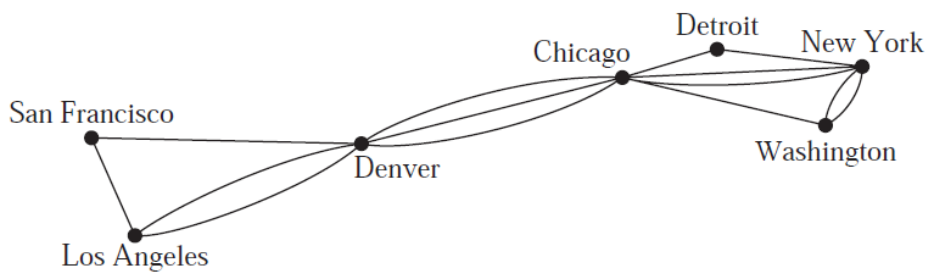


Various Graph Models

A simple Computer Network

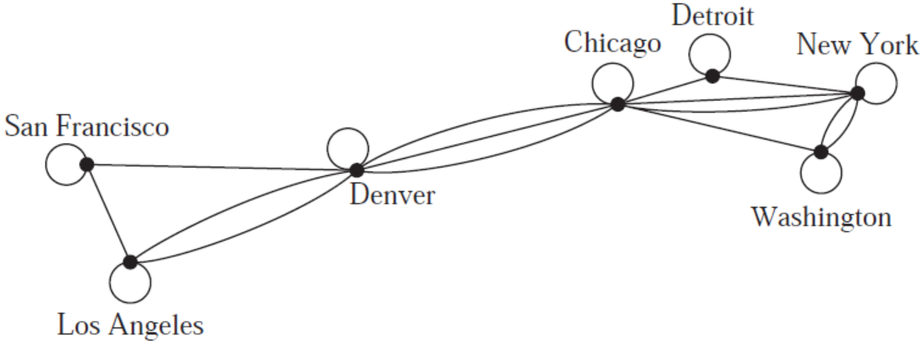


A Computer Network with multiple links

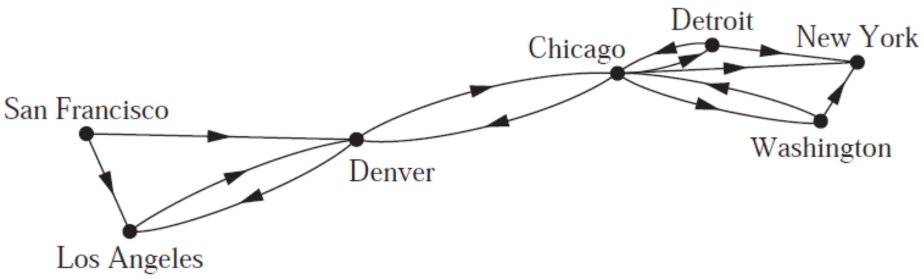


Various Graph Models

A Computer Network with diagnostic links denoted by loops



A Computer Network with one-way communication links



Graph Terminologies

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Module

Matrix Representation of Graphs: Adjacency Matrix, Incidence Matrix

Objective:

❖ To represent the graph into its corresponding adjacency and incident matrices

Outcome:

At the end of the session Student will be able to learn

❖ to understand the basic definitions and terminologies of a graph

❖ To identify the special graphs

Matrix Representation of Graphs

To determine whether two graphs are isomorphic, it will be easier to consider their matrix representations.

There are two types of matrices commonly used to represent graphs.

1. Adjacency Matrix
2. Incidence Matrix

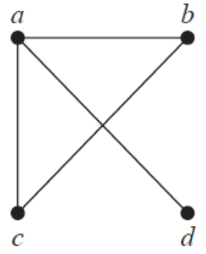
Adjacency Matrix

- When G is a simple graph with n vertices $v_1, v_2, v_3, \ldots, v_n$, the matrix

$$A = A_G = [a_{ij}] \quad where \quad a_{ij} = \begin{cases} 1\ , if\ v_i v_j\ is\ an\ edge\ of\ G \\ 0, otherwise \end{cases}$$

Is called the adjacency matrix of G

Ex: For the graph G ,the adjacency matrix is given by



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

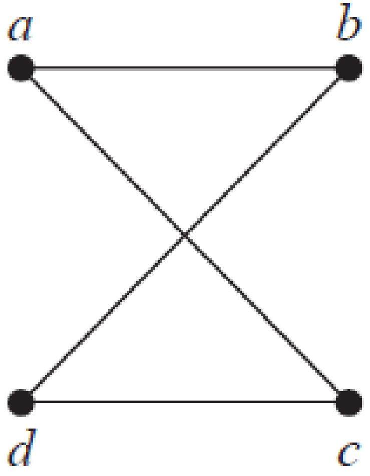
Adjacency Matrix

Ex: Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

Sol:



Properties of Adjacency Matrix

- Since a simple graph has no loops ,each diagonal entry of A is zero.
- The adjacency matrix of a simple graph is symmetric
- Degree of a vertex is equal to the number of one's in the corresponding row or column

Adjacency Matrix of a Pseudograph

Pseudograph:

A graph containing multiple edges and/or loops is called Pseudograph.

A Pseudograph can also be represented by an adjacency matrix using the following steps:

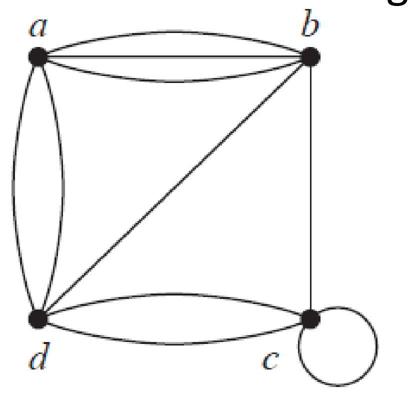
1)A loop at the vertex v_i is represented by a 1 at (i,I)th position

2)The (i,j)th entry equals the number of edges that are incident on v_i and v_j

Note :
The adjacency matrix of a Pseudograph is also a symmetric matrix

Adjacency Matrix of a Pseudograph

Ex: Find the adjacency matrix of the following Pseudo graph

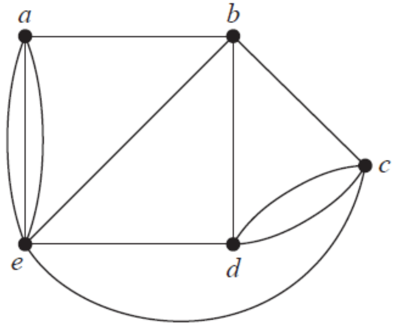


Sol: Taking the vertices in the order a, b, c,d , the adjacency matrix is given by

$$A= \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of a Multigraph

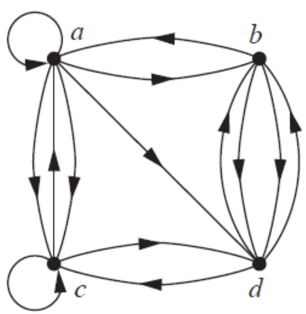
In a similar way, we can represent a multigraph using an adjacency matrix. These adjacency matrices are also symmetric.



$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

Adjacency Matrix of a directed graph

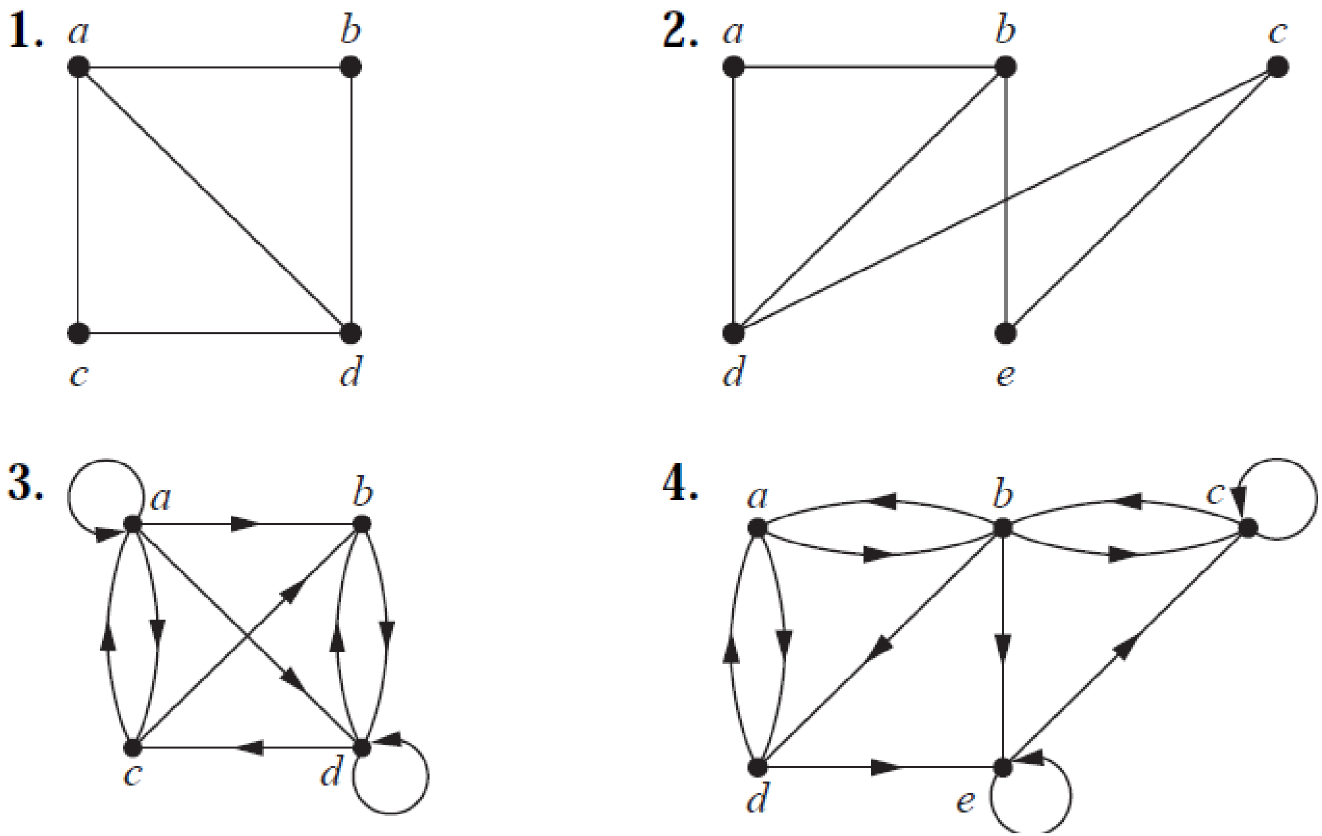
In a similar way, we can represent a directed graph using an adjacency matrix.
These adjacency matrices may or may not be symmetric.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

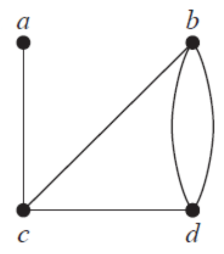
Practice Problems

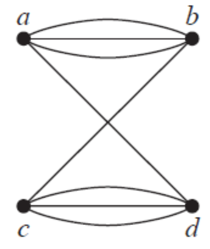
1.Represent the following graphs using adjacency matrix:

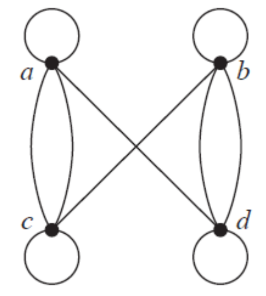


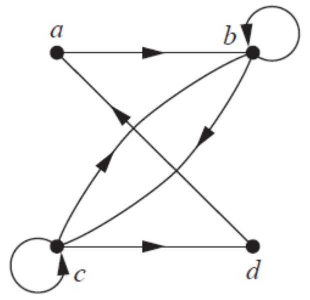
Practice Problems

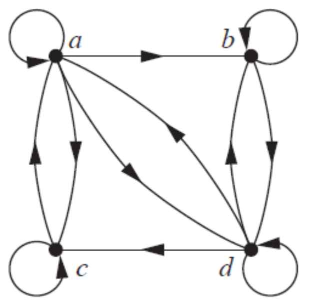
1.Represent the following graphs using adjacency matrix:

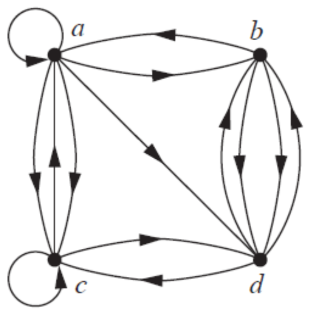
a) 

b) 

c) 

d) 

e) 

f) 

Practice Problems

2. Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$

d) C_4 e) W_4 f) Q_3

3.Represent the following adjacency matrices into graphs :

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

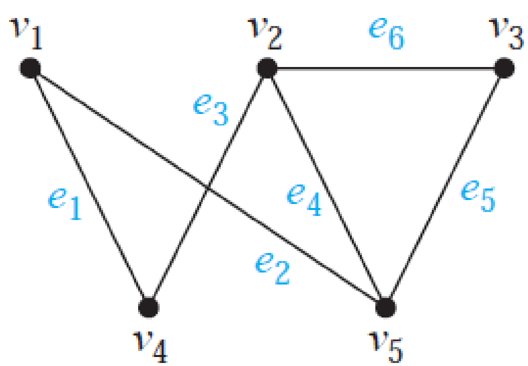
Incident Matrix

- If G=(V,E) is an undirected graph with n vertices $v_1, v_2, v_3, \ldots, v_n$, and m edges $e1, e2, e3, \ldots, en$ then the n x m matrix

$$B = [b_{ij}] \quad where \quad b_{ij} = \begin{cases} 1, & if\ edge\ e_j\ is\ incident\ on\ v_i \\ 0, & otherwise \end{cases}$$

Is called the Incidency matrix of G

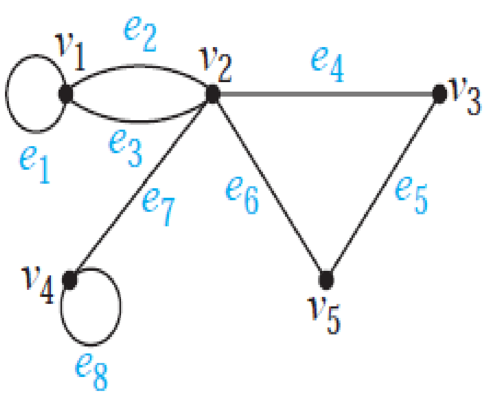
Ex: For the graph G ,the incidency matrix is given by



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incident Matrix

- Incident matrices can also be used to represent Pseudographs.
- Parallel edges are represented in the matrix using columns with identical entries, since these edges are incident on the same pair of vertices.
- Loop is represented by a column with exactly one unit entry ,corresponding to the concerned vertex.
- Ex: A pseudograph and its incident matrix are given as

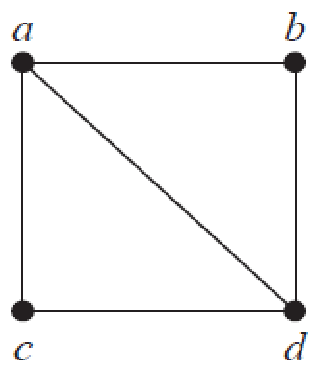


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

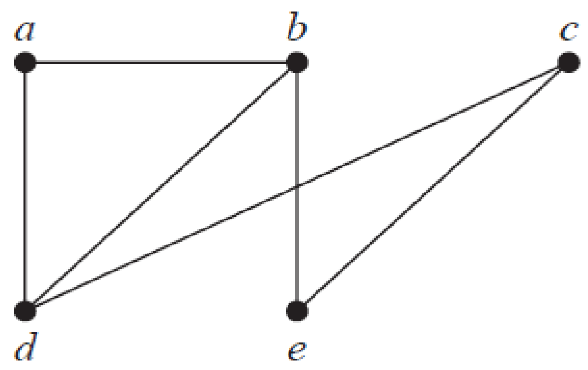
Practice Problems

1.Represent the following graphs using incidency matrix:

1.



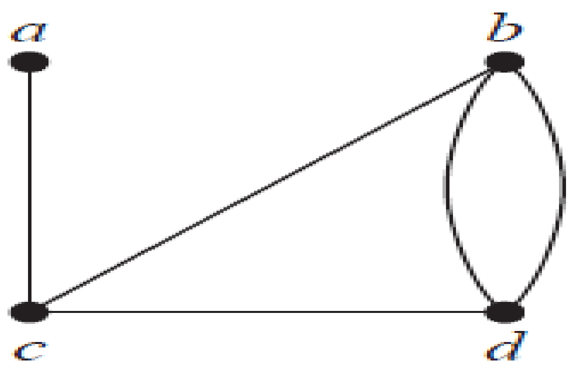
2.



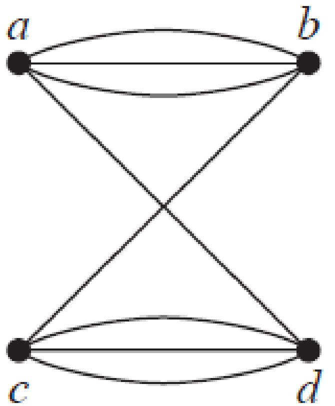
Practice Problems

2.Represent the following graphs using incidency matrix:

a)



b)



c)