UNIT-3

RECURRENCE RELATIONS

Topics

Recurrence Relations, Formation of Recurrence Relations, Solving Recurrence Relations by Substitution and Generating Functions, Method of Characteristic Roots.

Sequence:

- A discrete structure used to represent an ordered list
- 1,2,3,5,8 is a finite sequence with finite terms
- 1,3,9,27,…,3^n,… is an infinite sequence.
- A function from the subset of integers(usually natural numbers or whole numbers) to a set S
- a_n is the nth term of the sequence

Example of a sequence:

1. Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

2.

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* a and the *common ratio* r are real numbers.

The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1; 2 and 5; and 6 and $1/3$, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with

$$1, -1, 1, -1, 1, \ldots;$$

the list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with

$$2, 10, 50, 250, 1250, \ldots;$$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$

3. An *arithmetic progression* is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term* a and the *common difference* d are real numbers.

The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3, respectively, if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with

$$-1, 3, 7, 11, \ldots,$$

and the list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with

$$7, 4, 1, -2, \ldots.$$

Recurrence relation:

A recurrence relation is an equation for a sequence of numbers, where each number is given in terms of previous numbers in the sequence. These are very useful for determining the complexity of recursive algorithms. Recurrence relations have applications in many areas of Mathematics like number theory, group theory etc.

Eg:

0, 1, 1, 2, 3, 5, 8,……..

Fibonacci sequence: $F(n) = F(n-1) + F(n-2)$ for $n > 1$, $F(0) = 0$, $F(1) = 1$

Number of moves to solve the towers of Hanoi: $M(n) = 2M(n - 1) + 1$ for $n > 1$, $M(1) = 1$

Order of a recurrence relation:

The order of a recurrence relation is the difference between the largest

and smallest subscript appearing in the relation.

Eg:

$$i)\, a_n = -3a_{n-1} \text{ is a recurrence relation of order } 1.(i.e., n-(n-1)=1)$$

$$ii)\, a_{n+2} - a_{n+1} - 2a_n = 0 \text{ is a recurrence relation of order } 2.(i.e., n+2-n=2)$$

Problems:

Ex1. Find the first four terms of the recurrence relation

$$a_k = a_{k-1} + 3a_{k-2}, \text{ for all integers } k \geq 2, a_0 = 1, a_1 = 2$$

$$a_0 = 1$$

$$a_1 = 2$$

$$a_2 = a_1 + 3a_0 = 2 + 3.1 = 5$$

$$a_3 = a_2 + 3a_1 = 5 + 3.2 = 11$$

2. Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Fibonacci sequence: $F(n) = F(n-1) + F(n-2)$ for $n > 1$, $F(0) = 0$, $F(1) = 1$

3) $a_k = k(a_{k-1})^2$, *for all integers* $k \geq 1$, $a_0 = 1$

Solution:

$a_0 = 1$

k=1, $a_1 = 1.(a_0)^2 = 1.1 = 1$

k=2, $a_2 = 2.(a_1)^2 = 2.1 = 2$

k=3, $a_3 = 3.(a_2)^2 = 3.4 = 12$

4)Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n=a_{n-1}+3$ for $n=1,2,3\ldots$ and suppose that $a_0=2$.What are a_1,a_2,a_3?

Solution:

Given $a_n=a_{n-1}+3$ for $n=1,2,3\ldots$ and $a_0=2$

$$a_1=a_0+3=2+3=5$$

$$a_2=a_1+3=5+3=8$$

$$a_3=a_2+3=8+3=11$$

5. *Show that the sequence 2, 3, 4, 5, ..., 2 + n, ...*
for $n \geq 0$ satisfies the recurrence relation $a_k = 2\,a_{k-1} - a_{k-2}$, $k \geq 2$.

Solution. Let a_n (nth term of the sequence) $= 2 + n$

$$a_k = 2 + k$$
$$a_{k-1} = 2 + (k-1) = 1 + k$$
$$a_{k-2} = 2 + (k-2) = k$$

Now
$$2a_{k-1} - a_{k-2} = 2(1 + k) - k = 2 + k = a_k$$

$\therefore$
$$a_k = 2a_{k-1} - a_{k-2}$$

(or) *consider*

$$a_k - 2a_{k-1} + a_{k-2} = 2 + k - 2(1+k) + k$$
$$= 2 + k - 2 - 2k + k$$
$$= 0$$
$$a_k - 2a_{k-1} + a_{k-2} = 0$$

6. Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer n. Note that $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every nonnegative integer n. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation. ◀

Remark:

A recurrence relation for a particular sequence can be written in more than one way:

Consider the sequence 4,12,36,108,…

Here the numbers form a geometric sequence with common ratio 3.

Ie.,

$$\frac{a_{n+1}}{a_n} = 3 \Rightarrow a_{n+1} = 3a_n$$

is the recurrence relation.

But it is also generating the sequence 5,15,45,…

Hence one of the terms in the sequence should also be given or known to us.i.e., the initial condition makes the difference.

If $a_0 = 4$ then we get the sequence using the recurrence relation

$a_{n+1} = 4.3^n \;, n \geq 0$ called the general solution of the recurrence relation.

Formation of Recurrence Relation Model:

$a_0 = 10,000/-$

1. A person invests Rs.10,000 at the interest of 12% compounded annually. How much will be there at the end of 15 years.

$a_1 = 10000 + 12\% \times 10000$
$= a_0 + 0.12 a_0$
$a_1 = 1.12 a_0$
$a_2 = a_1 + .12 a_1$
$\therefore a_2 = 1.12 a_1$
$a_3 = a_2 + .12 a_2$
$a_3 = 1.12 a_2$
$a_4 = 1.12 a_3$

Sol:

Let a_n represents the amount at the end of n years.

Let a_{n-1} represents the amount at the end of n-1 years.

Since amount after n years is equal to the amount after n-1 years plus the interest

$$a_n = a_{n-1} + 0.12 a_{n-1}$$
$$a_n = 1.12 a_{n-1} \quad , n \geq 1$$

With initial condition a_0=10000.

$$\boxed{\begin{array}{l} a_n = 1.12 a_{n-1} \\ a_0 = 10000 \end{array}} \quad n \geq 1$$

Recurrence Relation Model:

$$a_1 = 1.12a_0$$

$$a_2 = 1.12a_1 = (1.12)^2 a_0$$

$$a_3 = 1.12a_2 = (1.12)^3 a_0$$

$$\ldots\ldots$$

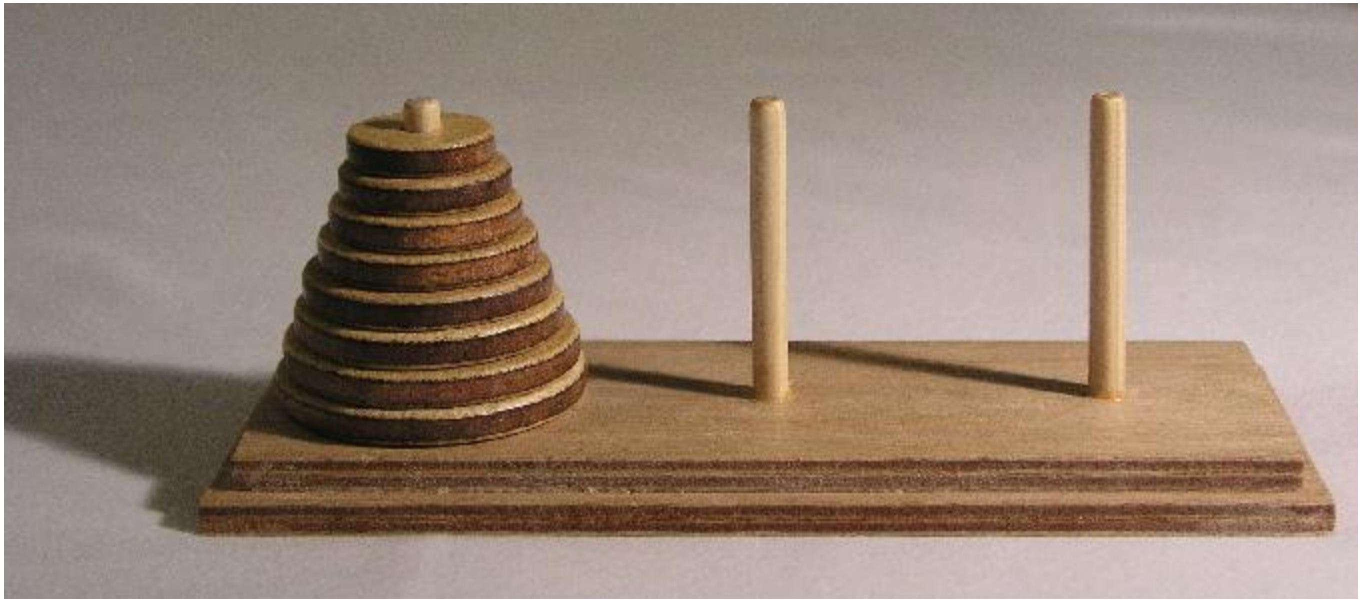
$$a_n = (1.12)^n a_0 \; is\, the \; required \; formula.$$

$$after \;\; 15 \;\; years,$$

$$a_{15} = (1.12)^{15}(10000)$$

Towers of Hanoi problem:

The **Tower of Hanoi** puzzle consists of three vertical pegs and several disks of various sizes. Each disk has a hole in its center for the pegs to go through.



The rules of the puzzle are as follows:

❖The puzzle begins with all disks placed on one of the pegs. They are placed in order of largest to smallest, bottom to top.

❖The goal of the puzzle is to move all of the disks onto another peg.

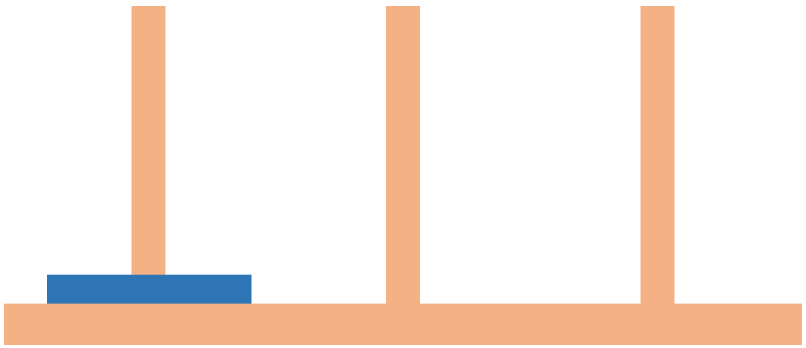
❖Only one disk may be moved at a time, and disks are always placed onto pegs.

❖Disks may only be moved onto an empty peg or onto a larger disk.

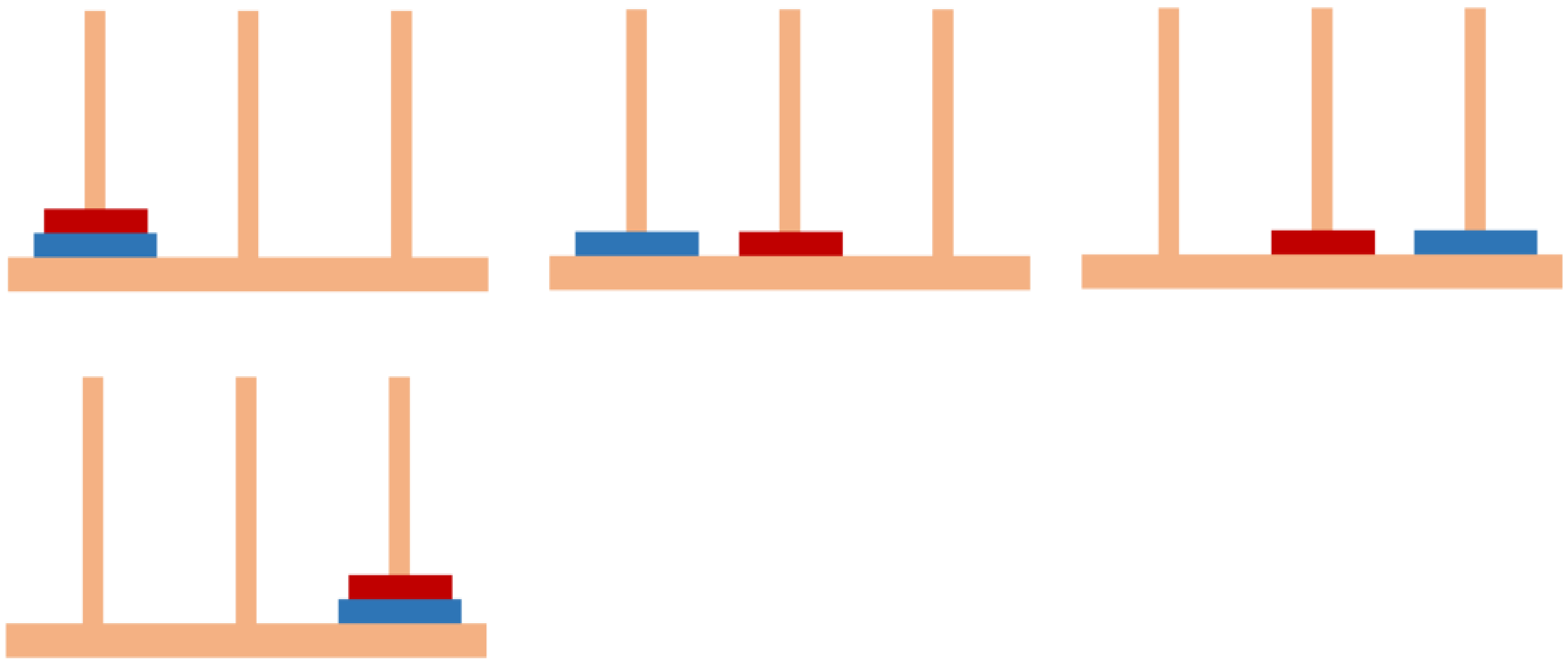
Let T_n be defined as the minimum number of moves needed to solve a puzzle with n disks.

In terms of the recurrence relation, n=1.

T_1=1, because it would only take 1 move to move all the disks to another peg.

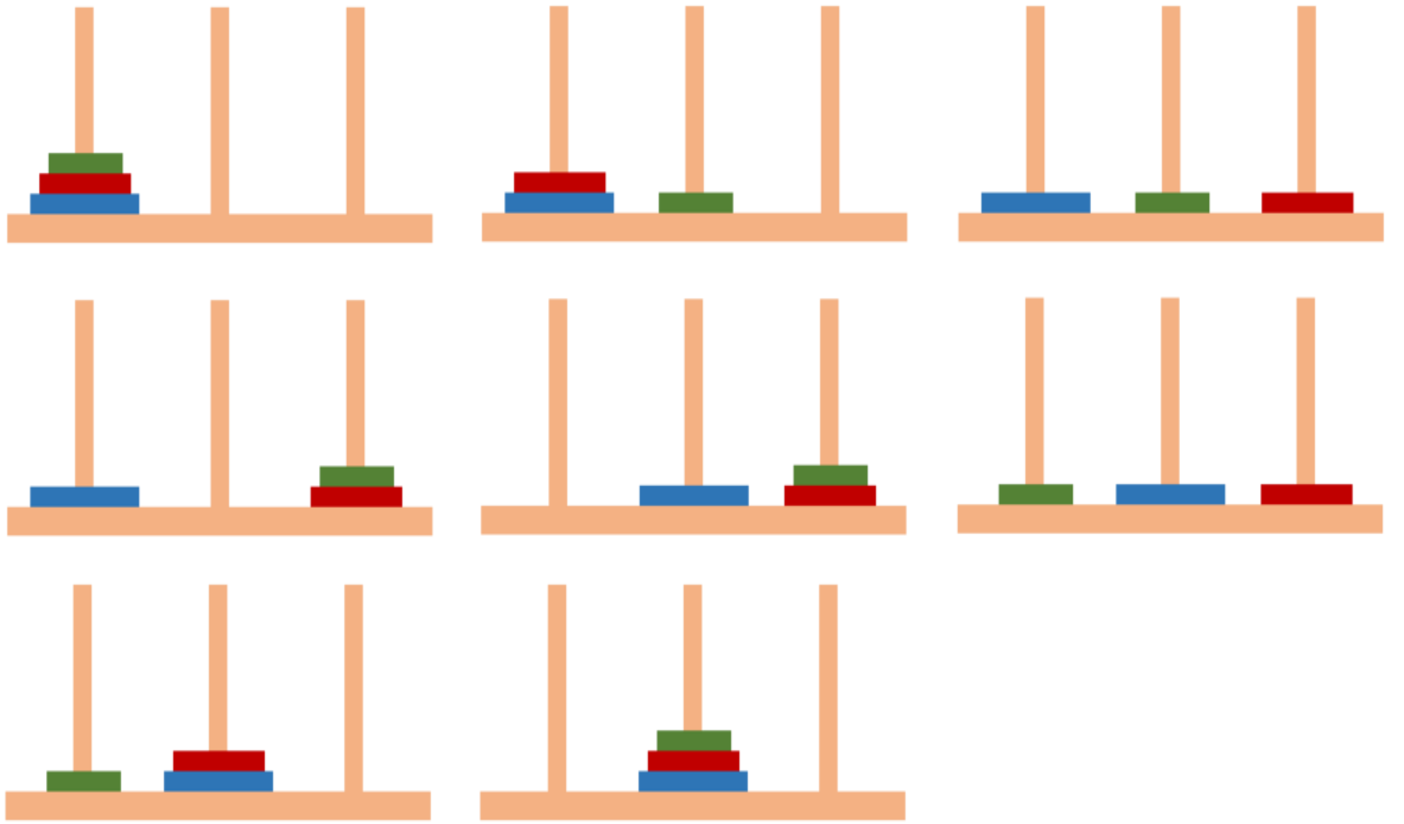


Below is the solution to a Tower of Hanoi puzzle with n=2.



It can be seen from above that $T2=3$.

Below is a solution to a Tower of Hanoi puzzle with n=3.



$T_n=2T_{n-1}+1$,T(0)=0 is the recurrence relation.

$T_0 = 0$

$T_1 = 1$ i.e., $2.0+1=2.\,T_0+1$

$T_2=3$ i.e., $2.1+1=2.\,T_1+1$

$T_3=7$ i.e., $2.3+1=2.\,T_2+1$

$T_n=2T_{n-1}+1$, T(0)=0 is the recurrence relation.

Linear Homogeneous recurrence relation:

A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$$

Where c_i's are constants is called recurrence relation of kth order provided that both c_0 and c_k are non zero.

It is linear since every term containing a_i ,of first order.

If f(n)=0 then the relation is known as homogeneous.

A recurrence relation is said to be non homogeneous if f(n)≠0.

Examples:

1. The recurrence relation $a_n - a_{n-1} = 3$

It is linear non homogenous reccurrence relation with degree 1 and order 1

The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. Order 1 i.e., $P_n-(1.11)P_{n-1}=0$

The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of Order 2.

The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of order 5.

The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear.

The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous.

$a_n = 2a_{n/2} + n$ does not have a constant order.

Its very hard to solve this type of equations.

Methods to Solve linear recurrence relations:

To solve recurrence relations, there are three methods

1. Substitution or iteration method
2. Characteristic roots
3. Generating functions

Substitution or iteration method:

In the substitution method the recurrence relation is used repeatedly to solve for a general expression for an in terms of n. We desire that this expression involve no other terms of the sequence except those given by boundary conditions.

Problems:

1. Solve the recurrence relation S(k)-0.25S(k-1)=0, S(0)=6

Solution:

S(k)=0.25S(k-1) and S(0)=6

K=1, S(1)=0.25 S(0)

K=2, S(2)=0.25*S(1) =0.25*0.25*S(0)

$=(0.25)^2$ S(0)

K=3, S(3)=0.25*S(2)

$=0.25*(0.25)^2$ S(0)

$=(0.25)^3$ S(0)

---- ----- ------------

S(k) $=(0.25)^k$ S(0)

$=6.(0.25)^k$

S(k) $=6.(0.25)^k$

2. Solve the recurrence relation

$$a_n = a_{n-1} + \frac{1}{n(n+1)}, a_0 = 1$$

$$solution : a_n = a_{n-1} + \frac{1}{n(n+1)}$$

$$n = 1,\ a_1 = a_0 + \frac{1}{1.2}$$

$$n = 2,\ a_2 = a_1 + \frac{1}{2.3} = a_0 + \frac{1}{1.2} + \frac{1}{2.3}$$

$$n = 3,\ a_3 = a_2 + \frac{1}{3.4} = a_0 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$$

$$------------$$

$$----------$$

$$a_n = a_0 + \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ---- + \frac{1}{n(n+1)}\right)$$

$$a_n = a_0 + \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$we\,have \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{(1+k)-k}{k(k+1)}$$

$$= 1 - \frac{1}{n+1}$$

$$a_n = a_0 + 1 - \frac{1}{n+1}$$

$$= 1 + 1 - \frac{1}{n+1}$$

$$= 2 - \frac{1}{n+1}$$

$$= \frac{2(n+1)-1}{n+1}$$

$$= \frac{2n+1}{n+1}$$

$$a_n = \frac{2n+1}{n+1}$$

3. Solve the recurrence relation

$$S(k)-10S(k-1)+9S(k-2)=0, S(0)=3, S(1)=11$$

$$Solution:$$

$$S(k)-10S(k-1)+9S(k-2)=0$$

$$S(k)=10S(k-1)-9S(k-2)$$

$$k=2,\ S(2)=10S(1)-9S(0)$$

$$=10.11-9.3$$

$$=110-27=83$$

$$S(2)=83$$

$$k=3,\ S(3)=10S(2)-9S(1)$$

$$=10.83-9.11$$

$$=830-99=731$$

$$S(3)=731$$

$$-------------$$

$$----------$$

$$S(0) = 3 = 9^0 + 2$$
$$S(1) = 11 = 9^1 + 2$$
$$S(2) = 83 = 9^2 + 2$$
$$S(3) = 731 = 9^3 + 2$$
$$-\,-\,-\,-\,-\,-\,-\,-\,-$$
$$-\,-\,-\,-\,-\,-\,-\,-$$
$$S(k) = 9^k + 2$$

4. $Solve\ a_n = a_{n-1} + n^2, a_0 = 7$

5. $Solve\ a_n = a_{n-1} + 2n + 1, a_0 = 1$

$$a_n = a_{n-1} + n^2, a_0 = 7$$

$$n = 1, a_1 = a_0 + 1^2$$

$$n = 2, a_2 = a_1 + 2^2$$

$$\therefore a_2 = a_0 + 1^2 + 2^2$$

$$n = 3, a_3 = a_2 + 3^2$$

$$= a_0 + 1^2 + 2^2 + 3^2$$

$$\therefore a_3 = a_0 + 1^2 + 2^2 + 3^2$$

$$-------------$$

$$---------------$$

$$a_n = a_0 + 1^2 + 2^2 + 3^2 + ... + n^2$$

$$a_n = a_0 + \frac{n(n+1)(2n+1)}{6}$$

$$a_n = 7 + \frac{n(n+1)(2n+1)}{6}$$

Solution of Recurrence Relations by Using Characteristic Roots

In this method, the solution is obtained as the sum of two parts:

(i) the homogeneous solution ,which satisfy the recurrence relation when the RHS of the relation is set to 0. i.e., $f(n)=0$

(ii)The particular solution which satisfies the relation with $f(n)$ on the RHS.

(a) Solution of linear homogeneous recurrence relation:

The basic approach for solving homogeneous relation with $f(n)=0$ is to look for the solution of the form $a_n=r^n$.

Solution of linear homogeneous Recurrence Relations

$a_n = a_n^{(h)} + a_n^{(p)}$

Let $a_n = r^n$ is the solution of the recurrence relation

$a_n = a_n^{(h)}$

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} = 0$$

Dividing by c0,

$$a_n + \frac{c_1}{c_0} a_{n-1} + \frac{c_2}{c_0} a_{n-2} + \ldots + \frac{c_k}{c_0} a_{n-k} = 0$$

Put $a_n = r^n$

$$r^n + \frac{c_1}{c_0} r^{n-1} + \frac{c_2}{c_0} r^{n-2} + \ldots + \frac{c_k}{c_0} r^{n-k} = 0$$

Dividing by r^{n-k},

Dividing by r^{n-k},

$$r^k + \frac{c_1}{c_0} r^{k-1} + \frac{c_2}{c_0} r^{k-2} + \dots + \frac{c_k}{c_0} = 0$$

This equation is known as the characteristic equation of the recurrence relation.

The solution of this equation of the k^{th} degree has k characteristic roots.

order = 1, $r + d_1 = 0$

order = 2 $r^2 + d_1 r + d_2 = 0$

order = 3 $r^3 + d_1 r^2 + d_2 r + d_3 = 0$

The roots of this equation can be

(i) Distinct

(ii) Multiple roots

(iii) Mixed roots

Characteristic equation with distinct roots

If the characteristic equation has distinct roots $r_1, r_2, \ldots r_k$ then the general form of the solutions for homogeneous equation is

$$a_n = a_n^{(h)} = c_1 r_1^n + c_2 r_2^n + \ldots + c_k r_k^n$$

Where $c_1, c_2, \ldots, c_k$ are constants which may be chosen to satisfy any initial conditions.

Solving Recurrence Relation :

1. Solve the recurrence relation, $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$ $given$ $a_0 = 0, a_1 = 1.$

Sol: The given recurrence relation $a_n - a_{n-1} - 2a_{n-2} = 0 - - - - -(1)$

$d_1 = -1, \; d_2 = -2$ order $= n-(n-2) = 2$

is a second order linear homogeneous recurrence relation with constant coefficients.

$x^2 + d_1 x + d_2 = 0$

Let $a_n = r^n$ be the solution of (1).

The characteristic equation is $r^n - r^{n-1} - 2r^{n-2} = 0$ $\quad x^2 - x - 2 = 0$

$\div by \; r^{n-2}$ $\quad x = 2, -1.$

$$r^2 - r - 2 = 0$$

$$r^2 - 2r + r - 2 = 0$$

$$r(r-2) + 1(r-2) = 0$$

$$(r-2)(r+1) = 0$$

The roots are r=2,-1 are distinct real roots.
The general solution is

$$a_n = a_n^h = c_1(2)^n + c_2(-1)^n$$

$$a_0 = 0 \Rightarrow c_1 + c_2 = 0 ----(2)$$

$$a_1 = 1 \Rightarrow 2c_1 - c_2 = 1 ----(3)$$

Solving(2) and (3)

$$c_1 = \frac{1}{3}, c_2 = -\frac{1}{3}$$

$$a_n = \frac{1}{3}(2)^n - \frac{1}{3}(-1)^n$$

Is the explicit solution.

2.Solve the recurrence relation of the Fibonacci sequence of numbers

$$f_n = f_{n-1} + f_{n-2} \ , n \geq 2 \ given \ f_0 = 0, f_1 = 1.$$

Sol:The given recurrence relation $f_n - f_{n-1} - f_{n-2} = 0 ----(1)$

is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be the solution of (1).

$$r^n - r^{n-1} - r^{n-2} = 0$$

$$\div by \ r^{n-2}$$

$$r^2 - r - 1 = 0, which\, is\, characteristic\, equation\, of\, the\, given$$

$$recurrence\, relation$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

The roots are distinct real roots.
The general solution is

$$f_n = f_n^{(h)} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f_0 = 0 \Rightarrow c_1 + c_2 = 0 ----(2)$$

$$f_1 = 1 \Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 ----(3)$$

Solving(2) and (3)

$$c_2 = -c_1 \,[\, from\,(2)]$$

$$Substitute\; in\,(3)$$

$$c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$c_1(2.\frac{\sqrt{5}}{2}) = 1 \Rightarrow c_1.\sqrt{5} = 1$$

$$\therefore\, c_1 = \frac{1}{\sqrt{5}} \Rightarrow c_2 = -c_1 = -\frac{1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

is the explicit solution for nth Fibonacci number.

Characteristic equation with multiple roots

If the characteristic equation has a root 'r'of m^{th} order then the general form of the solutions for homogeneous equation is

$$a_n = a_n^{(h)} = (c_1 + nc_2 + n^2 c_3 + \ldots + n^{m-1} c_m) r^n$$

Where $c_1, c_2, \ldots, c_k$ are constants which may be chosen to satisfy any initial conditions.

Characteristic equation with mixed roots

If the characteristic equation has mixed roots i.e., combination of distinct and multiple roots, then the general solution is also written as a combination of these two functions.

$$1) a_n = c_1 r_1^n + c_2 r_2^n + \ldots + c_k r_k^n$$

$$2) a_n = (c_1 + nc_2 + n^2 c_3 + \ldots + n^{m-1} c_m) r^n$$

$$Eg : r_1, r_2, r_2, r_3$$

$$a_n = c_1 r_1^n + c_2 r_3^n + (c_3 + nc_4) r_2^n$$

Where $c_1, c_2, \ldots, c_k$ are constants which may be chosen to satisfy any initial conditions.

3.Solve the recurrence relation, $a_n = 4(a_{n-1} - a_{n-2})$, $n \geq 2$ given $a_0 = 1, a_1 = 1.$

Sol:The given recurrence relation

$$a_n - 4a_{n-1} + 4a_{n-2} = 0 ----(1)$$

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

$d_1 = -4, \quad d_2 = 4$

$r^2 + d_1 r + d_2 = 0$

is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be the solution of (1).

The characteristic equation is

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0 \quad r = 2, 2$

$a_n^{(h)} = (c_1 + nc_2)2^n$

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$\div by \ r^{n-2}$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

The roots are r=2,2 are real roots of multiplicity 2.
The general solution is

$$a_n = a_n^{(h)} = (c_1 + nc_2)(2)^n$$

$$a_0 = 1 \Rightarrow c_1 = 1 \qquad ----(2)$$

$$a_1 = 1 \Rightarrow 2(c_1 + c_2) = 1 ----(3)$$

Solving(2) and (3)

$$c_1 + c_2 = \frac{1}{2}$$

$$c_2 = \frac{1}{2} - c_1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\therefore c_1 = 1, c_2 = -\frac{1}{2}$$

$$a_n = (1 - \frac{n}{2})(2)^n$$

Is the explicit solution.

4. Solve the recurrence relation $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0.$

Sol: The given recurrence relation $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0 ----(1)$

$n-(n-3)=3$

is a third order linear homogeneous recurrence relation with constant coefficients.

$r^3 + d_1 r^2 + d_2 \cdot r + d_3 = 0$

Let $a_n = r^n$ be the solution of (1).

$r^3 - 8r^2 + 21r - 18 = 0$.

The characteristic equation is

$$r^n - 8r^{n-1} + 21r^{n-2} - 18r^{n-3} = 0$$

$$\div by\ r^{n-3}$$

$$r^3 - 8r^2 + 21r - 18 = 0$$

$$r = 2, 8 - 32 + 42 - 18 = 0$$

2	1 - 8 + 21- 18
	0 + 2 - 12 +18
	1 – 6 + 9 0

$r^2-6r+9=0$

$(r-3)^2=0$

$r=3,3$

The roots are r=2,3,3 are mixed real roots.

The general solution is

$$a_n = a_n^{(h)} = (c_1 + nc_2)(3)^n + c_3(2)^n$$

5. What is the solution of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?

Sol:The given recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 - - - - -(1)$$

is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be the solution of (1)

The characteristic equation is

$$r^n - 6r^{n-1} + 9r^{n-2} = 0$$

$$\div by\ r^{n-2}$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

The roots are r=3,3 are real roots of multiplicity 2.
The general solution is

$$a_n = (c_1 + nc_2)(3)^n$$

$$a_0 = 1 \Rightarrow c_1 = 1 \qquad ----(2)$$

$$a_1 = 6 \Rightarrow 3(c_1 + c_2) = 6 ----(3)$$

$$c_1 + c_2 = 2$$

$$c_2 = 2 - c_1 = 2 - 1 = 1$$

$$\therefore c_1 = 1, c_2 = 1$$

$$a_n = (1 + n)(3)^n$$

Non-homogeneous recurrence relation:

The general solution of a linear non-homogeneous recurrence relation

$$c_0\, a_n + c_1\, a_{n-1} + c_2\, a_{n-2} + \ldots\ldots\ldots\ldots + c_k\, a_{n-k} = f(n)$$

where $c_0, c_1, \ldots\ldots\ldots\ldots\ldots, c_k$ are constants is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

where $a_n^{(h)}$ corresponds to the solution of associated linear homogeneous recurrence relation and

$a_n^{(p)}$ is the particular solution obtained from the non homogeneous part of recurrence relation.

f(n)	Trial solution
b^n (if b is not a root of characteristic equation) Polynomial P(n) of degree m	Ab^n $A_0+A_1n+A_2n^2+. \ . \ . \ .+A_mn^m$
c^n P(n) (if c is not a root of characteristic equation)	$c^n(A_0+A_1n+A_2n^2+. \ . \ . \ .+A_mn^m)$
b^n (if b is a root of characteristic equation of multiplicity s)	$A_0n^sb^n$

$f(n)$	Particular solution
C (polynomial of deg 0)	A
n or $c_0 n + c_1$ (poly of deg 1) Ex [$f(n) = n$ or $3n$ or $n+5$, $3n+7$]	$A_0 + A_1 n$
poly of deg 2 (n^2 or $5n^2$ or n^2+2n+1 or n^2+7n)	$A_0 + A_1 n + A_2 n^2$

$$a_{n+2} - a_{n+1} - 6a_n = 2^n$$

$d_1 =$

1) $a_{n+2} - a_{n+1} - 6a_n = 2^n$

$d_1 = -1,\ d_2 = -6$

$r^2 + d_1 r + d_2 = 0 \Rightarrow r^2 - r - 6 = 0 \Rightarrow r^2 - 3r + 2r - 6 = 0$

$(r-3)(r+2) = 0$

$r = -2, 3$

$f(n) = 2^n \quad b = 2$ which is not a root of characteristic eqn.

Let $a_n^{(p)} = A2^n$

2) $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \quad r = 2, 2 \quad s = 2$

$f(n) = 2^n$ Let $a_n^{(p)} = An^2 2^n$

3) $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$

$r^2 - 5r + 6 = 0 \Rightarrow r^2 - 3r - 2r + 6 = 0 \Rightarrow r = 2, 3 \quad s = 1$

$f(n) = 2^n \quad b = 2$ Let $a_n^{(p)} = An2^n$

4) $a_{n+2} - \overset{-2,3}{a_{n+1}} - 6a_n = 2^n + n^2$

$a_n^{p_1} = A2^n$

$a_n^{p_2} = A_0 + A_1 n + A_2 n^2$

$a_n^{(P)} = a_n^{(P_1)} + a_n^{(P_2)}$

5) $f(n) = 2^n \cdot n^2$

$a_n^p = 2^n (A_0 + A_1 n + A_2 n^2)$

6) $f(n) = n^2 + 4n + 6$

$a_n^{(P)} = A_0 + A_1 n + A_2 n^2$

Note:

1. If f(n) is constant i.e., a polynomial of degree zero, then trial solution is taken as A.
2. If f(n) is a linear combination of these, the trial solution is taken as the sum of corresponding trial functions with different unknown constant coefficients to be determined.

Problems:

$$1.\, Solve\; a_{n+2} - 5a_{n+1} + 6a_n = 2 \; with\; initial\; condition$$

$$a_0 = 1, a_1 = -1.$$

$$Solution: Given\;\; recurrence\; relation\; is$$

$$a_{n+2} - 5a_{n+1} + 6a_n = 2$$

$$The\; associated\; \hom ogeneous\; recurrence\; relation\; is$$

$$a_{n+2} - 5a_{n+1} + 6a_n = 0. \ldots (1)$$

$$let\; a_n = r^n\; be\; a\; solution\; of\;\; (1)$$

$$r^{n+2}-5r^{n+1}+6r^{n}=0$$

$$Divide\,by\,r^{n},$$

$$r^{2}-5r+6=0$$

$$r^{2}-3r-2r+6=0$$

$$r(r-3)-2(r-3)=0$$

$$(r-3)(r-2)=0$$

$$r=2,3.$$

$$a_n^{(h)}=c_1 3^n+c_2 2^n$$

$d_1=-5$, $d_2=6$

$x^2+d_1x+d_2=0$

$x^2-5x+6=0$

Particular solution :

Let $a_n^{(p)} = A$ *(since* $f(n)$ *is constant).* then $a_{n+1}^{(p)} = A$,

substitute this in the given equation, we have $a_{n+2}^{(p)} = A$

$A - 5A + 6A = 2$

$2A = 2$

$\therefore A = 1$

$\therefore a_n^{(p)} = 1$, *which is the particular solution.*

The general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1 3^n + c_2 2^n + 1$ —— ①

Given that $a_0 = 1$ i.e., at $n=0$, $a_0 = 1$, then ① becomes

$c_1 + c_2 + 1 = 1$

$c_1 + c_2 = 0$

$c_1 = -c_2$

$$and\ a_1 = -1$$ i.e., at n = 1, $a_1 = -1$, then ① becomes

$$3c_1 + 2c_2 + 1 = -1$$

$$-3c_2 + 2c_2 = -2$$

$$-c_2 = -2$$

$$c_2 = 2, c_1 = -c_2 = -2$$

$$\therefore c_1 = -2, c_2 = 2$$

$$\therefore a_n = -2.3^n + 2.2^n + 1, which\ is\ the\ solution$$

$$of\ the\ recurrence\ relation.$$

Solve the recurrence relation

$a_n = 3a_{n-1} + 2^n$, $a_0 = 1$.

Solution:- $a_n - 3a_{n-1} = 2^n$ — (1)

order $= 1$, $d_1 = -3$

characteristic equation is

$r + d_1 = 0 \Rightarrow r - 3 = 0$

$\therefore r = 3$

$a_n^{(h)} = c_1 3^n$

$f(n) = 2^n$ 2 is not a root of characteristic eqn

Let $a_n^{(p)} = A2^n \Rightarrow a_{n-1}^{(p)} = A2^{n-1} = \frac{A2^n}{2}$.

sub these values in — (1)

$A2^n - 3 \cdot \frac{A2^n}{2} = 2^n \Rightarrow 2^n\left(A - \frac{3A}{2}\right) = 2^n \cdot 1$

$A - \frac{3 \cdot A}{2} = 1 \Rightarrow -\frac{A}{2} = 1 \Rightarrow A = -2$

$\therefore a_n^{(P)} = A2^n = -2.2^n = -2^{n+1}$.

solution of recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$\therefore a_n = c_1 3^n - 2^{n+1}$.

Given

$a_0 = 1$ i.e., at $n=0$, $a_0 = 1$

$c_1 - 2 = 1 \Rightarrow c_1 = 3$.

$\therefore a_n = 3.3^n - 2^{n+1}$

$$a_n = 3^{n+1} - 2^{n+1}$$

Solve the recurrence relation

$a_n - 2a_{n-1} + a_{n-2} = 2,$

Solution: Given recurrence relation is

$a_n - 2a_{n-1} + a_{n-2} = 2$ —— (1)

$d_1 = -2, d_2 = 1$, order $= n-(n-2) = 2$

characteristic equation is $x^2 + d_1 x + d_2 = 0$

$x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0$

$x = 1, 1$

$a_n^{(h)} = (c_1 + nc_2)(1)^n = c_1 + nc_2.$

$f(n) = 2$ which is a constant

Let $a_n^{(p)} = A$ be the trial solution

$a_{n-1}^{(p)} = A$, $a_{n-2}^{(p)} = A$

Substitute these in ①,

$A - 2A + A = 2$

$0 = 2$, which is not possible.

So let $a_n^{(p)} = An$, $a_{n-1}^{(p)} = A(n-1) = nA - A$

$a_{n-2}^{(p)} = (n-2)A = nA - 2A$

① becomes

$An - 2.(nA - A) + (nA - 2A) = 2$

$0 = 2$, not possible (case of failure)

let $a_n^{(p)} = An^2$

$a_{n-1}^{(p)} = A(n-1)^2 = An^2 - 2An + A$

$a_{n-2}^{(p)} = A(n-2)^2 = An^2 - 4An + 4A$

Sub in (1)

$$An^2 - 2(An^2 - 2An + A) + An^2 - 4An + 4A = 2$$

$$4\cancel{An} - 2A - \cancel{4An} + 4A = 2$$

$$2A = 2 \Rightarrow A = 2/2 = 1.$$

$$\therefore a_n^{(P)} = An^2 = n^2.$$

Solution of given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = c_1 + nc_2 + n^2$$

2. Solve the recurrence relation

$$y_{n+2} - y_{n+1} - 2y_n = n^2$$

solution :

Given recurrence relation is $y_{n+2} - y_{n+1} - 2y_n = n^2$

The associated homogeneous recurrence relation is

$$y_{n+2} - y_{n+1} - 2y_n = 0 \text{-----}(1)$$

$d_1 = -1, d_2 = -2$

Let $y_n = r^n$ be the solution of (1)

characteristic eqn is

The characteristic equation is

$x^2 + d_1 x + d_2 = 0$

$x^2 - x - 2 = 0$

$$r^2 - r - 2 = 0$$

$$r^2 - 2r + r - 2 = 0$$

$$r(r-2) + 1(r-2) = 0$$

$$(r-2)(r+1) = 0$$

$$r = -1, 2.$$

$$\therefore y_n^{(h)} = c_1(-1)^n + c_2(2)^n$$

$f(n) = n^2$, poly of deg 2

Let $y_n^{(P)} = A_0 + A_1 n + A_2 n^2$

$y_{n+1}^{(P)} = A_0 + A_1(n+1) + A_2(n+1)^2$

$= A_0 + nA_1 + A_1 + n^2 A_2 + 2nA_2 + A_2$

$y_{n+1}^{(P)} = (A_0 + A_1 + A_2) + (A_1 + 2A_2)n + A_2 n^2$

$y_{(n+2)}^{P} = A_0 + A_1(n+2) + A_2(n+2)^2$

$= A_0 + nA_1 + 2A_1 + n^2 A_2 + 4nA_2 + 4A_2$

$= (A_0 + 2A_1 + 4A_2) + (A_1 + 4A_2)n + A_2 n^2$

Sub these values in given eqn

$y_{n+2} - y_{n+1} - 2y_n = n^2$

Particular solution :

Let the particular solution of the given equation

$be\ y_n^{(p)} = A_0 + A_1 n + A_2 n^2$ (since $f(n)$ is a polynomial of degree 2)

substituting this in the given equation, we have

$$A_0 + A_1(n+2) + A_2(n+2)^2 - [A_0 + A_1(n+1) + A_2(n+1)^2]$$
$$-2[A_0 + A_1 n + A_2 n^2] = n^2$$
$$A_0 + A_1(n+2) + A_2(n^2+4n+4) - A_0 - A_1(n+1) - A_2(n^2+2n+1)$$
$$-2A_0 - 2A_1 n - 2A_2 n^2 = n^2$$
$$(-2A_0 + A_1 + 3A_2) + (-2A_1 + 2A_2)n - 2A_2 n^2 = 1.n^2 + 0.n + 0$$

comparing the coefficients of like powers, we have

$$-2A_0 + A_1 + 3A_2 = 0$$

$$-2A_1 + 2A_2 = 0$$

$$-2A_2 = 1$$

$$\therefore A_2 = -\frac{1}{2}$$

$$-2A_1 + 2A_2 = 0 \Rightarrow -2A_1 = -2A_2$$

$$A_1 = A_2 = -\frac{1}{2}$$

$$-2A_0 + A_1 + 3A_2 = 0 \Rightarrow 2A_0 = A_1 + 3A_2$$

$$= -\frac{1}{2} + 3.-\frac{1}{2} = -2$$

$$2A_0 = -2 \Rightarrow A_0 = -1$$

$$\therefore A_0 = -1, A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}$$

$$y_n^{(p)} = -1 - \frac{n}{2} - \frac{n^2}{2}$$

$$y_n = y_n^{(h)} + y_n^{(p)}$$

$$y_n = c_1(-1)^n + c_2(2)^n - 1 - \frac{n}{2} - \frac{n^2}{2}$$

$3.\ Solve\ a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

$Solution:$

$Given\ recurrence\ relation\ a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

$\hom ogeneous\ recurrence\ relation\ is$

$a_{n+2} - 4a_{n+1} + 4a_n = 0 ------(1)$

$d_1 = -4,\ d_2 = 4$

$Let\ a_n = r^n\ be\ the\ solution\ of\ (1)$

order = 2

$r^{n+2} - 4r^{n+1} + 4r^n = 0$

characteristic eqn is

$x^2 + d_1 x + d_2 = 0$

$divide\ by\ r^n$

$characteristic\ equation\ is\ r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2,2$

$a_n^{(h)} = (c_1 + nc_2)2^n$

$Particular\ solution:$

$Here\ b = 2\ is\ a\ root\ of\ characteristic\ eqution\ with$

$multiplicity\ s = 2$

$So\ particular\ solution\ a_n^{(p)} = A_0 n^2 2^n$, $a_{n+1}^{(p)} = A_0(n+1)^2 2^{n+1}$

$= 2.2^n(n^2A_0 + 2nA_0 + A_0)$

$= 2^n(2n^2A_0 + 4nA_0 + 2A_0)$

$Substitute\ this\ in\ the\ given\ relation$

$$A_0(n+2)^2 2^{n+2} - 4A_0(n+1)^2 2^{n+1} + 4A_0 n^2 2^n = 2^n$$

$$2^n[4A_0(n^2+4n+4) - 8A_0(n^2+2n+1) + 4A_0 n^2] = 2^n . 1$$

$$16A_0 - 8A_0 = 1 \Rightarrow 8A_0 = 1$$

$a_{n+2}^{p} = A_0(n+2)^2 2^{n+2}$

$= 4\, 2^n(n^2A_0 + 4nA_0 + 4A_0)$

$= 2^n(4n^2A_0 + 16nA_0 + 16A_0)$

$$A_0 = \frac{1}{8}$$

$$a_n^{(p)} = \frac{1}{8} n^2 2^n.$$

$General\ solution\ is\ a_n = a_n^{(h)} + a_n^{(p)}$

$$\therefore a_n = (c_1 + nc_2)2^n + \frac{1}{8} n^2 2^n$$

Practice problems:

1. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
2. Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

3. Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

4. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

(1) $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$

$x^2 - 6x + 9 = 0 \quad x = 3, 3 \qquad (c_1 + nc_2)3^n.$

$a_n^{(p)} = A3^n n^2 \qquad a_{n-1}^{(p)} = A3^{n-1}(n-1)^2 \qquad a_{n-2}^{(p)} = A3^{n-2}(n-2)^2$

2) $a_n - 6a_{n-1} + 9a_{n-2} = n3^n$

$a_n^{(p)} = 3^n n^2 (A_0 + A_1 n). = 3^n (A_1 n^3 + A_0 n^2)$

$a_{n-1}^{(p)} = 3^{n-1}(n-1)^2 (A_0 + A_1(n-1)) = \frac{3^n}{3}(n^2 - 2n + 1)(A_0 + A_1 n - A_1)$

$= \frac{3^n}{3}(A_0 n^2 + A_1 n^3 - A_1 n^2 - 2A_0 n - 2A_1 n^2 + 2A_1 n + A_0 + A_1 n - A_1)$

$= \frac{3^n}{3}\left[A_1 n^3 + (A_0 - 3A_1)n^2 + (-2A_0 + 3A_1)n + (A_0 - A_1)\right]$

$$a^{(p)}_{n-2} = 3^{n-2} \cdot (n-2)^2 \left(A_0 + A_1(n-2)\right)$$

$$= \frac{3^n}{9}(n^2-4n+4)(A_0+A_1 n-2A_1)$$

$$= \frac{3^n}{9}\left(A_0 n^2 + A_1 n^3 - 2A_1 n^2 - 4A_0 n - 4A_1 n^2 + 8A_1 n + 4A_0 + 4A_1 n - 8A_1\right)$$

$$a^{(p)}_{n-2} = \frac{3^n}{9}\left[A_1 n^3 + (A_0 - 6A_1)n^2 + (-4A_0+12A_1)n + (4A_0-8A_1)\right]$$

sub these in given eqn

$$a_n - 6a_{n-1} + 9a_{n-2} = n3^n$$

$$3^n(A_1 n^3 + A_0 n^2) - \cancel{6}^{\,2} \cdot \frac{3^n}{\cancel{3}}\left[A_1 n^3 + (A_0-3A_1)n^2 + (-2A_0+3A_1)n + (A_0-A_1)\right] + \cancel{9} \cdot \frac{3^n}{\cancel{9}}\left[A_1 n^3 + (A_0-6A_1)n^2 + (-4A_0+12A_1)n + (4A_0-8A_1)\right] = n \cdot 3^n$$

$$3^n\left[\cancel{A_1 n^3} + \cancel{A_0 n^2} - \cancel{2A_1 n^3} - \cancel{2A_0 n^2} + \cancel{6A_1 n^2} + \cancel{4A_0 n} - 6A_1 n\right.$$

$$-\cancel{2A_0} + 2A_1 + \cancel{A_1 n^3} + \cancel{A_0 n^2} - \cancel{6A_1 n^2} - \cancel{4A_0 n} + 12A_1 n$$

$$\left.+4A_0 - 8A_1\right] = n \cdot 3^n$$

$$3^n\left[6A_1 n + (2A_0 - 6A_1)\right] = n \cdot 3^n$$

compare 3^n coeff on both sides

$$6A_1 n + (2A_0 - 6A_1) = n \cdot 1 + 0$$

n coeff:

$$6A_1 = 1 \Rightarrow A_1 = \frac{1}{6}$$

constants

$$2A_0 - 6A_1 = 0 \Rightarrow 2A_0 = 6A_1$$

$$A_0 = \frac{\cancel{6}}{2} \times \frac{1}{\cancel{6}} = \frac{1}{2}$$

$A_0 = \frac{1}{2}$, $A_1 = \frac{1}{6}$.

$$a_n^{(P)} = 3^n n^2\left(\frac{1}{2} + \frac{n}{6}\right) = 3^n n^2\left(\frac{3+n}{6}\right) = \frac{1}{6} 3^n n^2 (n+3)$$

3) $a_n - 6a_{n-1} + 9a_{n-2} = n^2 3^n$

$a_n^{(P)} = 3^n n^2 (A_0 + A_1 n + A_2 n^2)$

4) $a_n - 6a_{n-1} + 9a_{n-2} = (n^2+1) 3^n$

$a_n^{(P)} = 3^n n^2 (A_0 + A_1 n + A_2 n^2)$

5) $a_n - 6a_{n-1} + 9a_{n-2} = n^2 + 3^n$.

$a_n^{(P_1)} = A_0 + A_1 n + A_2 n^2$ be trial solution

$a_{n-1}^{(P_1)} = A_0 + A_1(n-1) + A_2(n-1)^2$

$a_{n-2}^{(P_1)} = A_0 + A_1(n-2) + A_2(n-2)^2$

sub these in $a_n - 6a_{n-1} + 9a_{n-2} = n^2$

$f(n) = 3^n$

$a_n^{(P_2)} = A\, 3^n n^2$ be the trial solution

$a_{n-1}^{(P_2)} = A\, 3^{n-1} (n-1)^2$

$a_{n-2}^{(P_2)} = A\, 3^{n-2} (n-2)^2$

sub these values in

$a_n - 6a_{n-1} + 9a_{n-2} = 3^n$

compare 3^n coeff, we get A -

$\therefore a_n^{P} = a_n^{(P_1)} + a_n^{(P_2)}$

Solution of Recurrence Relations by Using Generating Functions

The generating function for a sequence $a_0,a_1,a_2,\ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

$$Note: G(x) = \sum_{n=0}^{\infty} a_n x^n (or) \sum_{n=1}^{\infty} a_{n-1} x^{n-1} (or) \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

Example :

1)The generating function for the sequence 1,1,1,… is given by

$$G(x) = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Example :

2) The generating function for the sequence 1,2,3,… is given by

$$G(x) = 1 + 2x + 3x^2 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

3) The generating function for the sequence 1,a,a^2,… is given by

$$G(x) = 1 + ax + a^2x^2 + \ldots = \sum_{n=0}^{\infty} (ax)^n = \frac{1}{1-ax} \quad for \ |ax| < 1$$

Working Rule

To solve a recurrence relation (both homogeneous and non-homogeneous)with initial conditions,

- Multiply the relation by an appropriate power of x
- Sum up suitably so as to get an explicit formula for the associated generating function
- The solution is then obtained as the coefficient of x^n in the expansion of the generating function

Solving Recurrence Relation :

1. Use the method of generating function to solve the recurrence relation, $a_n = 3a_{n-1} + 1 \ , n \geq 1 \ given \ a_0 = 1$

Sol: Let the generating function of $\{a_n\}$ be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The recurrence relation is $a_n = 3a_{n-1} + 1 \ ------(1)$

On multiplying (1)both sides by x^n and apply summation on both sides,

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n$$

$$\left(\sum_{n=1}^{\infty} a_n x^n + a_0 \right) - a_0 = 3\sum_{n=1}^{\infty} a_{n-1} x^{n-1+1} + \sum_{n=1}^{\infty} x^{n-1+1}$$

$$i.e., G(x) - a_0 = 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} x^{n-1}$$

$$G(x) - a_0 = 3xG(x) + \frac{x}{1-x}$$

$$G(x) - 1 = 3xG(x) + \frac{x}{1-x} \quad (\because a_0 = 1)$$

$$(1-3x)G(x) = 1 + \frac{x}{1-x} = \frac{1-x+x}{1-x} = \frac{1}{1-x}$$

$$G(x) = \frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

Applying partial fractions method,

$a_0, a_1, a_2, a_3, \ldots \quad \text{— (1)}$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} a_n x^n \quad (\text{or}) \quad \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \quad (\text{or}) \quad \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\underline{a_1 x} + a_2 x^2 + a_3 x^3 + \ldots = \underline{G(x) - a_0}$$

$$a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots = G(x) - a_0 - a_1 x$$

$$a_3 x^3 + a_4 x^4 + \ldots = G(x) - a_0 - a_1 x - a_2 x^2$$

Taking LCM and equating the numerators

$$1 = A(1-3x) + B(1-x)$$

$$put\ \ x = 1,\ 1 = -2A \Rightarrow A = \frac{-1}{2}$$

$$put\ \ x = 1/3,\ \ 1 = \frac{2}{3}B \Rightarrow B = \frac{3}{2}$$

$$G(x) = \frac{-1/2}{1-x} + \frac{3/2}{1-3x}$$

$$G(x) = \frac{-1}{2}(1-x)^{-1} + \frac{3}{2}(1-3x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{-1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n$$

a_n=Coefficient of x^n in G(x)

$$a_n = \frac{-1}{2} + \frac{3}{2}3^n = \frac{1}{2}(3^{n+1} - 1)$$

2.Use the method of generating function to solve the recurrence relation,

$$a_n = 4a_{n-1} - 4a_{n-2} + 4^n \quad , n \geq 2 \ given \ a_0 = 2 \ \ and \ \ a_1 = 8$$

Sol:Let the generating function of $\{a_n\}$ be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The recurrence relation is $a_n = 4a_{n-1} - 4a_{n-2} + 4^n \text{ ------}(1)$

On multiplying (1)both sides by x^n and apply summation on both sides

$$\sum_{n=2}^{\infty} a_n x^n = 4\sum_{n=2}^{\infty} a_{n-1} x^n - 4\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} 4^n x^n$$

$$(\sum_{n=2}^{\infty} a_n x^n + a_0 + a_1 x) - a_0 - a_1 x = 4x(\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + a_0 - a_0)$$

$$-4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + (\sum_{n=2}^{\infty} 4^n x^n + 1 + 4x) - 1 - 4x$$

$$\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x = 4x\left(\sum_{n=1}^{\infty} a_{n-1} x^{n-1} - a_0\right) - 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + \sum_{n=0}^{\infty} 4^n x^n - 1 - 4x$$

$$G(x) - a_0 - a_1 x = 4x\left(G(x) - a_0\right) - 4x^2 G(x) + \frac{1}{1-4x} - 1 - 4x$$

$$sub\ a_0 = 2\ \&\ a_1 = 8$$

$$G(x) - 2 - 8x = 4x\left(G(x) - 2\right) - 4x^2 G(x) + \frac{1}{1-4x} - 1 - 4x$$

$$G(x)(1 - 4x + 4x^2) = \frac{1}{1-4x} - 1 - 4x + 2 = \frac{1}{1-4x} + 1 - 4x$$

$$G(x) = \frac{1 + (1-4x)^2}{(1 - 4x + 4x^2)(1-4x)}$$

Applying partial fractions method,

$$G(x)=\frac{1+(1-4x)^2}{(1-2x)^2(1-4x)}$$

$$G(x)=\frac{1+(1-4x)^2}{(1-2x)^2(1-4x)}=\frac{A}{1-4x}+\frac{B}{1-2x}+\frac{C}{(1-2x)^2}$$

$$1+(1-4x)^2=A(1-2x)^2+B(1-4x)(1-2x)+C(1-4x)$$

$$put\ \ x=1/4$$

$$1=A.\frac{1}{4}\Rightarrow A=4$$

$$put\ \ x=1/2$$

$$2=-C\Rightarrow C=-2$$

$$equating\ \ co-efficients\ \ of\ \ x^2,$$

$$16=4A+8B$$

$$16=16+8B\Rightarrow B=0$$

$$G(x)=\frac{4}{1-4x}-\frac{2}{(1-2x)^2}$$

$$G(x)=4(1-4x)^{-1}-2(1-2x)^{-2}$$

$$\sum_{n=0}^{\infty}a_nx^n=4(1+4x+(4x)^2+\ldots+(4x)^n+\ldots)$$

$$-2(1+2.2x+3.(2x)^2+\ldots+(n+1)(2x)^n+\ldots)$$

$$\sum_{n=0}^{\infty}a_nx^n=4.\sum_{n=0}^{\infty}4^nx^n-2\sum_{n=0}^{\infty}(n+1)2^nx^n$$

Equating the coefficients of x^n both sides,

$$a_n=4.4^n-2.(n+1)2^n$$

$$a_n=4^{n+1}-(n+1)2^{n+1}$$

which is the solution of given recurrence relation.

3.Use the method of generating function to solve the recurrence

relation, $a_{n+1} - 8a_n + 16a_{n-1} = 4^n \ , n \geq 1 \ given \ a_0 = 1 \ and \ a_1 = 8$

Sol:Let the generating function of {an} be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The recurrence relation is $a_{n+1} - 8a_n + 16a_{n-1} = 4^n \ ------(1)$

On multiplying (1)both sides by x^n and apply summation on both sides

$$\sum_{n=1}^{\infty} a_{n+1} x^n - 8\sum_{n=1}^{\infty} a_n x^n + 16\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 4^n x^n$$

$$\sum_{n=1}^{\infty} a_{n+1}x^n - 8\sum_{n=1}^{\infty} a_n x^n + 16\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} 4^n x^n$$

$$\frac{1}{x}\sum_{n=1}^{\infty} a_{n+1}x^{n+1} - 8\left(\sum_{n=1}^{\infty} a_n x^n + a_0 - a_0\right) + 16x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \left(\sum_{n=1}^{\infty} 4^n x^n + 1 - 1\right)$$

$$\frac{1}{x}\left\{\sum_{n=1}^{\infty} a_{n+1}x^{n+1} + a_0 + a_1 x - a_0 - a_1 x\right\} - 8\left(G(x) - a_0\right) + 16xG(x) = \frac{1}{1-4x} - 1$$

$$put\ \ a_0 = 1, a_1 = 8$$

$$\frac{1}{x}\left\{G(x) - 1 - 8x\right\} - 8\left(G(x) - 1\right) + 16xG(x) = \frac{4x}{1-4x}$$

$$G(x)\left\{\frac{1}{x}-8+16x\right\}=\frac{4x}{1-4x}+\frac{1}{x}=\frac{4x^2-4x+1}{x(1-4x)}$$

$$G(x)\left\{\frac{1-8x+16x^2}{x}\right\}=\frac{4x^2-4x+1}{x(1-4x)}$$

$$G(x)=\frac{4x^2-4x+1}{(1-8x+16x^2)(1-4x)}=\frac{4x^2+1-4x}{(1-4x)^3}$$

$$G(x)=(1-4x+4x^2)(1-4x)^{-3}$$

$$G(x)=(1-4x+4x^2)\{1+3.(4x)+6(4x)^2+........\}$$

$$G(x)=(1-4x+4x^2)\frac{1}{2}(1.2+2.3(4x)+3.4(4x)^2+...+(n-1)n(4x)^{n-2}$$

$$+n(n+1)(4x)^{n-1}+(n+1)(n+2)(4x)^n+...$$

Equating the coefficients of x^n,

$$a_n = \frac{1}{2}\left((n+1)(n+2)(4)^n - 4n(n+1)(4)^{n-1} + 4(n-1)n(4)^{n-2}\right)$$

$$a_n = \frac{1}{2}\left((n+1)(n+2)(4)^{n-1}.4 - 4n(n+1)(4)^{n-1} + 4(n-1)n(4)^{n-1}.\frac{1}{4}\right)$$

$$a_n = \frac{1}{2}\left(4(n+1)(n+2) - 4n(n+1) + (n-1)n\right)(4)^{n-1}$$

$$a_n = \frac{1}{2}\{4(n^2+3n+2) - 4n^2 - 4n + n^2 - n\}(4)^{n-1}$$

$$a_n = \frac{1}{2}\left(n^2+7n+8\right)(4)^{n-1}$$

Solving Recurrence Relation :

4.Use the method of generating function to solve the recurrence

relation, $a_n = 4a_{n-1} + 3n.2^n, n \geq 1 \;\; given \;\; a_0 = 4$

Sol: Let the generating function of $\{a_n\}$ be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The recurrence relation is $\quad a_n = 4a_{n-1} + 3n.2^n ------ -(1)$

On multiplying (1)both sides by x^n and summing up

$$\sum_{n=1}^{\infty} a_n x^n = 4\sum_{n=1}^{\infty} a_{n-1} x^n + 3\sum_{n=1}^{\infty} n.2^n x^n$$

$$\sum_{n=1}^{\infty} a_n x^n + a_0 - a_0 - 4\sum_{n=1}^{\infty} a_{n-1}x^n = 3\sum_{n=1}^{\infty} n.2^n x^n$$

$$(\sum_{n=0}^{\infty} a_n x^n - a_0) - 4x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = 3\sum_{n=1}^{\infty} n.2^n x^n$$,Taking 2x out

$$G(x) - a_0 - 4xG(x) = 6x\sum_{n=1}^{\infty} n.(2x)^{n-1}$$

$$put a_0 = 4$$

$$G(x) - 4 - 4xG(x) = 6x\sum_{n=1}^{\infty} n.(2x)^{n-1}$$

$$(1-4x)G(x) = \frac{6x}{(1-2x)^2} + 4$$

$$G(x) = \frac{6x}{(1-4x)(1-2x)^2} + \frac{4}{1-4x}$$

Applying partial fractions method,

$$\frac{6x}{(1-4x)(1-2x)^2} = \frac{A}{1-4x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

$$6x = A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x)$$

$$put\ x = \frac{1}{2}, \qquad 3 = -C \Rightarrow C = -3$$

$$put\ x = \frac{1}{4}, \qquad \frac{3}{2} = A\frac{1}{4} \Rightarrow A = 6$$

$$equating \quad x^2, 0 = 4A + 8B \Rightarrow 0 = 24 + 8B \Rightarrow B = -3$$

$$G(x) = \frac{6}{1-4x} - \frac{3}{1-2x} - \frac{3}{(1-2x)^2} + \frac{4}{1-4x}$$

$$G(x) = \frac{10}{1-4x} - \frac{3}{1-2x} - \frac{3}{(1-2x)^2}$$

$$G(x)=\frac{10}{1-4x}-\frac{3}{1-2x}-\frac{3}{(1-2x)^2}$$

$$G(x)=10(1+4x+\ldots+(4x)^n+\ldots)-3(1+2x+\ldots+(2x)^n+\ldots)$$

$$-3(1+2.(2x)+\ldots+(n+1)(2x)^n+\ldots)$$

$$G(x)=10(1+4x+\ldots+4^n x^n+\ldots)-3(1+2x+\ldots+2^n x^n+\ldots)$$

$$-3(1+2.(2x)+\ldots+(n+1)2^n x^n+\ldots)$$

$$\sum_{n=0}^{\infty}a_n x^n=10.\sum_{n=0}^{\infty}4^n x^n-3.\sum_{n=0}^{\infty}2^n x^n-3.\sum_{n=0}^{\infty}(n+1)2^n x^n$$

Equating coefficients of x^n,

$$a_n=10.4^n-3.2^n-3(n+1)2^n$$

$$a_n=10.4^n-(3n+3+3)2^n$$

$$a_n=10.4^n-(3n+6)2^n$$

solve the recurrence relation

$$a_{n+2} - 2a_{n+1} + a_n = 2^n, \quad a_0 = 2, \quad a_1 = 1.$$

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 2\sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2-2} - 2\sum_{n=0}^{\infty} a_{n+1} x^{n+1-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$x^{n+2} \cdot x^{-2}$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2} x^{n+2} - \frac{2}{x}\sum_{n=0}^{\infty} a_{n+1} x^{n+1} + \text{— — — —}$$