UNIT-4

GRAPH THEORY

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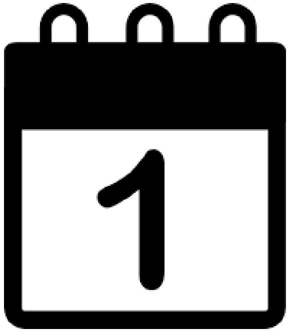
Contents

Basic Concepts of Graphs, Matrix Representation of Graphs: Adjacency Matrix, Incidence Matrix, Isomorphic Graphs, Paths and Circuits, Euler and Hamilton Graphs, Planar Graphs and Euler's Formula.

Module



Basic Concepts of Graphs

Objective:

❖ To learn the basic definitions ,terminologies of a graph

❖To learn special type of graphs

Outcome:

At the end of the session Student will be able to learn

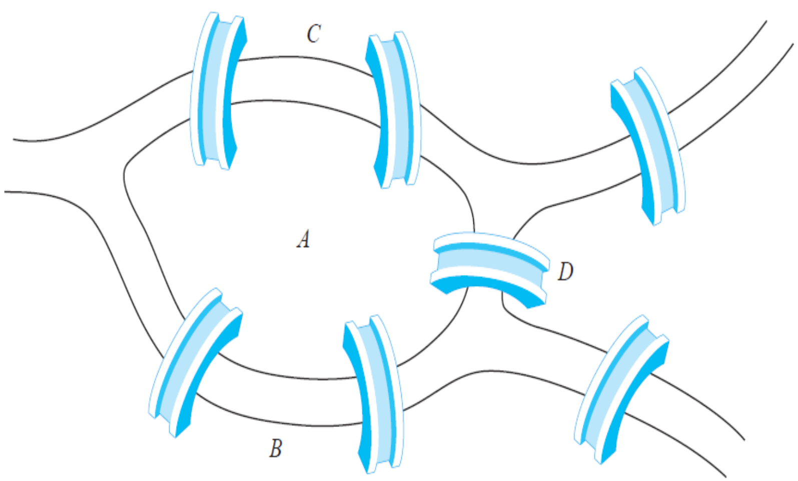
❖ to understand the basic definitions and terminologies of a graph

❖To identify the special graphs

Origin of Graph Theory

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



The Seven Bridges of Königsberg.

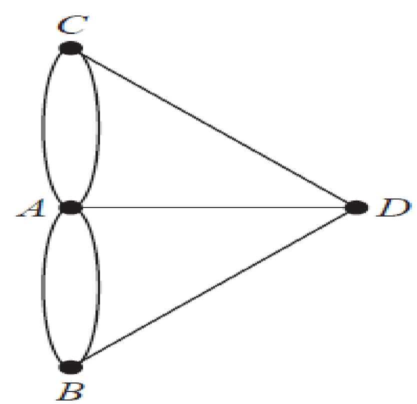


FIGURE 2 Multigraph Model of the Town of Königsberg.

Graphs

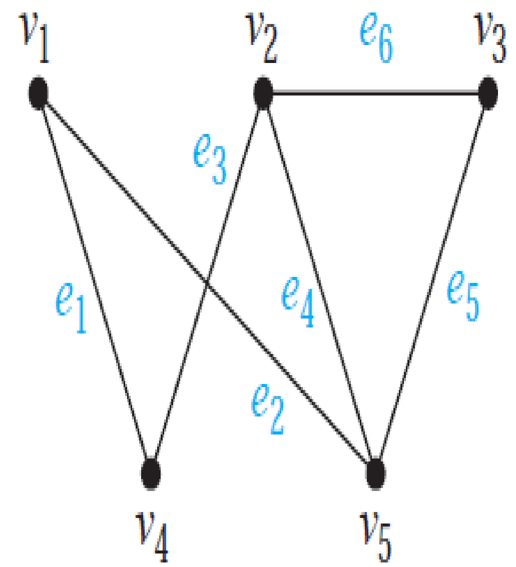
- Discrete structures consisting of vertices and edges that connect these vertices
- Depending on the type and number of edges that can connect a pair of vertices, there are many kinds of different graphs.
- Can be used to model a variety of areas
- Modelling road maps, assignment of jobs to employees of an organization, links between websites, modelling computer networks, social networks, outcomes of a round-robin tournament etc.,

Basic Definitions

Graph:

A graph G consists of two sets:
(1)A set V=V(G) whose elements are called vertices or nodes or points of G
(2)A set E=E(G) of unordered pairs of distinct vertices called edges of G.
A graph is denoted by G(V,E).

- A vertex is denoted by a dot or a small circle
- An edge is denoted by curve or a line



Adjacent:

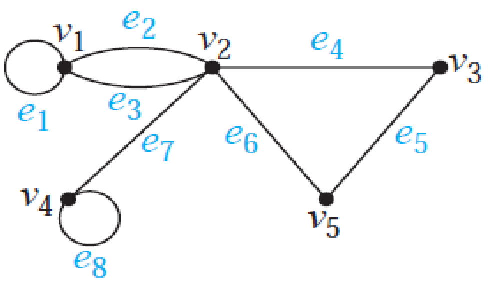
Two vertices are said to be adjacent if the two vertices are connected by an edge. The vertices are called endpoints of the edge.

Incident:

The edge is said to incident on each of its endpoints.

Multiple or Parallel Edges:

If two vertices in a graph G are connected by more than one edge, then such edges are called multiple or parallel edges.



Ex: e2,e3 are multiple edges.

Loop:

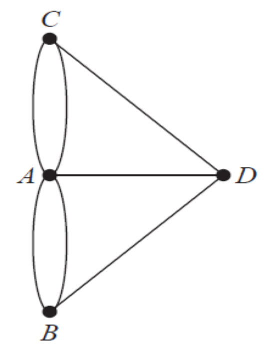
Any edge starting and ending at the same vertex is called a loop.

Ex:e1,e8 are loops.

Types of Graphs

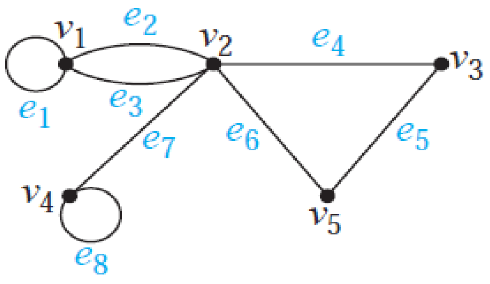
Multigraph:

A graph containing multiple edges is called Multigraph.



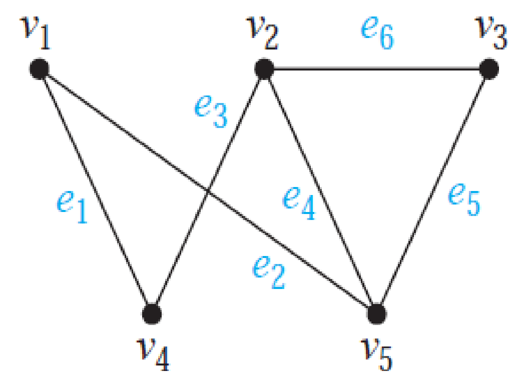
Pseudograph:

A graph containing multiple edges and/or loops
is called Pseudo graph.



Simple Graph:

A graph without multiple edges and loops is called a simple graph.

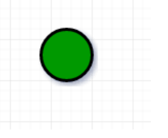


Finite Graph:

A graph is said to be finite if it has finite number of vertices and finite number of edges.

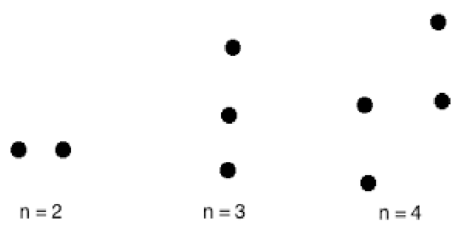
Trivial graph:

A graph with single vertex and no edges is called trivial graph.



Null graph:

A graph with finite number of vertices and no edges is called a Null graph.



Types of Vertices

Degree of a vertex:

In a graph, the degree of a vertex is defined as the number of edges that are incident on vertex v, denoted by deg(v).

Based on the degree of the vertex ,the vertices can be classified into

1.Odd vertex, if the degree of the vertex is an odd number

2.Even vertex, if the degree of the vertex is an even number

3.Isolated vertex, if the degree of the vertex is 0.

4.Pendant vertex, if the degree of the vertex is 1.

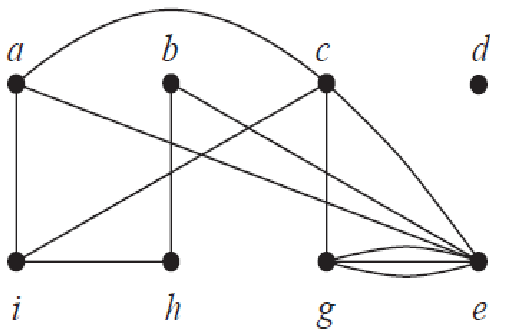
Note: The degree of the vertex having a loop is 2.

Ex:

i. The graph is a multigraph, since the vertices g,e have multiple edges.

ii. Degrees of the vertices:

Deg(a)=3
Deg(b)=2
Deg(c)=4
Deg(d)=0
Deg(e)=6
Deg(g)=4
Deg(h)=2
Deg(i)=3



iii. The vertices a, i are odd vertices
The vertices b, c, e, g,h are even vertices
d is an isolated vertex.

Some Useful Theorems

Theorem1(Handshaking Theorem):

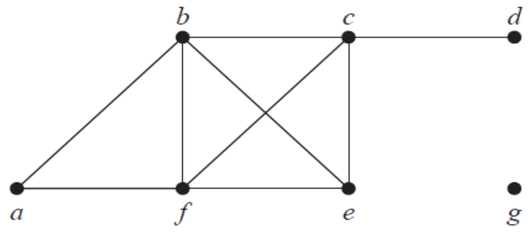
If G=(V,E) is an undirected graph with m edges then $\sum_i \deg(v_i) = 2m$

i.e., the sum of the degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence a even number.

Theorem2:

The number of vertices of odd degree in an undirected graph is even.

Verify Handshaking Theorem for the following graph



Sol:No.of vertices=7

No.of edges=m=9

Deg(a)=2,deg(b)=4,deg(c)=4,deg(d)=1,deg(e)=3,deg(f)=4,deg(g)=0

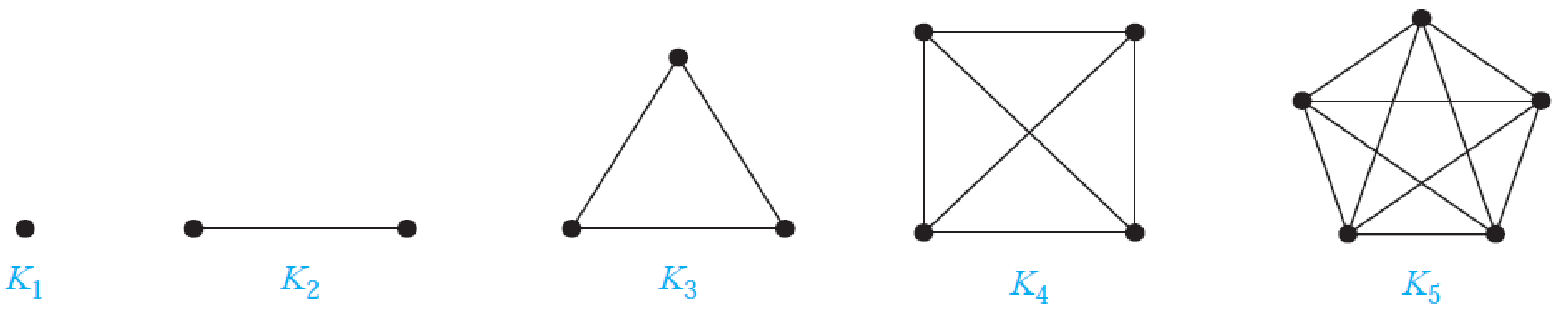
By Handshaking theorem,

$\sum_i \deg(v_i) = 2 + 4 + 4 + 1 + 3 + 4 + 0 = 18 = 2m$.Theorem is verified.

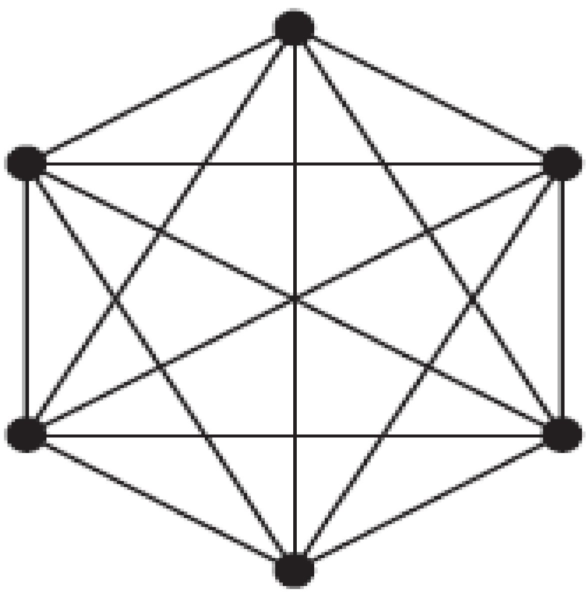
Some Special Simple Graphs

Complete Graph

A simple graph in which there is exactly one edge between each pair of distinct vertices is called a Complete Graph ,denoted by K_n.



Draw the Complete Graph K_6.



K_6

Note:

1.The number of edges in K_n is $\frac{n(n-1)}{2}$

2.The maximum number of edges in a simple graph with

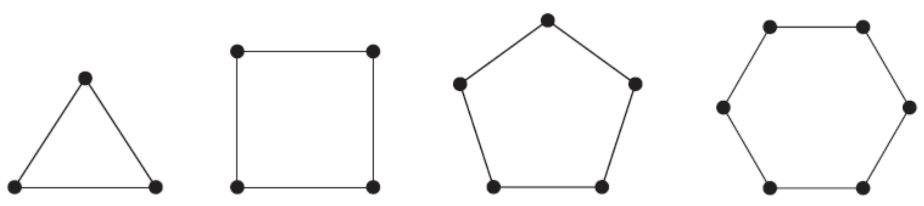
n vertices is $\frac{n(n-1)}{2}$

Regular Graph

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

If every vertex in a regular graph has degree n, then the graph is called n-regular.

Ex:2-regular graphs



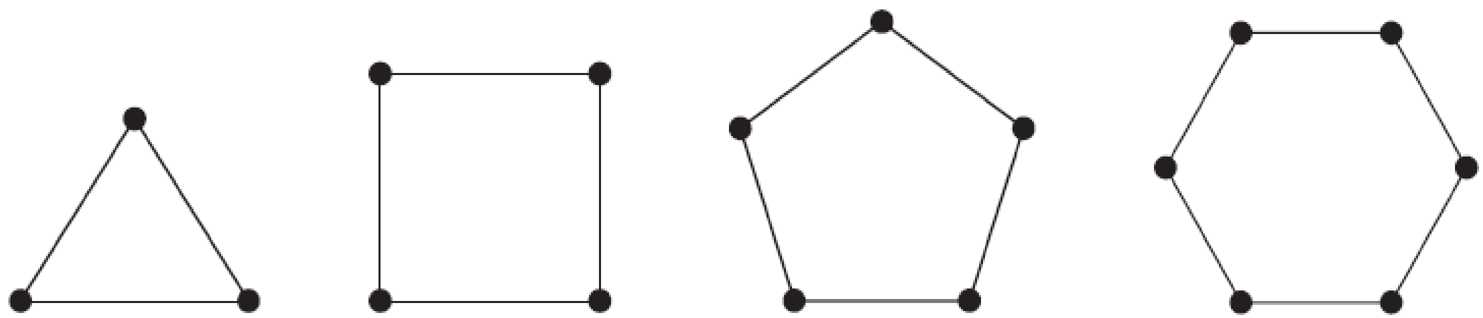
Some Special Simple Graphs

Cycles

A cycle Cn,n≥3 consists of n vertices v1,v2,…,vn and edges

{v1,v2},{v2,v3},…,{vn-1,vn},{vn,v1}.

A Cycle is a 2-regular graph



C_3 C_4 C_5 C_6

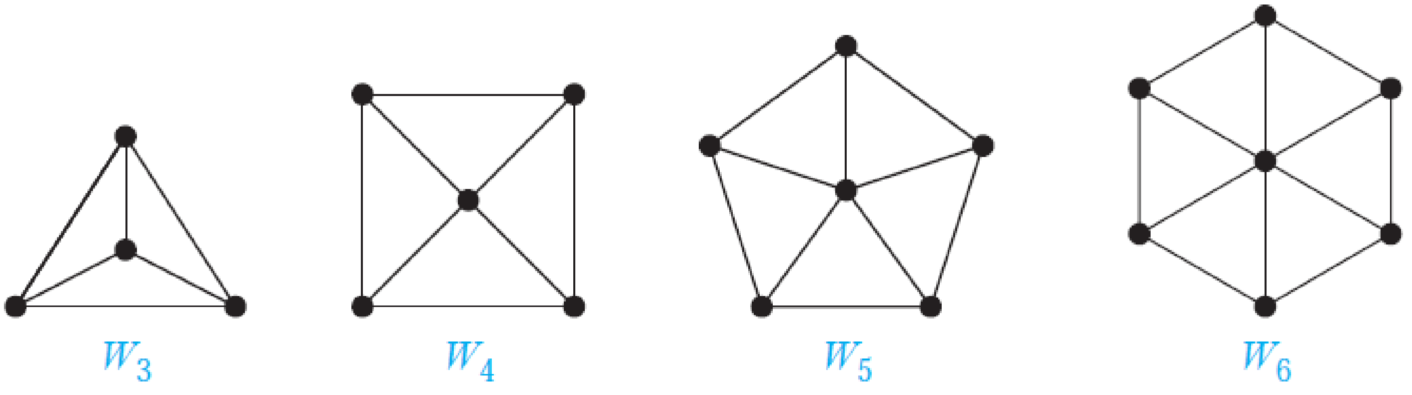
Wheels

A Wheel Wn is obtained by adding a additional vertex to a cycle

Cn,n≥3 and connect this new vertex to each of the other vertices

by new edges.

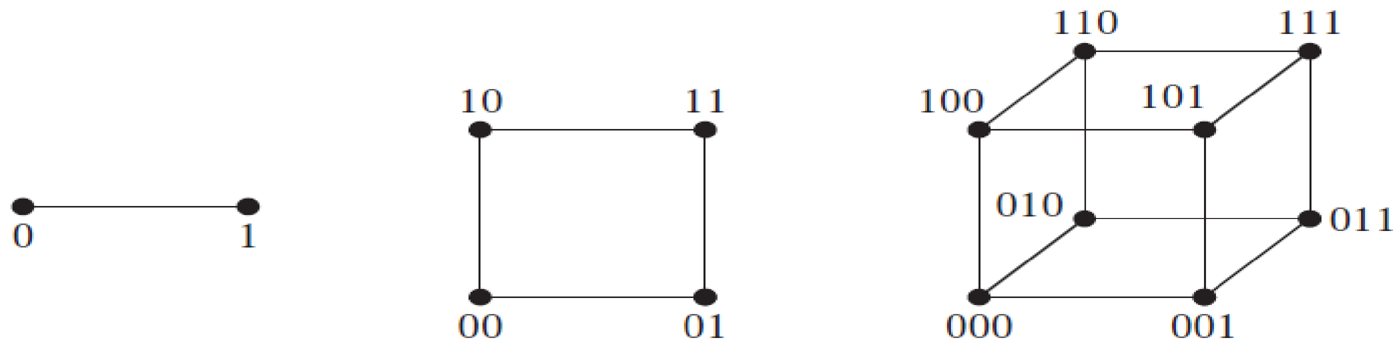
A Wheel is a 3-regular graph



Some Special Simple Graphs

n-Cubes

An n-cube is a graph that has vertices representing the 2^n bit strings of length n.Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position,denoted by Q_n.



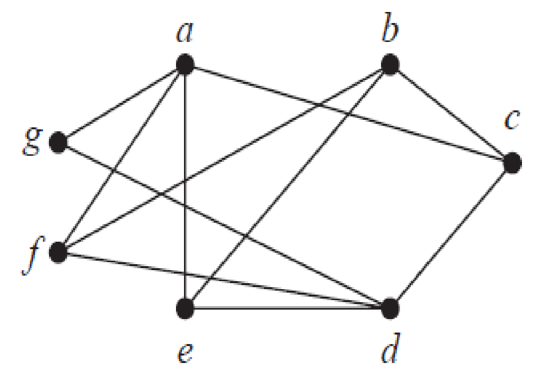
Q_1 Q_2 Q_3

Some Special Simple Graphs

Bipartite Graphs:

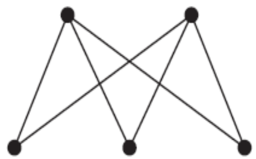
If the vertex set of a simple graph G=(V,E) can be partitioned into two subsets V1 and V2 such that every edge of G connects a vertex in V1 and a vertex in V2(so that no edge in G connects either two vertices in V1 or two vertices in V2),then G is called a bipartite graph

Ex:The vertex set can be divided into two disjoint sets {a,b,d} and {c,e,f,g} such that there is an edge from one set to another and not inside the sets.

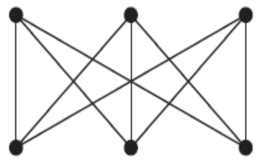


Complete Bipartite Graphs:

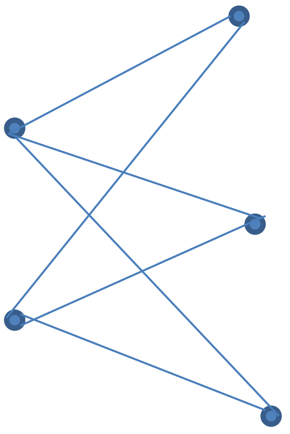
If each vertex of V_1 is connected with each vertex of V_2 by an edge, then G is called a completely bipartite graph denoted by $K_{m,n}$ where m is no. of vertices in V_1 set and n is no. of vertices in v_2 set.

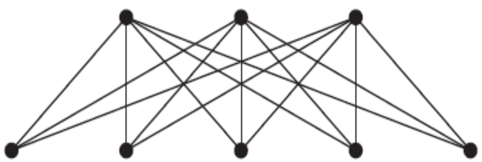


$K_{2,3}$

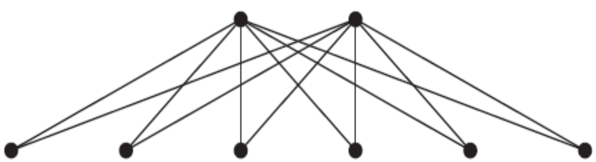


$K_{3,3}$





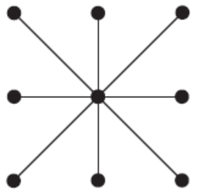
$K_{3,5}$



$K_{2,6}$

Applications of Special Graphs

1. A local area network can be represented as $K_{1,n}$

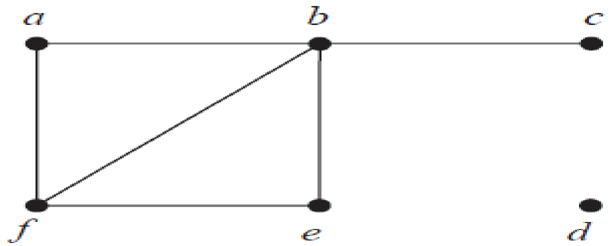


2.

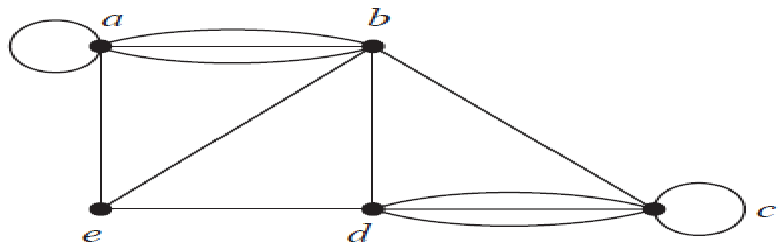
Practice Problems

1. In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

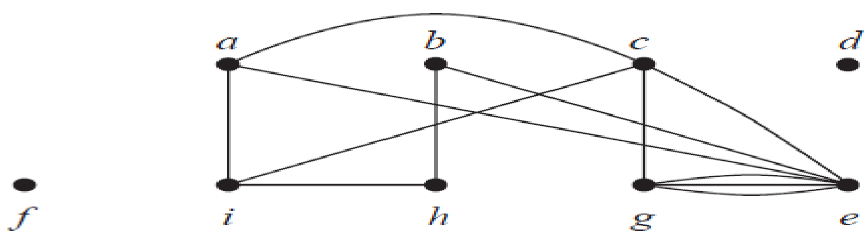
1.



2.



3.



Also check the handshaking theorem in each case

Practice Problems

2. Draw these graphs.

a) K_7 **b)** $K_{1,8}$ **c)** $K_{4,4}$

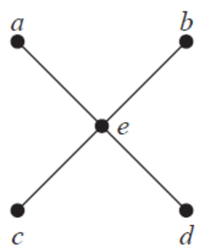
d) C_7 **e)** W_7

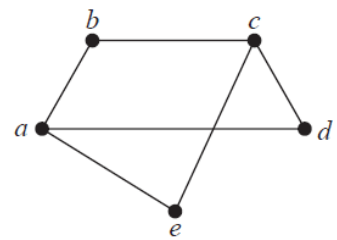
3. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

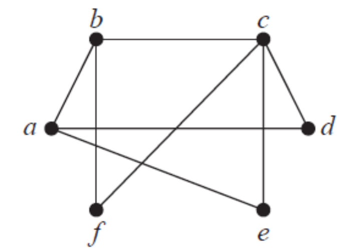
   a) Model the possible marriages on the island using a bipartite graph.

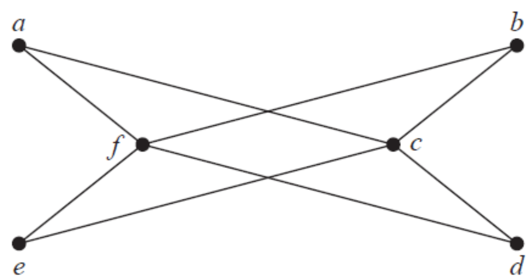
Practice Problems

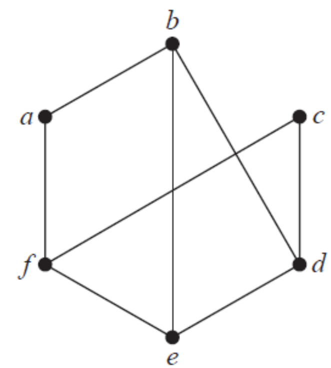
4.Check whether the following graphs are bipartite?

a) 

b) 

c) 

d) 

e) 

a) V={a,b,c,d,e} v1={a,b,c,d }, v2={ e}

Basic Definitions

Directed Graph or digraph:

A digraph G consists of two sets:

(1)A set V=V(G) whose elements are called vertices or nodes or points of G

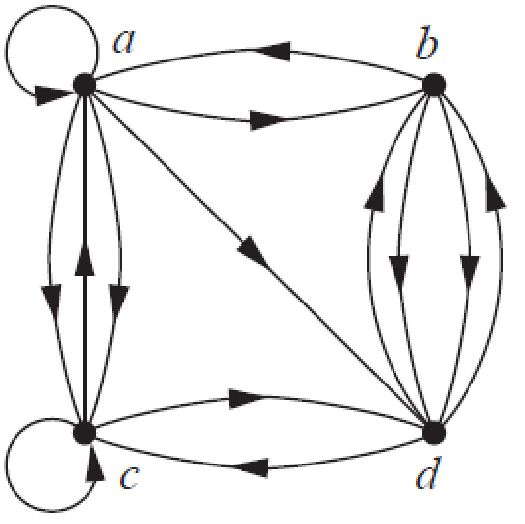
(2)A set E=E(G) of ordered pairs of vertices called edges or directed edges or arcs of G.

- A vertex is denoted by a dot or a small circle
- An edge is denoted by an arrow indicating the direction
- If a directed edge e =(u,v) starts at 'u' and ends at 'v' then u is called origin or initial point of the edge 'e' and v is called destination or terminal point of e
- u is adjacent to v and v is adjacent from u

Parallel Edges:

Two directed edges are said to be parallel if they both begin at vertex u and end at vertex v.

Ex: Edges between the vertices (a,c) and (b,d) are parallel.
Edges between the vertices (a,b) and(c,d) are <u>not parallel</u>, Since they don't have the same direction



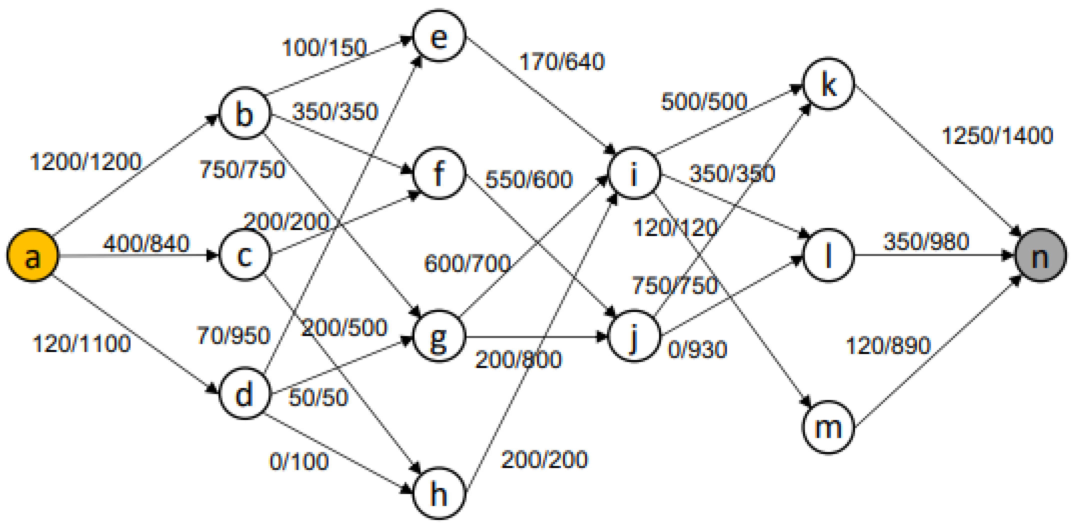
Loop:

Any directed edge starting and ending at the same vertex is called a loop.

Ex: The directed edges at the vertices a and c are loops.

Labelled Directed Graph:

If the edges and/or vertices of a directed graph are labelled with some type of data then 'G' is called a labelled directed graph.



Types of Vertices in digraphs

Indegree of a vertex:

The indegree of a vertex v is defined as the number of edges ending at the vertex v or entering into the vertex v, denoted by indeg(v).

Outdegree of a vertex:

The outdegree of a vertex v is defined as the number of edges beginning at the vertex v or leaving the vertex v, denoted by outdeg(v).

Source:

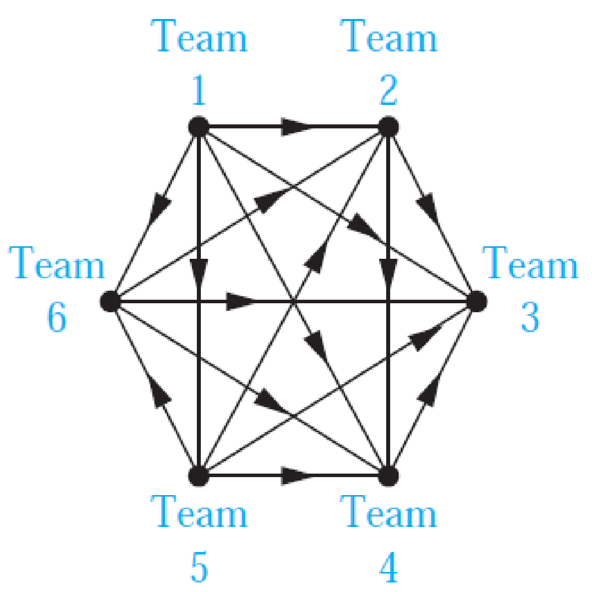
A vertex with zero indegree is called a source.

Sink:

A vertex with zero outdegree is called a sink.

Various Graph Models

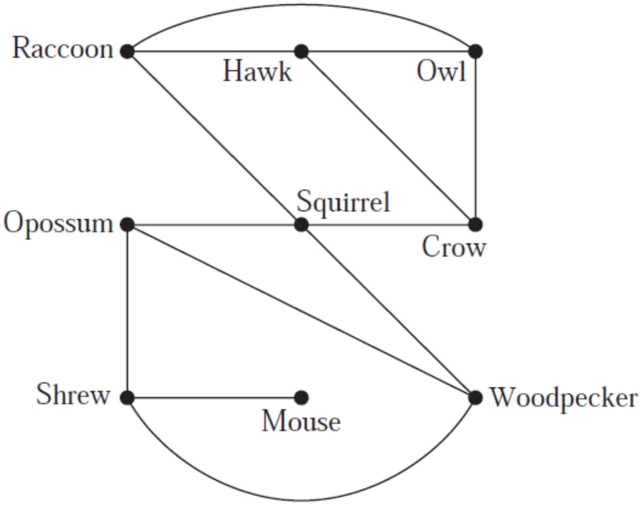
Outcomes of a round-robin tournament



Various Graph Models

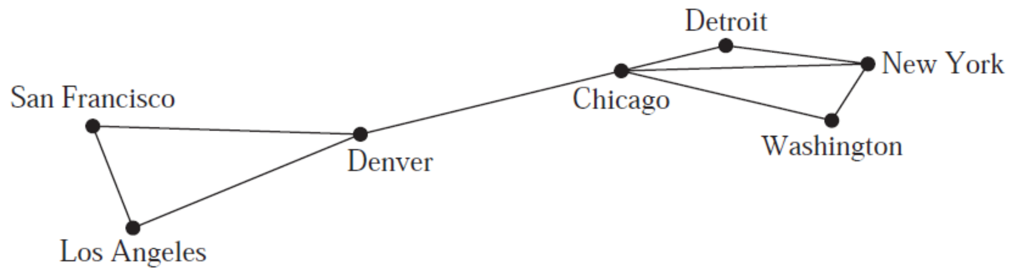
Niche Overlap Graph

The competition between the species in ecosystem can be modelled using these graphs.

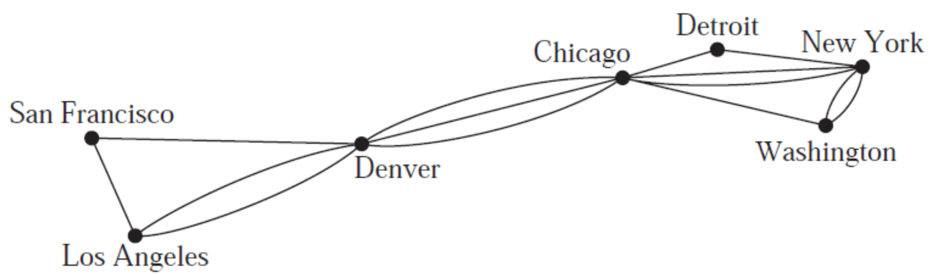


Various Graph Models

A simple Computer Network

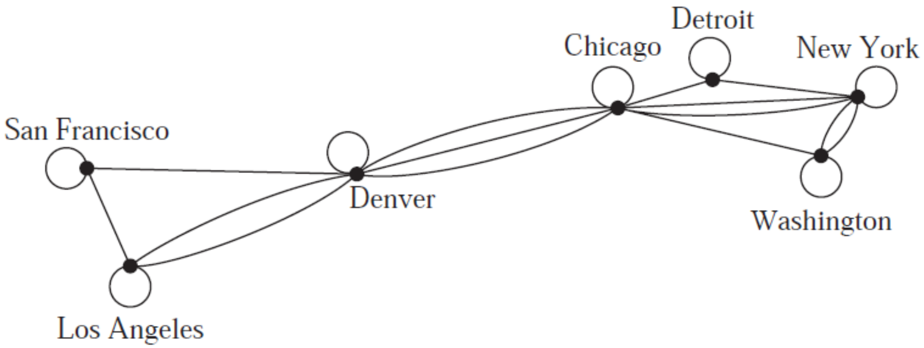


A Computer Network with multiple links

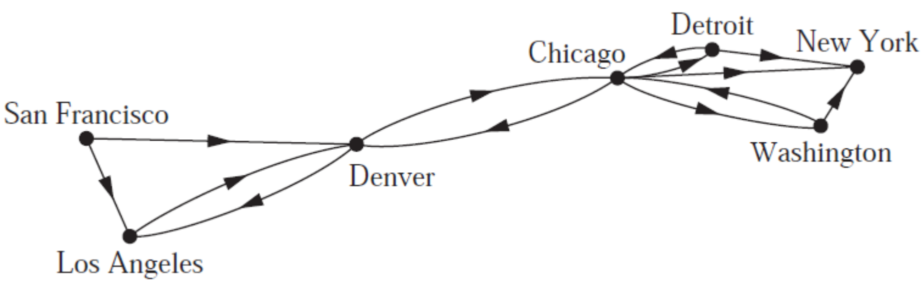


Various Graph Models

A Computer Network with diagnostic links denoted by loops



A Computer Network with one-way communication links



Graph Terminologies

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Module

Matrix Representation of Graphs: Adjacency Matrix, Incidence Matrix

Matrix Representation of Graphs

To determine whether two graphs are isomorphic, it will be easier to consider their matrix representations.

There are two types of matrices commonly used to represent graphs.

1. Adjacency Matrix
2. Incidence Matrix

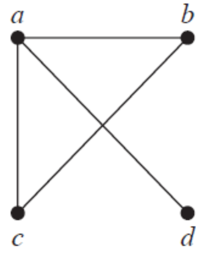
Adjacency Matrix

- When G is a simple graph with n vertices $v_1, v_2, v_3, \ldots, v_n$, the matrix

$$A = A_G = [a_{ij}] \;\; where \;\; a_{ij} = \begin{cases} 1\,, if \; v_i v_j \; is \; an \; edge \; of \; G \\ 0, otherwise \end{cases}$$

Is called the adjacency matrix of G

Ex: For the graph G ,the adjacency matrix is given by



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

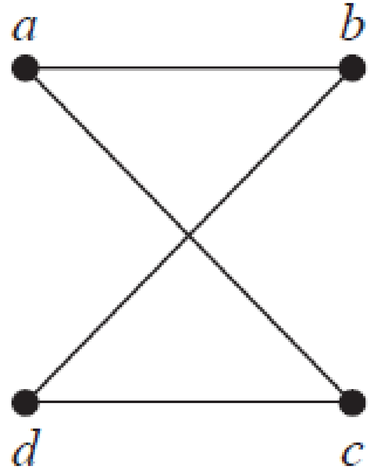
Adjacency Matrix

Ex: Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

Sol:



Properties of Adjacency Matrix

- Since a simple graph has no loops ,each diagonal entry of A is zero.
- The adjacency matrix of a simple graph is symmetric
- Degree of a vertex is equal to the number of one’s in the corresponding row or column

Adjacency Matrix of a Pseudograph

Pseudograph:

A graph containing multiple edges and/or loops is called Pseudograph.

A Pseudograph can also be represented by an adjacency matrix using the following steps:

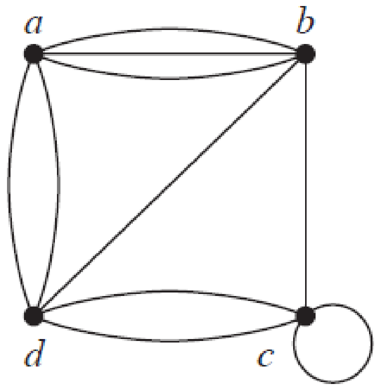
1)A loop at the vertex v_i is represented by a 1 at (i,I)th position

2)The (i,j)th entry equals the number of edges that are incident on v_i and v_j

Note :
The adjacency matrix of a Pseudograph is also a symmetric matrix

Adjacency Matrix of a Pseudograph

Ex: Find the adjacency matrix of the following Pseudo graph

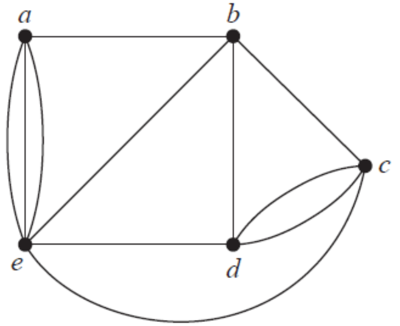


Sol: Taking the vertices in the order a, b, c,d , the adjacency matrix is given by

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of a Multigraph

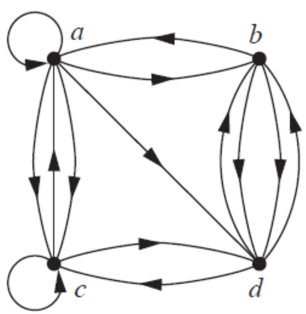
In a similar way, we can represent a multigraph using an adjacency matrix. These adjacency matrices are also symmetric.



$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of a directed graph

In a similar way, we can represent a directed graph using an adjacency matrix.
These adjacency matrices may or may not be symmetric.

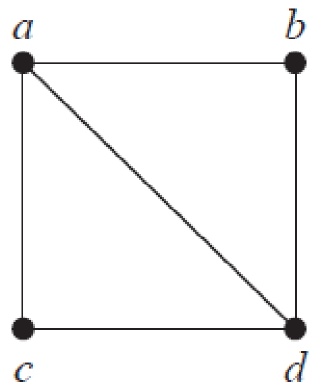


$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

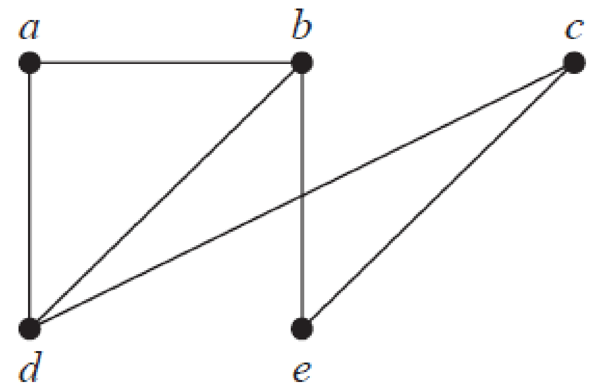
Practice Problems

1.Represent the following graphs using adjacency matrix:

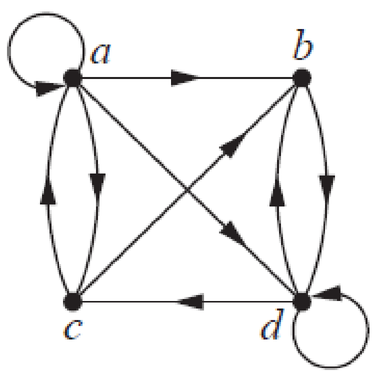
1.



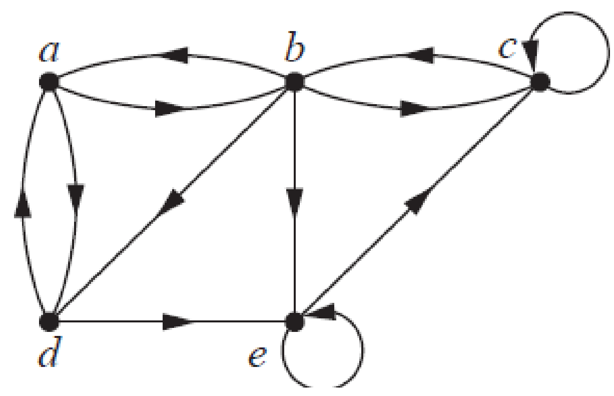
2.



3.



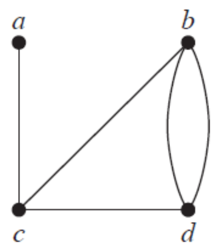
4.



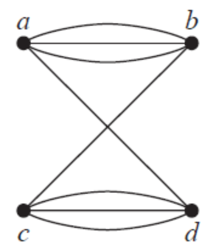
Practice Problems

1.Represent the following graphs using adjacency matrix:

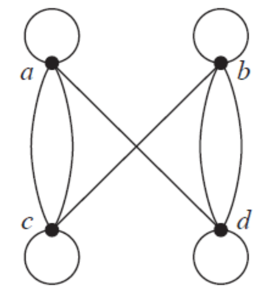
a)



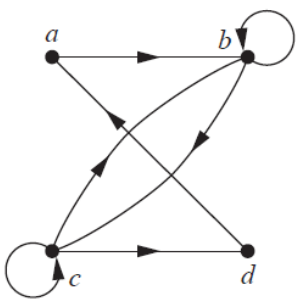
b)



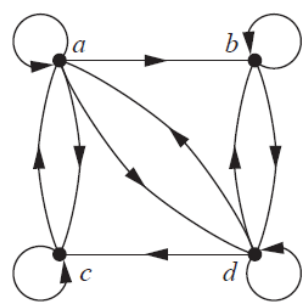
c)



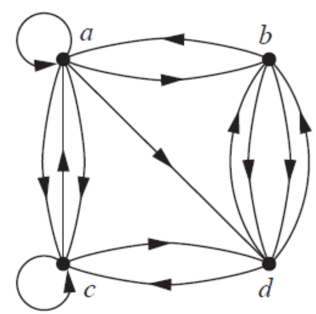
d)



e)



f)



Practice Problems

2. Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$

d) C_4 e) W_4 f) Q_3

3. Represent the following adjacency matrices into graphs :

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

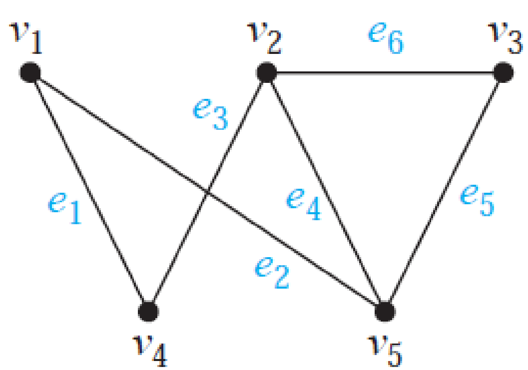
Incident Matrix

- If G=(V,E) is an undirected graph with n vertices $v_1,v_2,v_3,\dots,v_n$, and m edges $e1,e2,e3,\dots,en$ thenthe n x m matrix

$$B = [b_{ij}] \quad where \;\; b_{ij} = \begin{cases} 1\,, if\ edge\ e_j\ \ is\ incident\ on\ \ v_i \\ 0, \qquad\qquad otherwise \end{cases}$$

Is called the Incidency matrix of G

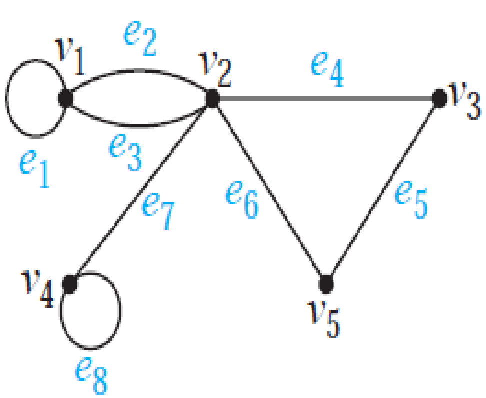
Ex: For the graph G ,the incidency matrix is given by



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incident Matrix

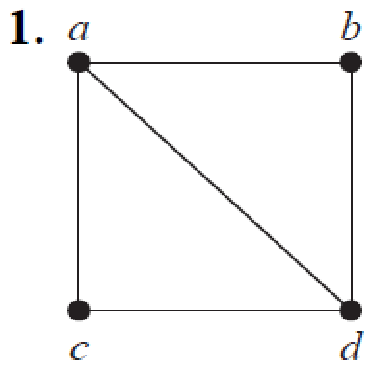
- Incident matrices can also be used to represent Pseudographs.
- Parallel edges are represented in the matrix using columns with identical entries, since these edges are incident on the same pair of vertices.
- Loop is represented by a column with exactly one unit entry ,corresponding to the concerned vertex.
- Ex: A pseudograph and its incident matrix are given as

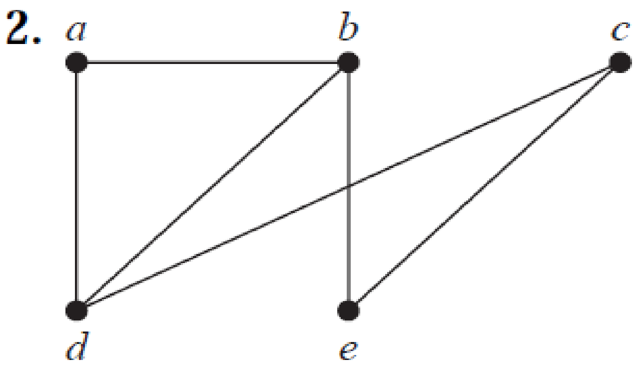


$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.
$$

Practice Problems

1.Represent the following graphs using incidency matrix:

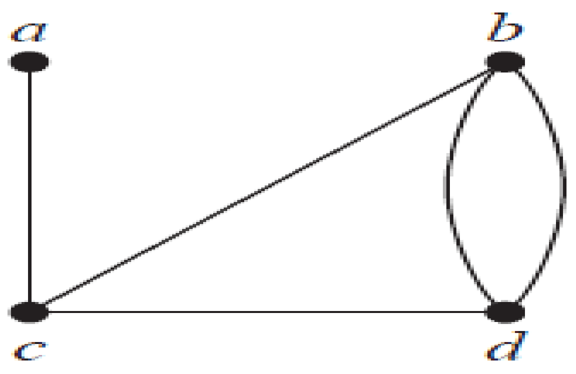




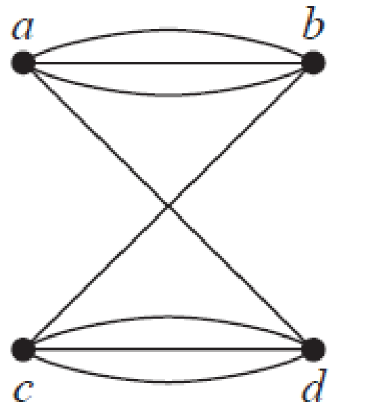
Practice Problems

2.Represent the following graphs using incidency matrix:

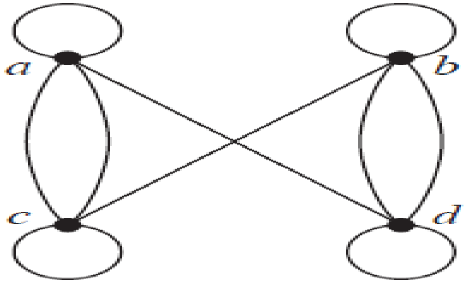
a)



b)



c)

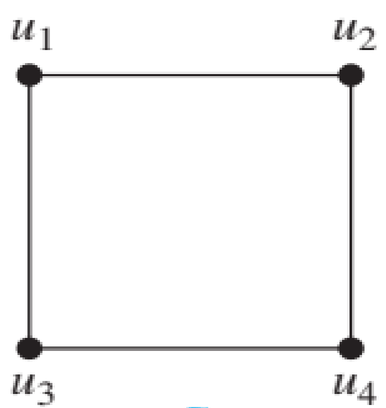


Isomorphism of Graphs

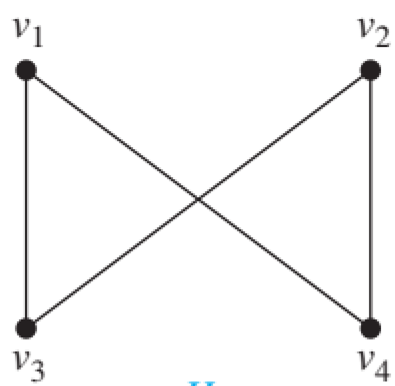
The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*.* Two simple graphs that are not isomorphic are called *nonisomorphic*.

Problems:

Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure 8, are isomorphic.



G



H

Solution:

No. of vertices in graph G=4

No. of edges in graph G=4

No. of vertices in graph H=4

No. of edges in graph H=4

No. of vertices in G=No. of vertices in graph H

No. of edges in graph G=No. of edges in graph H

$\deg(u_1)= 2$	$\deg(v_1)= 2$
$\deg(u_2)= 2$	$\deg(v_2)= 2$
$\deg(u_3)= 2$	$\deg(v_3)= 2$
$\deg(u_4)= 2$	$\deg(v_4)= 2$

No.of Same degree vertices of G= no.of Same degree vertices of H

$V=\{u_1, u_2, u_3, u_4\}$ $W=\{v_1, v_2, v_3, v_4\}$

Define $f: V \to W$

$f(u_1)= v_1$

$f(u_2)= v_4$

$f(u_3)= v_3$

$f(u_4)= v_2$

Let $(u_1, u_2) \in V$

$\{f(u_1),f(u_2)\}= \{v_1,v_4\} \in W$

$(u_2,u_4) \in V$

$\{f(u_2),f(u_4)\}=\{v_4,v_2\} \in W$

$(u_3,u_4) \in V$

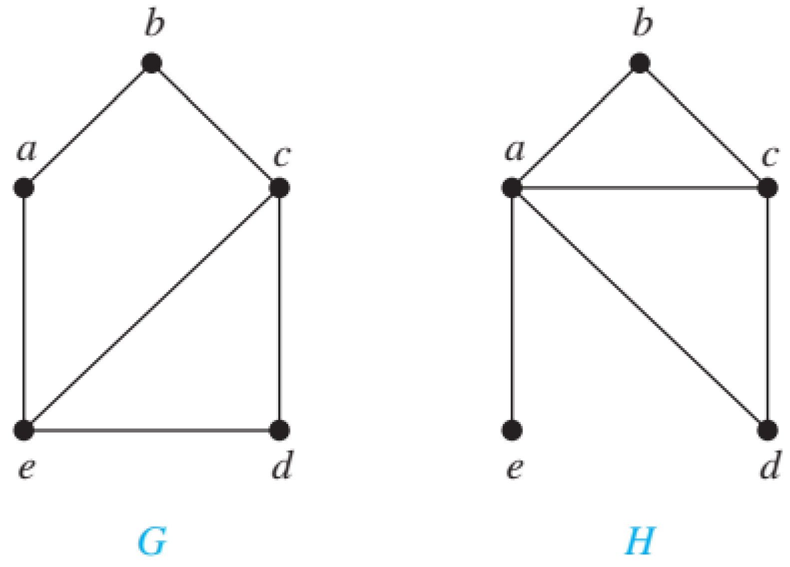
$\{f(u_3),f(u_4)\}=\{v_3,v_2\} \in W$

{u1,u3} ЄV

{f(u1),f(u3)}={v1, v3} ЄW

Therefore the graphs G and H are isomorphic to each other.

2. Check whether the given graphs are isomorphic or not.



Solution:

No. of vertices in graph G=5

No. of edges in graph G=6

No. of vertices in graph H=5

No. of edges in graph H= 6

No. of vertices in G=No. of vertices in graph H

No. of edges in graph G=No. of edges in graph H

Graph G	Graph H
deg(a)= 2	deg(a)= 4
deg(b)= 2	deg(b)= 2
deg(c)= 3	deg(c)= 3

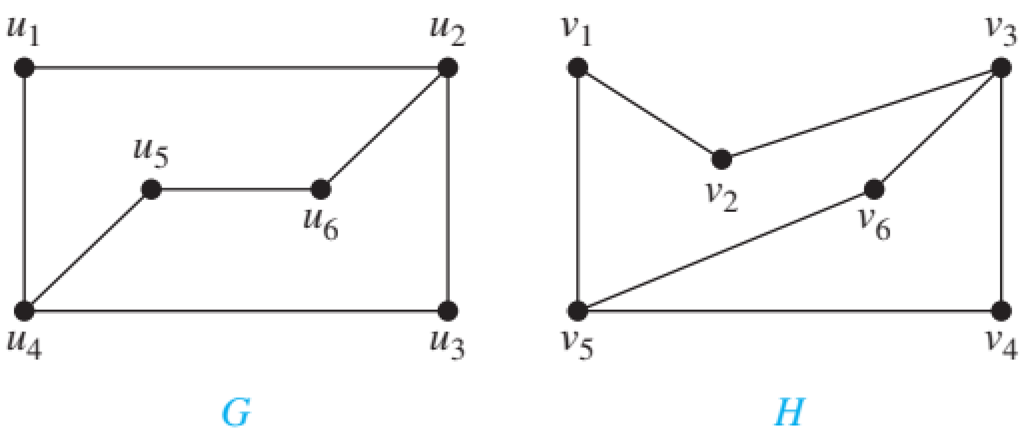
deg(d)= 2 deg(d)= 2

deg(e)= 3 deg(e)=1

Same degree vertices of G≠ Same degree vertices of H

So the given graphs are not isomorphic.

3. . Check whether the given graphs are isomorphic or not.



Solution:

No. of vertices in graph G=6

No. of edges in graph G=7

No. of vertices in graph H=6

No. of edges in graph H= 7

No. of vertices in G=No. of vertices in graph H

No. of edges in graph G=No. of edges in graph H

Graph G	Graph H
$deg(u_1)= 2$	$deg(v_1)= 2$
$deg(u_2)= 3$	$deg(v_2)= 2$
$deg(u_3)= 2$	$deg(v_3)= 3$

$deg(u_4) = 3$ $deg(v_4) = 2$

$deg(u_5) = 2$ $deg(v_5) = 3$

$deg(u_6) = 2$ $deg(v_6) = 2$

Same degree vertices in G= same degree vertices in H

$V=\{u_1, u_2, u_3, u_4, u_5, u_6\}$ $W=\{v_1, v_2, v_3, v_4, v_5, v_6\}$

Define $f: V \rightarrow W$

$f(u_1) = v_6$

$f(u_2) = v_3$

$f(u_3) = v_4$

$f(u_4) = v_5$

$f(u_5) = v_1$

$f(u_6) = v_2$

Let $\{u_1, u_2\} \in V$

$\{f(u_1),f(u_2)\}= \{v_6,v_3\} \in W$

$\{u_2,u_3\} \in V$

$\{f(u_2),f(u_3)\}=\{v_3,v_4\} \in W$

$\{u_3,u_4\} \in V$

$\{f\{u_3),f(u_4)\}=\{v_4,v_5\} \in W$

$\{u_4,u_1\}\ \} \in V$

$\{f(u_4),f(u_1)\}=\{v_5,v_6\} \in W$

The two graphs are isomorphic to each other.

$\{u_5, u_6\} \in V$

$\{f(u_5), f(u_6)\} = \{v_1, v_2\} \in W$

$\{u_6, u_2\} \in V$

$\{f(u_6), f(u_2)\} = \{v_2, v_3\} \in W$

Therefore, given two graphs are isomorphic.

Another method: up to assigning images same as previous method

Let us now examine the adjacency matrices of G and H

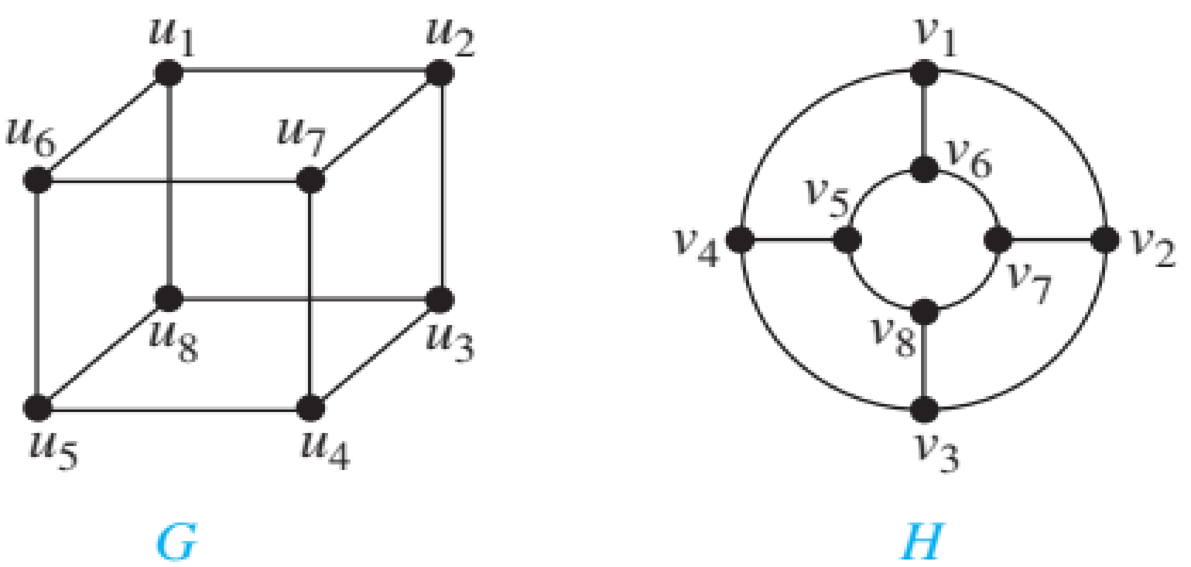
$$\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

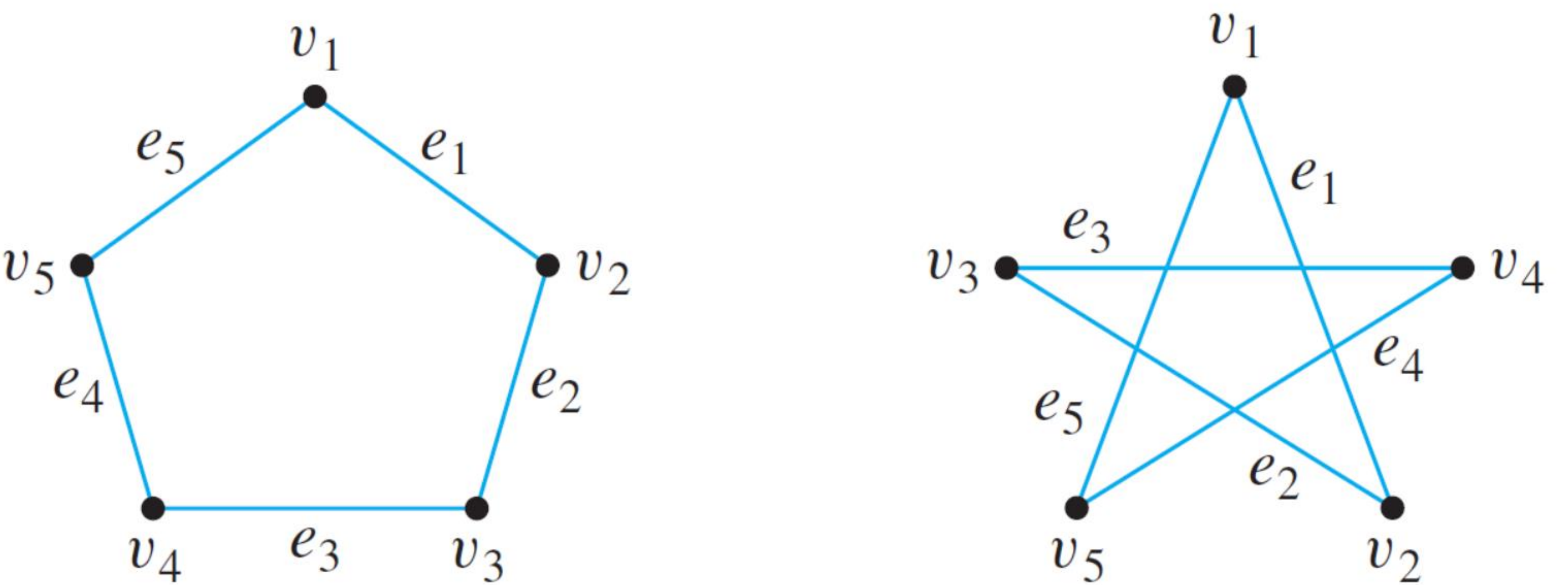
Therefore, given two graphs are isomorphic.

Practice problems:

1. Check whether the following graphs are isomorphic or not?

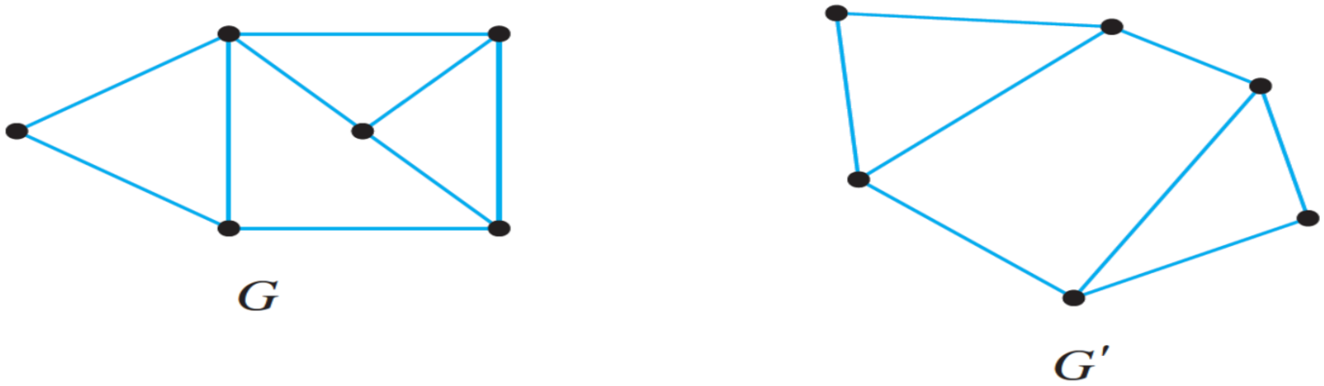


2. Check whether the following graphs are isomorphic or not?

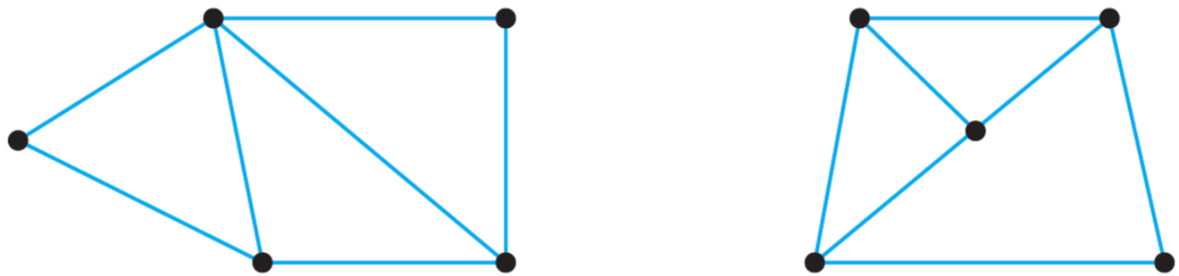


3.

a.



b.



Module

Paths and Circuits, Euler and Hamilton Graphs,

Paths

Many problems can be modeled with paths formed by travelling along the edges of graphs

Ex:

- ❖ Determining whether a message can be sent between two computers using intermediate links
- ❖ Planning efficient routes for mail delivery
- ❖ Garbage pickup
- ❖ Diagnostics in computer networks

  and so on

Paths

Informally, a **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.

Let n be a non-negative integer. Let G be an undirected graph. A path of length n from a vertex u to a vertex v in G is a sequence of n edges $e_1,e_2,e_3,\ldots,e_n$ of G for which there exists a sequence $x_0=u,x_1,\ldots,x_{n-1},x_n=v$ of vertices such that e_i has endpoints x_{i-1} and x_i for each $i=1,2,\ldots,n$.

Length of a path:

The number of edges in a path is called its length.

Note: we denote a path by its sequence of vertices

$$(v_0, v_1, v_2, \ldots\ldots, v_n)$$

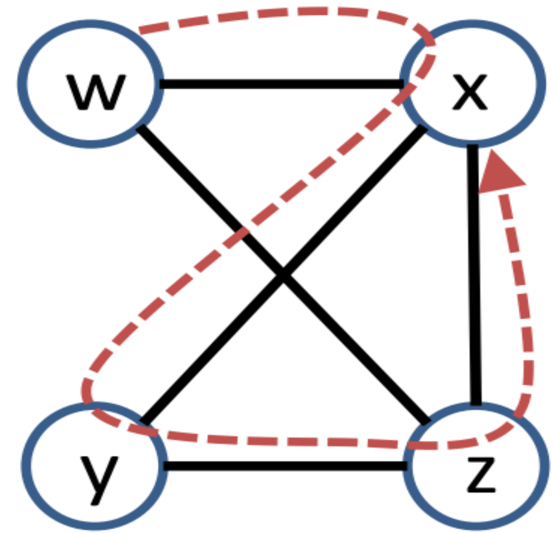
Closed path:

A path starts and ends in the same vertex is called a closed path.

Ex : Consider the graph on the right.

$w \to x \to y \to z \to x$ corresponds to a path of length 4

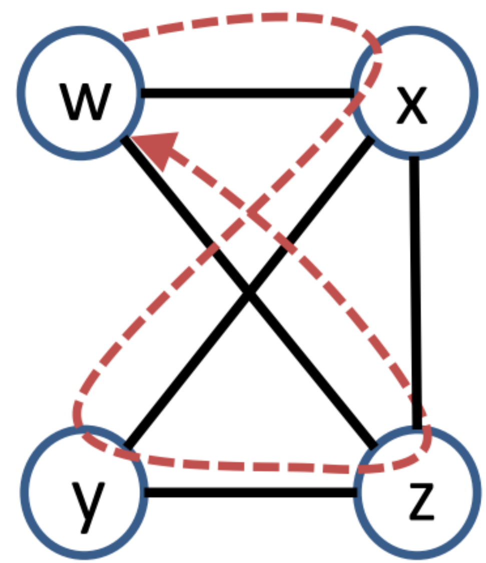
(W,X,Y,Z,X)



Ex : Consider the graph on the right.

$w \to x \to y \to z \to w$

gives to a circuit of length 4



Simple path:

A simple path is a path in which all vertices are distinct.

Trail:

A trail is a path in which all edges are distinct.

Cycle(Circuit):

A cycle or circuit is a closed path in which all vertices are distinct except the first and last vertices.

K-cycle:

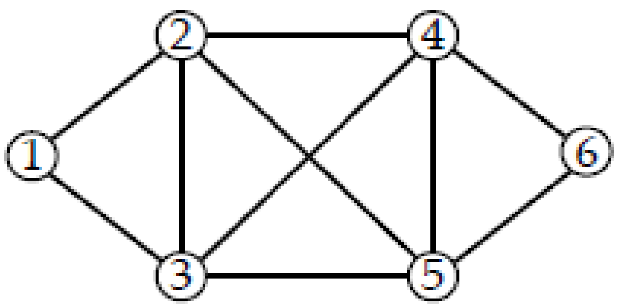
A cycle of length k is called a k-cycle.

Note:

Any cycle must have length 3 or more.

Walk:

A **walk** in a graph is a sequence of alternating vertices and edges $v_1 e_1 v_2 e_2 \ldots v_n e_n v_{n+1}$ with $n \geq 0$. If $v_1 = v_{n+1}$ then the walk is **closed**. The **length** of the walk is the number of edges in the walk. A walk of length zero is a **trivial walk**.



Paths:

(1,2,5),(1,2,3,4,5),

(1,3,5)—simple paths

(1,2,3,2,4,5)-not a simple path

Cycle:(1,2,3,1)--- 3-Cycle

(1,2,4,5,3,1)---5-cycle

- 124523 is walk and a trail, but not a path;
- 124231 is a walk and a closed walk;
- 1231 is a walk, trail, closed walk, circuit and cycle.

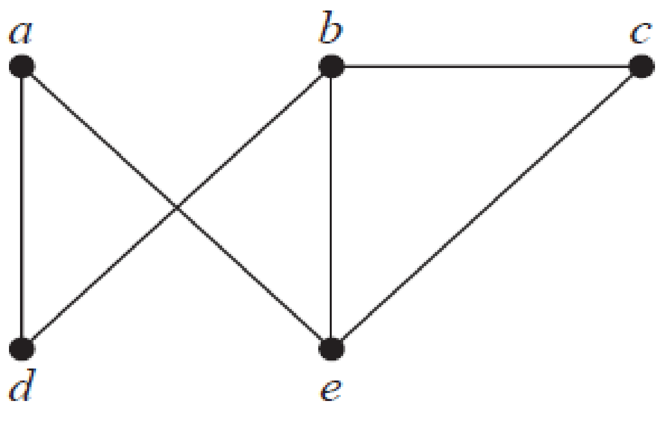
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b

b) a, e, a, d, b, c, a

c) e, b, a, d, b, e

d) c, b, d, a, e, c



a) Path of length 4

   not a circuit

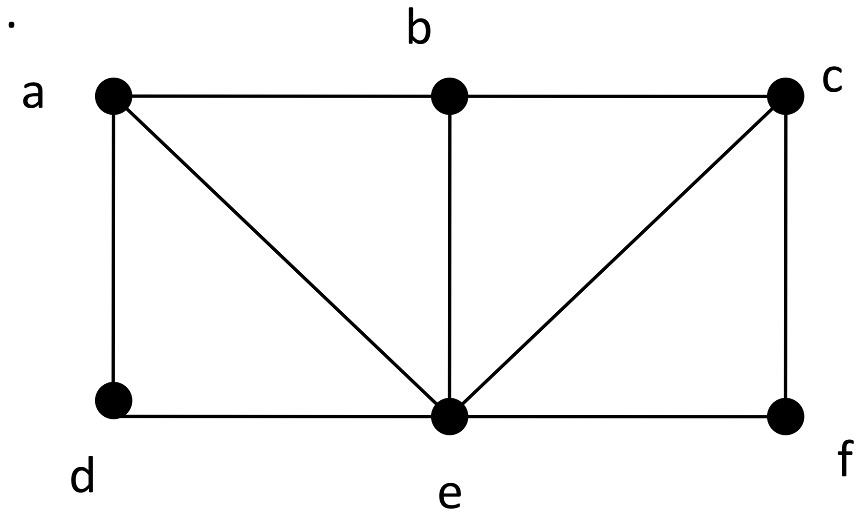
   not a simple path and not a trail.

b) Not a path

c) Not a path

d) Simple circuit of length 5

2. Consider graph G and discuss the following paths



P=(d,a,b,e,a,b,c,b)

Q=(d,a,e,b,f)

R=(d,a,e,b,c,e,f)

S=(d,a,e,c,f)

Solution:

P is not a trail, not a simple path, not a cycle.

Length of path=7

Q is not a path since there is no edge from b to f.

R is a trail but not simple path(since edge e is repeated)

S is trail and simple path of length 4.

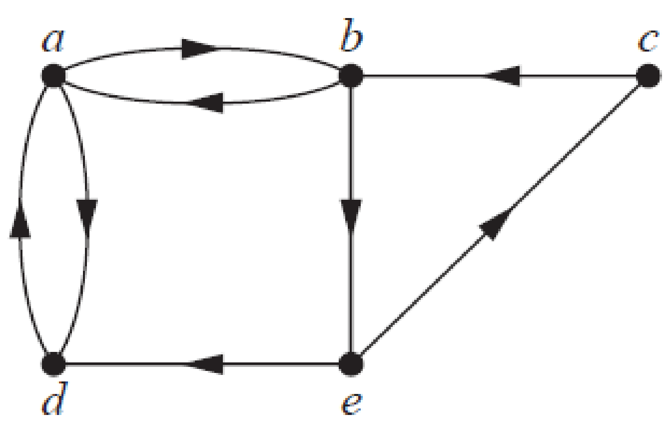
2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b

b) a, d, a, d, a

c) a, d, b, e, a

d) a, b, e, c, b, d, a

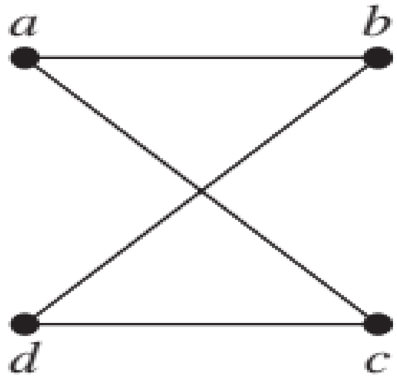


THEOREM

Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

Problem:

How many paths of length four are there from a to d in the simple graph G



Length of the path (r)=4

Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Hence, the number of paths of length four from a to d is the $(1, 4)$th entry of $\mathbf{A}^4$. Because

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

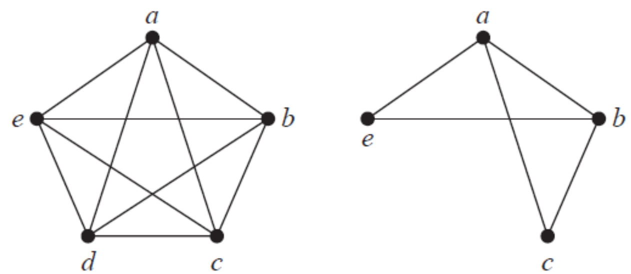
there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d.

Subgraph:

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

Induced Subgraph:

Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.



Connected graph:

A graph G is connected if there is a path between any two of its vertices otherwise it is said to be disconnected.

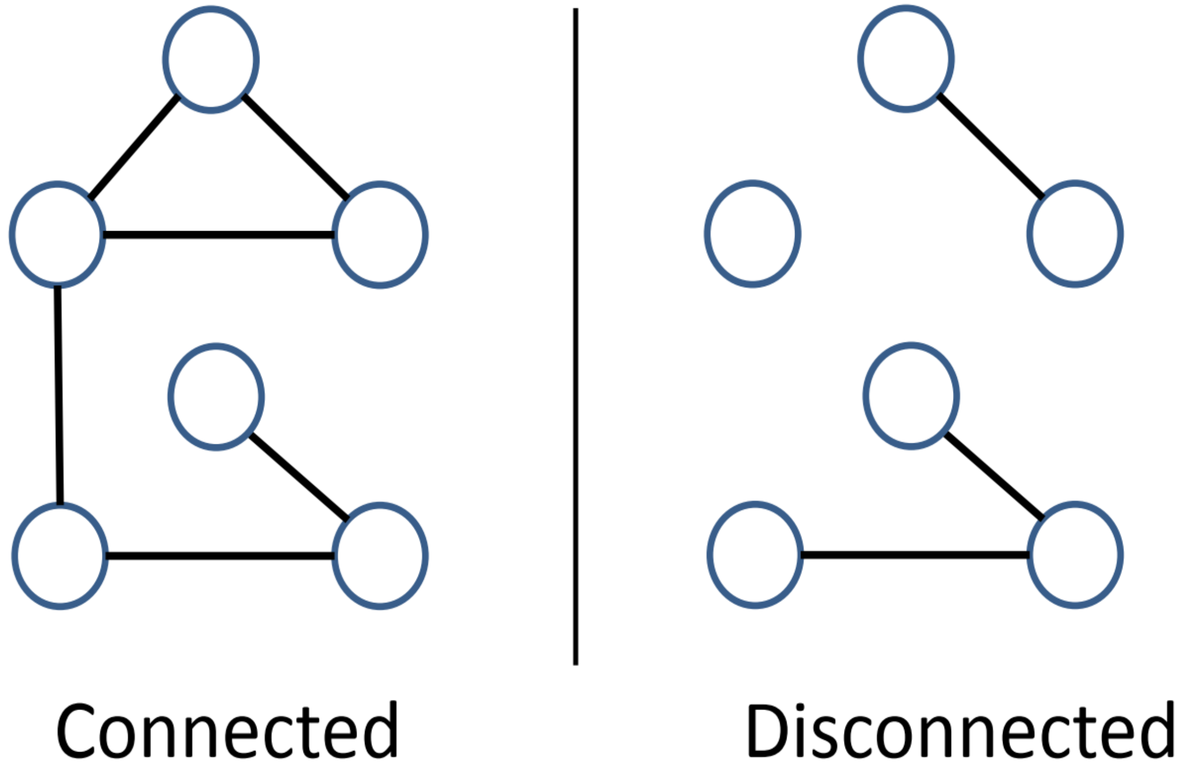
Distance:

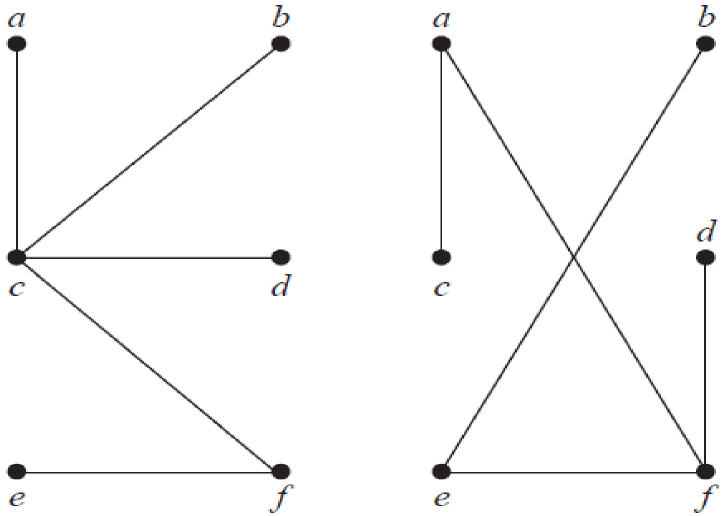
The distance between two vertices u, v in a connected graph G is the length of the shortest path between u and v and is denoted by d(u, v)

Diameter:

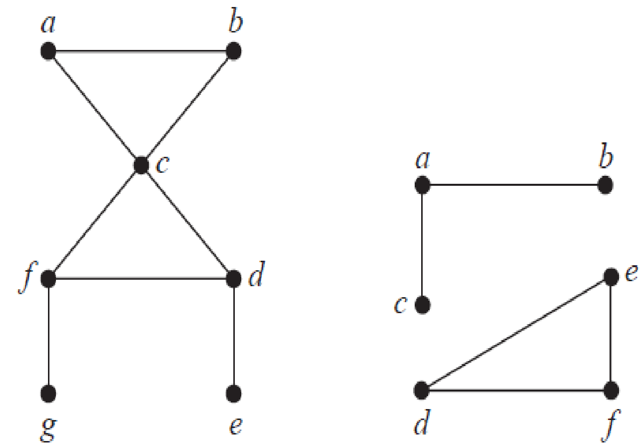
The diameter of a connected graph G is the maximum distance between any two points in G.

- Ex :





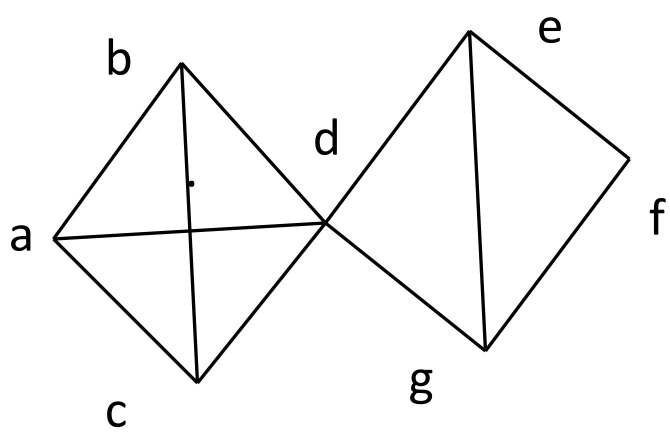
G_1 G_2



G_1 G_2

Theorem:

If a graph(either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.



d(a, f)= 3

d(a, c)= 1

d(a,g)=2

diam(G)=3

Cut Vertex:

A vertex v in a connected graph G is called cut vertex if G-v is disconnected.

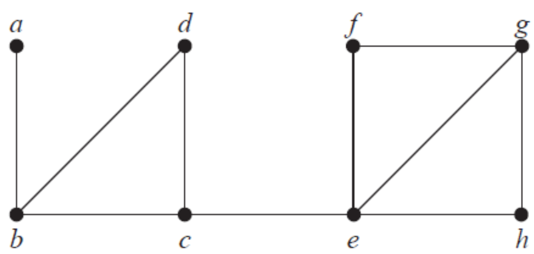
Note: While removing the vertex the edges incidenting on that vertex must also be removed.

Cut Edge or Bridge:

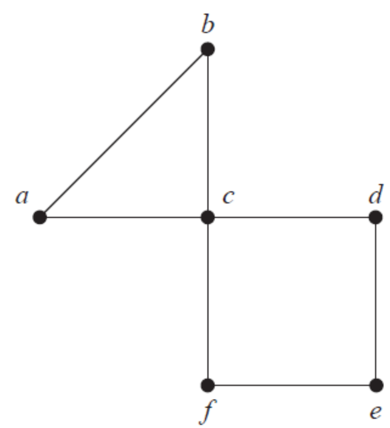
An edge e in a connected graph G is called cut edge or bridge if G-e is disconnected.

Cut Set:

The set of all minimum number of edges of G whose removal disconnects a graph G is called a cut set of G



G_1



G_2

In the above graph c or e is the cut vertex and (c,e) is the bridge

Eulerian Graphs

A path of graph G is called an ***Eulerian trail***, if it includes all the edges of G exactly once.

A closed Eulerian trail is called an **Eulerian Circuit.**

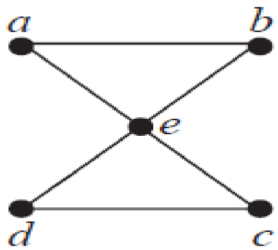
A graph containing an Eulerian circuit is called an **Eulerian Graph.**

Properties of Eulerian Graphs:

1)A non-empty connected graph is Eulerian if and only if all its vertices are all even degree.

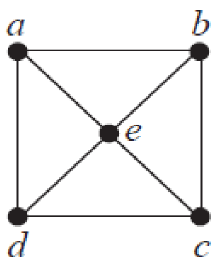
2) A connected graph G has an Eulerian path if and only if it has exactly two odd vertices or it has no odd vertices.

Eulerian Graphs

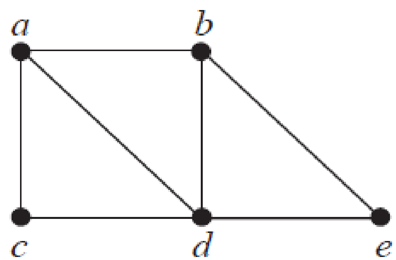


G_1

In graph G1 , there is an Euler circuit a,e,c,d,e,b,a.



G_2



G_3

Graph G2 does not have an Euler circuit or path
Graph G3 has an Euler path a,c,d,a,b,e,d,b.

Traversable trail:

A multigraph is said to be traversable if there is a path which includes all vertices and uses each edge exactly once. Such a path must be trail and is known as traversable trail.

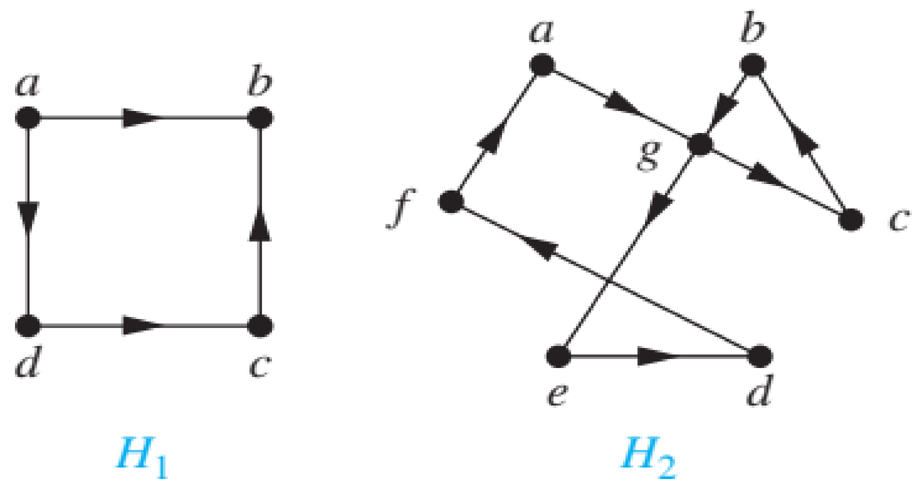
Eulerian circuit:

An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

Eulerian graph:

A graph G is called an Eulerian graph if there exists a closed traversable trail called an Eulerian trail.

2.Which of the directed graphs have an Eulerian circuit?



Solution: The graph H_2 has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$. Neither H_1 nor H_3 has an Euler circuit (as the reader should verify). H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not

Eulerian Graphs

For which values of n do these graphs have an Euler circuit?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

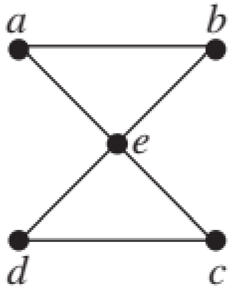
a) Euler circuit?
b) Euler path?

How to determine whether a graph G has an Euler circuit or path

1)List all the degree of all vertices in the graph
2)If any vertex has degree 0, then the graph is not connected .It cannot have Euler path or Euler circuit
3)If all the degrees are even ,then G has Euler path and Euler circuit
4)If exactly two vertices are of odd degree, the G has Euler path but no Euler circuit.

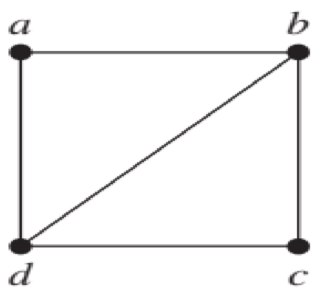
THEOREM 1

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.



THEOREM 2

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.



G_1

Explain Konigsberg bridge Problem. Represent the problem by means of graph. Does the problem have a solution?

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.

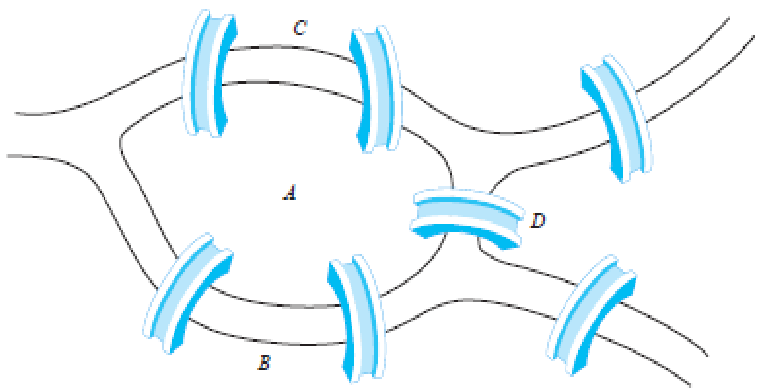
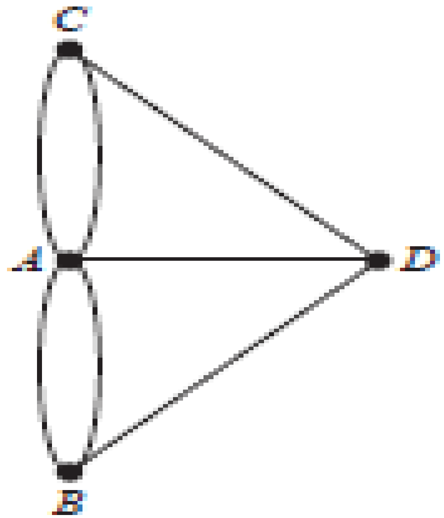


FIGURE 1 The Seven Bridges of Königsberg.

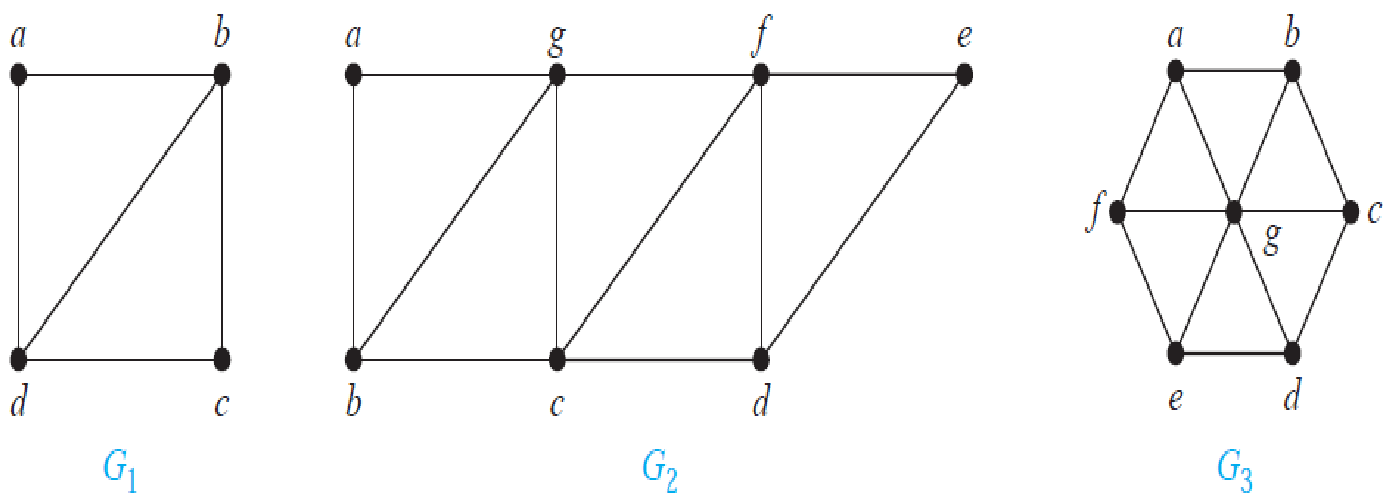
This problem can be modelled into a multigraph as



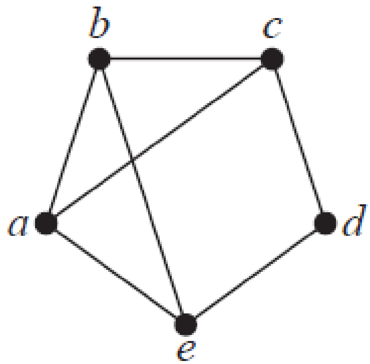
Listing the degrees of the vertices, d(A)=5,d(B)=3,d©=3,d(D)=3.
All the vertices are odd vertices.
By the property, If exactly two vertices are of odd degree, the G has Euler path but no Euler circuit.
Therefore, this graph does not have Euler Circuit.
i.e., There is no way to start at a given point, cross each bridge exactly once and return to the starting point.

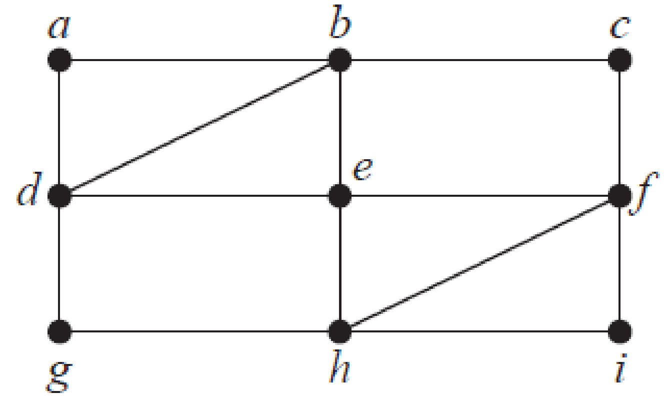
Which of the following graphs have Euler's path?

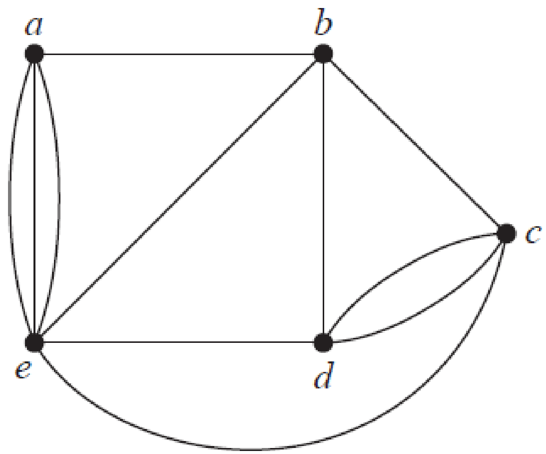
Solution: G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b. Similarly, G_2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is $b, a, g, f, e, d, c, g, b, c, f, d$. G_3 has no Euler path because it has six vertices of odd degree.

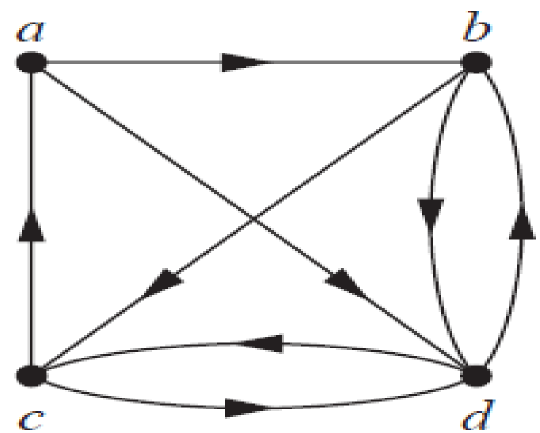


Which of the following graphs have Euler’s circuit or at least Euler’s path?









Hamiltonian path:

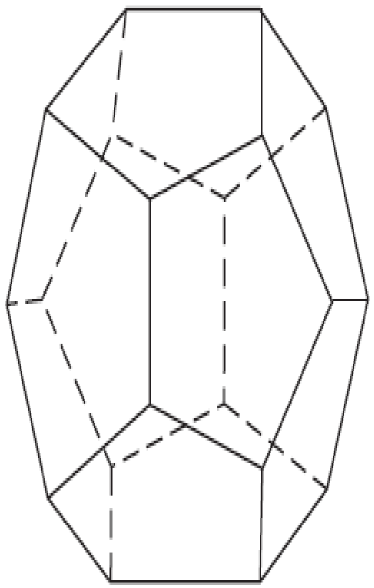
A path of a graph G is called a Hamiltonian path if it includes each vertex of G exactly once.

Hamiltonian circuit:

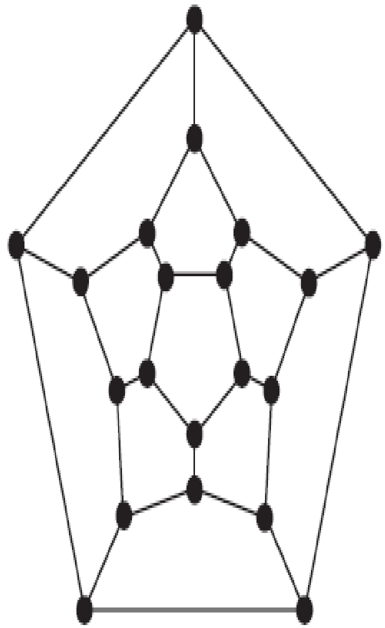
A circuit of a graph G is called Hamiltonian Circuit ,if it includes each vertex of G exactly once, except the starting and ending vertices (both are same) which appear twice.

Hamiltonian graph:

A graph containing a Hamiltonian circuit is called a Hamiltonian graph.



(a)



(b)

FIGURE 8 Hamilton's "A Voyage Round the World" Puzzle.

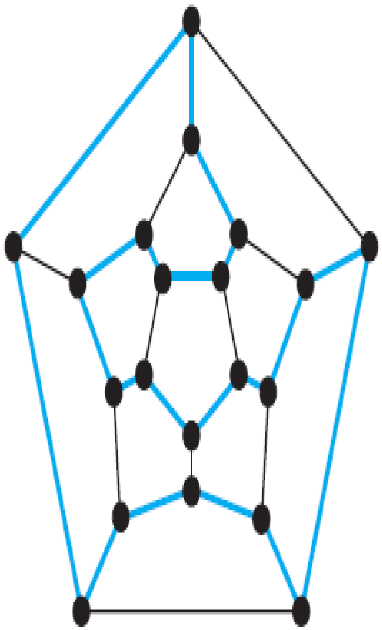


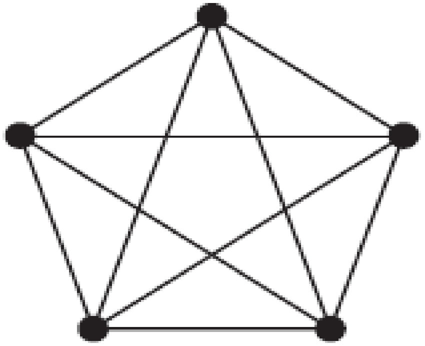
FIGURE 9 A Solution to the "A Voyage Round the World" Puzzle.

Properties of Hamiltonian Graph:

1)Let G be a simple graph with $n \geq 3$ vertices. If deg(v) $\geq n/2$ for all vertices of G ,then G is Hamiltonian.

2)Let G be a simple graph with $n \geq 3$ vertices .If deg(u)+deg(v) $\geq n$ for every pair of non-adjacent vertices u and v , then G is Hamiltonian.

Example: Consider the complete graph K5.

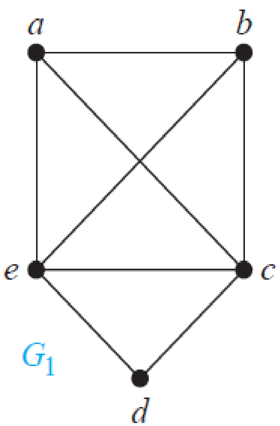


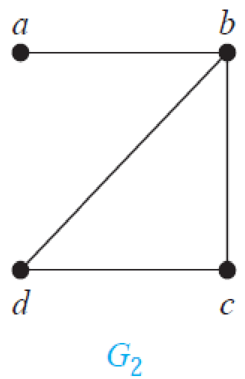
K_5

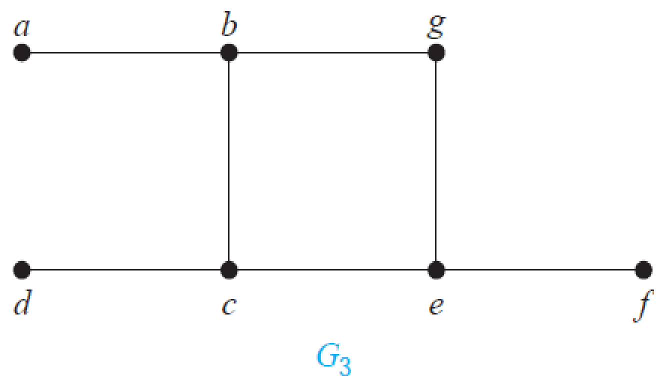
It is a simple graph with 5 vertices and the degree of each of the vertices is 4.So K5 contains a Hamiltonian circuit and it is Hamiltonian.

Which of the following graphs have a Hamiltonian circuit ,or if not Hamiltonian path?

Solution: G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.



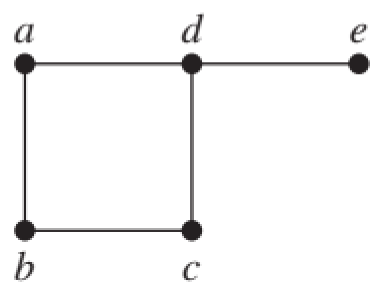




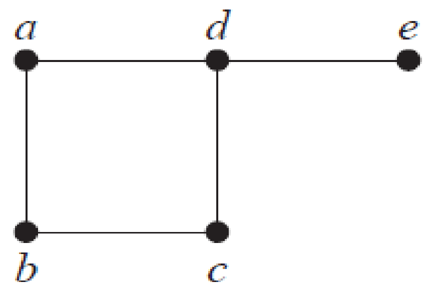
DIRAC'S THEOREM If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

CONDITIONS FOR THE EXISTENCE OF HAMILTON CIRCUITS

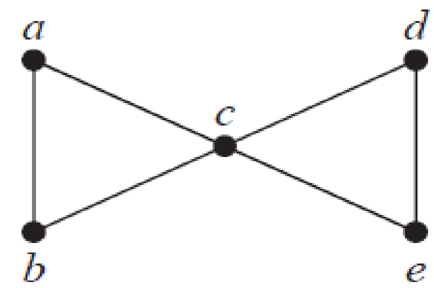
A graph with a vertex of degree one cannot have a hamiltonian circuit.



Show that the following graphs don't have Hamiltonian Circuit?



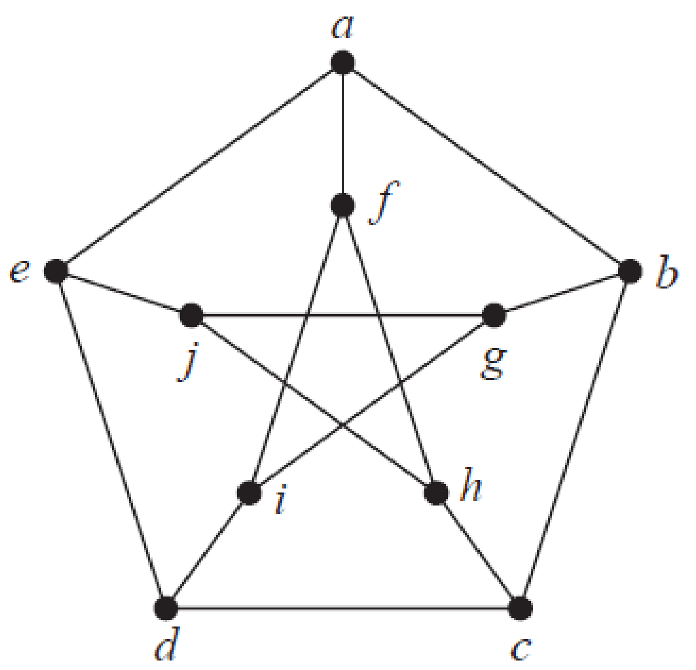
G



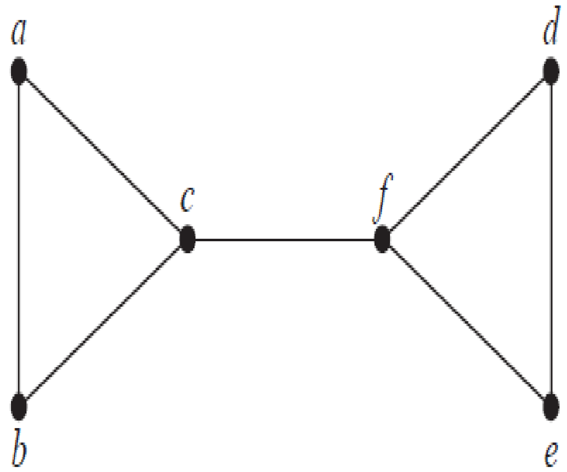
H

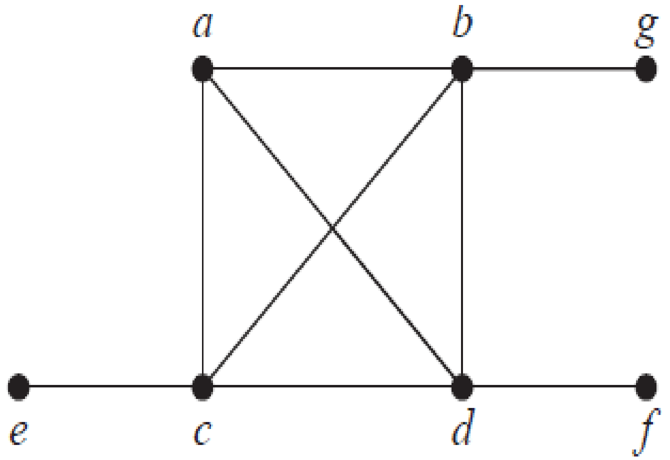
Solution: There is no Hamilton circuit in G because G has a vertex of degree one, namely, e. Now consider H. Because the degrees of the vertices a, b, d, and e are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with c, which is impossible.

Petersen Graph does not have a Hamiltonian circuit.



Does the graph have a Hamilton path? If so, find such a path. If it does not give an argument to show why no such path exists.





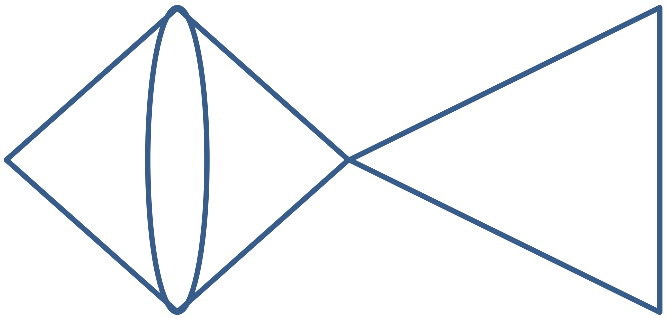
For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

$m = n \geq 2$

HAMILTONIAN and EULERIAN

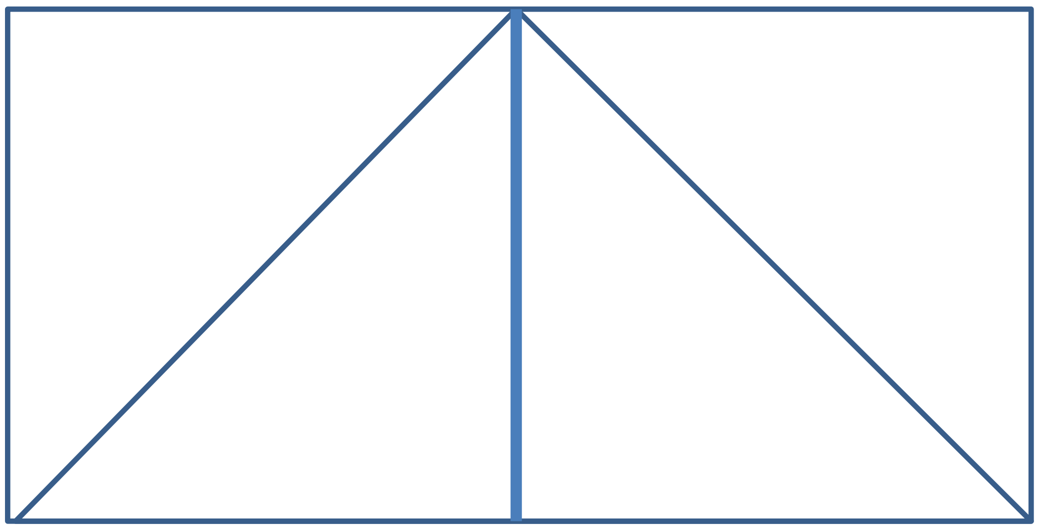
Hamiltonian circuit 1,2,3,4,1 exist which pass through all vertices exactly once.
Since all the vertices are with even degree, the graph is Eulerian.

EULERIAN BUT NOT HAMILTONIAN



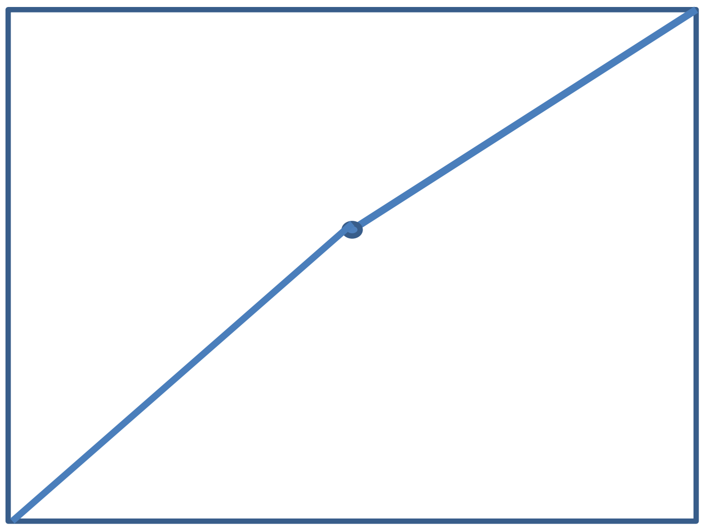
Since each vertex is of even degree, it is Eulerian but not having a Hamiltonian circuit

HAMILTONIAN BUT NOT EULERIAN



Contains a Hamiltonian circuit v1e1v2e2v3e3v4e3v5e5v6e6v1.
Since there are vertices with odd degree, the graph is not Eulerian.

NEITHER HAMILTONIAN NOR EULERIAN



No Hamiltonian circuit exist which pass through all vertices exactly once.
Since there are vertices with odd degree, the graph is not Eulerian.

PLANAR GRAPHS

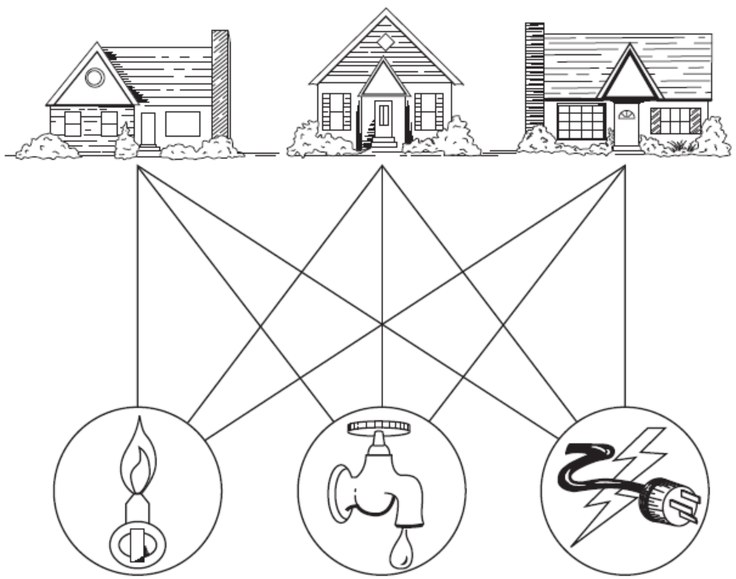


FIGURE 1 Three Houses and Three Utilities.

Can we redraw the graph so that no two edges cross each other?

PLANAR GRAPHS

A graph is called *planar* if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a *planar representation* of the graph.

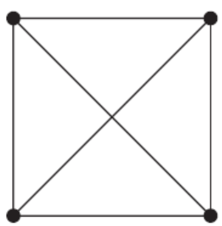


FIGURE 2 The Graph K_4.

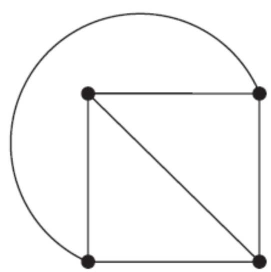


FIGURE 3 K_4 Drawn with No Crossings.

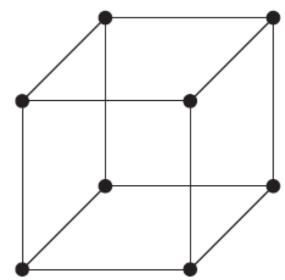


FIGURE 4 The Graph Q_3.

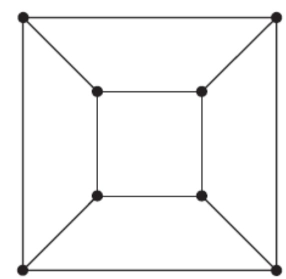
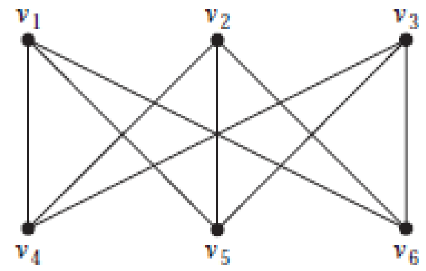


FIGURE 5 A Planar Representation of Q_3.

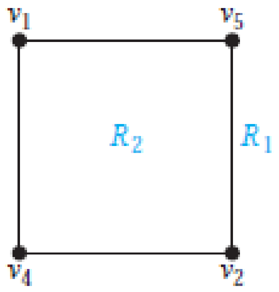
Is $K_{3,3}$ planar?



Solution: No, $K_{3,3}$ is not planar.

In any planar representation of $K_{3,3}$,

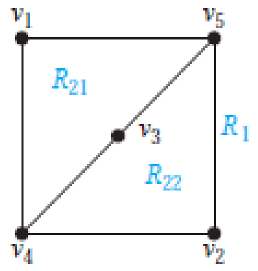
(i)the vertices v1 and v2 must be connected to v4 and v5.These 4 edges form a closed path and splits the plane into two regions R1 and R2.



(ii)The vertex v3 should be either in R1 or R2.

Assume v3 is in R2.

(iii) Join v3 to v4 and v5 with out crossing the edges.

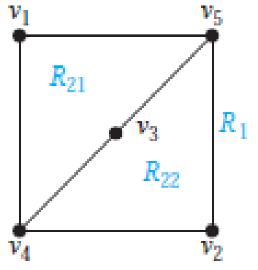


(iv) The region R2 is subdivided into two sub regions R21 and R22.

(v)Now place the vertex v6 without a crossing.

case1: v6 is in R1

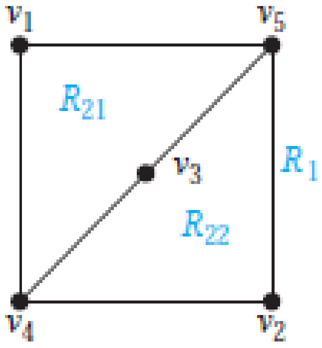
The edge between v3 and v6 cannot be drawn without crossing



Is K3,3 planar?

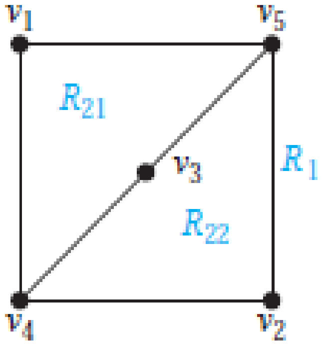
Case 2:v6 is in R21

The edges between v6 and v2 cannot be drawn without crossing



Case3: v6 is in R22

The edge between v6 and v1 cannot be drawn without a crossing.

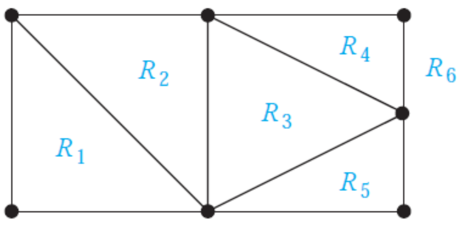


So,K3,3, is non-planar

EULER'S FORMULA

- A planar representation of a graph divides the plane into regions , including an unbounded region.
- In the following figure, the planar representation of the graph splits the plane into 6 regions
- Euler's formula gives a relation to find the number of regions of a graph using the number of vertices and edges.

EULER'S FORMULA Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.



r=6,v=7,e=11

r=6=11-7+2=e-v+2

Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution:

Number of vertices= v=20

Given each vertex is of degree 3,

Sum of degrees of vertices=2 x number of edges

3 x 20= 2.e

∴e=30

By Euler's formula,

r= e-v+2

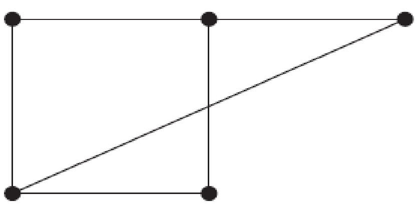
r=30-20+2=12

Practice problems:

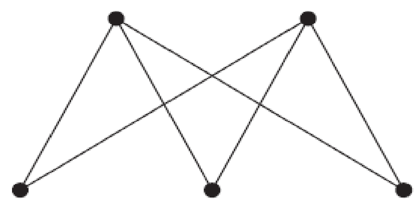
1. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?
2. Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?
3. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

Draw the graphs without any crossing or draw the planar representations of following graphs.

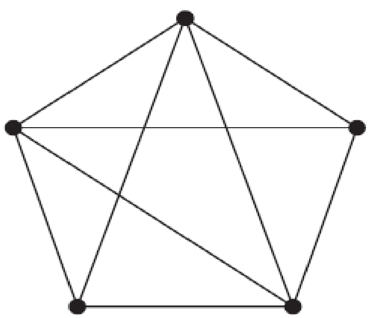
2.



3.



4.

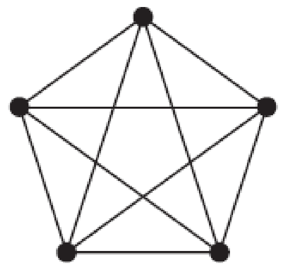


Corollary :

If G is a connected planar simple graph with e edges and $v (v \geq 3)$ vertices then

$$e \leq 3v-6.$$

Show that K_5 is not planar?



K_5

Solution: Assume K_5 is planar.

Then using Euler's formula,

$r=e-v+2$

$r=10-5+2=7$

Also for a connected simple planar graph, $e \leq 3v-6$

$\therefore 10 \leq 3(5)-6$

$10 \leq 9$, which is absurd.

There is a contradiction. Hence the assumption is wrong.

$\therefore$ K5 is planar is wrong. K5 is not planar.

Show that K3,3 is non-planar