2) use Generating function to solve the recurrence relation $a_{n+2}-2a_{n+1}+a_n=2^n$ with initial conditions $a_0=2, a_1=1$

Sol:

Let $G(x)=\sum_{n=0}^{\infty} a_n x^n = a_0+a_1x+a_2x^2+a_3x^3+\cdots$

Given: $a_{n+2}-2a_{n+1}+a_n=2^n$

Multiply by x^n on both side

$$a_{n+2}\cdot x^n-2a_{n+1}\cdot x^n+a_n\cdot x^n=2^n\cdot x^n$$

Multiply by $\sum_{n=0}^{\infty}$ on both side.

$$\sum_{n=0}^{\infty} a_{n+2}\cdot x^n-\sum_{n=0}^{\infty}2a_{n+1}\cdot x^n+\sum_{n=0}^{\infty}a_n\cdot x^n=\sum_{n=0}^{\infty}2^n\cdot x^n$$

W.K.T

$$G(x)=\sum_{n=0}^{\infty}a_nx^n=a_0x^0+a_1x^1+a_2x^2+a_3x^3+\cdots$$
$$=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

$$[a_2\cdot x^0+a_3x^1+a_4x^2+\cdots]-2[a_1\cdot x^0+a_2x^1+a_3x^2+\cdots]+[a_0\cdot x^0+a_1x^1+a_2x^2+\cdots]=[2^0x^0+2^1x^1+2^2x^2+\cdots]$$

x^2/x^2 $\quad$ x/x

Multiply by $\frac{x^2}{x^2}$ & $\frac{x}{x}$

$$\left[\frac{a_2x^2+a_3x^3+a_4x^4}{x^2}\right]-2\left[\frac{a_1x^1+a_2x^2+a_3x^3+\cdots}{x}\right]+\underbrace{G(x)}=\underbrace{[1+2x+(2x)^2+(2x)^3+\cdots]}$$

Binamial formula. $(1-2x)^{-1}$

$$\left(\frac{G(x)-a_0-a_1x}{x^2}\right)-2\left(\frac{G(x)-a_0}{x}\right)+G(x)=(1-2x)^{-1}$$

Take L.CM

$$\left(\frac{G(x)-a_0-a_1x}{x^2}\right)-2x\left(\frac{G(x)-a_0}{x^2}\right)+\frac{x^2G(x)}{x^2}=(1-2x)^{-1}$$

$$a_0=2,\ a_1=1$$

$$\left(\frac{G(x)-2-x}{x^2}\right)-2x\left(\frac{G(x)-2}{x^2}\right)+\frac{x^2G(x)}{x^2}=\frac{1}{(1-2x)}$$

$$G(x)-2-x-2xG(x)+4x+x^2G(x)=\frac{x^2}{(1-2x)}$$

$$G(x)\left[1-2x+x^2\right]+3x-2=\frac{x^2}{(1-2x)}$$

$$G(x)\left[x^2-2x+1\right]+3x-2=\frac{x^2}{(1-2x)}$$

$$G(x)(1-x)^2+3x-2=\frac{x^2}{(1-2x)}$$

1
-1 -1
-2

$$G(x)(1-x)^2=\frac{x^2}{(1-2x)}-3x+2$$

$$G(x)=\frac{x^2}{(1-2x)(1-x)^2}-3\frac{x}{(1-x)^2}+\frac{2}{(1-x)^2}$$

$$\frac{x^2}{(1-2x)(1-x)^2}=\frac{A}{(1-2x)}+\frac{B}{(1-x)}+\frac{C}{(1-x)^2}$$

$$\frac{x^2}{(1-2x)\cancel{(1-x)^2}}=\frac{A(1-x)^2+B(1-x)(1-2x)+C(1-2x)}{(1-2x)\cancel{(1-x)^2}}$$

$$x^2=A(1-x)^2+B(1-x)(1-2x)+C(1-2x)$$

put $x=1$

$$1=A(1-1)^2+B(1-1)(1-2(1))+C(1-2(1))$$

$$1=0+0+C(-1)$$

$$\boxed{C=-1}$$

put $x=0$

$0 = A+B+C \Rightarrow A+B = -C \Rightarrow \boxed{A+B=1} \Rightarrow (1)$

put $x=2$

$4 = A + B(-1)(1-4) + C(1-4)$

$4 = A + 3B + C(-3)$

$4 = A + 3B + 3$

$\boxed{A+3B=1} \Rightarrow (2)$

$(2) \Rightarrow A + 3B = 1$

$(1) \Rightarrow A + B = 1$

$(-) \quad (-) \quad (-)$

$2B = 0$

$B = 0.$

$\boxed{B=0}, \Rightarrow A+B=1$

$\boxed{A=1}$

$$\therefore \frac{x^2}{(1-2x)(1-x)^2} = \frac{1}{1-2x} + \frac{0}{1-x} - \frac{1}{(1-x)^2}$$

$$= \frac{1}{1-2x} - \frac{1}{(1-x)^2} \;//.$$

$$G(x) \Rightarrow \frac{x^2}{(1-2x)(1-x)^2} - 3\frac{x}{(1-x)^2} + \frac{2}{(1-x)^2}$$

$$= \frac{1}{1-2x} - \frac{1}{(1-x)^2} - \frac{3x}{(1-x)^2} + \frac{2}{(1-x)^2}$$

$$= \frac{1}{1-2x} + \frac{1}{(1-x)^2} - \frac{3x}{(1-x)^2}$$

$$= (1-2x)^{-1} + (1-x)^{-2} - 3x(1-x)^{-2} \quad \cancel{x(1-x)}^{-2}$$

$$G(x) = [1 + 2x + (2x)^2 + \dots] + [1 + 2x + 3x^2 + \dots] - 3x[1 + 2x + 3x^2 + \dots]$$

$$G(x) = \sum_{n=0}^{\infty} 2^n \cdot x^n + \sum_{n=0}^{\infty} (n+1)x^n - 3\sum_{n=0}^{\infty} n x^n$$

$$= \sum_{n=0}^{\infty} (2^n + (n+1) - 3n)\, x^n.$$

Hence, $a_n = 2^n + (n+1) - 3n.$

$= 1 - 2n + 2^n \;//.$