Medical Robotics

Some definitions

Sivakumar Balasubramanian
Department of Bioengineering
Christian Medical College Vellore

What is a Robot?

Coming soon.

Configuration Space: Describing the state of a robot

Configuration Space or C-space

Degrees of Freedom

Coming soon.

Task space and Workspace

Coming soon.

Mathematical Preliminary

Concepts in Vector Spaces

Vectors

- **Vectors** are ordered list of numbers (scalars). $\mathbf{v} = \begin{bmatrix} 1.2 \\ -0.1 \\ \vdots \\ -1.24 \end{bmatrix}$.

  Note: Small bold letter will represent vectors. e.g. $\mathbf{a}, \mathbf{x}, \ldots$

- Scalars can be any *field* $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$. Scalars will be represented using lower case normal font, e.g. $x, y, \alpha, \beta, \ldots$

- Addition/multiplication operations performed on vectors will follow the rules of addition/multiplication of the corresponding scalar fields.

- We will typically encounter only $\mathbb{R}$ and $\mathbb{C}$ in this course.

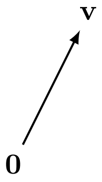
Vectors

- Individual elements of a vector $\mathbf{v}$ are indexed. The i^{th} element of $\mathbf{v}$ is referred to as v_i.
- *Dimension* or *size* of a vector is number of elements in the vector.
- Set of n-real vectors is denoted by $\mathbb{R}^n$ (similarly, $\mathbb{C}^n$)
- Vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, if
  - both have the same size; and
  - $a_i = b_i, i \in \{1, 2, 3, \dots n\}$

Vectors

- **Unit vector** $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **Zero vector** $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **One vector** $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

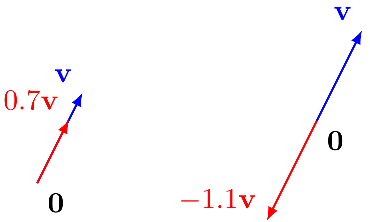
- Geometrically, real n-vectors can be thought of as points in $\mathbb{R}^n$ space.



Vectors

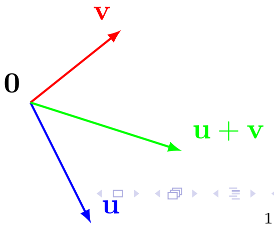
- **Vector scaling**: Multiplication of a scalar and a vector.

$$\mathbf{w} = a\mathbf{v} = a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ av_3 \\ \vdots \\ av_n \end{bmatrix} \quad a \in \mathbb{R};\ \mathbf{w}, \mathbf{v} \in \mathbb{R}^n$$



- **Vector addition**: Adding two vectors of the same dimension, element by element.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$



Vector spaces

- A set of vectors V that is closed under **vector addition** and **vector scaling**.

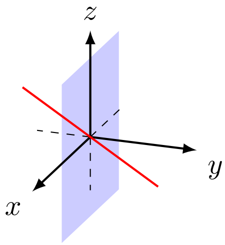
$$\forall \mathbf{x}, \mathbf{y} \in V, \quad \mathbf{x} + \mathbf{y} \in V$$

$$\forall \mathbf{x} \in V, \text{ and } \alpha \in F, \quad \alpha \mathbf{x} \in V$$

- A **subspace** S of a vector space V is a subset of V and is itself a vector space.

$$S \subset V, \quad \forall \mathbf{x}, \mathbf{y} \in S, \alpha \mathbf{x} + \beta \mathbf{y} \in S, \quad \alpha, \beta \in F$$

- For example, in $\mathbb{R}^3$ all planes and lines passing through the origin are subspaces of $\mathbb{R}^3$.



Linear independence

A collection of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots \mathbf{x}_n\}$, $\mathbf{x}_i \in \mathbb{R}^m \;\; i \in \{1, 2, 3, \dots n\}$ is called *linearly dependent* if,

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0, \text{ hold for some } \alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{R}, \text{ such that } \exists \alpha_i \neq 0$$

Another way to state this: A collection of vectors is *linearly dependent* if at least one of the vectors in the collection can be expressed as a linear combination of the other vectors in the collection, i.e.

$$\mathbf{x}_i = -\sum_{j=1, j\neq i}^{n} \left(\frac{\alpha_j}{\alpha_i}\right) \mathbf{x}_j$$

A collection of vectors is *linearly independent* if it is **not** *linearly dependent.*

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 \dots = \alpha_n = 0$$

Span of a set of vectors

- Consider a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \dots \mathbf{v}_r\}$ where $\mathbf{v}_i \in \mathbb{R}^n, 1 \leq i \leq r$.
- The **span** of the set S is defined as the set of all linear combination of the vectors $\mathbf{v}_i$,

$$span\,(S) = \{\alpha_1\mathbf{v}_1 = \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r\}\,,\ \alpha_i \in \mathbb{R}$$

- Is $span\,(S)$ a subspace of $\mathbb{R}^n$?
- We say that the subspace $span\,(S)$ is spanned by the *spanning set* S. $\longrightarrow$ S *spans* $span\,(S)$.
- **Sum of subspaces** X, Y is defined as the sum of all possible vectors from X and Y.

$$X + Y = \{\mathbf{x} + \mathbf{y} \,|\, \mathbf{x} \in X, \mathbf{y} \in Y\}$$

- Sum of two subspace is also a subspace.

Standard Inner Product

Standard inner product is defined as the following,

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

For complex vectors: $\mathbf{x}^* \mathbf{y} = \sum_{i=1}^{n} \overline{x}_i y_i, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$

Orthogonality. Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ are orthogonal or perpendicular to each other if,

$$\mathbf{x}^\top \mathbf{y} = 0 \iff \mathbf{x} \perp \mathbf{y}$$

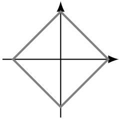
Norm of a vector

Norm is a measure of the size or length of a vector.

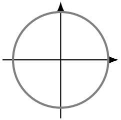
Euclidean norm of a n-vector $\mathbf{x} \in \mathbb{R}^n$ is defined as,
$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}$.

$\|\mathbf{x}\|_2$ is a measure of the length of the vector $\mathbf{x}$.

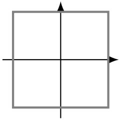
p-norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$



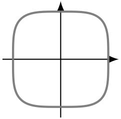
$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$$



$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$$



$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$



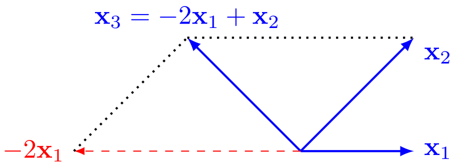
$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$$

Basis of a vector space

Consider a vector $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. What can we say about the coefficients α_is when the collection $\{\mathbf{x}_i\}_{i=1}^{n}$ is,

- linearly independent $\implies$ α_is are *unique*.
- linearly dependent $\implies$ α_is are not *unique*.

Consider $\mathbb{R}^2$ vector space. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

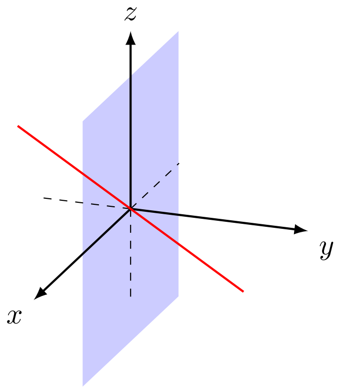


Independence-Dimension inequality: What is the maximum possible size of a linearly independent collection?

A linearly independent collection of n-vectors can at most have n vectors.

Basis of a vector space

How many vectors can we choose from the following vectors spaces before the set becomes linearly dependent?



Basis of a vector space

- A linearly independent set of n-vectors from $\mathbb{R}^n$, of size n, is called a *basis* for $\mathbb{R}^n$.

- Any n-vector from $\mathbb{R}^n$ can be represented as a *unique* linear combination of the elements of the basis.

- Consider the basis $\{\mathbf{x}_i\}_{i=1}^n$, $\mathbf{x}_i \in \mathbb{R}^n$. Any vector $\mathbf{y} \in \mathbb{R}^n$ can be represented as a linear combination of $\mathbf{x}_i$s, $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$. This is called the *expansion of* $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^n$ basis.

Basis of a vector space

- The numbers α_i are called the *coefficients* of the expansion of $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^n$ basis.

- **Orthogonal vectors**: A set of vectors $\{\mathbf{x}_i\}_{i=1}^n$ is *(mutually) orthogonal* if $\mathbf{x}_i \perp \mathbf{x}_j$ for all $i, j \in \{1, 2, 3, \dots n\}$ and $i \neq j$.

- This set is called **orthonormal** if its elements are all of unit length $\|\mathbf{x}_i\|_2 = 1$ for all $i \in \{1, 2, 3, \dots n\}$.

$$\mathbf{x}_i^\top \mathbf{x}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Representing a Vector in an Orthonormal Basis

- An orthonormal collection of vectors is linearly independent.

- Consider an orthonormal basis $\{\mathbf{x}_i\}_{i=1}^{n}$. The expansion of a vector $\mathbf{y}$ is given by,

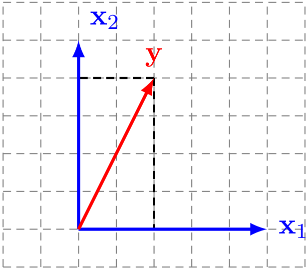
$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 + \cdots + \alpha_n \mathbf{x}_n$$

$$\mathbf{x}_i^\top \mathbf{y} = \alpha_1 \mathbf{x}_i^\top \mathbf{x}_1 + \alpha_2 \mathbf{x}_i^\top \mathbf{x}_2 + \alpha_3 \mathbf{x}_i^\top \mathbf{x}_3 + \cdots + \alpha_n \mathbf{x}_i^\top \mathbf{x}_n = \alpha_i$$

Representing a Vector in an Orthonormal Basis

- Thus, we can rewrite this as,

$$\mathbf{y} = \left(\mathbf{y}^\top \mathbf{x}_1\right) \mathbf{x}_1 + \left(\mathbf{y}^\top \mathbf{x}_2\right) \mathbf{x}_2 + \left(\mathbf{y}^\top \mathbf{x}_3\right) \mathbf{x}_3 + \cdots + \left(\mathbf{y}^\top \mathbf{x}_n\right) \mathbf{x}_n$$



Dimension of a Vector Space

- There an infinite number of bases for a vector space.
- There is one thing that is common among all these bases – the number of bases vectors.
- This number is a property of the vector space, and represents the "degrees of freedom" of the space. This is called the **dimension** of the vector space.
- A subspace of dimension m can have at most m independent vectors.
- Notice that the word "dimension" of a vector space is different from the "dimension" of a vector.
- E.g. Vectors from $\mathbb{R}^3$ are three dimensional vectors. But the yz-plane in $\mathbb{R}^3$ is a 2 dimensional subspace of $\mathbb{R}^3$.