

# Medical Robotics

## Rigid Body Kinematics

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# What is a rigid body?

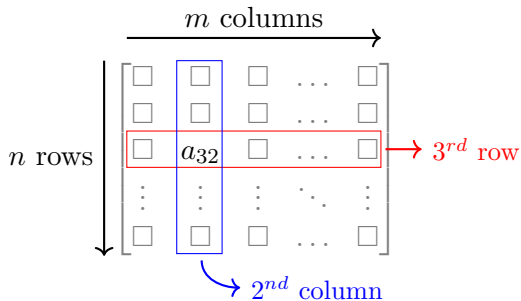
Coming soon.

# Mathematical Preliminary

## Matrices

# Matrices

- **Matrices** are rectangular array of numbers.  $\begin{bmatrix} 1.1 & -24 & \sqrt{2} \\ 0 & 1.12 & -5.24 \end{bmatrix}$



- Consider a matrix  $A$  with  $n$  rows and  $m$  columns.  $\begin{cases} \text{Tall/Skinny} & n > m \\ \text{Square} & n = m \\ \text{Wide/Fat} & n < m \end{cases}$

# Matrices

- ▶  $n$ -vectors can be interpreted as  $n \times 1$  matrices. These are called *column vectors*.
- ▶ A matrix with only one row is called a *row vector*, which can be referred to as  $n$ -row-vector.  $\mathbf{x} = [1.45 \quad -3.1 \quad 12.4]$
- ▶ **Block matrices & Submatrices:**  $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$ . What are the dimensions of the different matrices?

# Matrices

- ▶ Matrices are also compact way to give a set of indexed column  $n$ -vectors,  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \dots \mathbf{x}_m$ .

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \dots \quad \mathbf{x}_m]$$

- ▶ **Zero matrix**  $= \mathbf{0}_{n \times m} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

- ▶ **Identity matrix** is a square  $n \times n$  matrix with all zero elements, except the diagonals where all elements are 1.

$$i_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

# Matrices

- **Diagonal matrices** is a square matrix with non-zero elements on its diagonal.

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & -11 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 9.3 \end{bmatrix} = \mathbf{diag}(0.4, -11, 21, 9.3)$$

- **Triangular matrices:** Are square matrices. *Upper triangular*  $a_{ij} = 0, \forall i > j$ ;  
*Lower triangular*  $a_{ij} = 0, \forall i < j$ .

# Matrix operations: Transpose

- **Transpose** switches the rows and columns of a matrix.  $\mathbf{A}$  is a  $n \times m$  matrix, then its transpose is represented by  $\mathbf{A}^\top$ , which is a  $m \times n$  matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \implies \mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Transpose converts between column and row vectors.

What is the transpose of a block matrix?  $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$



# Matrix operations: Matrix Addition

- ▶ **Matrix addition** can only be carried out with matrices of same size. Like vectors we perform element wise addition.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

- ▶ **Properties of matrix addition:**

- ▶ *Commutative:*  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- ▶ *Associative:*  $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- ▶ *Addition with zero matrix:*  $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
- ▶ *Transpose of sum:*  $(\mathbf{A} + \mathbf{B})^{\top} = \mathbf{A}^{\top} + \mathbf{B}^{\top}$

# Matrix operations: Scalar multiplication

- **Scalar multiplication** Each element of the matrix gets multiplied by the scalar.

$$\alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

- We will mostly only deal with matrices with real entries. Such matrices are elements of the set  $\mathbb{R}^{n \times m}$ .
- Given the aforementioned matrix operations and their properties, is  $\mathbb{R}^{n \times m}$  a vector space?

# Matrix operations: Matrix multiplication

- ▶ A useful multiplication operation can be defined for matrices.
- ▶ It is possible to *multiply* two matrices  $\mathbf{A} \in \mathbb{R}^{n \times p}$  and  $\mathbf{B} \in \mathbb{R}^{p \times m}$  through this *matrix multiplication* procedure.
- ▶ The product matrix  $\mathbf{C} := \mathbf{AB} \in \mathbb{R}^{n \times m}$ , if the number of columns of  $\mathbf{A}$  is equal to the number of rows of  $\mathbf{B}$ .

$$c_{ij} := \sum_{k=1}^p a_{ik} b_{kj} \quad \forall i \in \{1, \dots, n\} \quad , j \in \{1 \dots m\}$$

# Matrix multiplication

- *Inner product* is a special case of matrix multiplication between a *row vector* and a *column vector*.

$$\mathbf{x}^\top \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n x_i y_i$$

# Matrix multiplication: Post-multiplication by a column vector

- Consider a matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$  and a  $m$ -vector  $\mathbf{x} \in \mathbb{R}^m$ . We can multiply  $\mathbf{A}$  and  $\mathbf{x}$  to obtain  $\mathbf{y} = \mathbf{Ax} \in \mathbb{R}^n$ .

$$\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i}x_i \\ \sum_{i=1}^m a_{2i}x_i \\ \vdots \\ \sum_{i=1}^m a_{ni}x_i \end{bmatrix} = \sum_{i=1}^m x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = \sum_{i=1}^m x_i \mathbf{a}_i$$

- Post-multiplying a matrix  $\mathbf{A}$  by a column vector  $\mathbf{x}$  results in a linear combination of the columns of matrix  $\mathbf{A}$ .
- $\mathbf{x}$  provides the column mixture.

# Matrix multiplication: Pre-multiplication by a row vector

- ▶ Let  $\mathbf{x}^\top \in \mathbb{R}^n$  and  $\mathbf{A} \in \mathbb{R}^{n \times m}$ , then  $\mathbf{y} = \mathbf{x}^\top \mathbf{A}$ .

$$\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i a_{i1} & \dots & \sum_{i=1}^n x_i a_{im} \end{bmatrix} = \sum_{i=1}^n x_i \tilde{\mathbf{a}}_i^\top$$

where,  $\tilde{\mathbf{a}}_i^\top = \begin{bmatrix} a_{i1} & \dots & a_{im} \end{bmatrix}$

- ▶ Pre-multiplying a matrix  $\mathbf{A}$  by a row vector  $\mathbf{x}$  results in a linear combination of the rows of  $\mathbf{A}$ .
- ▶  $\mathbf{x}^\top$  provides the row mixture.

# Matrix multiplication

- Multiplying two matrices  $\mathbf{A} \in \mathbb{R}^{n \times p}$  and  $\mathbf{B} \in \mathbb{R}^{p \times m}$  produces  $\mathbf{C} \in \mathbb{R}^{n \times m}$ ,

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{n2} & \dots & c_{nm} \end{bmatrix}$$

- **Four interpretations of matrix multiplication.**

1. Inner-Product interpretation
2. Column interpretation
3. Row interpretation
4. Outer product interpretation.

# Matrix multiplication: Inner-product Interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \mathbf{B} \in \mathbb{R}^{p \times m}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- $ij^{th}$  element of  $\mathbf{C}$  is the inner product of the  $i^{th}$  row of  $\mathbf{A}$  and the  $j^{th}$  column of  $\mathbf{B}$ .

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj} = \tilde{\mathbf{a}}_i^\top \mathbf{b}_j$$

where,  $i \in \{1 \dots n\}, j \in \{1 \dots m\}$



# Matrix multiplication: Column interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \mathbf{B} \in \mathbb{R}^{p \times m}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- Columns of  $\mathbf{C}$  are the linear combinations of the columns of  $\mathbf{A}$ .

$$\mathbf{C} = \mathbf{A} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \dots & \mathbf{Ab}_m \end{bmatrix}$$

- $j^{th}$  column of  $\mathbf{C}$  is the linear combination of the columns of  $\mathbf{A}$

$$\mathbf{c}_j = \sum_{k=1}^p b_{kj} \mathbf{a}_k$$

# Matrix multiplication: Row interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \quad \mathbf{B} \in \mathbb{R}^{p \times m}, \quad \mathbf{C} \in \mathbb{R}^{n \times m}$$

- Rows of  $\mathbf{C}$  are the linear combinations of the rows of  $\mathbf{B}$ .

$$\mathbf{C} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \\ \tilde{\mathbf{a}}_2^\top \\ \vdots \\ \tilde{\mathbf{a}}_n^\top \end{bmatrix} \mathbf{B} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \mathbf{B} \\ \tilde{\mathbf{a}}_2^\top \mathbf{B} \\ \vdots \\ \tilde{\mathbf{a}}_n^\top \mathbf{B} \end{bmatrix}$$

- $i^{th}$  row of  $\mathbf{C}$  is the linear combination of the rows of  $\mathbf{B}$

$$\tilde{\mathbf{c}}_i^\top = \sum_{k=1}^p a_{ik} \tilde{\mathbf{b}}_k^\top$$

# Matrix multiplication: Outer product interpretation

- **Outer product:** Product between a column vector and a row vector. Let  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^m$ . The *outer product* is defined as,

$$\mathbf{xy}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_m \\ x_2y_1 & x_2y_2 & \cdots & x_2y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_m \end{bmatrix} \in \mathbb{R}^{n \times m}$$

# Matrix multiplication: Outer product interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \mathbf{B} \in \mathbb{R}^{p \times m}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- $\mathbf{C}$  can be written as the sum of  $p$  outer products of columns of  $\mathbf{A}$  and rows of  $\mathbf{B}$ .

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \dots & \mathbf{a}_p \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \\ \tilde{\mathbf{b}}_2^\top \\ \tilde{\mathbf{b}}_3^\top \\ \vdots \\ \tilde{\mathbf{b}}_p^\top \end{bmatrix} = \sum_{i=1}^p \mathbf{a}_i \tilde{\mathbf{b}}_i^\top$$

# Properties of matrix multiplication

- ▶ **Not commutative:**  $\mathbf{AB} \neq \mathbf{BA}$

The product of two matrices might not always be defined. When it is defined,  $\mathbf{AB}$  and  $\mathbf{BA}$  need not match.

- ▶ **Distributive:**  $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$  and  $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

- ▶ **Associative:**  $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

- ▶ **Transpose:**  $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$

- ▶ **Scalar product:**  $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$

# Linear transformations

- ▶ Linear functions  $f : \mathbb{R}^m \mapsto \mathbb{R}$ ,

$$y = f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}; \quad \mathbf{w}, \mathbf{x} \in \mathbb{R}^m, \quad y \in \mathbb{R}$$

- ▶ Generalization of the linear function is when its range  $\mathbb{R}^n$ :

$$\mathbf{y} = f(\mathbf{x}); \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{y} \in \mathbb{R}^n$$

- ▶ These can be represented as,  $\mathbf{y} = \mathbf{A}\mathbf{x}$ ,  $\mathbf{A} \in \mathbb{R}^{n \times m}$ .

- ▶ Matrices can be thought of as representing a particular linear transformation.

# Why does matrix multiplication have this strange definition?

Consider the following two functions,

$$\mathbf{y} = f(\mathbf{x}) = \mathbf{A}\mathbf{x} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{v} = g(\mathbf{u}) = \mathbf{B}\mathbf{u} \longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{z} = h(\mathbf{u}) = f(g(\mathbf{u})) &= f\left(\begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix}\right) = \begin{bmatrix} a\alpha u_1 + a\beta u_2 + b\gamma u_1 + b\delta u_2 \\ c\alpha u_1 + c\beta u_2 + d\gamma u_1 + d\delta u_2 \end{bmatrix} \\ &= \begin{bmatrix} (a\alpha + b\gamma) u_1 + (a\beta + b\delta) u_2 \\ (c\alpha + d\gamma) u_1 + (c\beta + d\delta) u_2 \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

$$\mathbf{z} = \mathbf{A}(\mathbf{B}\mathbf{u}) = (\mathbf{A}\mathbf{B})\mathbf{u} \implies \mathbf{A}\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

Matrix multiplication represents the composition of linear transformations.

# Rank of a matrix $\mathbf{A}$

- **Rank of a matrix  $\mathbf{A}$ :** dimension of the subspace spanned by the columns of  $\mathbf{A}$  or the rows of  $\mathbf{A} \in \mathbb{R}^{n \times m}$ .

$$\begin{aligned} \text{rank}(\mathbf{A}) &= \dim \text{span}(\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}) \rightarrow \text{Column rank} \\ &= \dim \text{span}\left(\left\{\tilde{\mathbf{a}}_1^\top, \tilde{\mathbf{a}}_2^\top, \dots, \tilde{\mathbf{a}}_n^\top\right\}\right) \rightarrow \text{Row rank} \end{aligned}$$

- Column Rank is always equal to the row rank.
- Rank tells us the number of independent columns/row in the matrix.
- **Full rank matrix  $\mathbf{A}$ :**  $\text{rank}(\mathbf{A}) = \min(n, m)$   
**Rank deficient matrix  $\mathbf{A}$ :**  $\text{rank}(\mathbf{A}) < \min(n, m)$



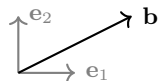
# Matrix Inverse

- ▶ Consider the square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .  $\mathbf{B} \in \mathbb{R}^{n \times n}$  is the inverse of  $\mathbf{A}$ , if  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$ , and  $\mathbf{B}$  is represented as  $\mathbf{A}^{-1}$ .
- ▶ Not all matrices have inverses. A matrix with an inverse is called **non-singular**, otherwise it is called **singular**.
- ▶ For a non-singular matrix  $\mathbf{A}$ ,  $\mathbf{A}^{-1}$  is unique.  $\mathbf{A}^{-1}$  is both the left and right inverse.
- ▶ A matrix  $\mathbf{A}$  has an inverse, if and only if  $\mathbf{A}$  is full rank, i.e.  $\text{rank}(\mathbf{A}) = n$
- ▶  $\mathbf{Ax} = \mathbf{b}$  can be solved as follows,  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ . *It is never solved like this in practice.*
- ▶ Inverse of product of matrices,  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ .
- ▶  $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$  and  $(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$

# Representation of vectors in a basis

- Consider the vector space  $\mathbb{R}^n$  with basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ . Any vector in  $\mathbf{b} \in \mathbb{R}^n$  can be represented as a linear combination of  $\mathbf{v}_i$ s,

$$\mathbf{b} = \sum_{i=1}^n \mathbf{v}_i \mathbf{a}_i = \mathbf{V} \mathbf{a}; \quad \mathbf{a} \in \mathbb{R}^n, \quad \mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \in \mathbb{R}^{n \times n}$$



$\{\mathbf{v}_1, \mathbf{v}_2\}$ ,  $\{\mathbf{u}_1, \mathbf{u}_2\}$  and  $\{\mathbf{e}_1, \mathbf{e}_2\}$  are valid basis for  $\mathbb{R}^2$ , and the presentation for  $\mathbf{b}$  in each one of them is different.

# Matrix Inverse

- ▶ Consider the equation  $\mathbf{Ax} = \mathbf{y}$ , where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ .
- ▶ Assume  $\mathbf{A}$  is non-singular  $\implies$  columns of  $\mathbf{A}$ ,  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ , represent a basis for  $\mathbb{R}^n$ .
- ▶ Then  $\mathbf{x}$  represents  $\mathbf{y}$  in the basis consisting of the columns of  $\mathbf{A}$ .

$$\mathbf{y} = \mathbf{Ax} = \sum_{i=1}^n \mathbf{a}_i x_i \implies \mathbf{x} = \mathbf{A}^{-1} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \\ \tilde{\mathbf{b}}_2^\top \\ \vdots \\ \tilde{\mathbf{b}}_n^\top \end{bmatrix} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \mathbf{y} \\ \tilde{\mathbf{b}}_2^\top \mathbf{y} \\ \vdots \\ \tilde{\mathbf{b}}_n^\top \mathbf{y} \end{bmatrix}$$

- ▶  $\mathbf{A}^{-1}$  is the matrix that allows change of basis to the columns of  $\mathbf{A}$  from the standard basis!

## Some more definitions and conventions

When studying kinematics we will always assume the following:

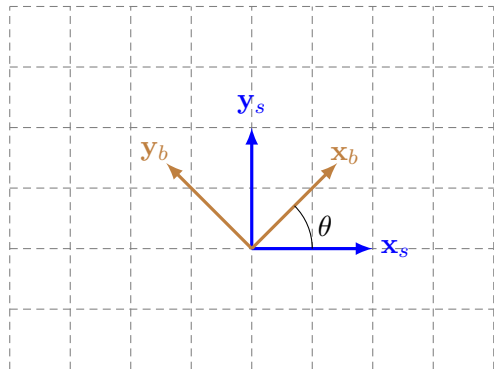
- ▶ **Fixed frame  $F$ :** This is a stationary frame of reference attached to the corner of a room, the base of a robot, etc.
- ▶ **Body frame  $B$ :** A frame attached to a rigid body of interest. E.g. frame of a robot's moving link, frame on a mobile robot's chasis, etc.
- ▶ All frames are assumed to be right-handed frames. If  $\mathbf{x}$  and  $\mathbf{y}$  are the  $x$  and  $y$  axes of a reference frame, then  $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ , where ' $\times$ ' is the cross product between two vectors.

# Rigid Body Motions in a Plane: Rotation

Let  $\{s\}$  and  $\{b\}$  be the fixed and body frames respectively. The body frame is rotated by an angle  $\theta$  with respect to the fixed frame.

The vectors  $\{\mathbf{x}_s, \mathbf{y}_s\}$  and  $\{\mathbf{x}_b, \mathbf{y}_b\}$  form an orthonormal basis for  $\{s\}$  and  $\{b\}$ , respectively.

- Representation of  $\mathbf{x}_b$  in  $\{s\}$ : 
$$\begin{bmatrix} \mathbf{x}_s^\top \mathbf{x}_b \\ \mathbf{y}_s^\top \mathbf{x}_b \end{bmatrix}$$
- Representation of  $\mathbf{y}_b$  in  $\{s\}$ : 
$$\begin{bmatrix} \mathbf{x}_s^\top \mathbf{y}_b \\ \mathbf{y}_s^\top \mathbf{y}_b \end{bmatrix}$$



# Rigid Body Motions in a Plane: Rotation

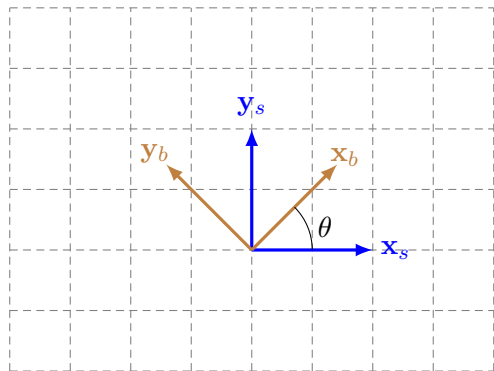
This representation can be compactly expressed as the following,

$$\begin{bmatrix} \mathbf{x}_s^\top \mathbf{x}_b & \mathbf{x}_s^\top \mathbf{y}_b \\ \mathbf{y}_s^\top \mathbf{x}_b & \mathbf{y}_s^\top \mathbf{y}_b \end{bmatrix} \triangleq \mathbf{R}_{sb}$$

From the definition of the standard inner product, we know that,

$$\mathbf{R}_{sb} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$\mathbf{R}_{sb}$  is rotation matrix representing  $\{b\}$  in  $\{s\}$ . It is parametrized by a single parameter  $\theta$ .

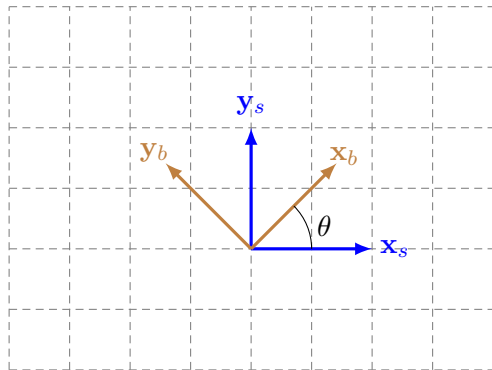


# Rigid Body Motions in a Plane: Rotation

All 2D Rotation matrices  $\mathbf{R} \in \mathbb{R}^{2 \times 2}$  are orthogonal matrices, i.e.  $\mathbf{R}^{-1} = \mathbf{R}^\top$ .

Rotation matrices have three different purposes:

- ①  $\mathbf{R}_{sb}$ : Represents the orientation of  $\{b\}$  in  $\{s\}$ .



# Rigid Body Motions in a Plane: Rotation

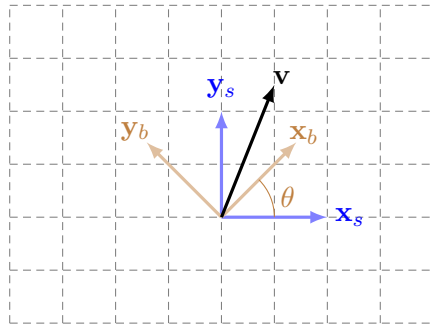
Rotation matrices have three different purposes:

② Transforming vectors between reference frames.

$$\mathbf{v}_s = \mathbf{R}_{sb} \mathbf{v}_b \quad \mathbf{v}_b = \mathbf{R}_{bs} \mathbf{v}_s = \mathbf{R}_{sb}^\top \mathbf{v}_s$$

Note that the second letter in the subscript of the rotation matrix on left gets cancelled by the first letter in the subscript of the vector on the right.

$$\mathbf{v}_l = \mathbf{R}_{lm} \mathbf{v}_m$$



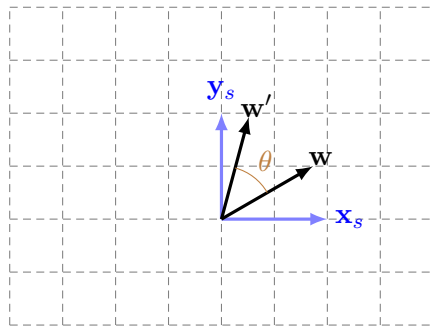


# Rigid Body Motions in a Plane: Rotation

Rotation matrices have three different purposes:

③ Rotate a vector  $\mathbf{w}_s$  by an angle  $\theta$  about the origin.

$$\mathbf{w}'_s = \mathbf{R}_{sb} \mathbf{w}_s$$



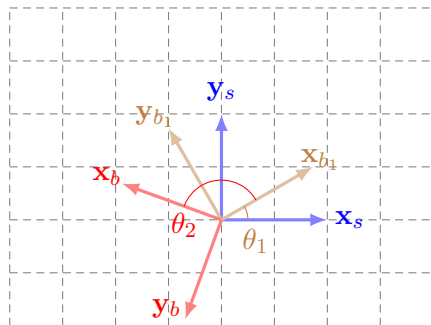
# Rigid Body Motions in a Plane: Rotation

Consider two body reference frames  $\{b_1\}$  and  $\{b_2\}$ . Let  $\mathbf{R}_{sb_1}$  be the rotation matrix representing  $\{b_1\}$  in  $\{s\}$ . And let  $\mathbf{R}_{b_1b_2}$  be the rotation matrix representing  $\{b_2\}$  in  $\{b_1\}$ . Then,

$$\mathbf{R}_{sb_2} = \mathbf{R}_{sb_1} \mathbf{R}_{b_1b_2}$$

Show that:

$$\mathbf{R}_{b_1b_2} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$



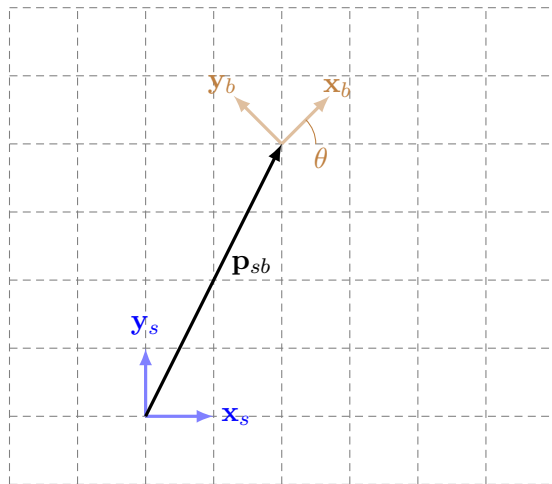
# Rigid Body Motions in a Plane: Rotation and Translation

What do we do when have the body frame  $\{b\}$  translated by a vector  $\mathbf{p}_{sb}$  and rotated by a certain angle with respect to the space frame  $\{s\}$ ?

**Note:**  $\mathbf{p}_{sb}$  is the representation of the position of the origin of  $\{b\}$  with respect to the  $\{s\}$ .

Let  $\mathbf{R}_{sb}$  be the representation of  $\{b\}$  in  $\{s\}$ . Then, the complete representation of  $\{b\}$  in  $\{s\}$  is given by the pair  $(\mathbf{R}_{sb}, \mathbf{p}_{sb})$ .

$$\mathbf{p}_{sb} = \begin{bmatrix} p_{sb_x} \\ p_{sb_y} \end{bmatrix} \quad \mathbf{R}_{sb} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$



# Rigid Body Motions in a Plane: Rotation and Translation

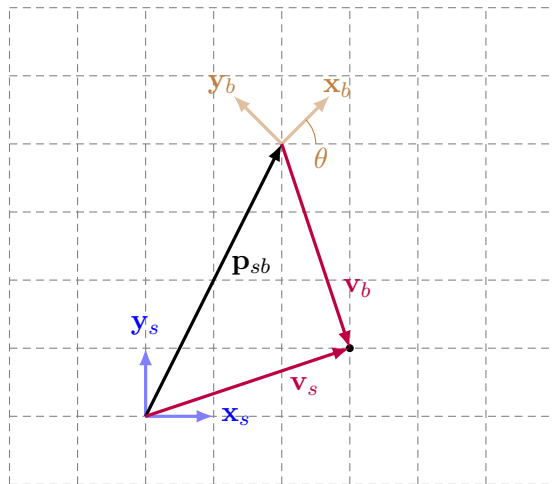
Consider the point represented by the small black circle, whose position is given by  $\mathbf{v}_s$  and  $\mathbf{v}_b$  in  $\{s\}$  and  $\{b\}$ , respectively. The transformation between  $\mathbf{v}_s$  and  $\mathbf{v}_b$  is given by ,

$$\mathbf{v}_s = \mathbf{R}_{sb}\mathbf{v}_b + \mathbf{p}_{sb}$$

$$\begin{aligned}\mathbf{v}_b &= \mathbf{R}_{sb}^\top (\mathbf{v}_s - \mathbf{p}_{sb}) \\ &= \mathbf{R}_{bs}\mathbf{v}_s - \mathbf{R}_{bs}\mathbf{p}_{sb} \\ &= \mathbf{R}_{bs}\mathbf{v}_s + \mathbf{p}_{bs}\end{aligned}$$

$(\mathbf{R}_{sb}, \mathbf{p}_{sb})$  transforms  $\{b\} \mapsto \{s\}$ .

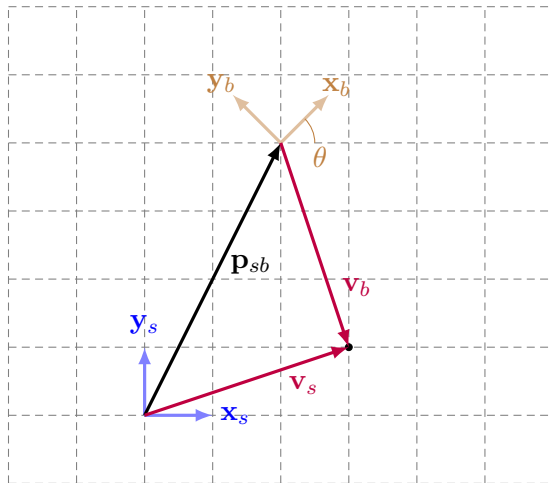
$(\mathbf{R}_{sb}^\top, -\mathbf{R}_{sb}^\top \mathbf{p}_{sb})$  transforms  $\{s\} \mapsto \{b\}$ .



# Rigid Body Motions in a Plane: Rotation and Translation

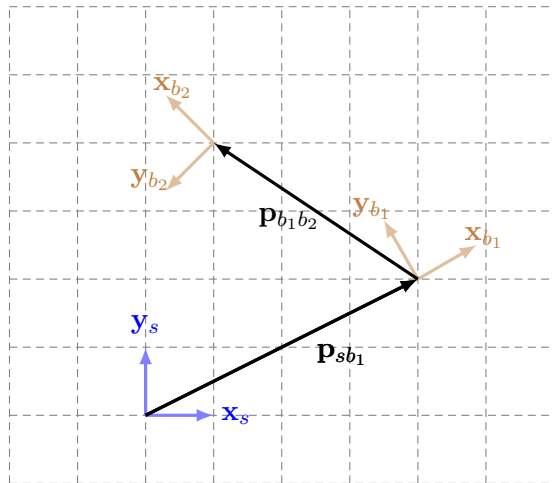
The pair  $(\mathbf{R}_{sb}, \mathbf{p}_{sb})$  allows:

1. Representation of a rotated and translated frame  $\{b\}$  w.r.t  $\{s\}$ .
2. Transformation of vectors between  $\{s\}$  and  $\{b\}$ .
3. Performs rotation and translation of a give vector.



# Rigid Body Motions in a Plane: Rotation and Translation

Find the pair  $(\mathbf{R}_{sb_2}, \mathbf{p}_{sb_2})$ , given the pairs  $(\mathbf{R}_{sb_1}, \mathbf{p}_{sb_1})$  and  $(\mathbf{R}_{b_1b_2}, \mathbf{p}_{b_1b_2})$ .



# Rigid Body Motions in a Plane: Homogenous representation

- ▶ The combined transformation of rotation and translation can be represented as a simple linear operation by using the homogenous representation for vectors and the transformations.
- ▶ For rotations and translations in  $\mathbb{R}^2$ , these rigid body transformations have a simple representation for the matrices and vectors in  $\mathbb{R}^3$ . The homogenous representation of a vector  $\mathbf{r} \in \mathbb{R}^2$  is given by,

$$\bar{\mathbf{r}} = \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

- ▶ The homogenous representation of rigid body transformation involving a rotation  $\mathbf{R} \in \mathbb{R}^{2 \times 2}$  and translation  $\mathbf{p} \in \mathbb{R}^2$  is given by,

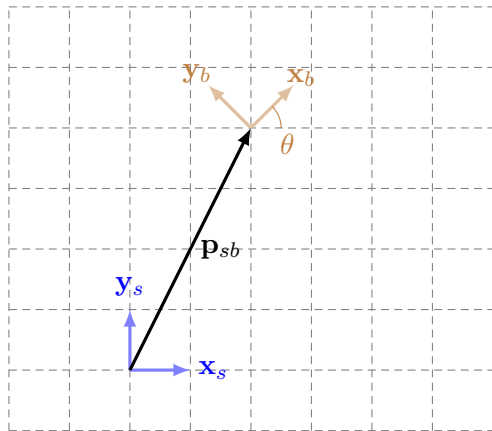
$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^3$$

# Rigid Body Motions in a Plane: Homogenous representation

$$\mathbf{p}_{sb} := \begin{bmatrix} p_{sb_x} \\ p_{sb_y} \end{bmatrix} \quad \mathbf{R}_{sb} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

The homogenous representation of the pair  $(\mathbf{R}_{sb}, \mathbf{p}_{sb})$  is given by,

$$\overline{\mathbf{H}}_{sb} = \begin{bmatrix} \mathbf{R}_{sb} & \mathbf{p}_{sb} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & p_{sb_x} \\ -\sin \theta & \cos \theta & p_{sb_y} \\ 0 & 0 & 1 \end{bmatrix}$$





# Rigid Body Motions in a Plane: Rotation and Translation

- ▶ The inverse of the rigid body transformation  $\overline{\mathbf{H}}_{sb}$  is given by,

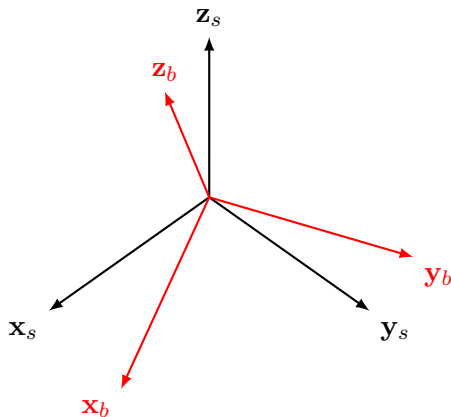
$$\overline{\mathbf{H}}_{bs} = \overline{\mathbf{H}}_{sb}^{-1} = \begin{bmatrix} \mathbf{R}_{sb}^{\top} & -\mathbf{R}_{sb}^{\top} \mathbf{p}_{sb} \\ \mathbf{0} & 1 \end{bmatrix}$$

- ▶ Verify that  $\overline{\mathbf{H}}_{sb} \overline{\mathbf{H}}_{bs} = \overline{\mathbf{H}}_{bs} \overline{\mathbf{H}}_{sb} = \mathbf{I}_3$ .
- ▶ What is the interpretation of  $-\mathbf{R}_{sb}^{\top} \mathbf{p}_{sb}$ ?
- ▶ What are the homogenous representations of a pure rotation and a pure translation?

# Rigid Body Motions in 3D: Rotation

The representation of  $\{b\}$  in  $\{s\}$  is given the following,

$$\mathbf{R}_{sb} = \begin{bmatrix} \mathbf{x}_s^\top \mathbf{x}_b & \mathbf{x}_s^\top \mathbf{y}_b & \mathbf{x}_s^\top \mathbf{z}_b \\ \mathbf{y}_s^\top \mathbf{x}_b & \mathbf{y}_s^\top \mathbf{y}_b & \mathbf{y}_s^\top \mathbf{z}_b \\ \mathbf{z}_s^\top \mathbf{x}_b & \mathbf{z}_s^\top \mathbf{y}_b & \mathbf{z}_s^\top \mathbf{z}_b \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



# Rigid Body Motions in 3D: Rotation

## Properties of rotation matrices:

- ▶ Inverse of a rotation matrix  $\mathbf{R}$  is its transpose.  $\mathbf{R}^\top = \mathbf{R}^{-1}$ . This means that the columns of a rotation matrix are orthonormal.
- ▶ The determinant of a rotation matrix is always 1.  $\det(\mathbf{R}) = 1$ .
- ▶ The set of all 3D rotation matrices form a *group* called the ***Special Orthogonal*** group called the  $SO(3)$  group.

$$SO(3) := \left\{ \mathbf{R} \in \mathbb{R}^{3 \times 3} : \mathbf{R}^\top \mathbf{R} = \mathbf{I}, \det \mathbf{R} = +1 \right\}$$

$SO(3)$  group is also referred to as the *rotation group* of  $\mathbb{R}^3$ .