Medical Robotics

Rigid Body Kinematics

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What is a rigid body?

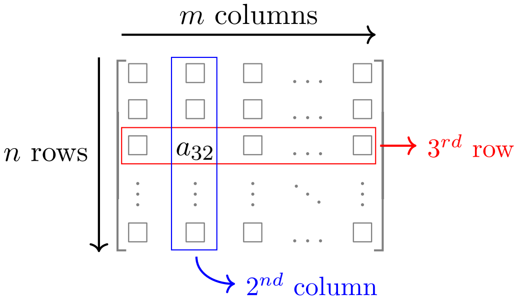
Coming soon.

Mathematical Preliminary

Matrices

Matrices

- **Matrices** are rectangular array of numbers. $\begin{bmatrix} 1.1 & -24 & \sqrt{2} \\ 0 & 1.12 & -5.24 \end{bmatrix}$



- Consider a matrix A with n rows and m columns. $\begin{cases} \textbf{Tall/Skinny} & n > m \\ \textbf{Square} & n = m \\ \textbf{Wide/Fat} & n < m \end{cases}$

Matrices

- n-vectors can be interpreted as $n \times 1$ matrices. These are called *column vectors*.

- A matrix with only one row is called a *row vector*, which can be referred to as n-row-vector. $\mathbf{x} = \begin{bmatrix} 1.45 & -3.1 & 12.4 \end{bmatrix}$

- **Block matrices & Submatrices**: $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$. What are the dimensions of the different matrices?

Matrices

- Matrices are also compact way to give a set of indexed column n-vectors, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \dots \mathbf{x}_m$.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \dots & \mathbf{x}_m \end{bmatrix}$$

- **Zero matrix**$= \mathbf{0}_{n \times m} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$
- **Identity matrix** is a square $n \times n$ matrix with all zero elements, except the diagonals where all elements are 1.

$$i_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}$$

Matrices

- **Diagonal matrices** is a square matrix with non-zero elements on its diagonal.

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & -11 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 9.3 \end{bmatrix} = \mathbf{diag}\,(0.4, -11, 21, 9.3)$$

- **Triangular matrices**: Are square matrices. *Upper triangular* $a_{ij} = 0, \forall i > j$; *Lower triangular* $a_{ij} = 0, \forall i < j$.

Matrix operations: Transpose

- **Transpose** switches the rows and columns of a matrix. $\mathbf{A}$ is a $n \times m$ matrix, then its transpose is represented by $\mathbf{A}^\top$, which is a $m \times n$ matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \implies \mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Transpose converts between column and row vectors.

What is the transpose of a block matrix? $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$

Matrix operations: Matrix Addition

- **Matrix addition** can only be carried out with matrices of same size. Like vectors we perform element wise addition.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

- **Properties of matrix addition**:
  - *Commutative*: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - *Associative*: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
  - *Addition with zero matrix*: $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
  - *Transpose of sum*: $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$

Matrix operations: Scalar multiplication

- **Scalar multiplication** Each element of the matrix gets multiplied by the scalar.

$$\alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

- We will mostly only deal with matrices with real entries. Such matrices are elements of the set $\mathbb{R}^{n \times m}$.

- Given the aforementioned matrix operations and their properties, is $\mathbb{R}^{n \times m}$ a vector space?

Matrix operations: Matrix multiplication

- A useful multiplication operation can be defined for matrices.

- It is possible to *multiply* two matrices $\mathbf{A} \in \mathbb{R}^{n \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times m}$ through this *matrix multiplication* procedure.

- The product matrix $\mathbf{C} := \mathbf{AB} \in \mathbb{R}^{n \times m}$, if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

$$c_{ij} := \sum_{k=1}^{p} a_{ik} b_{kj} \quad \forall i \in \{1, \ldots n\} \quad , j \in \{1 \ldots m\}$$

Matrix multiplication

- *Inner product* is a special case of matrix multiplication between a *row vector* and a *column vector.*

$$\mathbf{x}^\top \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Matrix multiplication: Post-multiplication by a column vector

- Consider a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ and a m-vector $\mathbf{x} \in \mathbb{R}^m$. We can multiply $\mathbf{A}$ and $\mathbf{x}$ to obtain $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^n$.

$$\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i}x_i \\ \sum_{i=1}^m a_{2i}x_i \\ \vdots \\ \sum_{i=1}^m a_{ni}x_i \end{bmatrix} = \sum_{i=1}^{m} x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = \sum_{i=1}^{m} x_i \mathbf{a}_i$$

- Post-multiplying a matrix $\mathbf{A}$ by a column vector $\mathbf{x}$ results in a linear combination of the columns of matrix $\mathbf{A}$.

- $\mathbf{x}$ provides the column mixture.

Matrix multiplication: Pre-multiplication by a row vector

- Let $\mathbf{x}^\top \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times m}$, then $\mathbf{y} = \mathbf{x}^\top \mathbf{A}$.

$$\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i a_{i1} & \dots & \sum_{i=1}^n x_i a_{im} \end{bmatrix} = \sum_{i=1}^n x_i \tilde{\mathbf{a}}_i^\top$$

where, $\tilde{\mathbf{a}}_i^\top = \begin{bmatrix} a_{i1} & \dots & a_{im} \end{bmatrix}$

- Pre-multiplying a matrix $\mathbf{A}$ by a row vector $\mathbf{x}$ results in a linear combination of the rows of $\mathbf{A}$.

- $\mathbf{x}^\top$ provides the row mixture.

Matrix multiplication

- Multiplying two matrices $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{B} \in \mathbb{R}^{p\times m}$ produces $\mathbf{C} \in \mathbb{R}^{n\times m}$,

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{n2} & \dots & c_{nm} \end{bmatrix}$$

- **Four interpretations of matrix multiplication.**
  1. Inner-Product interpretation
  2. Column interpretation
  3. Row interpretation
  4. Outer product interpretation.

Matrix multiplication: Inner-product Interpreation

$$\mathbf{C} = \mathbf{A}\mathbf{B}, \;\; \mathbf{A} \in \mathbb{R}^{n\times p}, \mathbf{B} \in \mathbb{R}^{p\times m}, \mathbf{C} \in \mathbb{R}^{n\times m}$$

- ij^{th} element of $\mathbf{C}$ is the inner product of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$.

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = \tilde{\mathbf{a}}_i^\top \mathbf{b}_j$$

where, $i \in \{1 \dots n\}, j \in \{1 \dots m\}$

Matrix multiplication: Column interpretation

$$\mathbf{C} = \mathbf{A}\mathbf{B}, \;\; \mathbf{A} \in \mathbb{R}^{n\times p}, \, \mathbf{B} \in \mathbb{R}^{p\times m}, \, \mathbf{C} \in \mathbb{R}^{n\times m}$$

- Columns of $\mathbf{C}$ are the linear combinations of the columns of $\mathbf{A}$.

$$\mathbf{C} = \mathbf{A}\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_m\end{bmatrix} = \begin{bmatrix}\mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \dots & \mathbf{A}\mathbf{b}_m\end{bmatrix}$$

- j^{th} column of $\mathbf{C}$ is the linear combination of the columns of $\mathbf{A}$

$$\mathbf{c}_j = \sum_{k=1}^{p} b_{kj}\mathbf{a}_k$$

Matrix multiplication: Row interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \, \mathbf{B} \in \mathbb{R}^{p \times m}, \, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- Rows of $\mathbf{C}$ are the linear combinations of the rows of $\mathbf{B}$.

$$\mathbf{C} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \\ \tilde{\mathbf{a}}_2^\top \\ \dots \\ \tilde{\mathbf{a}}_n^\top \end{bmatrix} \mathbf{B} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \mathbf{B} \\ \tilde{\mathbf{a}}_2^\top \mathbf{B} \\ \dots \\ \tilde{\mathbf{a}}_n^\top \mathbf{B} \end{bmatrix}$$

- i^{th} row of $\mathbf{C}$ is the linear combination of the rows of $\mathbf{B}$

$$\tilde{\mathbf{c}}_i^\top = \sum_{k=1}^{p} a_{ik} \tilde{\mathbf{b}}_k^\top$$

Matrix multiplication: Outer product interpretation

- **Outer product**: Product between a colum vector and a row vector. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. The *outer product* is defined as,

$$\mathbf{x}\mathbf{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_m \end{bmatrix} \in \mathbb{R}^{n \times m}$$

Matrix multiplication: Outer product interpretation

$$\mathbf{C} = \mathbf{A}\mathbf{B}, \quad \mathbf{A} \in \mathbb{R}^{n\times p}, \mathbf{B} \in \mathbb{R}^{p\times m}, \mathbf{C} \in \mathbb{R}^{n\times m}$$

- $\mathbf{C}$ can be written as the sum of p outer products of columns of $\mathbf{A}$ and rows of $\mathbf{B}$.

$$\mathbf{C} = \mathbf{A}\mathbf{B} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \dots & \mathbf{a}_p \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \\ \tilde{\mathbf{b}}_2^\top \\ \tilde{\mathbf{b}}_3^\top \\ \vdots \\ \tilde{\mathbf{b}}_p^\top \end{bmatrix} = \sum_{i=1}^{p} \mathbf{a}_i \tilde{\mathbf{b}}_i^\top$$

Properties of matrix multiplication

- **Not commutative:** $\mathbf{AB} \neq \mathbf{BA}$
  The product of two matrices might not alwasys be defined. When it is defined, $\mathbf{AB}$ and $\mathbf{BA}$ need not match.

- **Distributive:** $\mathbf{A}\left(\mathbf{B}+\mathbf{C}\right) = \mathbf{AB}+\mathbf{BC}$ and $\left(\mathbf{A}+\mathbf{B}\right)\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

- **Associative:** $\mathbf{A}\left(\mathbf{BC}\right) = \left(\mathbf{AB}\right)\mathbf{C}$

- **Transpose:** $\left(\mathbf{AB}\right)^{\top} = \mathbf{B}^{\top}\mathbf{A}^{\top}$

- **Scalar product:** $\alpha\left(\mathbf{AB}\right) = \left(\alpha\mathbf{A}\right)\mathbf{B} = \mathbf{A}\left(\alpha\mathbf{B}\right)$

Linear transformations

- Linear functions $f : \mathbb{R}^m \mapsto \mathbb{R}$,

$$y = f\left(\mathbf{x}\right) = \mathbf{w}^\top \mathbf{x}; \;\; \mathbf{w}, \mathbf{x} \in \mathbb{R}^m, \;\; y \in \mathbb{R}$$

- Generalization of the linear function is when its range $\mathbb{R}^n$:

$$\mathbf{y} = f\left(\mathbf{x}\right); \;\; \mathbf{x} \in \mathbb{R}^m, \;\; \mathbf{y} \in \mathbb{R}^n$$

- These can be represented as, $\mathbf{y} = \mathbf{A}\mathbf{x}, \;\; \mathbf{A} \in \mathbb{R}^{n \times m}$.

- Matrices can be thought of as representing a particular linear transformation.

Why does matrix multiplication have this strange definition?

Consider the following two functions,

$$\mathbf{y} = f(\mathbf{x}) = \mathbf{A}\mathbf{x} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{v} = g(\mathbf{u}) = \mathbf{B}\mathbf{u} \longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{z} = h(\mathbf{u}) = f(g(\mathbf{u})) = f\left(\begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix}\right) &= \begin{bmatrix} a\alpha u_1 + a\beta u_2 + b\gamma u_1 + b\delta u_2 \\ c\alpha u_1 + c\beta u_2 + d\gamma u_1 + d\delta u_2 \end{bmatrix} \\ &= \begin{bmatrix} (a\alpha + b\gamma) u_1 + (a\beta + b\delta) u_2 \\ (c\alpha + d\gamma) u_1 + (c\beta + d\delta) u_2 \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

$$\mathbf{z} = \mathbf{A}(\mathbf{B}\mathbf{u}) = (\mathbf{A}\mathbf{B})\mathbf{u} \implies \mathbf{A}\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

Matrix multiplication represents the composition of linear transformations.

Rank of a matrix A

- **Rank of a matrix A**: dimension of the subspace spanned by the columns of **A** or the rows of $\mathbf{A} \in \mathbb{R}^{n \times m}$.

$$\begin{aligned} rank\left(\mathbf{A}\right) &= \dim span\left(\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m\}\right) \rightarrow \text{Column rank} \\ &= \dim span\left(\left\{\tilde{\mathbf{a}}_1^\top, \tilde{\mathbf{a}}_2^\top, \ldots \tilde{\mathbf{a}}_n^\top\right\}\right) \rightarrow \text{Row rank} \end{aligned}$$

- Column Rank is always equal to the row rank.

- Rank tells us the number of independent columns/row in the matrix.

- **Full rank matrix A**: $rank\left(\mathbf{A}\right) = \min\left(n, m\right)$
  Rank deficient matrix A: $rank\left(\mathbf{A}\right) < \min\left(n, m\right)$

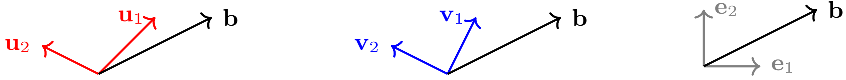
Matrix Inverse

- Consider the square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. $\mathbf{B} \in \mathbb{R}^{n \times n}$ is the inverse of $\mathbf{A}$, if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, and $\mathbf{B}$ is represented as $\mathbf{A}^{-1}$.
- Not all matrices have inverses. A matrix with an inverse is called **non-singular**, otherwise it is called **singular**.
- For a non-singular matrix $\mathbf{A}$, $\mathbf{A}^{-1}$ is unique. $\mathbf{A}^{-1}$ is both the left and right inverse.
- A matrix $\mathbf{A}$ has an inverse, if and only if $\mathbf{A}$ is full rank, i.e. $rank\left(\mathbf{A}\right) = n$
- $\mathbf{Ax} = \mathbf{b}$ can be solved as follows, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. *It is never solved like this in practice.*
- Inverse of product of matrices, $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$ and $\left(\mathbf{A}^{-1}\right)^{\top} = \left(\mathbf{A}^{\top}\right)^{-1}$

Representation of vectors in a basis

- Consider the vector space $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$. Any vector in $\mathbf{b} \in \mathbb{R}^n$ can be representated as a linear combination of $\mathbf{v}_i$s,

$$\mathbf{b} = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{a}_i = \mathbf{V}\mathbf{a}; \;\; \mathbf{a} \in \mathbb{R}^n, \;\; \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$



$\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ are valid basis for $\mathbb{R}^2$, and the presentation for $\mathbf{b}$ in each one of them is different.

Matrix Inverse

- Consider the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- Assume $\mathbf{A}$ is non-singular $\implies$ columns of $\mathbf{A}$, $\{\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n\}$, represent a basis for $\mathbb{R}^n$.
- Then $\mathbf{x}$ represents $\mathbf{y}$ in the basis consisitng of the columns of $\mathbf{A}$.

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \sum_{i=1}^{n} \mathbf{a}_i x_i \implies \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \\ \tilde{\mathbf{b}}_2^\top \\ \vdots \\ \tilde{\mathbf{b}}_n^\top \end{bmatrix} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \mathbf{y} \\ \tilde{\mathbf{b}}_2^\top \mathbf{y} \\ \vdots \\ \tilde{\mathbf{b}}_n^\top \mathbf{y} \end{bmatrix}$$

- $\mathbf{A}^{-1}$ is the matrix that allows change of basis to the columns of $\mathbf{A}$ from the standard basis!

Some more definitions and conventions

When studying kinematics we will always assume the following:

- **Fixed frame** F: This is a stationary frame of reference attached to the corner of a room, the base of a robot, etc.

- **Body frame** B: A frame attached to a rigid body of interest. E.g. frame of a robot's moving link, frame on a mobile robot's chasis, etc.

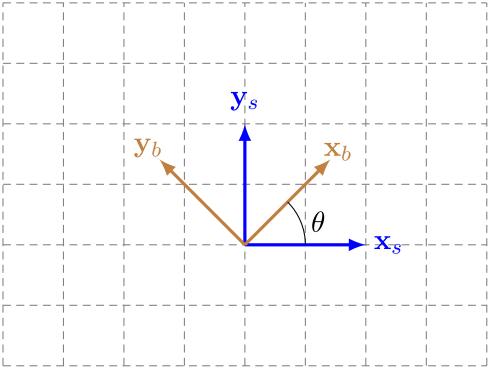
- All frames are assumed to be right-handed frames. If $\mathbf{x}$ and $\mathbf{y}$ are the x and x axes of a referene frame, then $\mathbf{z} = \mathbf{x} \times \mathbf{y}$, where '$\times$' is the cross product between two vectors.

Rigid Body Motions in a Plane: Rotation

Let $\{s\}$ and $\{b\}$ be the fixed and body frames respectively. The body frame is rotated by an angle θ with respect to the fixed frame.

The vectors $\{\mathbf{x}_s, \mathbf{y}_s\}$ and $\{\mathbf{x}_b, \mathbf{y}_b\}$ form an orthonormal basis for $\{s\}$ and $\{b\}$, respectively.

- Representation of $\mathbf{x}_b$ in $\{s\}$: $\begin{bmatrix} \mathbf{x}_s^\top \mathbf{x}_b \\ \mathbf{y}_s^\top \mathbf{x}_b \end{bmatrix}$
- Representation of $\mathbf{y}_b$ in $\{s\}$: $\begin{bmatrix} \mathbf{x}_s^\top \mathbf{y}_b \\ \mathbf{y}_s^\top \mathbf{y}_b \end{bmatrix}$



Rigid Body Motions in a Plane: Rotation

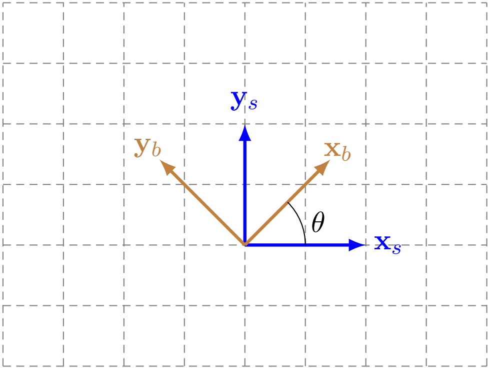
This representation can be compactly expressed as the following,

$$\begin{bmatrix} \mathbf{x}_s^\top \mathbf{x}_b & \mathbf{x}_s^\top \mathbf{y}_b \\ \mathbf{y}_s^\top \mathbf{x}_b & \mathbf{y}_s^\top \mathbf{y}_b \end{bmatrix} \triangleq \mathbf{R}_{sb}$$

From the definition of the standard inner product, we know that,

$$\mathbf{R}_{sb} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

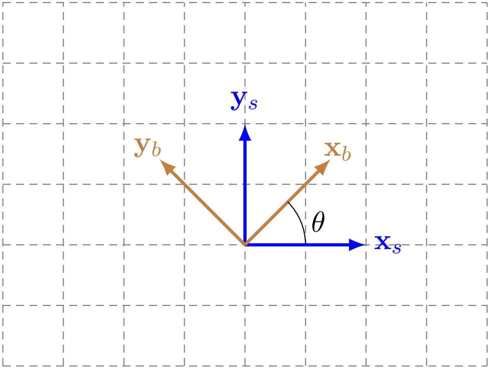
$\mathbf{R}_{sb}$ is rotation matrix representing $\{b\}$ in $\{s\}$. It is parametrized by a single parameter θ.



Rigid Body Motions in a Plane: Rotation

All 2D Rotaiton matrices $\mathbf{R} \in \mathbb{R}^{2\times 2}$ are orthogonal matrices, i.e. $\mathbf{R}^{-1} = \mathbf{R}^\top$.
Rotation matrices have three different purposes:

1 $\mathbf{R}_{sb}$: Represents the orientation of $\{b\}$ in $\{s\}$.



Rigid Body Motions in a Plane: Rotation

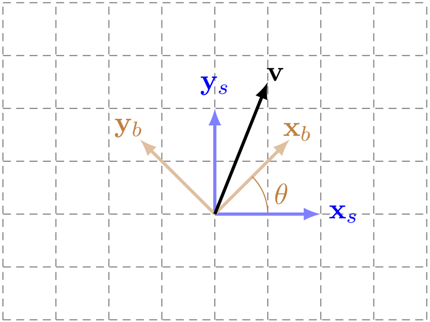
Rotation matrices have three different purposes:

(2) Transforming vectors between reference frames.

$$\mathbf{v}_s = \mathbf{R}_{sb}\mathbf{v}_b \quad \mathbf{v}_b = \mathbf{R}_{bs}\mathbf{v}_s = \mathbf{R}_{sb}^{\top}\mathbf{v}_s$$

Note that the second letter in the subscript of the rotation matrix on left gets cancelled by the first letter in the subscript of the vector on the right.

$$\mathbf{v}_l = \mathbf{R}_{l\cancel{m}}\mathbf{v}_{\cancel{m}}$$

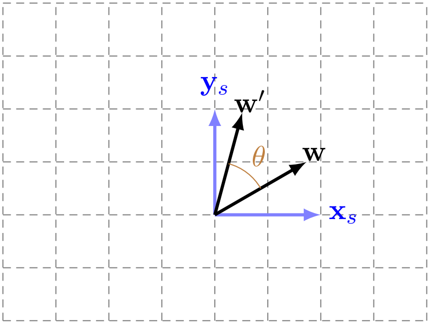


Rigid Body Motions in a Plane: Rotation

Rotation matrices have three different purposes:

3 Rotate a vector $\mathbf{w}_s$ by an angle θ about the origin.

$$\mathbf{w}'_s = \mathbf{R}_{sb}\mathbf{w}_s$$



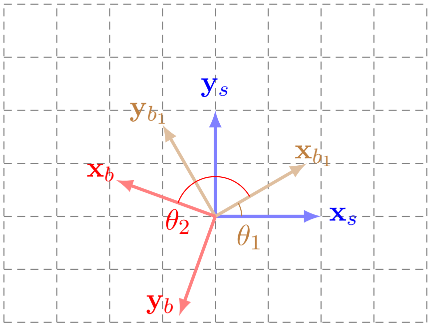
Rigid Body Motions in a Plane: Rotation

Consider two body reference frames $\{b_1\}$ and $\{b_2\}$. Let $\mathbf{R}_{sb_1}$ $\{b_1\}$ in $\{s\}$. And let $\mathbf{R}_{b_1 b_2}$ be the rotation matrix representing $\{b_2\}$ in $\{b_1\}$. Then,

$$\mathbf{R}_{sb_2} = \mathbf{R}_{s\cancel{b_1}} \mathbf{R}_{\cancel{b_1} b_2}$$

Show that:

$$\mathbf{R}_{b_1 b_2} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$



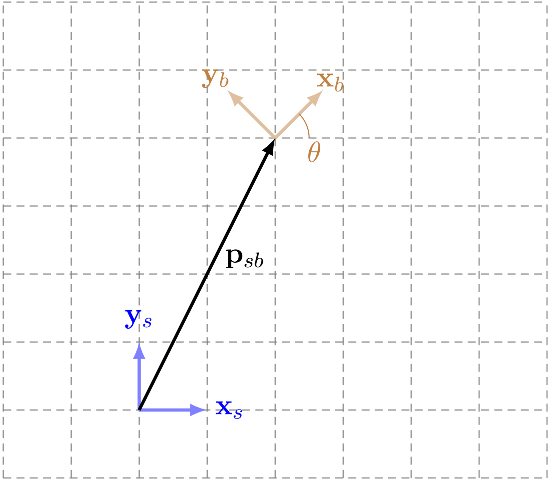
Rigid Body Motions in a Plane: Rotation and Translation

What do we do when have the body frame $\{b\}$ translated by a vector $\mathbf{p}_{sb}$ and rotated by a certain angle with repsect to the space frame $\{s\}$?

Note: $\mathbf{p}_{sb}$ is the representation of the position of the origin of $\{b\}$ with respect to the $\{s\}$.

Let $\mathbf{R}_{sb}$ be the represntation of $\{b\}$ in $\{s\}$. Then, the complete representation of $\{b\}$ in $\{s\}$ is given by the pair $(\mathbf{R}_{sb}, \mathbf{p}_{sb})$.

$$\mathbf{p}_{sb} = \begin{bmatrix} p_{sb_x} \\ p_{sb_y} \end{bmatrix} \qquad \mathbf{R}_{sb} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



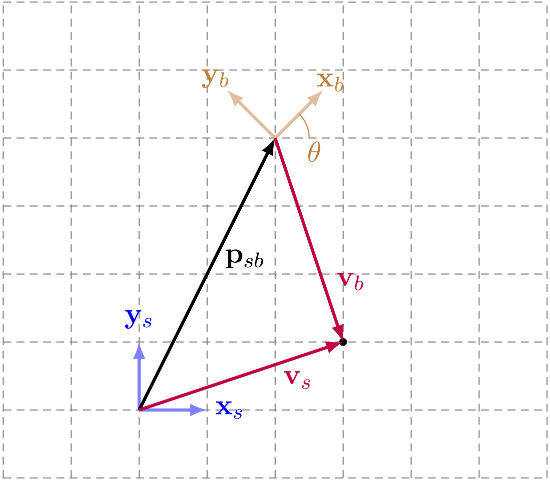
Rigid Body Motions in a Plane: Rotation and Translation

Consider the point represented by the small black circle, whose position is given by $\mathbf{v}_s$ and $\mathbf{v}_b$ in $\{s\}$ and $\{b\}$, respectively. The transformation between $\mathbf{v}_s$ and $\mathbf{v}_b$ is given by ,

$$\mathbf{v}_s = \mathbf{R}_{sb}\mathbf{v}_b + \mathbf{p}_{sb}$$

$$\begin{aligned}\mathbf{v}_b &= \mathbf{R}_{sb}^{\top}\left(\mathbf{v}_s - \mathbf{p}_{sb}\right) \\ &= \mathbf{R}_{bs}\mathbf{v}_s - \mathbf{R}_{bs}\mathbf{p}_{sb} \\ &= \mathbf{R}_{bs}\mathbf{v}_s + \mathbf{p}_{bs}\end{aligned}$$

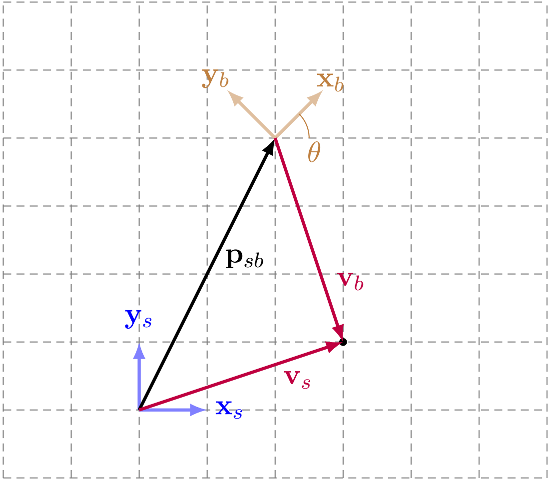
$(\mathbf{R}_{sb}, \mathbf{p}_{sb})$ transforms $\{b\} \mapsto \{s\}$.
$\left(\mathbf{R}_{sb}^{\top}, -\mathbf{R}_{sb}^{\top}\mathbf{p}_{sb}\right)$ transforms $\{s\} \mapsto \{b\}$.



Rigid Body Motions in a Plane: Rotation and Translation

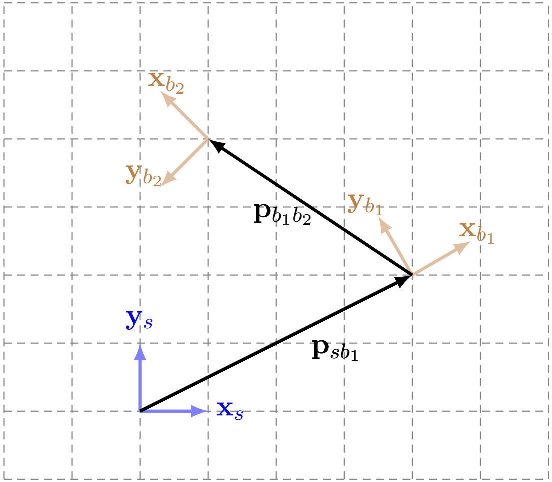
The pair $(\mathbf{R}_{sb}, \mathbf{p}_{sb})$ allows:

1. Representation of a rotated and translated frame $\{b\}$ w.r.t $\{s\}$.
2. Transformation of vectors between $\{s\}$ and $\{b\}$.
3. Performs rotation and transalation of a give vector.



Rigid Body Motions in a Plane: Rotation and Translation

Find the pair $(\mathbf{R}_{sb_2}, \mathbf{p}_{sb_2})$, given the pairs $(\mathbf{R}_{sb_1}, \mathbf{p}_{sb_1})$ and $(\mathbf{R}_{b_1 b_2}, \mathbf{p}_{b_1 b_2})$.



Rigid Body Motions in a Plane: Homogenous representation

- The combined transformation of rotation and translation can be represented as a simple linear operation by using the homogenous representation for vectors and the transformations.
- For rotations and translations in $\mathbb{R}^2$, these rigid body transformations have a simple representation for the matrices and vectors in $\mathbb{R}^3$. The homogenous representation of a vector $\mathbf{r} \in \mathbb{R}^2$ is given by,

$$\bar{\mathbf{r}} = \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

- The homogenous representation of rigid body transformation invovling a rotation $\mathbf{R} \in \mathbb{R}^{2\times2}$ and translation $\mathbf{p} \in \mathbb{R}^2$ is given by,

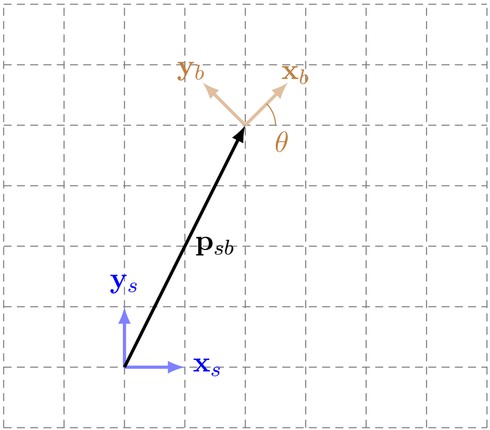
$$\overline{\mathbf{H}} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^3$$

Rigid Body Motions in a Plane: Homogenous representation

$$\mathbf{p}_{sb} = \begin{bmatrix} p_{sb_x} \\ p_{sb_y} \end{bmatrix} \qquad \mathbf{R}_{sb} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The homogenous representation of the pair $(\mathbf{R}_{sb}, \mathbf{p}_{sb})$ is given by,

$$\overline{\mathbf{H}}_{sb} = \begin{bmatrix} \mathbf{R}_{sb} & \mathbf{p}_{sb} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & p_{sb_x} \\ -\sin\theta & \cos\theta & p_{sb_y} \\ 0 & 0 & 1 \end{bmatrix}$$



Rigid Body Motions in a Plane: Rotation and Translation

- The inverse of the rigid body transformation $\overline{\mathbf{H}}_{sb}$ is given by,

$$\overline{\mathbf{H}}_{bs} = \overline{\mathbf{H}}_{sb}^{-1} = \begin{bmatrix} \mathbf{R}_{sb}^{\top} & -\mathbf{R}_{sb}^{\top}\mathbf{p}_{sb} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Verify that $\overline{\mathbf{H}}_{sb}\overline{\mathbf{H}}_{bs} = \overline{\mathbf{H}}_{bs}\overline{\mathbf{H}}_{sb} = \mathbf{I}_3$.

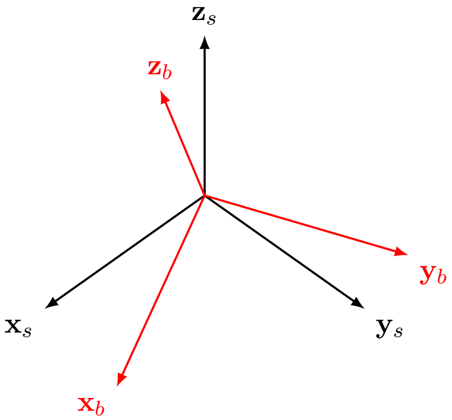
- What is the interpretation of $-\mathbf{R}_{sb}^{\top}\mathbf{p}_{sb}$?

- What are the homogenous representations of a pure rotation and a pure translation?

Rigid Body Motions in 3D: Rotation

The representation of $\{b\}$ in $\{s\}$ is given the following,

$$\mathbf{R}_{sb} = \begin{bmatrix} \mathbf{x}_s^\top \mathbf{x}_b & \mathbf{x}_s^\top \mathbf{y}_b & \mathbf{x}_s^\top \mathbf{z}_b \\ \mathbf{y}_s^\top \mathbf{x}_b & \mathbf{y}_s^\top \mathbf{y}_b & \mathbf{y}_s^\top \mathbf{z}_b \\ \mathbf{z}_s^\top \mathbf{x}_b & \mathbf{z}_s^\top \mathbf{y}_b & \mathbf{z}_s^\top \mathbf{z}_b \end{bmatrix} \in \mathbb{R}^{3\times 3}$$



Rigid Body Motions in 3D: Rotation

Properties of rotation matrices:

- Inverse of a rotation matrix $\mathbf{R}$ is its transpose. $\mathbf{R}^\top = \mathbf{R}^{-1}$. This means that the columns of a rotation matrix are orthonormal.
- The determinant of a rotation matrix is always 1. $\det(\mathbf{R}) = 1$.
- The set of all 3D rotation matrices form a *group* called the ***Special Orthogonal*** group called the $SO(3)$ group.

$$SO(3) := \left\{ \mathbf{R} \in \mathbb{R}^{3\times 3} \; : \; \mathbf{R}^\top \mathbf{R} = \mathbf{I}, \det \mathbf{R} = +1 \right\}$$

$SO(3)$ group is also referred to as the *rotation group* of $\mathbb{R}^3$.