Introduction to Digital Signal Processing

Geometric Signal Theory

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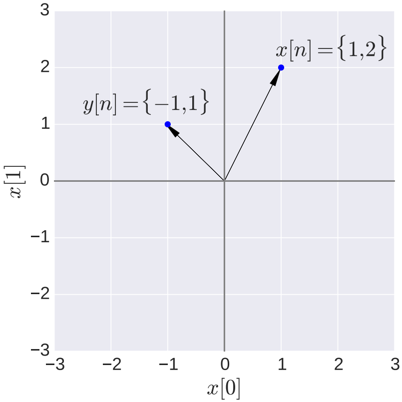
Geometric Signal Theory

- An interesting viewpoint that can help understand signal processing.
- Provide a geometric view of signals and some important signal processing operations.

Discrete-time Signals as Vector

All practical signals we deal with are of finite duration.
Consider two finite duration real discrete-time signal

$$x[n] = (x_0, x_1) = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top \qquad y[n] = (y_0, y_1) = \begin{bmatrix} y_0 & y_1 \end{bmatrix}^\top, \ n \in \{0, 1\}$$



Some familiar and useful geometric ideas

- **Length** of a vector.

$$\|x\| = \sqrt{x_0^2 + x_1^2} = \sqrt{\text{Energy of the signal}}$$

- **Distance** between vectors.

$$\|x - y\| = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}$$

- **Scalar product** or **Inner product** between vectors.

$$\langle x, y \rangle = x_0 y_0 + x_1 y_1 \implies \|x\| = \langle x, x \rangle$$

$$\langle x, y \rangle = \|x\| \|y\| \cos \theta$$

Extension to N dimensions

Consider the following signals of duration N,

$$x[n] = \{x_0, x_1, \ldots, x_{N-1}\} = \begin{bmatrix} x_0 & x_1 & \ldots & x_{N-1} \end{bmatrix}^\top$$

$$y[n] = \{y_0, y_1, \ldots, y_{N-1}\} = \begin{bmatrix} y_0 & y_1 & \ldots & y_{N-1} \end{bmatrix}^\top$$

where, $x_i, y_i \in \mathbb{C}$

- **Length** of a vector. $\|x\| = \left(\sum_{i=0}^{N-1} |x_i|^2\right)^{\frac{1}{2}}$
- **Distance** between vectors. $\|x - y\| = \left(\sum_{i=0}^{N-1} |x_i - y_i|^2\right)^{\frac{1}{2}}$
- **Inner product** between vectors. $\langle x, y \rangle = \sum_{i=0}^{N-1} x_i \overline{y_i}$

What is the inner product?

- A measure of the similarity of signal $x[n]$ with another signal $y[n]$ by looking at their relative orientations.

$$\langle x, y \rangle = \sum_{i \in \mathbb{Z}} x_i \overline{y_i} = \|x\| \|y\| \cos \theta$$

where, θ is the angle between the signals x and y.

- $\langle x, y \rangle$ tells us how much of x is in y and *vice versa*.

What is the inner product?

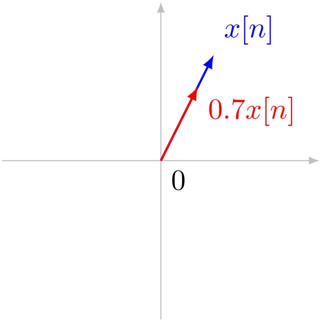
- x and y are orthogonal, when $\langle x, y \rangle = 0 \implies x \perp y$

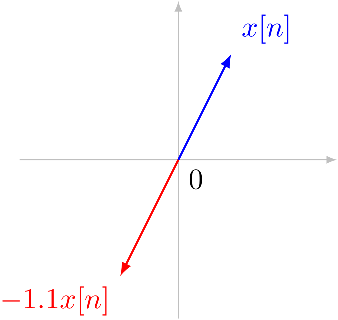
- For example, let $x = [1, 1]^T$ and $y = [1, -1]^T$. What is $\langle x, y \rangle$?

Some basic operations on signals

Scaling. Amplifying or attenuating a signal.

$$x[n] \longrightarrow \alpha_0 x[n] = (\alpha_0 x_0, \alpha_0 x_1, \ldots, \alpha_0 x_{N-1})$$

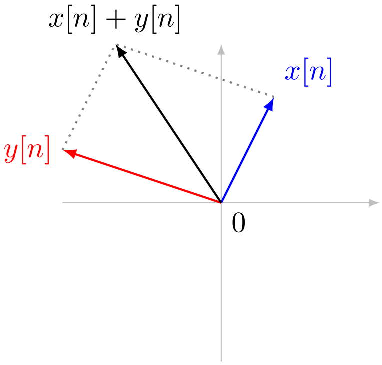




Some basic operations on signals

Signal addition. Combining two signals.

$$x[n] + y[n] = (x_0 + y_0, x_1 + y_1, \ldots, x_{N-1} + y_{N-1})$$

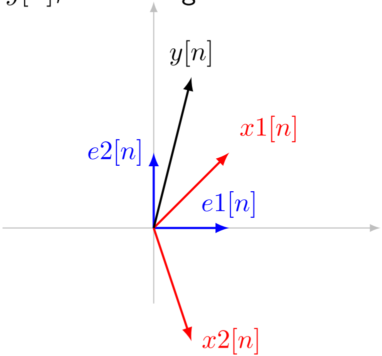


Representing signals in terms of other signals

- We can represent signals as linear combinations of other signals.

$$y[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] + \ldots + \alpha_m x_m[n] \implies y[n] \leftrightarrow (\alpha_i)_{i=1}^m$$

- The appropriate choice for $\{x_i[n]\}_{i=1}^m$ can provide a different perspective about the signal $y[n]$, which might not be immedaitely obvious in $y[n]$.



$$y[n] = \alpha_1 e_1[n] + \alpha_2 x_2[n] \rightarrow (\alpha_1, \alpha_2)$$

$$y[n] = \beta_1 x_1[n] + \beta_2 x_2[n] \rightarrow (\beta_1, \beta_2)$$