Linear Systems

Matrix Inverses

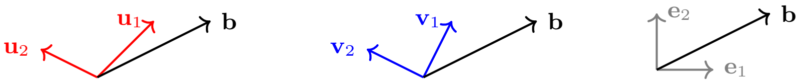
Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Representation of vectors in a basis

- Consider the vector space $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$. Any vector in $\mathbf{b} \in \mathbb{R}^n$ can be representated as a linear combination of $\mathbf{v}_i$s,

$$\mathbf{b} = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{a}_i = \mathbf{V}\mathbf{a}; \;\; \mathbf{a} \in \mathbb{R}^n, \;\; \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$



$\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ are valid basis for $\mathbb{R}^2$, and the presentation for $\mathbf{b}$ in each one of them is different.

Representation of vectors in a basis

- Finding out $\mathbf{a}$ is easiest when we are dealing with an orthonormal basis $\mathbf{U}$, in which case $\mathbf{a}$ is given by,

$$\mathbf{a} = \begin{bmatrix} \mathbf{u}_1^T b \\ \mathbf{u}_2^T b \\ \vdots \\ \mathbf{u}_n^T b \end{bmatrix} = \mathbf{U}^T \mathbf{b} = \mathbf{b}_U$$

Representation of vectors in a basis

Consider a vector $\mathbf{b}$ whose representation in the standard basis is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- Consider a basis $V = \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. Find out $\mathbf{b}_V$.

Representation of vectors in a basis

Consider a vector $\mathbf{b}$ whose representation in the standard basis is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- $U = \left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$. Find out $\mathbf{b}_U$.

Representation of vectors in a basis

Consider a vector $\mathbf{b}$ whose representation in the standard basis is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- $W = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} \right\}$. Find out $\mathbf{b}_W$.

Matrix Inverse

- Consider the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- Let us assume $\mathbf{A}$ is non-singular $\implies$ columns of $\mathbf{A}$ represent a basis for $\mathbb{R}^n$.
- What does $\mathbf{x}$ represent? It is the representation of $\mathbf{y}$ in the basis consisitng of the columns of $\mathbf{A}$.

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{a}_i x_i \implies \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \\ \tilde{\mathbf{b}}_2^T \\ \ldots \\ \tilde{\mathbf{b}}_n^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \mathbf{y} \\ \tilde{\mathbf{b}}_2^T \mathbf{y} \\ \ldots \\ \tilde{\mathbf{b}}_n^T \mathbf{y} \end{bmatrix}$$

Matrix Inverse

- $\mathbf{A}^{-1}$ is a matrix that allows change of basis to the columns of $\mathbf{A}$ from the standard basis!

• $W = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} \right\}$. Find $\mathbf{b}_W$ by calculating the inverse of the matrix $\mathbf{W} = \begin{bmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$.

• What about $V = \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. What is $\mathbf{b}_V$?

Left Inverse

- Consider a rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. There exists no inverse $\mathbf{A}^{-1}$ for this matrix.
- But, there exist two matrices $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$, such that,

$$\mathbf{CA} = \mathbf{I}_n \quad \text{or} \quad \mathbf{AB} = \mathbf{I}_m$$

- Both cannot be true for a rectangular matrix, only one can be true when the matrix is full rank.
- A rectangular matrix can only have either a left or a right inverse.

Left Inverse

Consider a matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Let $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{2 \times 3}$. Can you explain why only $\mathbf{CA} = \mathbf{I}_2$ can be true and not $\mathbf{AB} = \mathbf{I}_3$?

Can you also explain why $\mathbf{C}$ is not unique?

Left Inverse

- Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find a complete solution for the left inverse of $\mathbf{A}$ such that $\left(\mathbf{C} + \hat{\mathbf{C}}\right) = \mathbf{I}_n$.

Left Inverse

- Consider the system, $\mathbf{Ax} = \mathbf{b}$. $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find $\mathbf{x}$.

- What happens when $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. What is $\mathbf{x}$?

Right Inverse

- For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $n > m$ with full rank, $\mathbf{AB} = \mathbf{I}_m \longrightarrow \mathbf{B}$ is the right inverse.
- Right inverse of $\mathbf{A}$ exists only if the rows of $\mathbf{A}$ are independent, i.e. $rank\,(\mathbf{A}) = m \longrightarrow \mathbf{A}^T\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$
- $\mathbf{Ax} = \mathbf{b}$ can be solved for any $\mathbf{b}$. $\mathbf{x} = \mathbf{Bb} \implies \mathbf{A}\,(\mathbf{Bb}) = \mathbf{b}$.
- There are an infitnite number of $\mathbf{B}$s $\implies$ an infinite number of solutions $\mathbf{x}$.

Right Inverse

- Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Find a complete solution for the right inverse of $\mathbf{A}$.

Right Inverse

Solve $\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compare the solutions from Gauss-Jordan method and the ones obtained using right-inverses.

Right Inverse

Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of $\mathbf{A}$, such that $\mathbf{CA} = \mathbf{I}_n$. What is $rank\,(\mathbf{C})$?

Pseudo Inverse

- Consider a tall matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with independent columns. It turns out the Gram matrix $\mathbf{A}^T\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible. If that is the case then,

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{A} = \mathbf{I}_n; \quad \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T \text{ is a left inverse.}$$

- $\mathbf{A}^\dagger = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ is called the *pseudo inverse* or the *Moore-Penrose inverse*.

- For the case of a fat, wide matrix, we have $\mathbf{A}^\dagger = \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}$.

- When $\mathbf{A}$ is square and invertible, $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

Pseudo Inverse

- Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ using the $\mathbf{A}^{\dagger}$. $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find $\mathbf{x}$.

- Compare $\mathbf{A}^{\dagger}$ with that of the general left inverse $\mathbf{C}$. Calculate $\|\mathbf{C}\|^2$ and find out the $\min \|\mathbf{C}\|^2$. What is $\left\|\mathbf{A}^{\dagger}\right\|^2$?

Pseudo Inverse

- Solve $\mathbf{Ax} = \mathbf{b}$ using the $\mathbf{A}^\dagger$. $\mathbf{A} = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \end{bmatrix}$. Find $\mathbf{x}$.

- Write down all the possible solution $\mathbf{x}$. What is the $\mathbf{x}$ with the smallest length? What is $\mathbf{A}^\dagger \mathbf{b}$?

Matrix Inverse and Pseudo Inverse through $\mathbf{QR}$ factorization

- Consider an invertible, square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

$$\mathbf{A} = \mathbf{QR} \implies \mathbf{A}^{-1} = (\mathbf{QR})^{-1} = \mathbf{R}^{-1}\mathbf{Q}^{-1} = \mathbf{R}^{-1}\mathbf{Q}^{T}$$

  where, $\mathbf{R}, \mathbf{Q} \in \mathbb{R}^{n \times n}$. $\mathbf{R}$ is upper triangular, and $\mathbf{Q}$ is an orthogonal matrix.

- In the case of a left invertible rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, we can factorize $\mathbf{A} = \mathbf{QR}$, with $\mathbf{Q} \in \mathbb{R}^{m \times n}$ and $\mathbf{R} \in \mathbb{R}^{m \times m}$.

$$\mathbf{A}^{\dagger} = \left(\mathbf{A}^{T}\mathbf{A}\right)^{-1}\mathbf{A}^{T} = \left(\mathbf{R}^{T}\mathbf{Q}^{T}\mathbf{Q}\mathbf{R}\right)^{-1}\mathbf{R}^{T}\mathbf{Q}^{T} = \left(\mathbf{R}^{T}\mathbf{R}\right)^{-1}\mathbf{R}^{T}\mathbf{Q}^{T} = \mathbf{R}^{-1}\mathbf{Q}^{T}$$

Matrix Inverse and Pseudo Inverse through $\mathbf{QR}$ factorization

- For a right invertible wide, fat matrix, we can find out the pseudo-inverse of $\mathbf{A}^T$, and then take the transpose of the pseudo-inverse.

$$\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{I} \implies \left(\mathbf{A}^{\dagger}\right)^T \mathbf{A}^T = \left(\mathbf{A}^T\right)^{\dagger} \mathbf{A}^T = \mathbf{I}$$

$$\mathbf{A}^T = \mathbf{QR} \implies \left(\mathbf{A}^T\right)^{\dagger} = \mathbf{R}^{-1}\mathbf{Q}^T = \left(\mathbf{A}^{\dagger}\right)^T \implies \mathbf{A}^{\dagger} = \mathbf{Q}\mathbf{R}^{-T}$$