Transducers & Instrumentation

Module 01/02

(Sensor Dynamics Characteristics; LTI system; Convolution; Laplace Transform; Frequency Response; Zero, First, and Second Order LTI systems)

Sensor dynamic characterization

- Many sensors do not respond instantaneously to a given input.

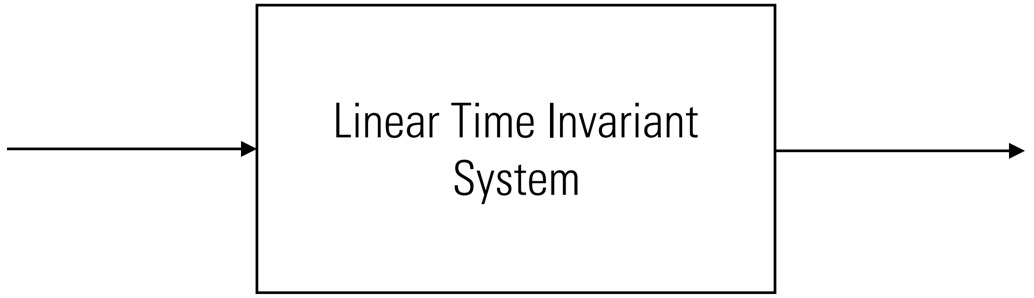
E.g., Contact thermometer

- Mathematically described using differential equations relating the measured and other inputs to the sensor output.

- A common and very useful model for such dynamical systems are linear time invariant systems.

Linear Time Invariant Systems

- Both Linear and Time Invariant.



Some useful signals

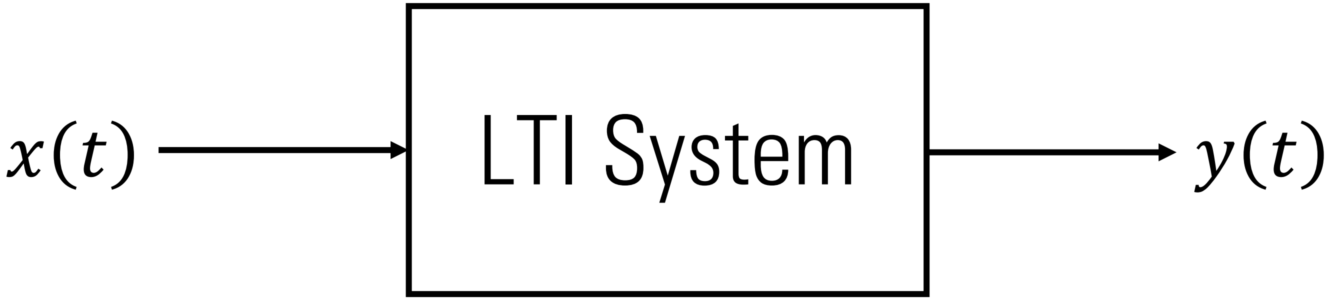
- Step signal: $1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$
- Exponential signal: $A \cdot e^{\beta t}$
- Sinusoidal signals: $A \cdot \sin(\omega t + \varphi)$

Some useful signals

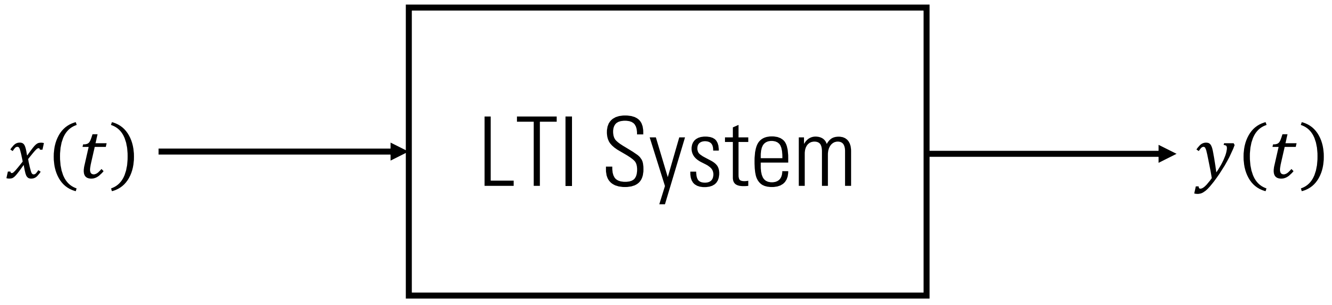
- Dirac Delta Function $\delta(t)$: $\int_a^b \delta(t)dt = \begin{cases} 1, & 0 \in [a,b] \\ 0, & 0 \notin [a,b] \end{cases}$

$$\int_a^b f(t)\delta(t-t_0)dt = \begin{cases} f(t_0), & t_0 \in [a,b] \\ 0, & t_0 \notin [a,b] \end{cases}$$

Input-Output Relationship of LTI systems



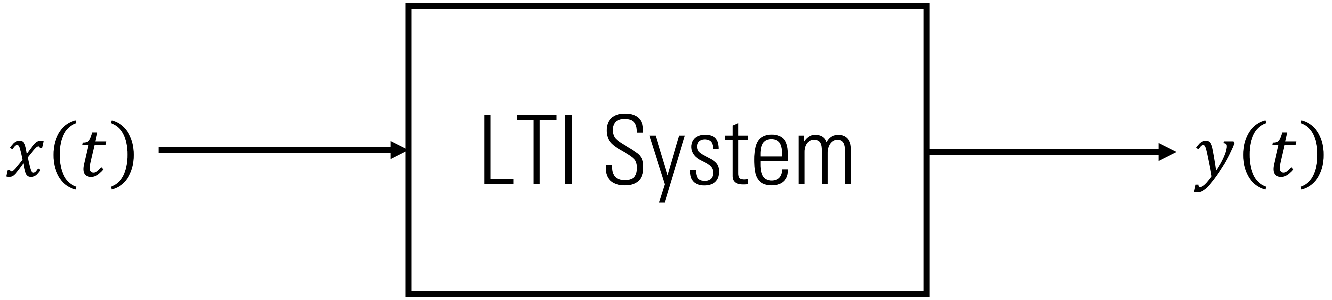
Convolution



Impulse Response

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow h(t)$$

Another Description of LTI systems



$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x$$

Laplace Transform

- A very popular and useful integral transform method for analysing LTI systems.

- Unilateral Laplace transform.

$$X(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt\,, \qquad s \in \mathbb{C}, s = \sigma + j\omega$$

- Laplace Transform Pairs:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

Laplace Transform Pairs

Time Domain Signal	Laplace Transform
$1(t)$	
$e^{at}1(t)$	
$\sin(\omega_0 t)\,1(t)$	
$\delta(t)$	

Laplace Transform Property

- $x(t)$ and $X(s)$ are Laplace transform pairs. Then,

$$y(t) \xleftrightarrow{\mathcal{L}} Y(s) \Rightarrow ax(t) + by(t) \xleftrightarrow{\mathcal{L}} a\,X(s) + bY(s)$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \Rightarrow \frac{dx}{dt} \xleftrightarrow{\mathcal{L}} sX(s) - x(0^-)$$

Why is the Laplace Transform useful for LTI systems?

$x(t) \longrightarrow$ $\boxed{\text{LTI System}}$ $\longrightarrow y(t)$

Transfer Function of an LTI system

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s) \Longrightarrow H(s) = \frac{Y(s)}{X(s)}$$

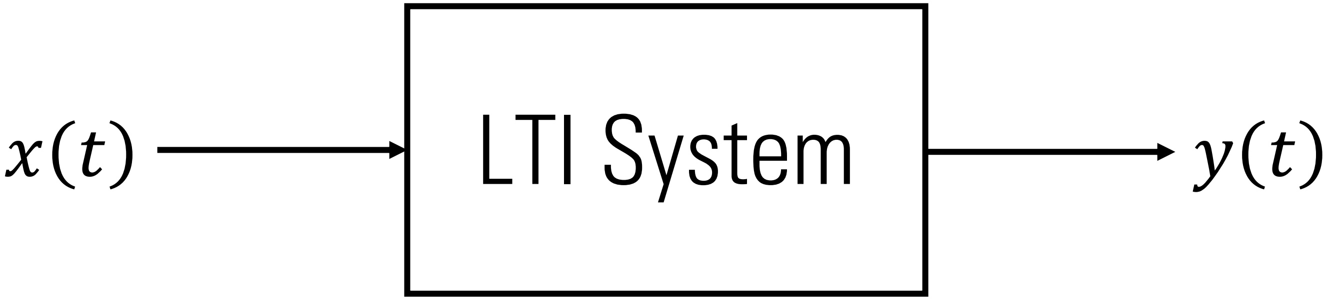
Frequency Response of an LTI System

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = h(t) * x(t)$$

$$H(j\omega) = H(s)\Big|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)}$$

$$H(j\omega) = |H(j\omega)|e^{j\,\arg(H(j\omega))} \rightarrow \begin{cases} |H(j\omega)| & \text{Magnitude Response} \\ \arg(H(j\omega)) & \text{Phase Response} \end{cases}$$

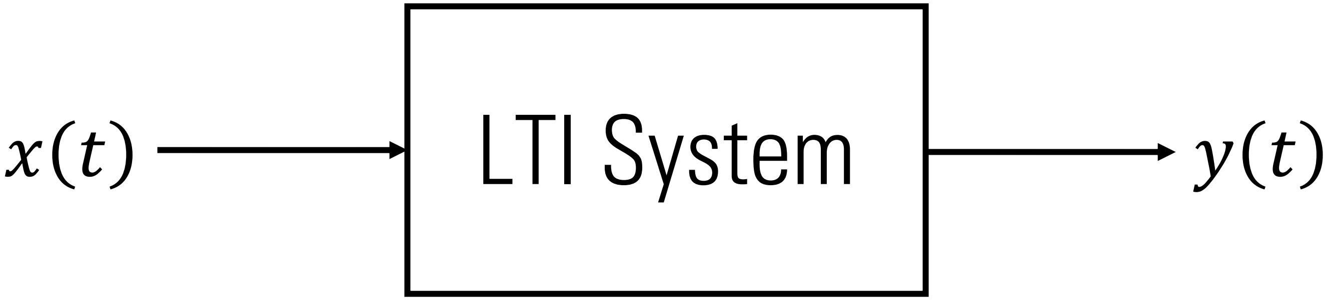
Zero Order System



$$a_0 y(t) = b_0 x(t)$$

Pure Linear Resistor
Pure Linear Spring

First Order System



$$a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

Time Constant: $\tau \triangleq \dfrac{a_1}{a_0}$

Static Sensitivity: $K \triangleq \dfrac{b_0}{a_0}$

First Order System

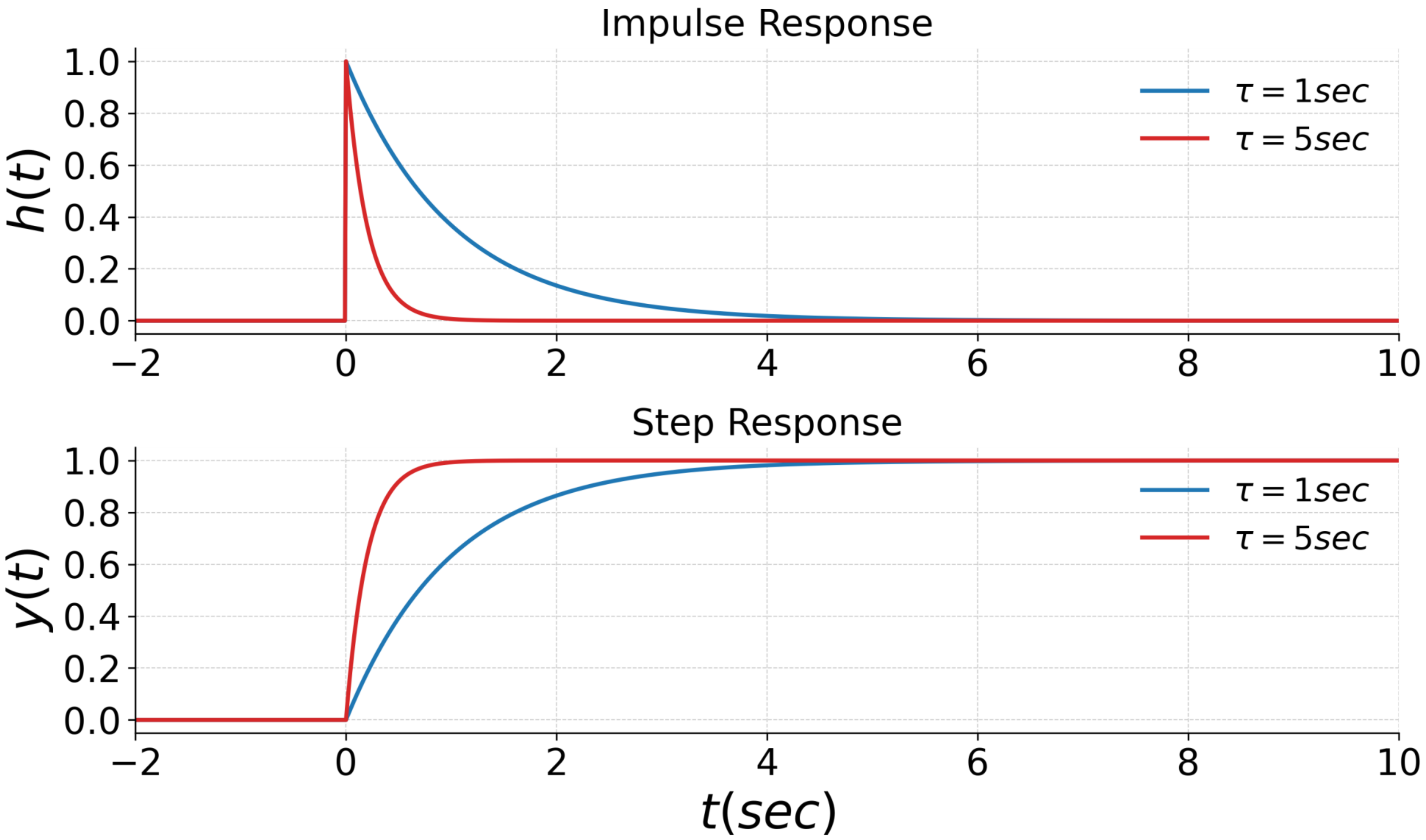
$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

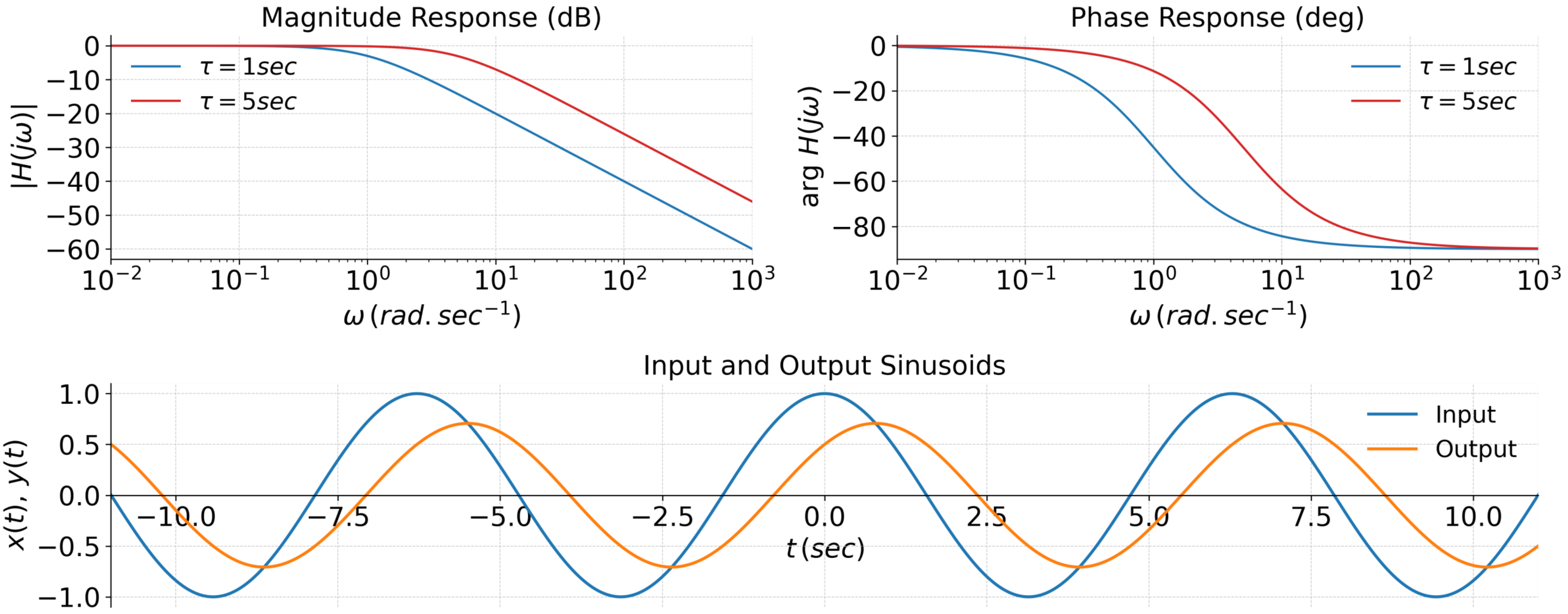
RC / RL circuits

Spring-damper/Mass-damper

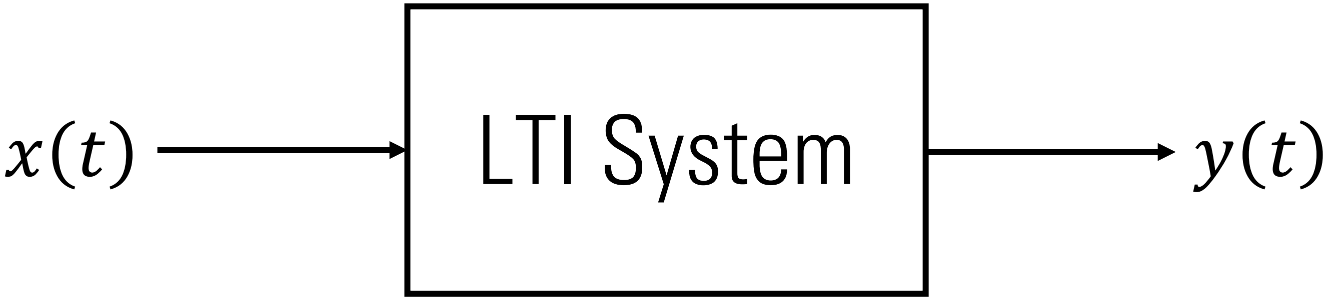
First Order System



First Order System



Second Order System



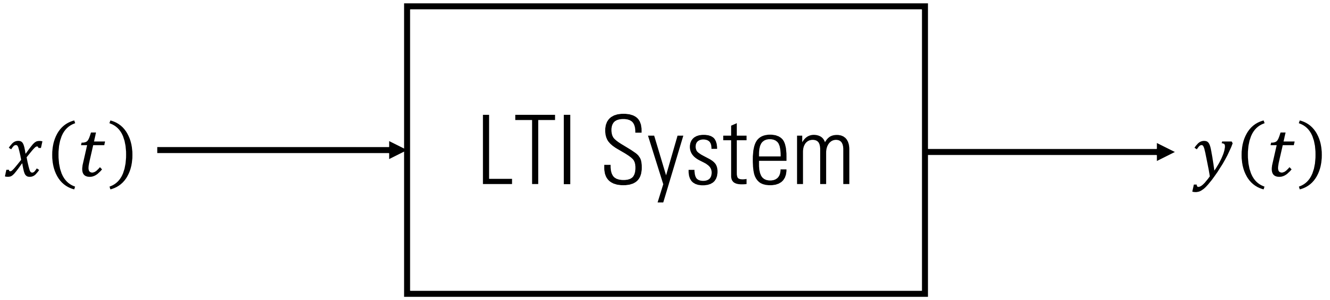
$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

Static Sensitivity: $K \triangleq \dfrac{b_0}{a_0}$

Natural Frequency: $\omega_n \triangleq \sqrt{\dfrac{a_0}{a_2}}$ $\implies \dfrac{1}{\omega_n^2}\dfrac{d^2 y}{dt^2} + \dfrac{2\zeta}{\omega_n^2}\dfrac{dy}{dt} + y = Kx$

Damping ratio: $\zeta \triangleq \dfrac{a_1}{2\sqrt{a_0 a_2}}$

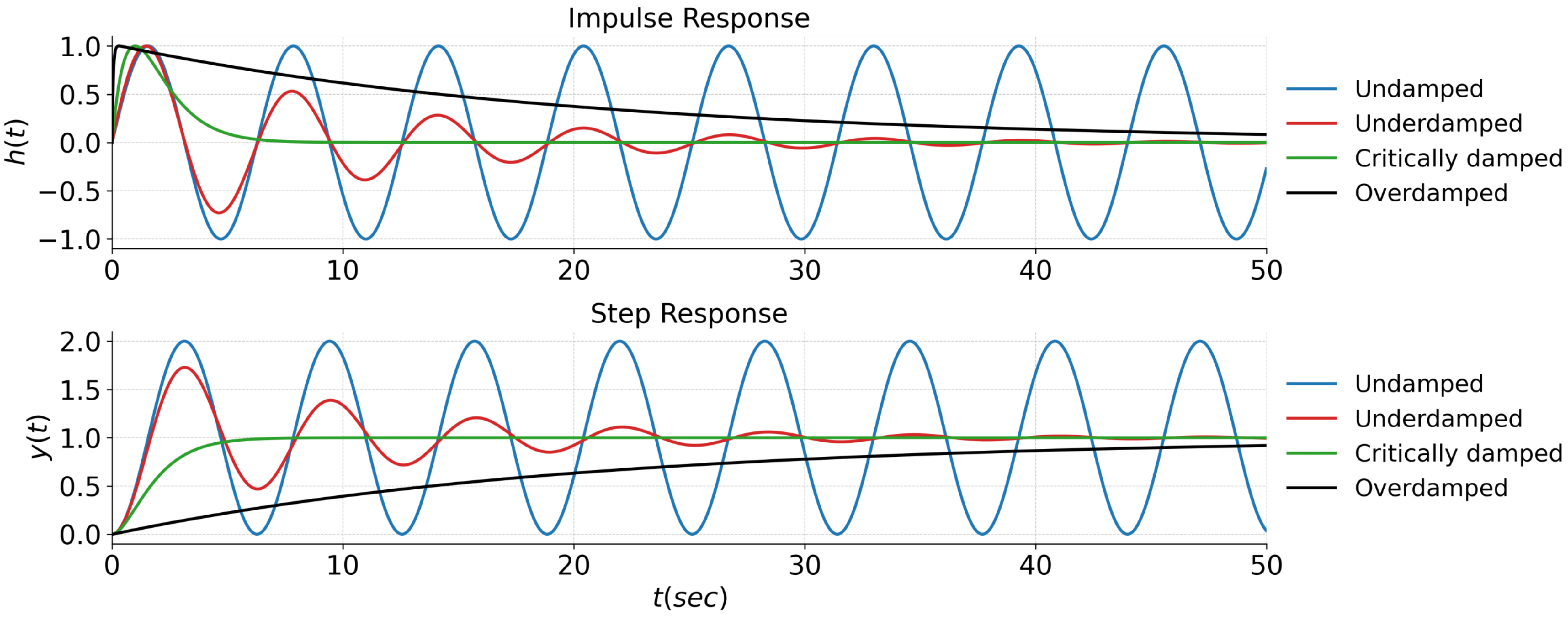
Second Order System



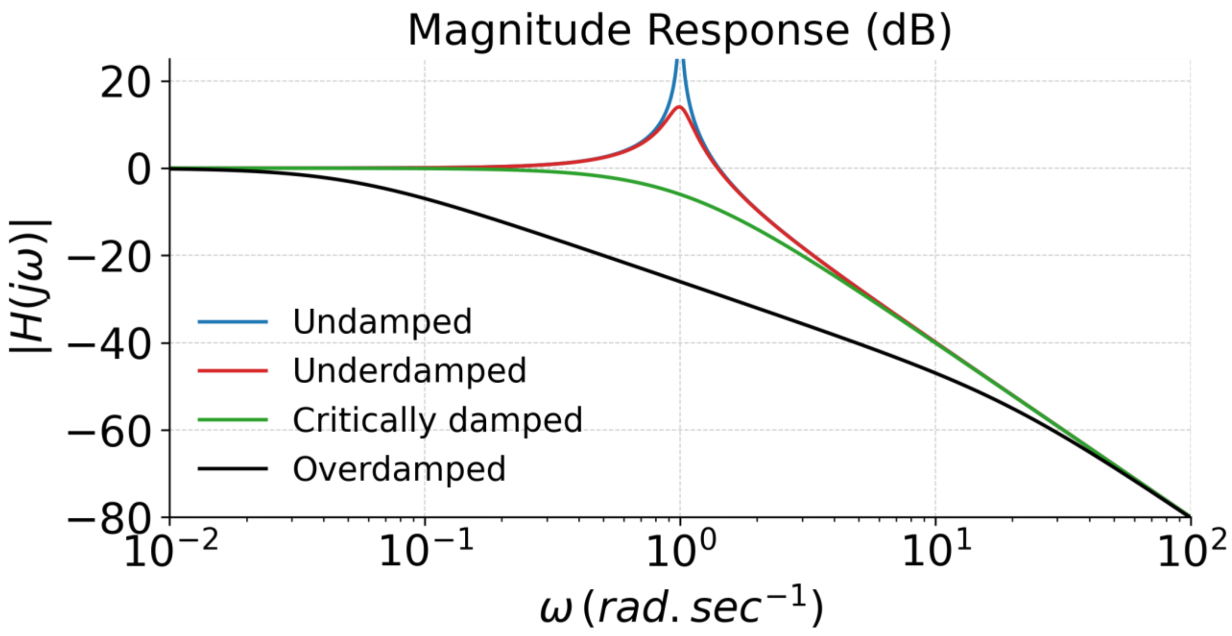
$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

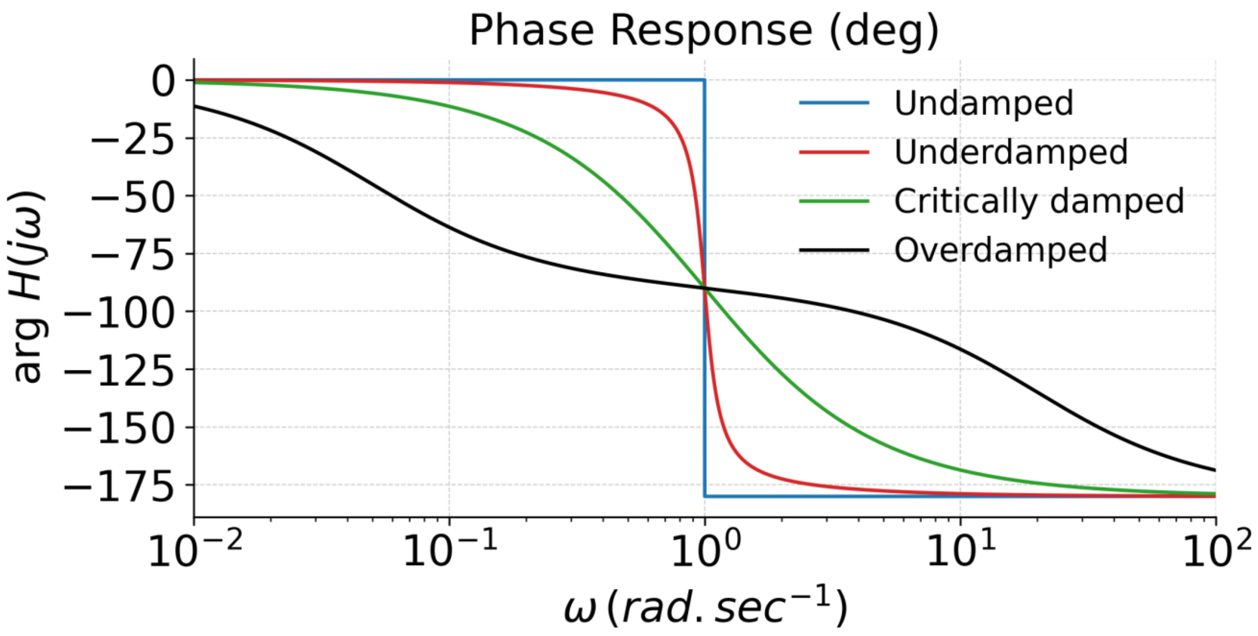
RLC circuits
Mass-Spring-damper

Second Order System Response



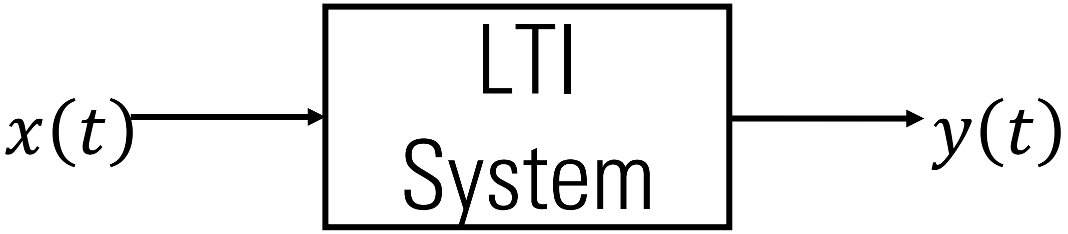
Second Order System – Frequency Response





Dynamic characterization of sensors

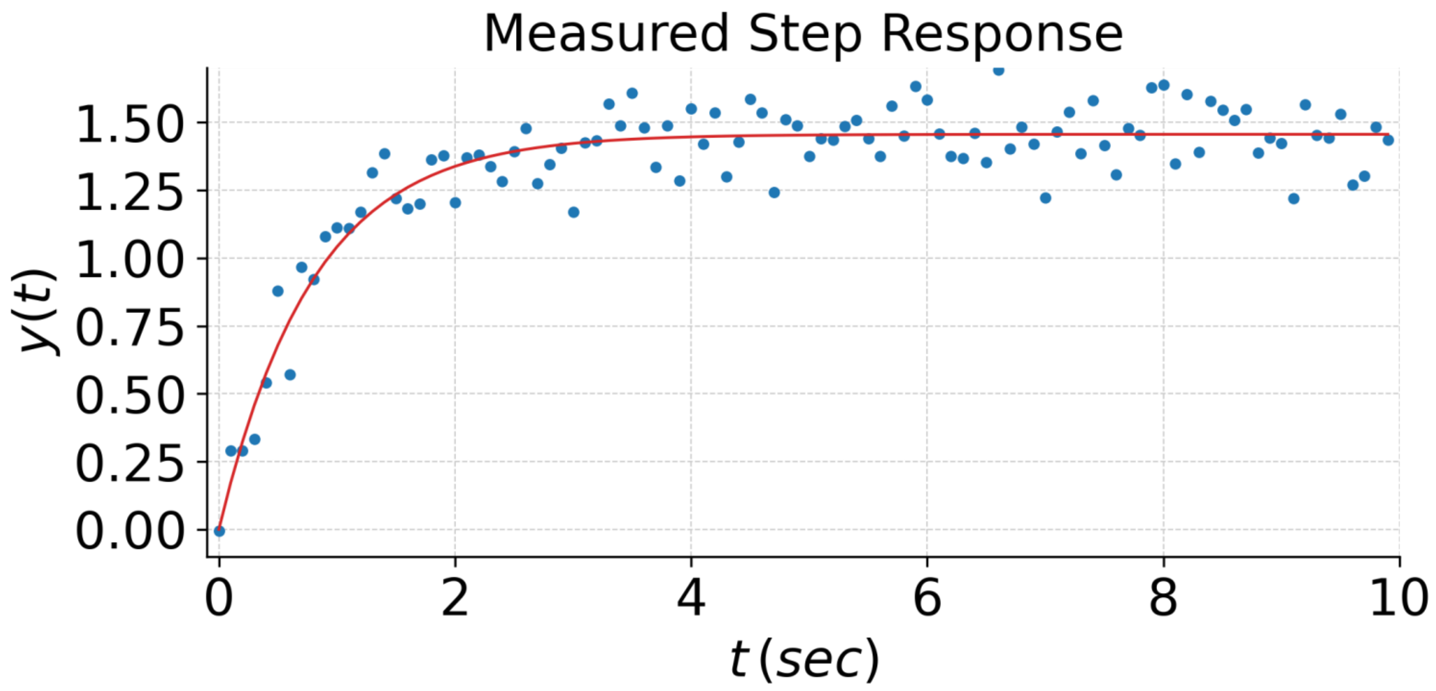
- Identifying sensor parameters from measured data.

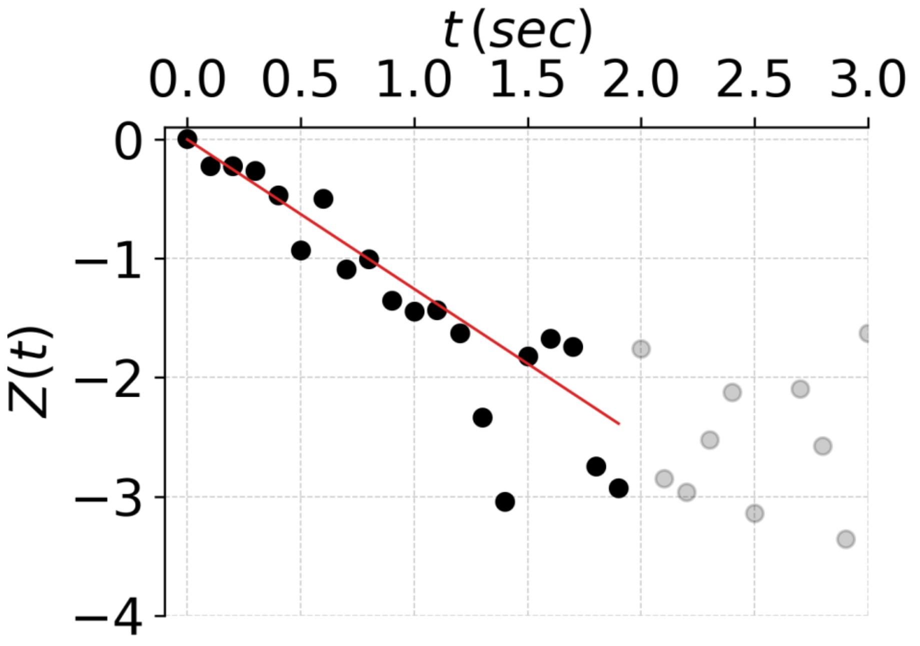


- System identification tools can be used for doing this.
- Simple procedure for first order system using a step response.

$$y(t) = K \cdot \left(1 - e^{-t/\tau}\right) \Longrightarrow \log_e \left(1 - \frac{y(t)}{K \cdot x(t)}\right) = -\frac{t}{\tau}$$

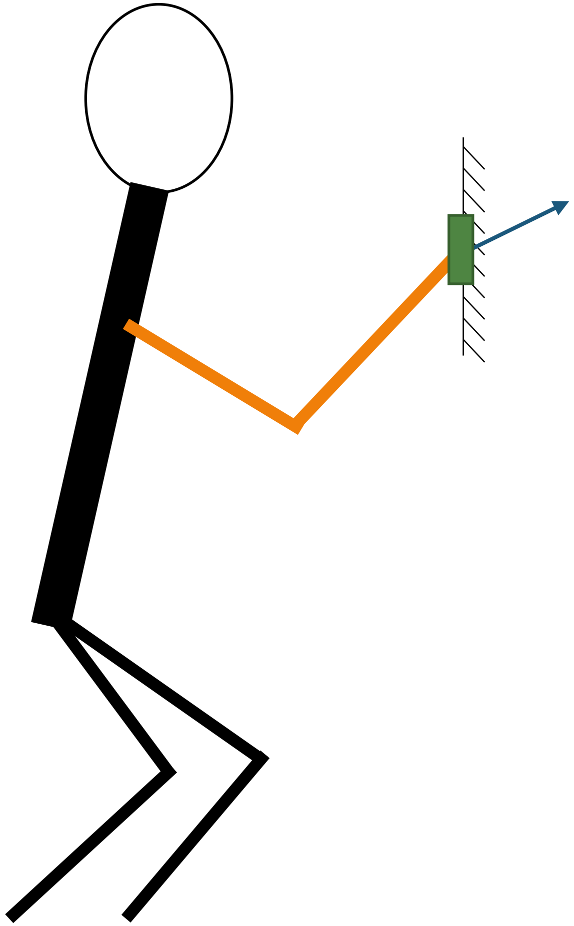
Dynamic characterization of sensors





Multi-Input Multi-Output Case

- We are often interested in measurands that have multiple components.



$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$\mathbf{x} \longrightarrow \boxed{\text{Sensor}} \longrightarrow \mathbf{y}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

$$x_i \in \mathbb{R}, 1 \leq i \leq m \qquad y_i \in \mathbb{R}, 1 \leq i \leq n$$

Multi-Input Multi-Output Case

- A general MIMO sensor

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$$

$f_i(\mathbf{x})$: Sensor output due to all the individual measurands at the input.

Linear MIMO Sensor

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \mathbf{Ax}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$: Sensitivity Matrix

$a_{ij} = \frac{\partial y_i}{\partial x_j}$: Sensitivity of the i^{th} output to the j^{th} input.

Linear MIMO Sensor

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} x_m = \mathbf{Ax}$$

$\mathbf{a}_i = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$: ith column of output to the sensitivity matrix $\mathbf{A}$.

Linear MIMO Sensor

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{A} \in \mathbb{R}^{n \times m} \quad \begin{cases} n < m: & \text{Fat/Wide matrix} \\ n = m: & \text{Square matrix} \\ n > m: & \text{Skinny/Tall matrix} \end{cases}$$

For sensing $\mathbf{A}$ must be square or tall, i.e., equal number of more outputs than inputs.

Square or Tall $\mathbf{A}$ is full rank → $\mathbf{A}\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$.

If $\mathbf{A}$ is square and full rank, then $\mathbf{A}^{-1}$ exists, where $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_{\boldsymbol{n}}$.

$$\Longrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

Real scenario: $\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\varepsilon} \implies \hat{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{y} = \mathbf{x} + \mathbf{A}^{-1}\boldsymbol{\varepsilon}$

Linear MIMO Sensor

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{A} \in \mathbb{R}^{n \times m} \quad \begin{cases} n < m: & \text{Fat/Wide matrix} \\ n = m: & \text{Square matrix} \\ n > m: & \text{Skinny/Tall matrix} \end{cases}$$

When $\mathbf{A}$ is tall, there is no $\mathbf{A}^{-1}$ but we can use the pseudoinverse,

$$\mathbf{A}^{\dagger} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}$$

Where, $\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{I}_m$

Real scenario: $\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\varepsilon} \implies \hat{\mathbf{x}} = \mathbf{A}^{\dagger}\mathbf{y} = \mathbf{x} + \mathbf{A}^{\dagger}\boldsymbol{\varepsilon}$

$\hat{\mathbf{x}} = \mathbf{A}^{\dagger}\mathbf{y}$ is the least squares estimate of $\mathbf{x}$, which allows to average out the contribution of the noise $\boldsymbol{\varepsilon}$.