Introduction to Digital Signal Processing

Fourier Representation of Discrete-Time Signals

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Discrete-Time Fourier Series

- Continuous-time complex sinusoids have frequencies $\omega \in (-\infty, \infty)$.
- Discrete-time sinusoids have frequencies $\Omega \in (-\pi, \pi]$
- A discrete-time periodic signal $x[n]$ with fundamental period N can be represented as a sum of discrete-time sinusoids,

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}$$

- Appropriate choice of c_k will let us represent $x[n]$ as a linear combination of $e^{\frac{j2\pi k}{N}n}$.
- The Fourier coefficient c_k can be determined by,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

Discrete-Time Fourier Series

- c_k is discrete like in the case of continuous-time Fourier series, and it is also periodic with fundamental period N, i.e. $c_k = c_{k+N}$.

$$\begin{tikzcd} x[n] \arrow[r, bend left, "\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}"] & \left(c_k\right)_{k=0}^{N-1} \arrow[l, bend left, "\sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}"] \end{tikzcd}$$

Properties of Discrete-Time Fourier Series

- Fourier representation is discrete and periodic. (c_k is period with fundamental period N)
- When $N = 2M$ is even, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \leq l < \frac{N}{2}$$

- When $N = 2M+1$ is odd, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \leq l < \frac{N-1}{2}$$

- Parseval's identity.

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

The distribution of $|c_k|^2$ as a function of $0 \leq k < N$ is the *power spectral density* of the periodic signal $x[n]$.

Discrete-Time Fourier Series

Find the DTFS of $x[n] = \begin{cases} 1, & 0 \leq n < M \\ 0, & M \leq n < N-1 \end{cases}$ with fundamental period N.

Discrete-time Fourier Transform

- Similar to the continus-time case, the Fourier representation of discrete -time aperiodic signals can be obtained as the limiting case of the a periodic signals with increasing period N.

- The discrete-time Fourier transform (DTFT) of an aperiodic signal $x[n]$ with finie energy is given by,
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}, \quad -\pi \leq \Omega < \pi$$

- $X(\Omega)$ is a continuous in Ω and periodic with period 2π.

- Inverse DTFT,
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Discrete-time Fourier Transform

- DTFT exists only if $x[n]$ is absolutely summable.

$$\sum_{n} |x[n]| < \infty \implies |X(\Omega)| < \infty$$

- When $x[n]$ is only square summable, then DTFT converges to the true DTFT only in the mean squared sense.
  E.g.,

$$x[n] = \begin{cases} \frac{\Omega_c}{n}, & n = 0 \\ \frac{\Omega_c}{n} \frac{\sin \Omega_c n}{\Omega_c n}, & n \neq 0 \end{cases} \longrightarrow X(\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$$

Properties of DTFT

- **Linearity**: $\alpha x[n] + \beta y[n] \xleftrightarrow{\text{DTFT}} \alpha X(\Omega) + \beta Y(\Omega)$
- **Shift in time**: $x[n - n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega)$
- **Shift in frequency**: $x[n]e^{j\Omega_0 n} \xleftrightarrow{\text{DTFT}} X(\Omega - \Omega_0)$
- **Time and frequency scaling**: $x(\alpha t) \xleftrightarrow{\text{DTFT}} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \ \alpha > 0$
- **Symmetry in time**: $x(t) \xleftrightarrow{\text{DTFT}} X(\Omega)$ is real.
- **Convolution in time**: $x[n] * y[n] \xleftrightarrow{\text{DTFT}} X(\Omega)Y(\Omega)$

Classification of signals based on the frequencty spectrum

- We can also classify signals based on how their energy is distributed across frequency.
- **Low frequency signal**. Most of the energy is concentrated around 0Hz and frequencies around 0Hz.
- **High frequency signal**. Very less is concentrated around 0Hz, and most of the energy is in the higher frequencies all the way upto $\omega \to \infty$.
- **Bandpass signal**. Very less concentration around 0Hz and at high frequencies. Most of the energy is concentrated within a band of finite frequencies.
- **Bandlimited signal**. Signal energy is uniformly zero beyond a particular frequency, i.e. $|X(\omega)| = 0, \ \forall |\omega| > \omega_b$.

Bandwidth of a signal

- The band of frequencies over which most of the energy is distributed is the *bandwidth* of the signal.

- Several ways to define the bandwidth of a signal.

- **3 dB bandwidth**. Frequency range over which the spectral density is above a particular threshold.
  Threshold value is often defined relative to the max. value of the spectal density.

Frequency-Domain and Time-Domain Properties

- **Continuous-Time, Periodic** $\longrightarrow$

- **Discrete-Time, Non-Periodic** $\longrightarrow$

- **Continuous-Frequency, Periodic** $\longrightarrow$

- **Discrete-Frequency, Non-Periodic** $\longrightarrow$