Introduction to Digital Signal Processing

Z-transform

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Z transform

- Exponential signals are *eigenfucntiopns* of LTI systems.

$$z^n \longrightarrow H(z)\, z^n$$

$H(z)$ is the eigenvalue corresponding to the eigenfunction z^n.

- If $x[n] = \sum_k \alpha_k z_k^n$, then $y[n] = \sum_k \alpha_k H(z_k)\, z_k^n$.

$$\begin{aligned}(\alpha_k)_{k\in\mathbb{Z}} &\longrightarrow \text{Representation of } x[n] \text{ using } z_k^n \\ (H(z_k)\,\alpha_k)_{k\in\mathbb{Z}} &\longrightarrow \text{Representation of } x[n] \text{ using } z_k^n\end{aligned}$$

- The z-transoform allows us to find the representation of any disctet-time signal $x[n]$ in terms of the set of complex exponentials $\{z^n\}_{z\in\mathbb{C}}$

z transform

The z-transform of a discrete time signal $x[n]$ is defined as the following power seris,

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}(x[n])$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

where, $z \in \mathbb{C}$.

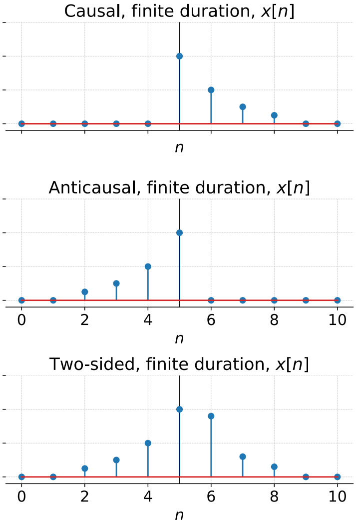
- The values of z for which the above summation covnerges is called the *region of conergence* of $X(z)$.

z transform

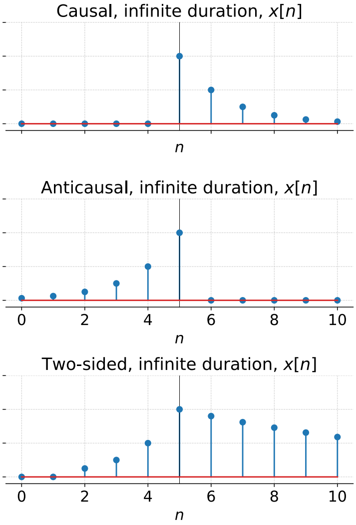
z-transform of some signals.

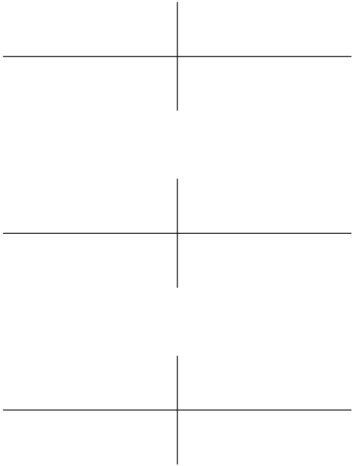
1. $\delta[n]$
2. $\delta[n-k]$
3. $\delta[n+k]$
4. $\sum_{k=0}^{5} \alpha_k \delta[n-k]$
5. $1[n]$
6. $a^k \cdot 1[n]$
7. $-a^k \cdot 1[-n-1]$

z-transform and ROCs



z-transform and ROCs





Properties of the z-transform

- Linearity
- Time-shifting
- Convolution in time
- Initital value theorem

Transfer function of an LTI system

The z-transform of the impulse response $h[n]$ is defined as the transfer function of a discrete-time LTI system.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

When the system is causal, then $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$.
The z-transforms of the input $x[n]$ and $y[n]$ are related to each other through the transfer function,

$$Y(z) = H(z) \cdot X(z)$$

Unilateral z-transform

When solving difference equations, we are interested in computing the output $y[n]$ from time $n = 0$ for an input $x[n]$ that is specified from time $n = 0$. Here we cannot use the regular z-transform (also called as two-sided or bilateral z-transform).

$$X^+(z) \triangleq \sum_{n=0}^{\infty} x[n]z^{-n}$$

This is useful when analysing linear difference equations.
When the time domain signal $x[n]$ is delayed by a sample, such that the signal is $x[n-1] \cdot 1[n]$, then we have

$$x[n] \overset{\mathcal{Z}^+}{\longleftrightarrow} X^+(z) \implies x[n-1] \overset{\mathcal{Z}^+}{\longleftrightarrow} z^{-1}X^+(z) + x[-1]$$

$X^+(z) = X(z)$ for causal an signal $x[n]$.

Rational z-transforms

- In practice, we often come across rational polynomial of z.
- Consider a LTI system described by the following different equation,

$$y[n]+a_1y[n-1]+a_2y[n-2]+\cdot+a_Ny[n-N]=b_0x[n]+b_1x[n-1]+\cdots+b_Mx[n-M]$$

We are interested in sovling this equation from time $n=0$ for an input specified from time $n\geq 0$. Taking the unilateral z-transform on both sides,

$$\begin{aligned}
y[n] &\overset{\mathcal{Z}^+}{\longleftrightarrow} Y^+(z) \\
y[n-1] &\overset{\mathcal{Z}^+}{\longleftrightarrow} z^{-1}Y^+(z)+y[-1] \\
y[n-2] &\overset{\mathcal{Z}^+}{\longleftrightarrow} z^{-2}Y^+(z)+z^{-1}y[-1]+y[-2] \\
&\vdots \\
y[n-N] &\overset{\mathcal{Z}^+}{\longleftrightarrow} z^{-N}Y^+(z)+z^{-(N-1)}y[-1]+z^{-(N-2)}y[-2]+\cdots+y[-N]
\end{aligned}$$

Rational z-transforms

If we assume a causal inout signal $x[n]$,

$$Y^+(z) + \sum_{k=1}^{N} a_k z^{-k} \left(Y^+(z) + \sum_{l=1}^{k} y[-l] z^l \right) = X(z) \left(b_0 + b_1 z^{-1} + \cdots + z^{-M} \right)$$

$$Y^+(z) \left(1 + \sum_{k=1}^{N} a_k z^{-k} \right) + \sum_{k=1}^{N} a_k z^{-k} \left(\sum_{l=1}^{k} y[-l] z^l \right) = X(z) \left(b_0 + b_1 z^{-1} + \cdots + z^{-M} \right)$$

$$Y^+(z) = \frac{B(z)}{A(z)} X(z) + \frac{N_0(z)}{A(z)}$$

where,

- $B(z) = \sum_{l=0}^{M} b_l z^- l$
- $A(z) = 1 + \sum_{k=1}^{N} a_k z^{-k}$
- $N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \left(\sum_{l=1}^{k} y[-l] z^l \right)$

Rational z-transform

Zero-state response, when the initial conditions are zero.

$$Y_{zs}(z) = \frac{B\left(z\right)}{A\left(z\right)} X\left(z\right) = H\left(z\right) X\left(z\right)$$

Zero-input reesponse, wheren the input is zero and the initial conditions are non-zero.

$$Y_{zi}^{+}\left(z\right) = \frac{N_0\left(z\right)}{A\left(z\right)}$$

Inverse z-transform of Rational Polynomials of z

Consider the following rational polynomial,

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

The rational polynomial $X(z)$ is called *proper* if $M < N$ and $a_N \neq 0$.
It is *improper* if $M \geq N$.
Improper rationla polynomial can be converted to the following form,

$$X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{N_0(z)}{D(z)}$$

Convert the rational polynomial as a function of z instead of z^{-1} by multiplying both the numerator and denominator by z^N.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Inverse z-transform of Rational Polynomials of z

Convert the rational polynomial as a function of z instead of z^{-1} by multiplying both the numerator and denominator by z^N.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Let the roots of the denominator of $\frac{X(z)}{z}$ be $p_1, p_2, \ldots p_N$.

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

The roots of the denominator $D(z)$ are the called the *poles* of $X(z)$ and the roots of the numerator $N(z)$ are called the *zeros* of $X(z)$.

Inverse z-transform of Rational Polynomials of z

Distinct Poles

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{(z - p_1)(z - p_2) \cdots (z - p_N)} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

Multipying both sides by $(z - p_k)$, and substituting $z = p_k$, we get

$$(z - p_k) \frac{X(z)}{z} \bigg|_{z = p_k} = A_k$$

Find the inverse z-transform of the following,

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

Inverse z-transform of Rational Polynomials of z

Inverse z-transform of Rational Polynomials of z

Multiple-order poles. Let $X(z)$ have a pole of multiplicity l, then the denominator has a term $(z - p_k)^l$

Find the inverse z-transform of the following,

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2}$$

Inverse z-transform of Rational Polynomials of z

$$\mathcal{Z}^{-1}\left(\frac{1}{1 - z^{-1}p_k}\right) = \begin{cases} p_k^n \cdot 1[n], & ROC : |z| > |p_k| \\ -p_k^n \cdot 1[-n-1], & ROC : |z| < |p_k| \end{cases}$$

When we have multiple poles,

$$\mathcal{Z}^{-1}\left(\frac{pz^{-1}}{(1 - z^{-1}p)^2}\right) = np^n \cdot 1[n]$$

Response of LTI systems

Let the transfer function of an LTI system be,

$$H(z) = \frac{B(z)}{A(z)}$$

Let the z-transform of the input signal be $X(z) = \frac{N(z)}{Q(z)}$.
The output of the system is given by,

$$Y(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Assuming there are no repeated poles in the denominator $A(z)Q(z)$,

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

$$y[n] = \sum_{k=1}^{N} A_k \cdot p_k^n \cdot 1[n] + \sum_{k=1}^{N} Q_k \cdot q_k^n \cdot 1[n]$$

Response of LTI systems

Find the output of the causal LTI system for zero initial conditions,

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

for the following inputs.

1. $\delta[n]$
2. $1[n]$
3. $0.1^n \cdot 1[n]$
4. $\cos(0.25\pi n) \cdot 1[n]$

Response to non-zero initial conditions

$$Y(z) = H(z)X(z) + \frac{N_0(z)}{A(z)}$$

- $Y_{zs}(z) = H(z)X(z)$ is the zero-state response.
- $Y_{zi}^{+}(z) = \frac{N_0(z)}{A(z)}$ is the zero-input response.

$$y_{zi}[n] = \sum_{k=1}^{N} D_k \cdot p_k^n \cdot 1[n]$$

Total response.

$$y[n] = \sum_{k=1}^{N} (A_k + D_k) \cdot p_k^n \cdot 1[n] + \sum_{k=1}^{N} Q_k \cdot q_k^n \cdot 1[n]$$

Stability of LTI systems

BIBO stability criteria.

$$\sum_n |h[n]| < \infty$$

For a causal LTI system, if all the poles of its transfer function lie within the unit circle, then the system is BIBO stable.
In general, an LTI system is BIBO stable if and only if the ROC of $H(z)$ includes the unit circle.