Transducers & Instrumentation

Module 01/02

(Sensor Dynamics Characteristics; LTI system; Convolution; Laplace Transform; Frequency Response; Zero, First, and Second Order LTI systems)

Sensor dynamic characterization

- Many sensors do not respond instantaneously to a given input.

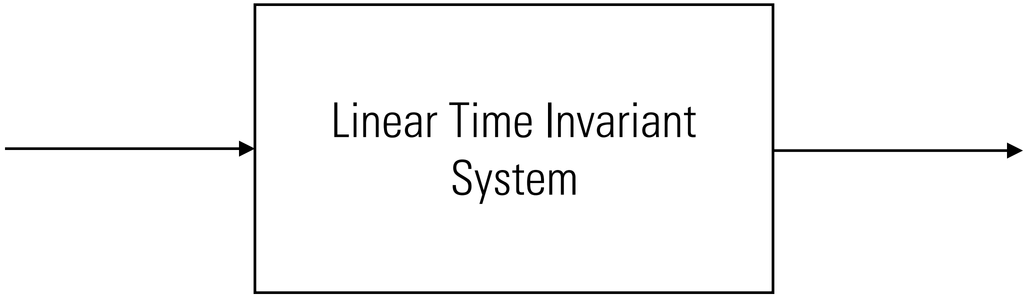
E.g., Contact thermometer

- Mathematically described using differential equations relating the measured and other inputs to the sensor output.

- A common and very useful model for such dynamical systems are linear time invariant systems.

Linear Time Invariant Systems

- Both Linear and Time Invariant.



Some useful signals

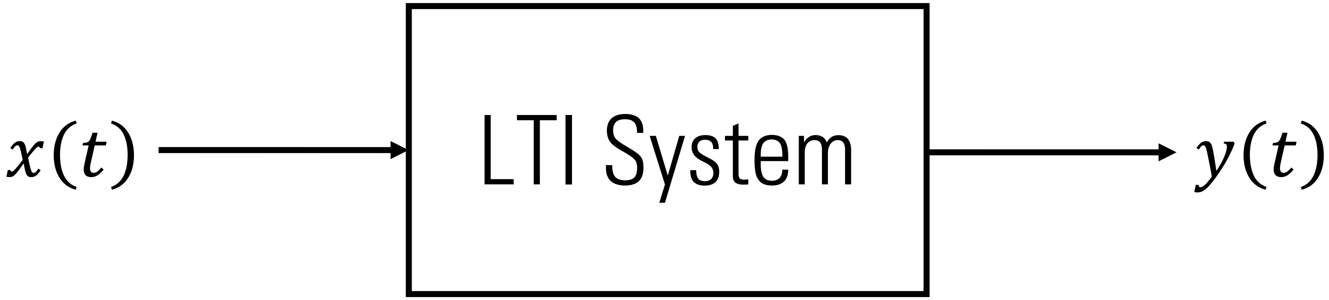
- Step signal: $1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$
- Exponential signal: $A \cdot e^{\beta t}$
- Sinusoidal signals: $A \cdot \sin(\omega t + \varphi)$

Some useful signals

- Dirac Delta Function $\delta(t)$: $\int_a^b \delta(t)dt = \begin{cases} 1, & 0 \in [a,b] \\ 0, & 0 \notin [a,b] \end{cases}$

$$\int_a^b f(t)\delta(t-t_0)dt = \begin{cases} f(t_0), & t_0 \in [a,b] \\ 0, & t_0 \notin [a,b] \end{cases}$$

Input-Output Relationship of LTI systems



Convolution

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

Impulse Response

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow h(t)$$

Another Description of LTI systems

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x$$

Laplace Transform

- A very popular and useful integral transform method for analysing LTI systems.

- Unilateral Laplace transform.

$$X(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt\,, \qquad s \in \mathbb{C}, s = \sigma + j\omega$$

- Laplace Transform Pairs:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

Laplace Transform Pairs

Time Domain Signal	Laplace Transform
$1(t)$	
$e^{at}1(t)$	
$\sin(\omega_0 t)\,1(t)$	
$\delta(t)$	

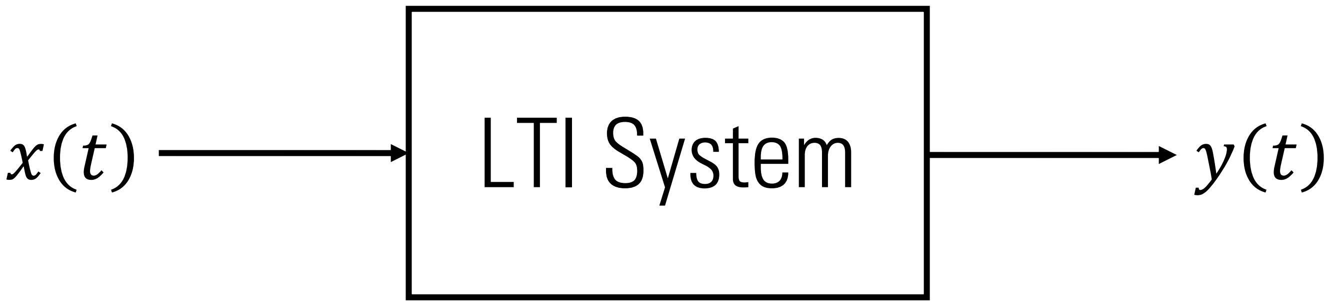
Laplace Transform Property

- $x(t)$ and $X(s)$ are Laplace transform pairs. Then,

$$y(t) \xleftrightarrow{\mathcal{L}} Y(s) \Longrightarrow ax(t) + by(t) \xleftrightarrow{\mathcal{L}} a\,X(s) + bY(s)$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \Longrightarrow \frac{dx}{dt} \xleftrightarrow{\mathcal{L}} sX(s) - x(0^{-})$$

Why is the Laplace Transform useful for LTI systems?

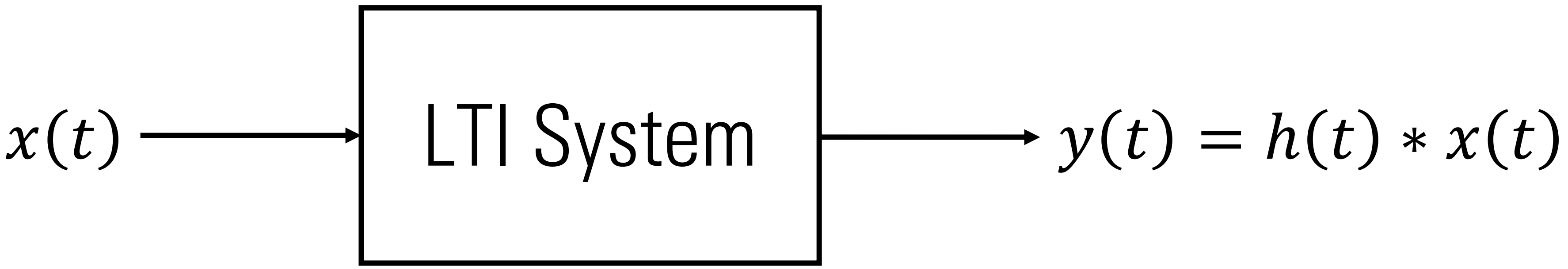


Transfer Function of an LTI system

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s) \Longrightarrow H(s) = \frac{Y(s)}{X(s)}$$

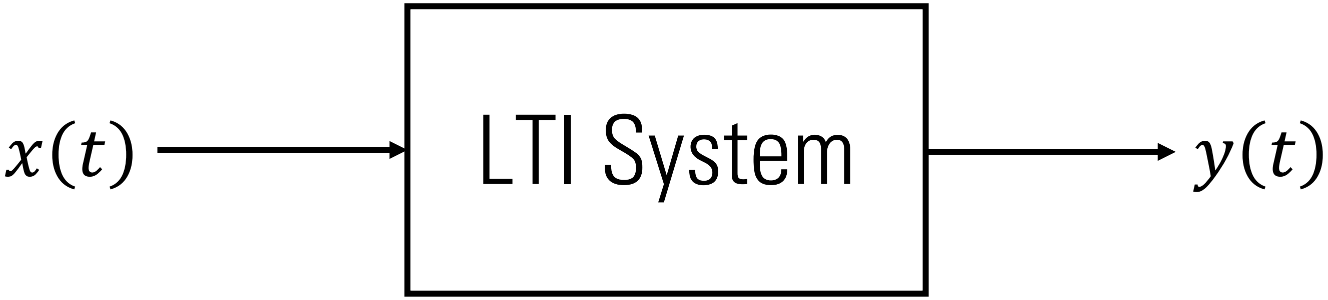
Frequency Response of an LTI System



$$H(j\omega) = H(s)\Big|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)}$$

$$H(j\omega) = |H(j\omega)|e^{j\,\arg(H(j\omega))} \rightarrow \begin{cases} |H(j\omega)| & \text{Magnitude Response} \\ \arg\big(H(j\omega)\big) & \text{Phase Response} \end{cases}$$

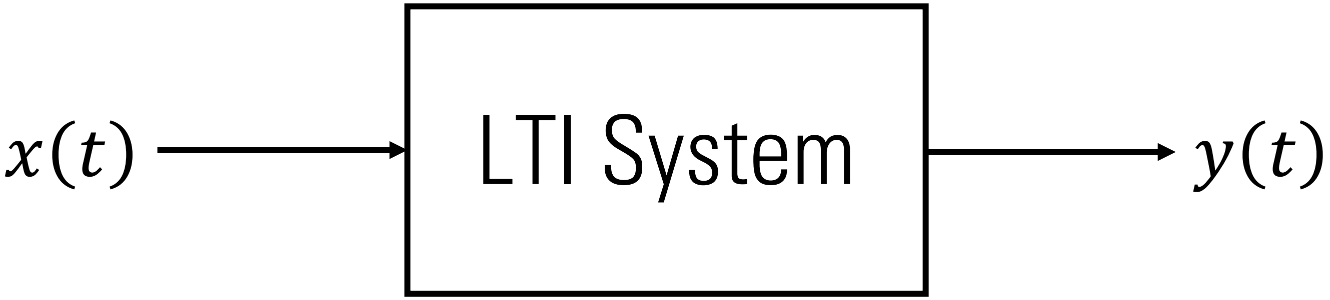
Zero Order System



$$a_0 y(t) = b_0 x(t)$$

Pure Linear Resistor
Pure Linear Spring

First Order System



$$a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

Time Constant: $\tau \triangleq \dfrac{a_1}{a_0}$

Static Sensitivity: $K \triangleq \dfrac{b_0}{a_0}$

First Order System

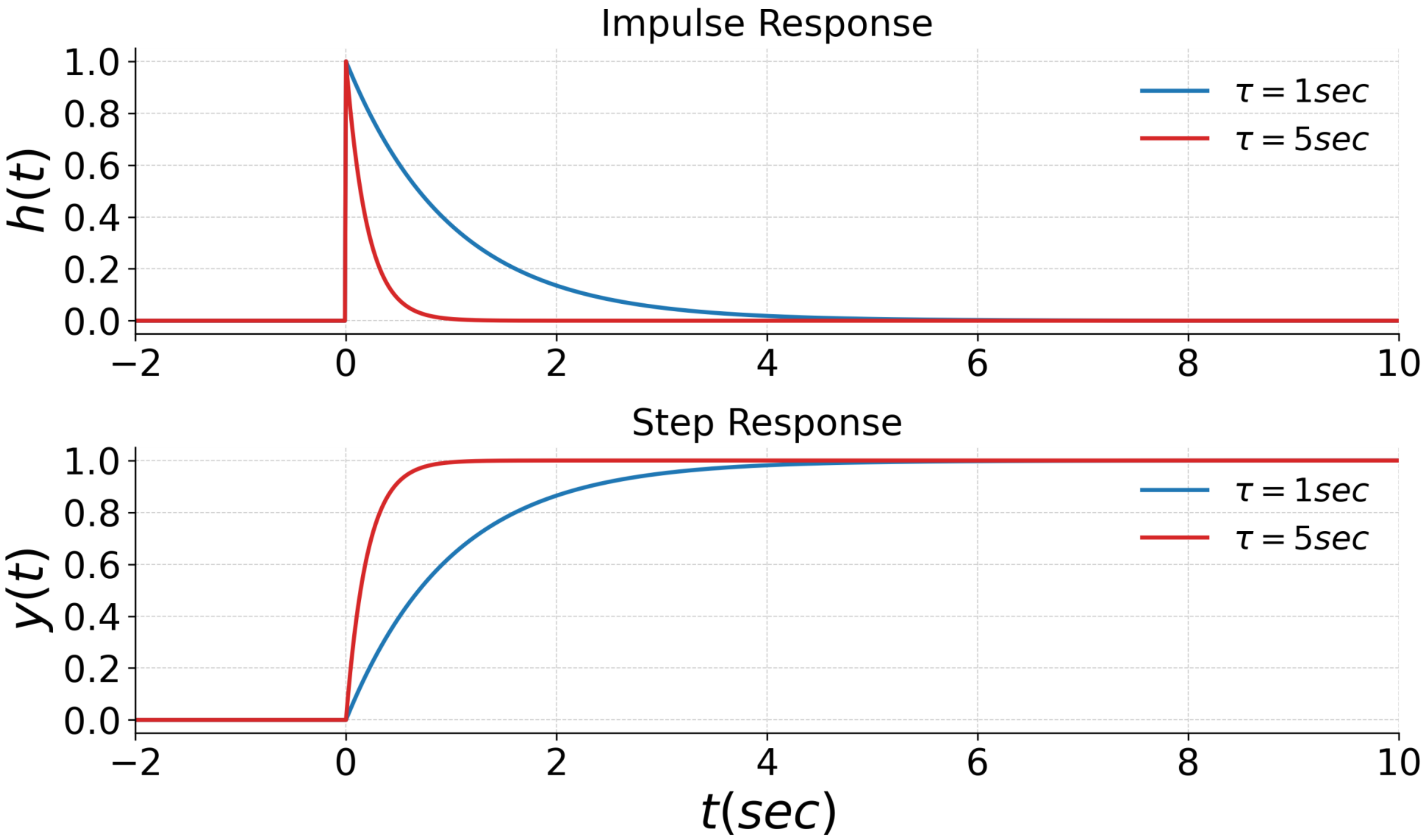
$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

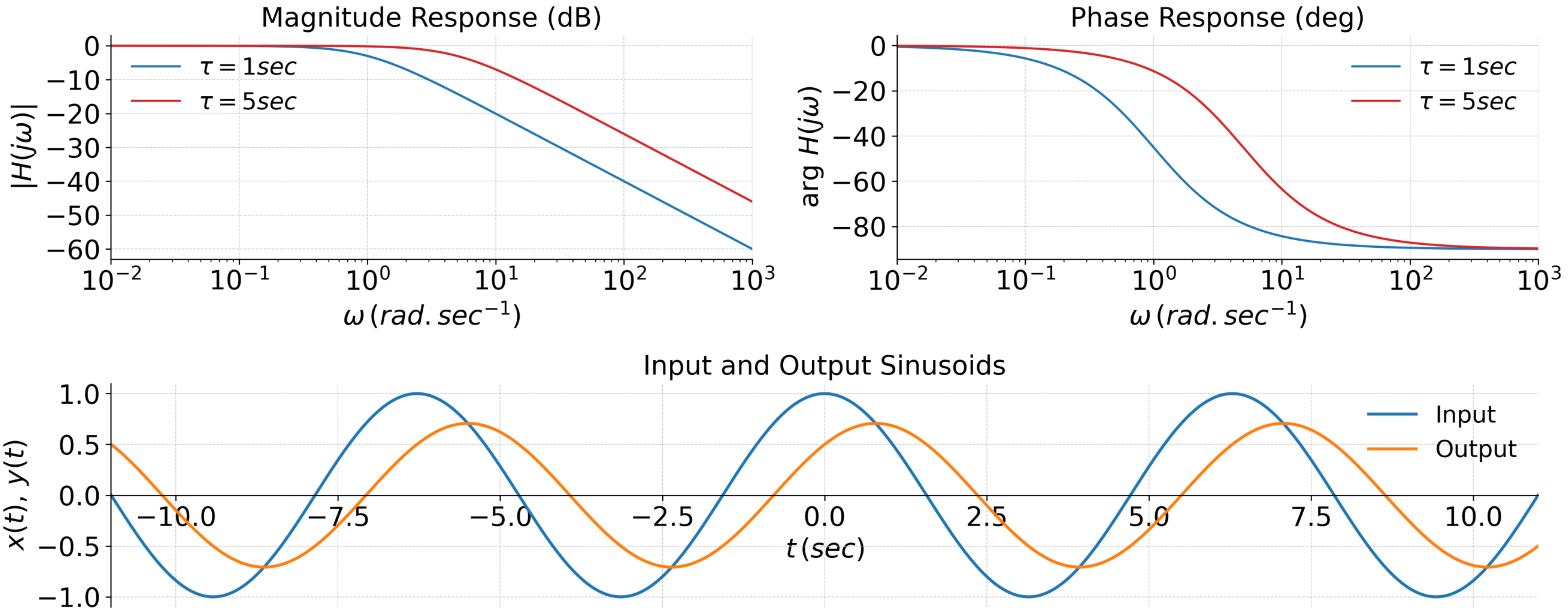
RC / RL circuits

Spring-damper/Mass-damper

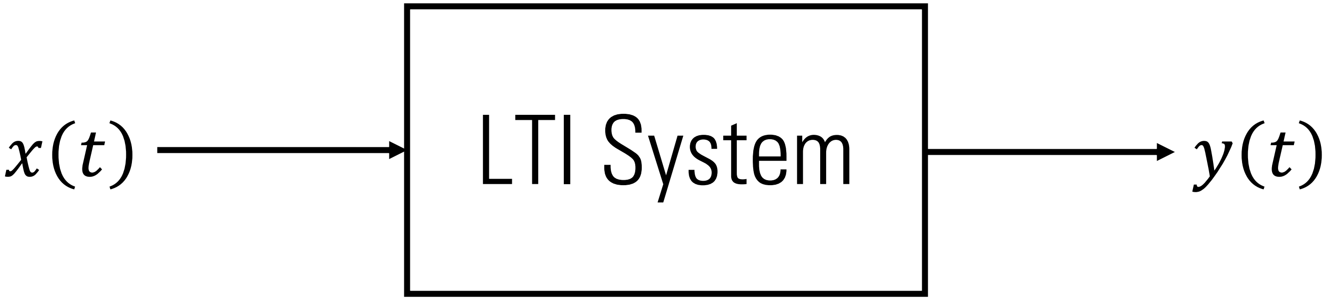
First Order System



First Order System



Second Order System



$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

Static Sensitivity: $K \triangleq \dfrac{b_0}{a_0}$

Natural Frequency: $\omega_n \triangleq \sqrt{\dfrac{a_0}{a_2}}$ $\implies \dfrac{1}{\omega_n^2}\dfrac{d^2y}{dt^2} + \dfrac{2\zeta}{\omega_n^2}\dfrac{dy}{dt} + y = Kx$

Damping ratio: $\zeta \triangleq \dfrac{a_1}{2\sqrt{a_0 a_2}}$

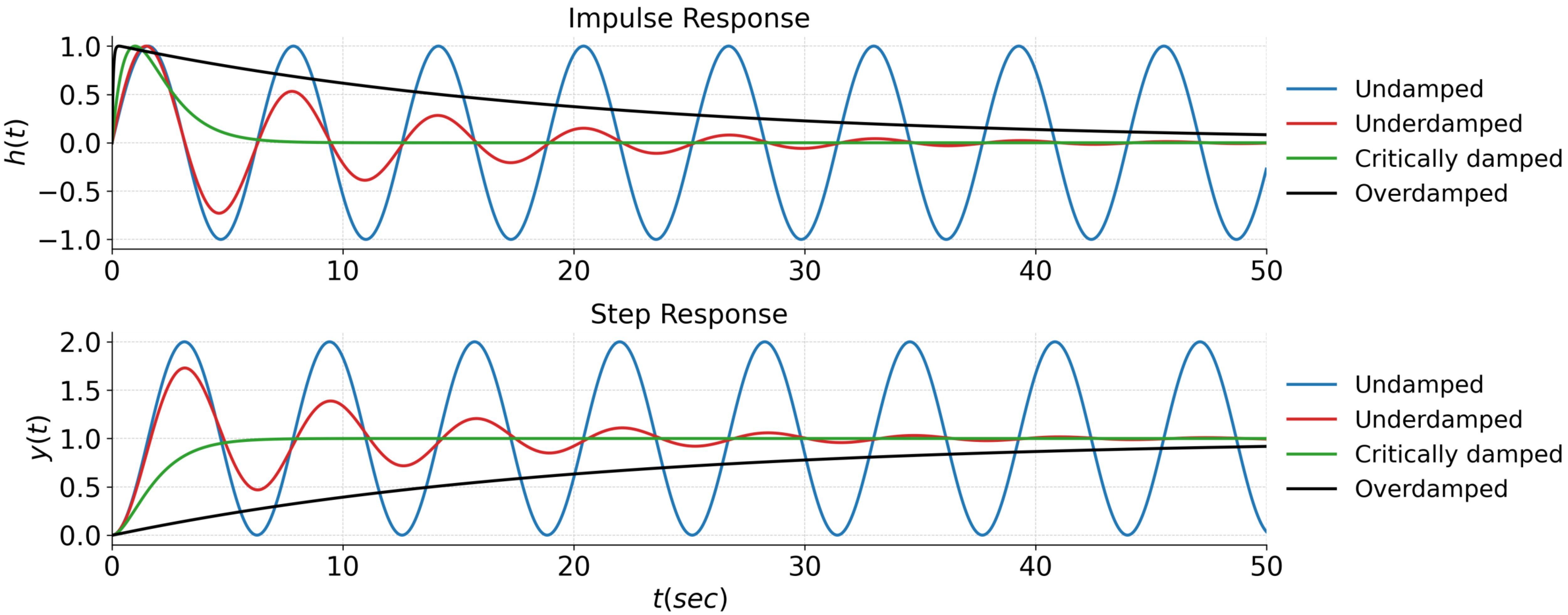
Second Order System

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

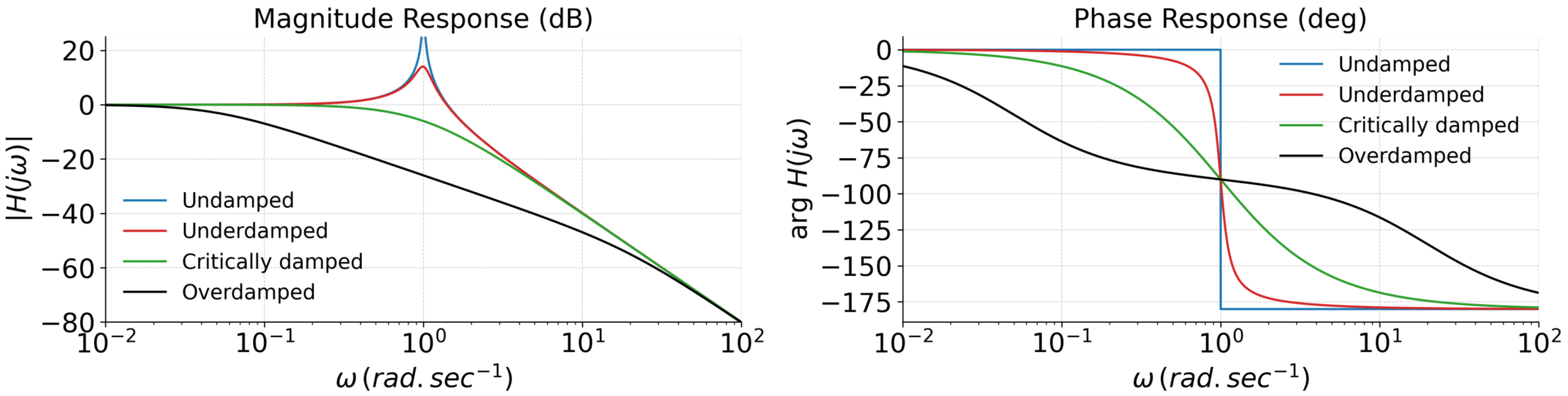
$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

RLC circuits
Mass-Spring-damper

Second Order System Response

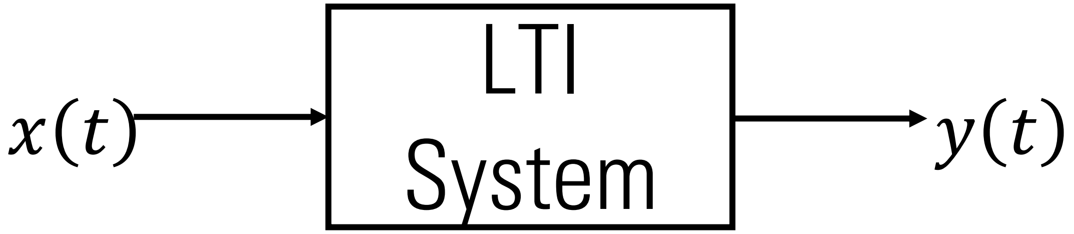


Second Order System – Frequency Response



Dynamic characterization of sensors

- Identifying sensor parameters from measured data.



- System identification tools can be used for doing this.
- Simple procedure for first order system using a step response.

$$y(t) = K \cdot \left(1 - e^{-t/\tau}\right) \Longrightarrow \log_e\left(1 - \frac{y(t)}{K \cdot x(t)}\right) = -\frac{t}{\tau}$$

Dynamic characterization of sensors