Linear Systems: Singular Value Decomposition Assignment

1. For a square $\mathbf{A} \in \mathbb{R}^{n\times n}$, the SVD tells us how a unit sphere in $\mathbb{R}^n$ is distorted by the linear transformation performed by $\mathbf{A}$. This degree of distortion can be quantified using the singular values of $\mathbf{A}$, which is the 2-norm *condition number*,

$$\kappa = \frac{\sigma_1}{\sigma_n}$$

   (a) Explain why $\kappa \geq 1$?

   (b) What is condition number of a singular matrix?

   (c) If $\mathbf{A}$ is non-singular, show that $\kappa = \|\mathbf{A}\|_2 \left\|\mathbf{A}^{-1}\right\|_2$

   (d) Condition numbers can also be defined based on other p-norms. The general p-norm condition number is given by, $\kappa_p = \|\mathbf{A}\|_p \left\|\mathbf{A}^{-1}\right\|_p$. Evaluate the 1-norm, 2-norm and ∞-norm condition numbers for the following matrices. How do these number compare with each other? (i) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; (ii) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 10 & -9 \end{bmatrix}$; (iii) $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}$.

   (e) Conditions numbers play an important role in practice. We had earlier an example of an ill-conditioned system $\mathbf{Ax} = \mathbf{b}$ (problem **??**). Consider the following systems, where: (i) $\mathbf{A}_1 = \begin{bmatrix} 1 & -1 \\ 10 & -9 \end{bmatrix}$; and (ii) $\mathbf{A}_2 = \begin{bmatrix} 1 & -10 \\ 1 & 10 \end{bmatrix}$.
   For $\mathbf{b} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, what are the solutions $\mathbf{x}_1 \left(= \mathbf{A}_1^{-1}\mathbf{b}\right)$ and $\mathbf{x}_2 \left(= \mathbf{A}_2^{-1}\mathbf{b}\right)$?
   Suppose there is an error in the measurement of $\mathbf{b}$, and we have $\tilde{\mathbf{b}} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$. The relative error in $\mathbf{b}$ is given by $\delta b = \frac{\|\mathbf{b}-\tilde{\mathbf{b}}\|_2}{\|\mathbf{b}\|_2}$. What are the new solutions $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$?
   Calculate δx_1 and δx_2, the relative errors in $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively? How do these compare to δb?
   Note: Through this problem, you should be able to see that an ill-conditioned system has a large condition number, which can amplify error and thus lead to large uncertainty in the solutions.