Introduction to Digital Signal Processing Fourier Representation of Discrete-Time Signals

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Discrete-Time Fourier Series

- ▶ Continuous-time complex sinusoids have frequencies $\omega \in (-\infty, \infty)$.
- ▶ Discrete-time sinusoids have frequencies $\Omega \in (-\pi, \pi]$
- lacktriangle A discrete-time periodic signal x [n] with fundamental period N can be represented as a sum of discrete-time sinusoids,

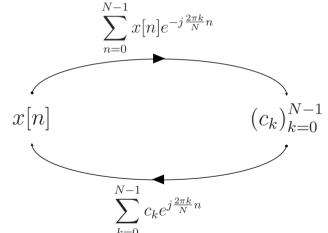
$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}$$

- ▶ Appropriate choice of c_k will let us represent x[n] as a linear combination of $e^{\frac{j2\pi k}{N}n}$.
- ightharpoonup The Fourier coefficient c_k can be determined by,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

Discrete-Time Fourier Series

 c_k is discrete like in the case of continuous-time Fourier series, and it is also periodic with fundamental period N, i.e. $c_k = c_{k+N}$.



Properties of Discrete-Time Fourier Series

- lacktriangle Fourier representation is discrete and periodic. (c_k is period with fundamental period N)
- ▶ When N = 2M is even, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \le l < \frac{N}{2}$$

▶ When N = 2M + 1 is odd, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \le l < \frac{N-1}{2}$$

Parseval's identity.

$$\frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

The distribution of $|c_k|^2$ as a function of $0 \le k < N$ is the *power spectral density* of the periodic signal x[n].

Discrete-Time Fourier Series

Find the DTFS of
$$x[n] = \begin{cases} 1, & 0 \leq n < M \\ 0, & M \leq n < N-1 \end{cases}$$
 with fundamental period $N.$

Discrete-time Fourier Transform

- Similar to the continus-time case, the Fourier representation of discrete -time aperiodic signals can be obtained as the limiting case of the a periodic signals with increasing period N.
- The discrete-time Fourier transform (DTFT) of an aperiodic signal x[n] with finie energy is given by,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, -\pi \le \Omega < \pi$$

- $ightharpoonup X\left(\Omega\right)$ is a continuous in Ω and periodic with period 2π .
- ► Inverse DTFT,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Discrete-time Fourier Transform

▶ DTFT exists only if x[n] is absolutely summable.

$$\sum_{n} |x[n]| < \infty \implies |X(\Omega)| < \infty$$

When x[n] is only square summable, then DTFT converges to the true DTFT only in the mean squared sense. E.g.,

$$x[n] = \begin{cases} \frac{\Omega_c}{n}, & n = 0\\ \frac{\Omega_c}{n} \frac{\sin \Omega_c n}{\Omega_c n}, & n \neq 0 \end{cases} \longrightarrow X(\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c\\ 0, & \Omega_c < |\Omega| \le \pi \end{cases}$$

Properties of DTFT

- ▶ Linearity: $\alpha x[n] + \beta y[n] \stackrel{\mathsf{DTFT}}{\longleftrightarrow} \alpha X(\Omega) + \beta Y(\Omega)$
- ▶ Shift in time: $x[n-n_0] \stackrel{\mathsf{DTFT}}{\longleftrightarrow} e^{-j\Omega n_0} X(\Omega)$
- ► Shift in frequency: $x[n]e^{j\Omega_0n} \xleftarrow{\mathsf{DTFT}} X(\Omega \Omega_0)$
- ▶ Time and frequency scaling: $x(\alpha t) \xleftarrow{\mathsf{DTFT}} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \ \alpha > 0$
- ▶ Symmetry in time: $x(t) \stackrel{\mathsf{DTFT}}{\longleftrightarrow} X(\Omega)$ is real.
- ► Convolution in time: $x[n] * y[n] \xleftarrow{\mathsf{DTFT}} X(\Omega)Y(\Omega)$

Classification of signals based on the frequencty spectrum

- ▶ We can also classify signals based on how their energy is distributed across frequency.
- ▶ Low frequency signal. Most of the energy is concentrated around 0Hz and frequencies around 0Hz.
- ▶ **High frequency signal**. Very less is concentrated around 0Hz, and most of the energy is in the higher frequencies all the way upto $\omega \to \infty$.
- ▶ Bandpass signal. Very less concentration around 0Hz and at high frequencies. Most of the energy is concentrated within a band of finite frequencies.
- **Bandlimited signal**. Signal energy is uniformly zero beyond a particular frequency, i.e. $|X(\omega)| = 0, \ \forall |\omega| > \omega_b$.

Bandwidth of a signal

▶ The band of frequencies over which most of the energy is distributed is the *bandwidth* of the signal.

Several ways to define the bandwidth of a signal.

▶ **3 dB bandwidth**. Frequency range over which the spectral density is above a particular threshold.

Threshold value is often defined relative to the max. value of the spectal density.

Frequency-Domain and Time-Domain Properties

▶ Continuous-Time, Periodic →

▶ Discrete-Time, Non-Periodic →

▶ Continuous-Frequency, Periodic →

▶ Discrete-Frequency, Non-Periodic →