Linear Systems: Matrix Inverse Assignment

1. Consider the following bases for $\mathbb{R}^3$.

$$A^S = \left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} \right\}$$

$$B^A = \left\{ \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

Where, X^Y is the basis X represented in another basis Y; S stands for the standard basis. Let $\mathbf{b}_X$ stand for the representation of vector in $\mathbb{R}^3$ in the basis X.

   (a) Consider a vector $\mathbf{b}_S = \begin{bmatrix}-1\\0\\2\end{bmatrix}$ represented in the standard basis. What is the representation of $\mathbf{b}_S$ in the other four basis A, and B?

   (b) Consider a vector $\mathbf{d}_B = \begin{bmatrix}-1\\-1\\0\end{bmatrix}$ represented in the basis B. What is the representation of this vector in the standard basis?

2. When does the following diagnoal matrix have an inverse?

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

Write down an expression for D^{-1}.

3. Prove that the inverse of a non-singular upper-triangular matrix is upper-triangular. Using this show that for a lower triangular matrix it is lower-triangular.

4. Consider a 2×2 block matrix, $\mathbf{A} = \begin{bmatrix}\mathbf{B} & \mathbf{C}\\ \mathbf{D} & \mathbf{E}\end{bmatrix}$, where $\mathbf{A} \in \mathbb{R}^{m\times m}$. Find an expression for the inverse $\mathbf{A}^{-1}$ interms of the block components and their inverses (if they exist) of $\mathbf{A}$. Hint: Consider $\mathbf{A}^{-1} = \begin{bmatrix}\mathbf{P} & \mathbf{Q}\\ \mathbf{R} & \mathbf{S}\end{bmatrix}$, and solve $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

5. Express the inverse of the following matrix in terms of $\mathbf{A}$ and $\mathbf{b}$.

$$\mathbf{H} = \begin{bmatrix}\mathbf{A} & \mathbf{b}\\ \mathbf{0} & 1\end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

where, $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$.

6. Consider a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ with linearly independent columns. Prove that the Gram matrix $\mathbf{A}^T\mathbf{A}$ is invertible.

7. Find all possible left/right inverses for the following matrices, if they exist.

   (a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix}$

   (b) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

   (c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -3 & 4 \end{bmatrix}$

   (d) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

For each of these matrices find the corresponding pseudo-inverse $\mathbf{A}^\dagger$, and verify that the pseudo-inverse has the minimum squared sum of its components.

8. Prove that the inverse of a non-singular symmetric matrix is symmetric.

9. Consider the scalar equation, $ax = ay$. Here we can cancel a from the equation when $a \neq 0$. When can we carry out similar cancellations for matrcies?

   (a) $\mathbf{AX} = \mathbf{AY}$. Prove that here $\mathbf{X} = \mathbf{Y}$ only when $\mathbf{A}$ is left invertible.

   (b) $\mathbf{XA} = \mathbf{YA}$. Prove that here $\mathbf{X} = \mathbf{Y}$ only when $\mathbf{A}$ is right invertible.

10. Consider two non-singular matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Explain whether or not the following matrices are invertible. If they are, then provide an expression for it inverse.

   (a) $\mathbf{C} = \mathbf{A} + \mathbf{B}$

   (b) $\mathbf{C} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}$

   (c) $\mathbf{C} = \begin{bmatrix}\mathbf{A} & \mathbf{A}+\mathbf{B}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}$

   (d) $\mathbf{C} = \mathbf{ABA}$

11. Consider the matrices $\mathbf{A} \in \mathbb{R}^{m\times l_1}$ and $\mathbf{B} \in \mathbb{R}^{l_2\times m}$. Can you find the requirements for matrices $\mathbf{A}$ and $\mathbf{B}$, such that $\mathbf{AXB} = \mathbf{I}$, where $\mathbf{X} \in \mathbb{R}^{l_1\times l_2}$? Assuming those conditions are satisfied, find an expression for $\mathbf{X}$?

12. Consider a matrix $\mathbf{C} = \mathbf{AB}$, where $\mathbf{A} \in \mathbf{R}^{m\times n}$ and $\mathbf{B} \in \mathbf{R}^{n\times m}$. Explain why $\mathbf{C}$ is not invertible when $m > n$. Suppose $m < n$, under what conditions is $\mathbf{C}$ invertible?

13. For a square matrix $\mathbf{A}$ with non-signular $\mathbf{I} - \mathbf{A}$, prove that $\mathbf{A}\left(\mathbf{I}-\mathbf{A}\right)^{-1} = \left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{A}$.

14. Consider the non-singular matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$. Prove that,

$$\mathbf{A}\left(\mathbf{A}+\mathbf{B}\right)^{-1}\mathbf{B} = \mathbf{B}\left(\mathbf{A}+\mathbf{B}\right)^{-1}\mathbf{A} = \left(\mathbf{A}^{-1}+\mathbf{B}^{-1}\right)^{-1}$$