Linear Systems: Extras

Matrices

1. If the augmented matrix $\left[\mathbf{A}|\mathbf{b}\right]$ is reduced to the matrix $\left[\mathbf{E}|\mathbf{c}\right]$.
   (a) Is $\left[\mathbf{E}|\mathbf{c}\right]$ in row echelon form if $\mathbf{E}$ is?
   (b) If $\left[\mathbf{E}|\mathbf{c}\right]$ is in row echelon form, is $\mathbf{E}$ also in row echelon form?
2. Reducing a matrix $\mathbf{A}$ to its reduced row echelon through row operations form reveals the relationship between the different columns of $\mathbf{A}$. Explain why the row operations on $\mathbf{A}$ leave the relationship between its columns unaffected.
3. Consider the following reduced row echelon form.

$$\mathbf{E_A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Is $\mathbf{A}$ unique? If it is, then find $\mathbf{A}$. If it is not unique, (a) Explain why $\mathbf{A}$ is not ; and (b) What additional information would you need to uniquely determine $\mathbf{A}$?
4. Can a parabola $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ pass through the points: $\left\{(0,1), (1,4), (2,11), (-1,2)\right\}$?
5. Explain why the system $\mathbf{Ax} = \mathbf{b}$ cannot be inconsistent if $rank\left(\mathbf{A}\right) < n$.
6. Consider a homogeneous system of equations which has n unknowns and l free variables. What is rank of $\mathbf{A}$?
7. Can a linear system $\mathbf{Ax} = \mathbf{b}$ have exactly 2 solutions? Explain you answer.
8. Consider a augmented matrix $\left[\mathbf{A}|\mathbf{b}\right]$ of a consistent system, with the number of equations greater than the number of unknowns, i.e. $m \geq n$. What will $\mathbf{E_A}$ look like for a consistent system?
9. Explain how the $\mathbf{LU}$ factors of a matrix $\mathbf{A}$ can be used to determine $\mathbf{A}^{-1}$.
10. Consider two matrices $\mathbf{A} \in \mathbb{R}^{m \times p}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$. Prove that,

$$C\left(\left[\mathbf{A}|\mathbf{B}\right]\right) = C\left(\mathbf{A}\right) + C\left(\mathbf{B}\right)$$

11. Are the following statements true? Explain your answer.
    (a) $C\left(\mathbf{AB}\right) \subseteq C\left(\mathbf{A}\right)$
    (b) $N\left(\mathbf{AB}\right) \supseteq C\left(\mathbf{B}\right)$
12. Consider a set of vector $B = \left\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\right\}$. Prove that the set $\mathbf{A}\left(B\right) = \left\{\mathbf{Ab}_1, \mathbf{Ab}_2, \ldots, \mathbf{Ab}_n\right\}$ spans $C\left(\mathbf{AB}\right)$.
13. Consider a consistent set of linear equations, $\mathbf{Ax} = \mathbf{b}$, and let $\mathbf{a} \in C\left(\mathbf{A}^T\right)$. Prove that $\mathbf{a}^T\mathbf{x}$ is constant for a all solutions x of the equation $\mathbf{Ax} = \mathbf{b}$.
14. If $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$ is a square matrix, with $N\left(\mathbf{A}_1\right) = C\left(\mathbf{A}_2^T\right)$. Then prove that $\mathbf{A}$ is non-singular.
15. Prove that the rank of a matrix $\mathbf{A}$ is invariant under multiplication by a non-singular matrix.

Miscellaneous

16.
17. Prove that $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite, if the columns of the $\mathbf{A}$ are independent.
18. Demonstrate that a full rank, symmetric matrix $\mathbf{A}$ can be expressed as a sum of a series of simple rank one matrices of the form,

$$\mathbf{A} = \sum_{i=1}^{n} \mathbf{l}_i d_i \mathbf{l}_i^T$$

19. Show that the quadratic form $\mathbf{x}^T\mathbf{Ax}$ can be reduced to the form $\mathbf{x}^T\tilde{\mathbf{A}}\mathbf{x}$, where $\tilde{\mathbf{A}} = \hat{\mathbf{A}}^T$.
20. Prove that $\mathbf{x}^T\mathbf{Ax} = 0$ when $\mathbf{A}$ is skew symmetric.
21. Consider a matrix $\mathbf{A}$ with m rows, $\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \\ \vdots \\ \tilde{\mathbf{a}}_m^T \end{bmatrix}$, where $\tilde{\mathbf{a}}_i \in \mathbb{R}^n$. If a new row $\tilde{\mathbf{a}}_{m+1}^T$ is added to $\mathbf{A}$, how does $\mathbf{A}^T\mathbf{A}$ change?