Introduction to Digital Signal Processing

Discrete Fourier Transform

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Fourier Representation of Discrete-time Signals

- Discrete-time Fourier Series (DTFS):

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n} \qquad x[n] = \sum_{k=0}^{N-1} X[k] \cdot e^{j\frac{2\pi k}{N}n}$$

  Sinusoids with discrete frequencies that are an integer multiple of $\Omega_0 = \frac{2\pi}{N}$ are considered, with $0 \leq k < N$.

- Discrete-time Fourier Transform (DTFT):

$$X(\Omega) = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\Omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

  Sinusoids with all possible frequencies are considered, with $-\pi \leq \Omega < \pi$.

Connection between ω and Ω

- Continuus-time frequency: $\omega \in \mathbb{R} \longrightarrow \cos(\omega t)$

- Discrete-time frequency: $\Omega \in [-\pi, \pi) \longrightarrow \cos(\Omega n)$

- Let a continuous-time $x_c(t)$ be sampled at $F_s = \frac{1}{T_s}$Hz to obtain the discrete-time signal $x_d[n] = x_c(n \cdot T_s)$.
$$\Omega = \frac{\omega}{F_s}$$

Connections between properties in time and frequency domains

Time Doman	Frequency Domain
Periodic, Continuous	Periodic, Continuous
Non-periodic, Continuous	Non-periodic, Continuous
Periodic, Discrete	Periodic, Discrete
Non-periodic, Discrete	Non-periodic, Discrete

We cannot use any of these for Fourier analysis of non-periodic signals on a computer.

Can we use a sampled version of the DTFT? How can we be sure that we have not missed any information?

Discrete Fourier Transform (DFT): Sampled DTFT

- Sampling a DTFT and reconstructing the time domain signal will results in a periodic time-domain signal.

$$x[n] \longrightarrow X(\Omega)$$

We sample $X(\Omega)$ at uniform intervals such that $\delta\Omega = \frac{2\pi}{N}$. If we reconstruct the time-domain signal from the sampled DTFT $X\left(\frac{2\pi k}{N}\right)\Big|_{0 \leq k < N}$.

$$x_r[n] = \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j\frac{2\pi k}{N}n}$$

$x_r[n]$ is periodic with fundamental period $N \longrightarrow x_r[n+N] = x_r[n]$.

If $x[n]$ is time-limited, i.e. $x[n] = 0, \,\, \forall 0 < n$ and $n \geq N$, then

$$x_r[n] = x[n], \,\, \forall 0 \leq x[n] < N$$

Discrete Fourier Transform (DFT)

Let $x[n]$ be a time-limited signal, such that $x[n] = 0, \; \forall 0 < n, \;$ and $n \geq N$, then

$$X[k] \triangleq \sum_{n=0}^{N-1} x[n] \cdot W_N^{-kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot W_N^{kn}, \qquad n = 0, 1, 2, \ldots, N-1$$

Discrete Fourier Transform (DFT)

Let's represent $x[n]$ as a column vector,

$$\mathbf{x}_N = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} \qquad \mathbf{X}_N = \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & W_N^3 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Discrete Fourier Transform (DFT)

$$\mathbf{X}_N = \mathbf{W}_N \cdot \mathbf{x}_N \qquad \text{and} \qquad \mathbf{x}_N = \mathbf{W}_N^{-1} \cdot \mathbf{X}_N = \frac{1}{N}\mathbf{W}_N^* \cdot \mathbf{X}_N$$

Thus, we have

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* \implies \mathbf{W}_N \cdot \mathbf{W}_N^H = n \cdot \mathbf{I}$$

The matrix $\tilde{\mathbf{W}}_N = \frac{1}{\sqrt{N}}\mathbf{W}_N$ is a unitary matrix, as $\tilde{\mathbf{W}}_N \cdot \tilde{\mathbf{W}}_N^H = \tilde{\mathbf{W}}_N^H \cdot \tilde{\mathbf{W}}_N = \mathbf{I}$.

The columns of $\tilde{\mathbf{W}}_N$ for an orthonornal basis for $\mathbb{C}^N$!

Geometry of the N-point DFT

2-point DFT

Let $x[n] = \{x[0], x[1]\} = \{2, 1\}$. Compute the 2-point DFT of this signal.

Computing the Inverse DFT for n beyond 0 and N

What would happen if we computed $x[n]$, $n < 0$ or $n \geq N-1$?

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot W_M^{kn}$$

Let $n = N + l$, $l \in \mathbb{Z}$. Then,

$$\begin{aligned} x[N+l] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot W_M^{k(N+l)} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot W_M^{kN} \cdot W_M^{kl} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot W_M^{kl} \\ &= x[l] \end{aligned}$$

Negative frequencies in DFT

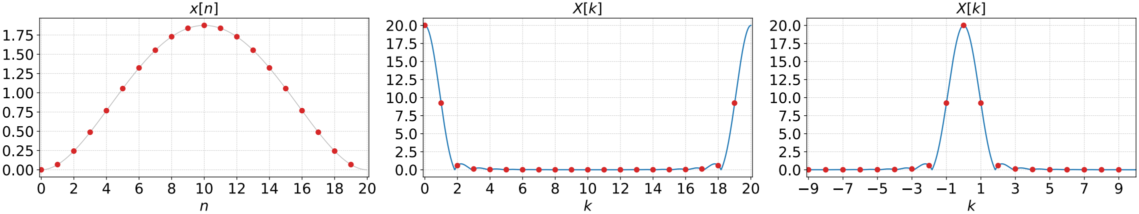
What is $X[k]$ for $k < 0$?

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{-kn} = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n}$$

Let N be even, and let $k = N - l$, such that $0 < l < \frac{N}{2}$,

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi(N-l)}{N}n} \\ &= \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi N}{N}n} \cdot e^{-j\frac{2\pi(-l)}{N}n} = \sum_{n=0}^{N-1} x[n] \cdot e^{j\frac{2\pi l}{N}n} \\ &= x[-l] \end{aligned}$$

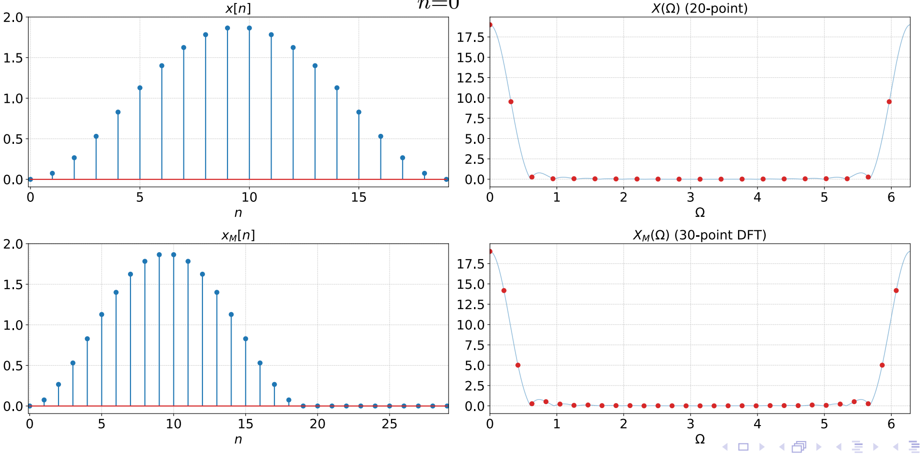
Negative frequencies in DFT



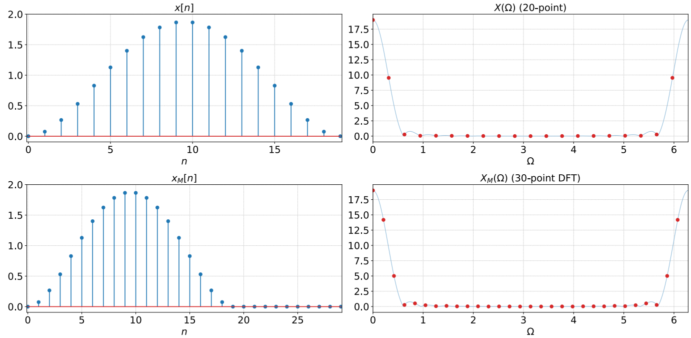
Zero-padding in the time-domain

Let N be the length of the signal $x[n]$. The M-point DFT, where $M > N$ is,

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_M^{kn}$$



Zero-padding in the time-domain



Mapping of the ω to Ω or k

Let $x[n]$ be obtained by sampling at F_s a continuous-time signal $x(t)$, whose Fourier transform is $X(\omega)$.

Length of $x[n]$ is $N \implies x(t)$ is time-limited with duration $N\frac{1}{F_s}$.

Mapping of the ω to Ω or k:

$$\omega \in (-2\pi F_s, 2\pi F_s) \quad \mapsto \quad \Omega \in [-\pi, \pi) \quad \mapsto \quad \begin{cases} -\frac{N}{2}+1 \leq k \leq \frac{N}{2}, & k \text{ is even} \\ -\frac{N-1}{2} \leq k \leq \frac{N-1}{2}, & k \text{ is odd} \end{cases}$$

$$\text{Frequency corresponding to } k: \left(\frac{F_s}{N}\right)k$$

$$\text{Frequency resolution of the } N\text{-point DFT}: \frac{F_s}{N}$$

You cannot increase frequency resolution by increasing sampling rate!!!!

Zero-padding in the Frequency

Let $x[n]$ be obtained by sampling at F_s a continuous-time signal $x(t)$, whose Fourier transform is $X(\omega)$.

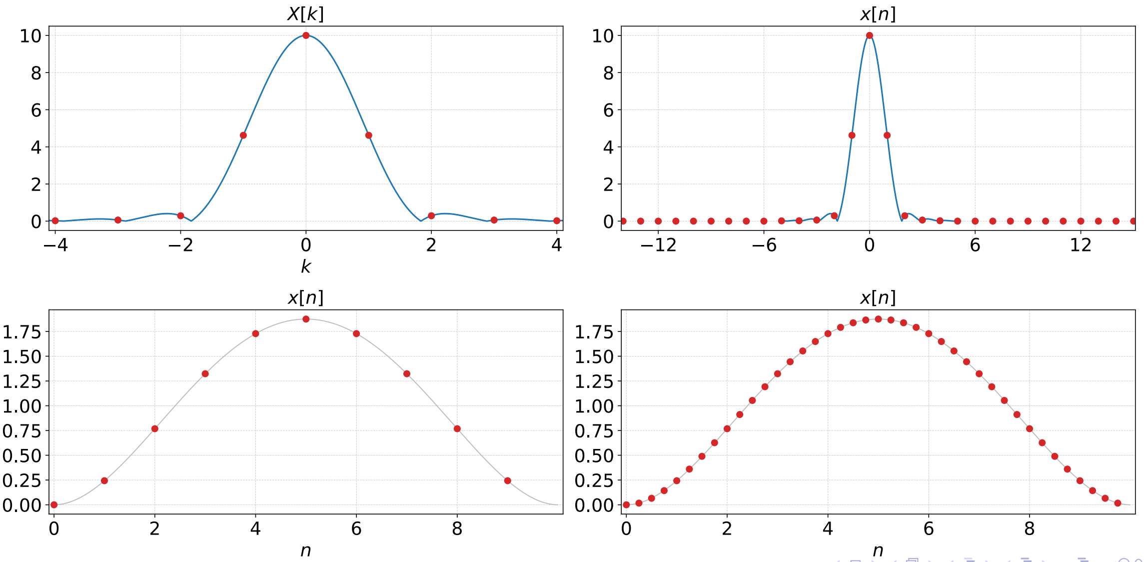
Length of $x[n]$ is $N \implies x(t)$ is time-limited with duration $N\frac{1}{F_s}$. Let's assume N to be even.

Let $X[k]$ be the N-point DFT of $x[n]$, such that $-\frac{N}{2}+1 \leq k \leq \frac{N}{2}$.

Zero-padding $X[k]$: Let's append zeros to $X[k]$ to increase its length to M; M is assumed to be even.

$$\tilde{X}[k] = \begin{cases} X[k], & -\frac{N}{2}+1 \leq k \leq \frac{N}{2} \\ 0, & -\frac{M}{2}+1 \leq k \leq -\frac{N}{2} \\ 0, & \frac{N}{2}+1 \leq k \leq \frac{M}{2} \end{cases}$$

Zero-padding in the Frequency



Properties of DFT

- **Periodicity**

Properties of DFT

- **Symmetry of DFT**

Properties of DFT

- **Multiplication and Circular Convolution**

$$x[n] \circledast y[n] \xleftrightarrow{\text{DFT}} X[k] \cdot Y[k]$$

Properties of DFT

- **Multiplication and Circular Convolution**

$$x[n] \cdot y[n] \stackrel{\text{DFT}}{\longleftrightarrow} \frac{1}{N} X[k] \circledast Y[k]$$

Properties of DFT

- **Parseval's theorem**

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{n-1} |X[k]|^2$$

Frequency analysis of signals using DFT

Let $x(t)$ be a continuous-time signal of interest, and let $X(\omega)$ be its frequency spectrum.

$$x(t) = \cos(2\pi f_0 t) \implies X(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

Sampling this signal at F_sHz we get $x[n] = \cos\left(2\pi f_0 \frac{n}{F_s}\right) = \cos\left(\frac{2\pi f_0}{F_s} n\right)$, where $\Omega_0 = \frac{2\pi f_0}{F_s}$.

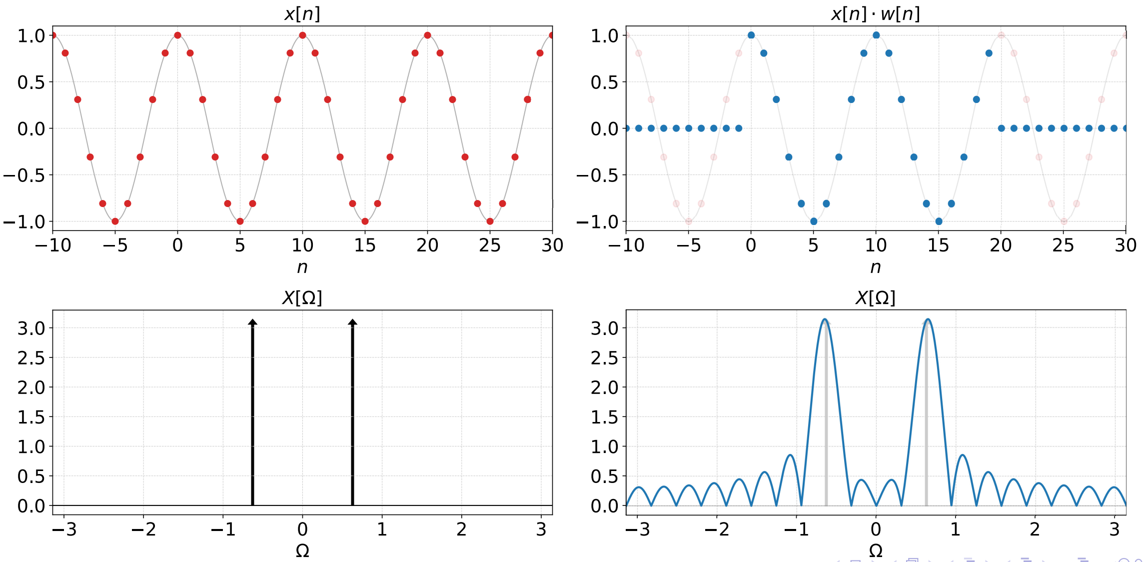
$$X(\Omega) = \pi\delta(\Omega + \Omega_0) + \pi\delta(\Omega - \Omega_0)$$

In practice, we only take a finite number of samples from $x[n]$ for analysis. Let the number of samples be N, and thus the duration of the signal considered for analysis if $T = \frac{N}{F_s}$.

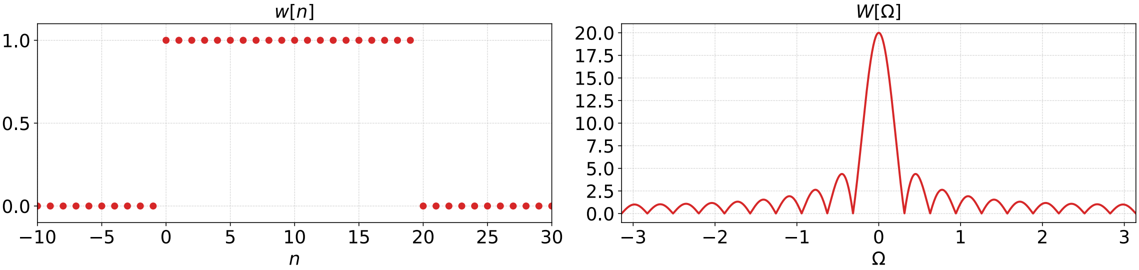
$$x_a[n] = \begin{cases} x[n], & 0 \leq n < N \\ 0, & \text{Otherwise} \end{cases} = x[n] \cdot w[n]$$

where, $w[n] = \begin{cases} 1, & 0 \leq n < N \\ 0, & \text{Otherwise} \end{cases}$.

Frequency analysis of signals using DFT

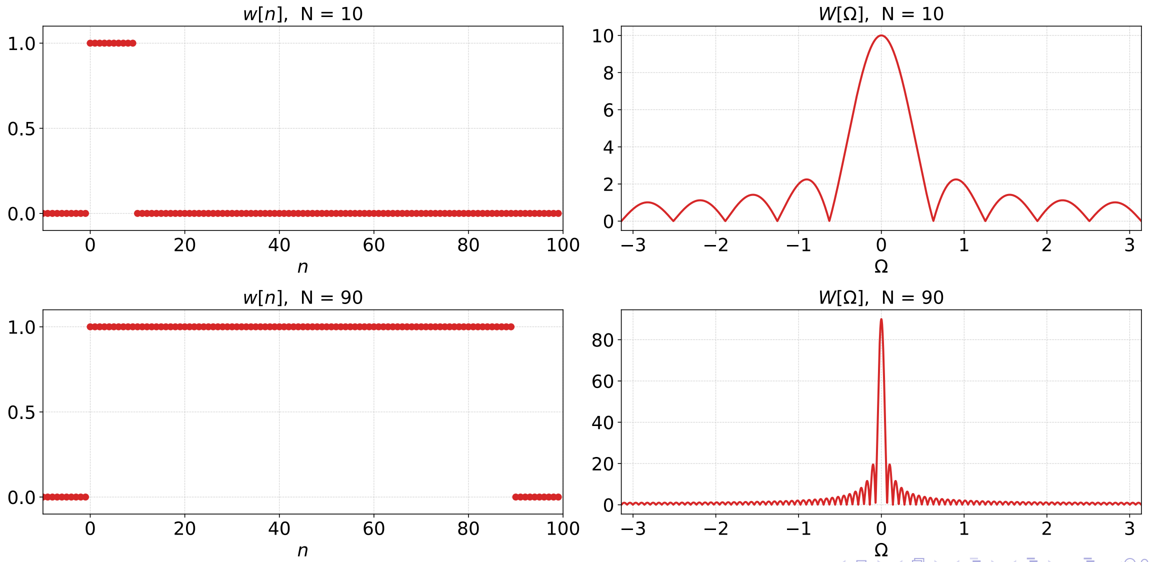


Frequency analysis of signals using DFT



$$W(\Omega) = \frac{\sin(\Omega N/2)}{\sin(\Omega/2)} e^{-j\Omega(N-1)/2}$$

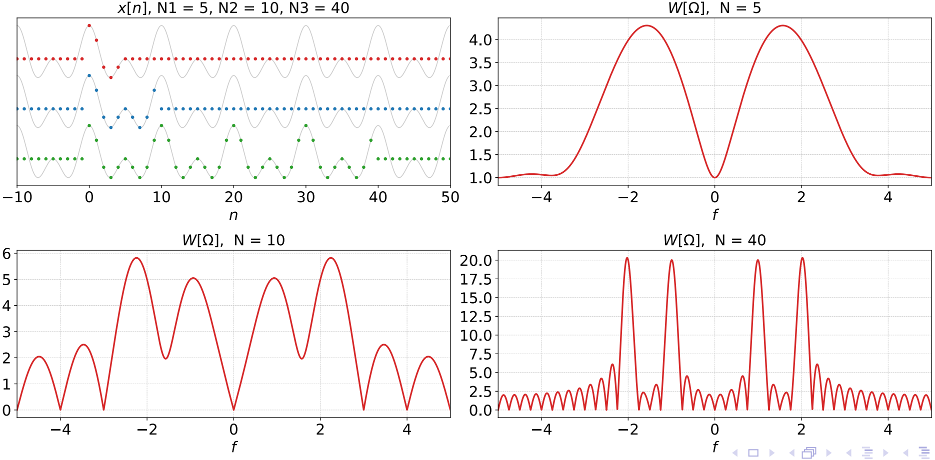
Frequency analysis of signals using DFT



Frequency analysis of signals using DFT

Consider the signal $x(t)$ sampled at $F_s = 10Hz$.

$$x(t) = \cos(2\pi t) + \cos(4\pi t)$$



Frequency analysis of signals using DFT

Resolving two frequencies depends on the size of the rectangular window.

$$W\left(\Omega\right) = \frac{\sin\left(\Omega N/2\right)}{\sin\left(\Omega/2\right)} e^{-j\Omega(N-1)/2}$$

Condition on the window size for resolving signals with close frequencies,

$$|\Omega_1 - \Omega_2| \geq \frac{2\pi}{N}$$

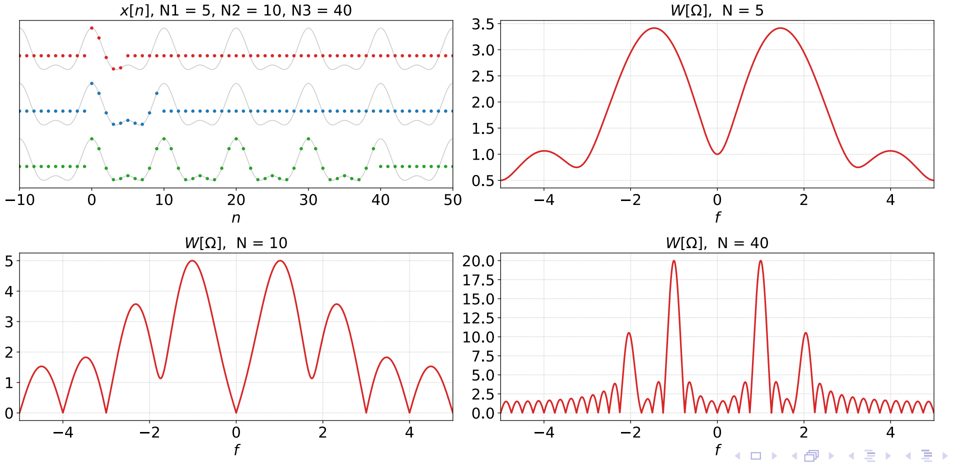
When data is sampled at F_s, then

$$|f_1 - f_2| \geq \frac{F_s}{N}$$

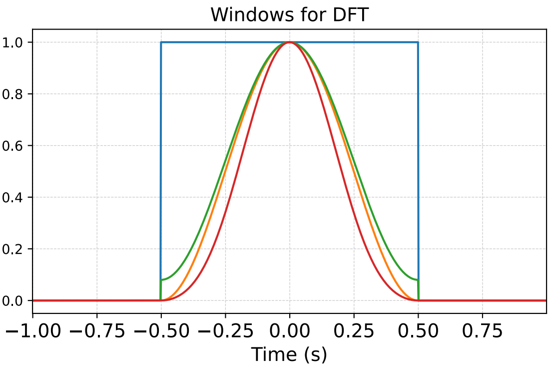
Frequency analysis of signals using DFT

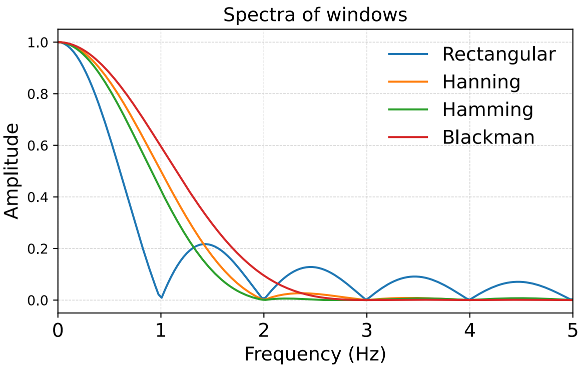
Consider the signal $x(t)$ sampled at $F_s = 10Hz$.

$$x(t) = \cos(2\pi t) + 0.5\cos(4\pi t)$$

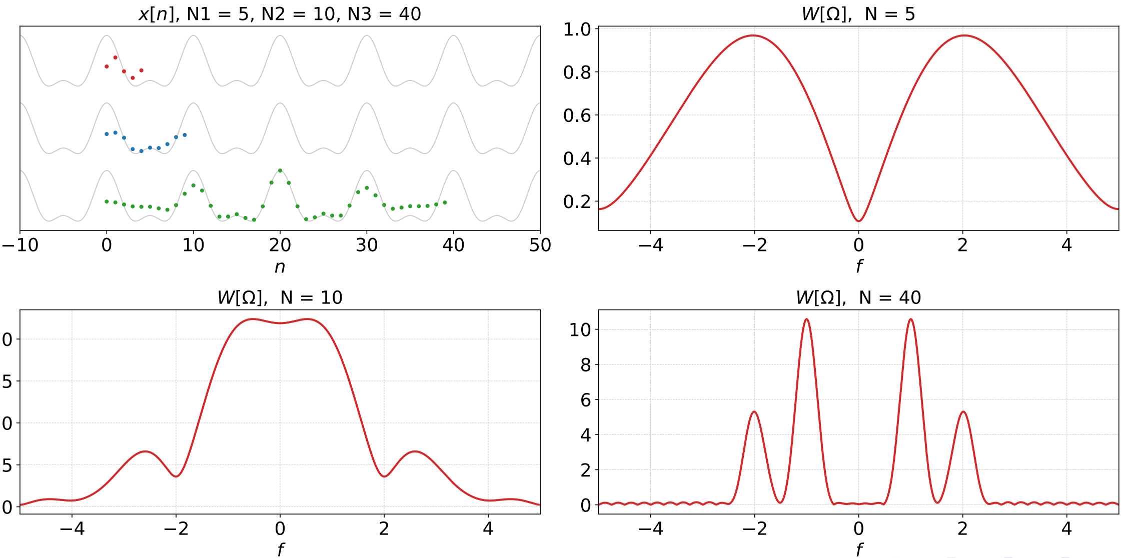


Other windows for DFT analysis

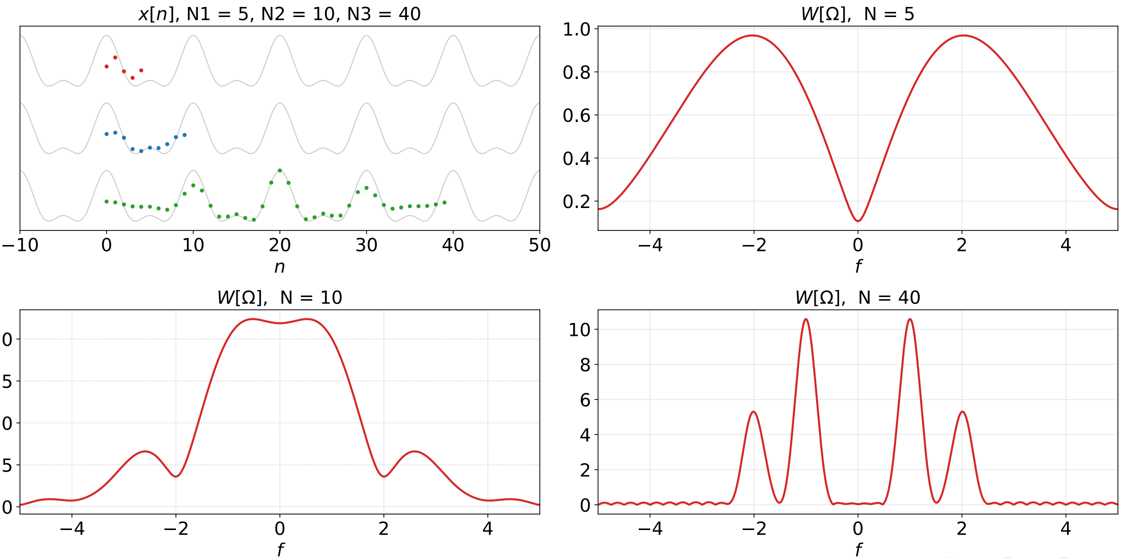




Frequency analysis of signals using DFT: Hamming Window



Frequency analysis of signals using DFT: Hamming Window



Another effect of window length