Introduction to Signal Processing

Lecture 5: **Laplace Transform and its Properties**

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Reading material

- Chapter 6 from reference [2]
- Chapter 9 from reference [3]

Laplace transform

- For an LTI system, we saw that the following is true,

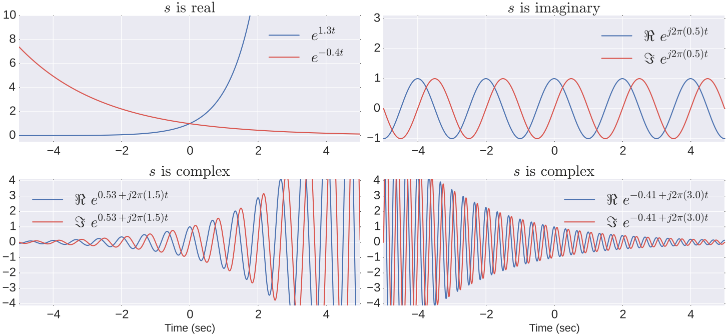
$$y(t) = e^{st}\left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)$$

- This naturally led to the Fourier transform, where $s = j\omega$.
- There is no restriction on s to be purely imaginary. One would obtain a more general transform by assuming, $s = \sigma + j\omega \quad \longrightarrow$ *Laplace transform.*
- The Fourier transform which was a tool to decompose a given signal $x(t)$ into different complex exponentials $e^{j\omega}$.
- What does the Laplace transform do? i.e. when $s = \sigma + j\omega$?

Laplace transform

- Laplace transform decomposes a given signal $x(t)$ into exponential growing or decaying sinusoids.

$$X(s) = \mathcal{L}\left\{x(t)\right\} \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt; \;\; x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$



- Fourier transform can be obtained from Laplace transform, $X(s)\Big|_{s=j\omega}$.

Laplace transform

- Another way to look at the Laplace transform is the following,

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t} \right) e^{-j\omega t} dt$$

- Laplace transform can be see as the Fourier transform of $x(t)e^{-\sigma t}$.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \overset{\mathcal{F}}{\longleftrightarrow} x(t)e^{-\sigma t}$$

- Just like the Fourier transform, the Laplace transform exists if the above integral converges. The convergence of this integral can be controlled by the factor σ.
- To understand this, evaluate the Laplace transform of $x(t) = e^{\alpha t}u(t)$. Does the Laplace transform exist for all values of s?

Laplace transform

- The set of all s (with the corresponding σ) for which the Laplace transform exists, is the called the *Region of convergence (ROC)*.
- A complete specification of a signal's Laplace transform requires the specification of $X(s)$ and the ROC.
- Why is this? You can understand this by evaluating the Laplace transforms of the following:

$$x_1(t) = e^{at}u(t) \text{ and } x_2(t) = -e^{-at}u(-t)$$

What are their Laplace transforms and their ROCs? This should make it clear why the ROC is essential.

Unilateral Laplace Transform

- Laplace transform is often used as a tool to analyse LTI systems, through the Laplace transform of their impulse response.

$$H(s) = \mathcal{L}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

  $H(s)$ is called the transfer function of the LTI system.

- We are often only interested in systems that are causal, i.e. impulse responses that are one-sided. $h(t) = 0, \forall t < 0$. For such cases, we define a modified version of the Laplace transform, called the *unilateral* Laplace transform,

$$H(s) = \mathcal{L}_{-}\{h(t)\} \triangleq \int_{0^-}^{\infty} h(t)e^{-st}dt$$

  The lower limit is set to 0^- because this enables the transform to take into the initial conditions in a system. This will be made more clear later.

Transfer Function

Evaluate the Laplace transforms of the following impulse responses:

- $h(t) = e^{-t/\tau}u(t),\ \tau > 0$
- $h(t) = a_1 e^{-t/\tau_1}u(t) + a_2 e^{-t/\tau_2}u(t),\ \tau_1 > 0 \text{ and } \tau_2 > 0$
- $h(t) = e^{-\alpha t}\sin 2\pi\omega t,\ \alpha > 0$
- $h(t) = e^{\beta t},\ \beta > 0$

A common feature you will find from the Laplace transforms of all these impulse responses is that they are all rational polynomials of the following form,

$$H(s) = \frac{B(s)}{A(s)}$$

The roots of the two polynomial $B(s)$ and $A(s)$ are called the *zeros* and *poles* of the transfer function.

Transfer Function

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots a_0}$$

$$H(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)}; \;\; K = \frac{b_M}{a_N}$$

Where the z_is and p_is are the zeros and poles of the transfer function, respectively.

Zeros: $\lim_{s \to z_i} H(s) = 0$ and **Poles**: $\lim_{s \to p_i} H(s) = \infty$

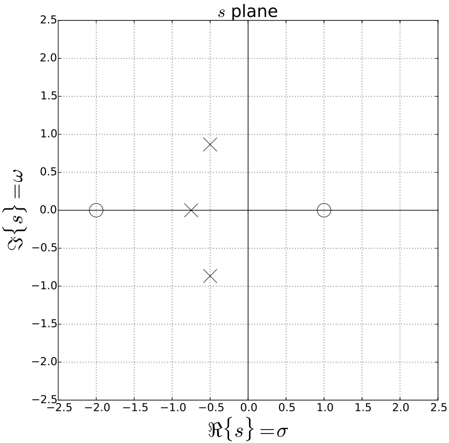
The poles and zeros of a transfer function can visualized on the complex s plane with σ as the abscissa and ω as the ordinate.

z_i is presented as a 'o' and p_i is represented as a '$\times$'.

s plane

Consider the following transfer function.

$$H(s) = \frac{(s-1)(s+2)}{(s-\frac{3}{4})(s^2+s+1)} = \frac{(s-5)(s+2)}{(s-\frac{3}{4})(s+\frac{1}{2}-j\frac{\sqrt{3}}{2})(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})}$$



Properties of the ROC

- The ROC is always a strip parallel to the imaginary axis ($j\omega$ axis).
  This this because the convergence properties of the Laplace transform for a signal are determined only by $\Re\{s\} = \sigma$.
- For rational Laplace transforms, the ROC does not contain any poles.
  If the ROC has any poles, then it would contain a point where the Laplace transform integral does not converge.
- For a right sided signal[1], if a point $\Re\{s\} = \sigma$ is in the ROC, then all points $\Re\{s\} > \sigma$ are also in the ROC.
- For a left sided signal[2], if a point $\Re\{s\} = \sigma$ is in the ROC, then all points $\Re\{s\} < \sigma$ are also in the ROC.
- If the signal is two sided[3], then the ROC is restricted to a strip, such that $\sigma_1 < \Re\{s\} < \sigma_2$.
- For a finite duration signal, if there is at least one value of s in the ROC, then the entire s-plane is in the ROC.

---

[1] The value of the signal is zero prior to a finite time value T.

[2] The value of the signal is zero after to a finite time value T.

[3] [illegible]

Inverse Laplace Transform

How does one would recover the time domain signal from a Laplace transform? $X(s)$ is the Fourier transform of $x(t)e^{-\sigma t}$ for some fixed $\sigma = \Re\{s\}, s \in \text{ROC}$.

So, the inverse Laplace transform can be obtained by taking the inverse Fourier transform of $X(\sigma + j\omega)$.

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{\sigma + j\omega t}d\omega$$

Let $\sigma + j\omega = s \implies jd\omega = ds$. Then the *inverse Laplace transform* is given by,

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

For most practical purposes, the above complex integral need not be computed. We simply make use of table of standard transform pairs and properties of Laplace transform to obtain the inverse.

Properties of Laplace transform

- **Linearity:** Let, $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, $s \in R_1$ and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, $s \in R_2$. Then,
$$a_1x_1(t) + a_2x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a_1X_1(s) + a_2X_2(s), \;\; s \in R_1 \cap R_2$$
- **Time shifting:** Let, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $s \in R$. Then,
$$x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} X(s)e^{st_0}, \;\; s \in R$$
- **Shifting in the s domain:** Let, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $s \in R$. Then,
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \;\; s \in R + \Re\{s_0\}$$
- **Time scaling:** Let, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $s \in R$. Then,
$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X(\frac{s}{a}), \;\; s \in \frac{R}{a}$$

Properties of Laplace transform

- **Convolution:** Let, $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, $s \in R_1$ and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, $s \in R_2$. Then, $x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s)$, with the ROC containing $R_1 \cap R_2$

- **Differentiation in time:** Let, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $s \in R$. Then,

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \text{ with the ROC containing } R$$

- **Differentiation in s:** Let, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $s \in R$. Then,

$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \quad s \in R$$

- **Integration in time:** Let, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $s \in R$. Then,

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s), \quad s \in R \cap R_u$$

  Where, R_u is the ROC of the Laplace transform of $u(t)$.

Properties of Laplace transform

Consider a signal $x(t)$, such that $x(t) = 0, \forall t < 0$, and does not contain any singularities of any order[4].

- **Initial value theorem:** $x\left(0_{+}\right) = \lim_{s \to \infty} sX(s)$
- **Final value theorem:** $\lim_{t \to \infty} x\left(t\right) = \lim_{s \to 0} sX(s)$

In the case of *unilateral Laplace transform*, the differentiation property is slightly different,

$$\mathcal{L}_{-}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0_{-})$$

In fact, this can repeated, to obtain the transform of higher order derivatives,

$$\mathcal{L}_{-}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2X(s) - sx(0_{-}) - \dot{x}(0_{-})$$

This property is very useful for taking into account the effect of initial conditions on the response of an LTI system.

[4]E.g. Impulse function and its higher order derivatives.

Properties of Laplace transform

Consider a signal $x(t)$, such that $x(t) = 0, \forall t < 0$, and does not contain any singularities of any order[5].

- **Initial value theorem:** $x(0_+) = \lim_{s \to \infty} sX(s)$
- **Final value theorem:** $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$

In the case of *unilateral Laplace transform*, the differentiation property is slightly different,

$$\mathcal{L}_- \left\{ \frac{dx(t)}{dt} \right\} = sX(s) - x(0_-)$$

In fact, this can repeated, to obtain the transform of higher order derivatives,

$$\mathcal{L}_- \left\{ \frac{d^2x(t)}{dt^2} \right\} = s^2X(s) - sx(0_-) - \dot{x}(0_-)$$

This property is very useful for taking into account the effect of initial conditions on the response of an LTI system.

[5] E.g. Impulse function and its higher order derivatives.