# Linear Systems: Positive Definite Matrices and Matrix Norm Assignment

1. Prove that $\mathbf{A}^T\mathbf{A}$ is positive semi-definite for any matrix $\mathbf{A}$. When is $\mathbf{A}^T\mathbf{A}$ guaranteed to be positive definite?

2. If $\mathbf{A}$ is positive definite, then prove that $\mathbf{A}^{-1}$ is also positive definite.

3. Show that a positive definite matrix cannot have a zero or a negative element along its diagonal.

4. Show that the following statements are true.

   (a) All positive definite matrices are inverstible.

   (b) The only positive definite projection matrix is $\mathbf{I}$.

5. Is the function $f(x_1, x_2, x_3) = 12x_1^2 + x_2^2 + 6x_3^2 + x_1x_2 - 2x_2x_3 + 4x_3x_1$ positive definite?

6. The $LU$ decomposition for symmetric matrices can be written as $\mathbf{A} = \mathbf{L}^T\mathbf{D}\mathbf{L}$, where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{L}$ is lower triangular with $1$ along its main diagonal. When $\mathbf{A}$ is postive definite, we can write, $\mathbf{A} = \mathbf{C}^T\mathbf{C} = \mathbf{L}^T\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{L}$. This is the *Cholesky decomposition*. Find $\mathbf{C}$ for the following,

   (a) $\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. Prove the following for $\mathbf{A} \in \mathbb{R}^{m \times n}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}_1^T} \\ \tilde{\mathbf{a}_1^T} \\ \vdots \\ \tilde{\mathbf{a}_m^T} \end{bmatrix}$$

   (a) $\|\mathbf{A}\|_1 = \max_{1 \le i \le n} \|\mathbf{a}_i\|_1$

   (b) $\|\mathbf{A}\|_\infty = \max_{1 \le i \le m} \|\tilde{\mathbf{a}}_i\|_1$

   (c) $\|\mathbf{A}\|_2 = \max_{1 \le i \le n} |\lambda_i|$, where $\lambda_i$ are the eigenvalues of $\mathbf{A}^T\mathbf{A}$.

   (d) $\|\mathbf{A}\|_F = trace\left(\mathbf{A}^T\mathbf{A}\right)$

8. Prove that the induced norm of a matrix product is bounded: $\|\mathbf{AB}\| \le \|\mathbf{A}\| \|\mathbf{B}\|$.

9. Verify the following inequalities on vector and matrix norms ($\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$):

   (a) $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$

   (b) $\|\mathbf{x}\|_2 \le \sqrt{m} \|\mathbf{x}\|_\infty$

   (c) $\|\mathbf{A}\|_\infty \le \sqrt{n} \|\mathbf{A}\|_2$

   (d) $\|\mathbf{A}\|_2 \le \sqrt{m} \|\mathbf{A}\|_\infty$

10. Find an expression for the induced 2-norm of an outer product, $\mathbf{A} = \mathbf{u}\mathbf{v}^T$, where $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$.