Introduction to Signal Processing

Lecture 7: **Analysis of Continuous-time LTI Systems**

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Transfer Functions

- LTI systems are described by linear constant coefficient differential equations,

$$\sum_{i=0}^{N} a_i \frac{d^i}{dt^i} y(t) = \sum_{j=1}^{M} b_j \frac{d^j}{dt^j} x(t)$$

- The transfer function is the ratio of the Laplace transform of $y(t)$ and $x(t)$ with zero initial conditions.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=1}^{M} b_i s^i}{\sum_{i=1}^{N} a_i s^i}$$

  where, $Y(s)$ and $X(s)$ are polynomials of s with order N and M, respectively, and the coefficients a_i and b_i are all real.

- In practice, we only encounter transfer functions that are proper rational fractions,i.e. $N \geq M$.

Decomposition of transfer functions

- A transfer function $H(s) = \frac{P(s)}{Q(s)}$ can be decomposed into two form, which are of practical importance: (a) *pole-zero* representation and (b) *partial fraction* representation.
- **Pole-zero** representation: $P(s)$ and $Q(s)$ are factorized, yielding

$$H(s) = \frac{(s - z_1)(s - z_2)\cdots(s - z_M)}{(s - p_1)(s - p_2)\cdots(s - p_M)}$$

  where, $z_i, p_i \in \mathbb{C}$ are the zeros and poles of the transfer function, respectively.
- **Partial fraction** representation: The following is the most general partial fraction representation,

$$H(s) = R(s) + \sum_{i=1}^{N} \sum_{r=1}^{j_i} \frac{A_i}{(s - p_i)^r}$$

  where, $p_i \in \mathbb{C}$ are the poles of the transfer function.

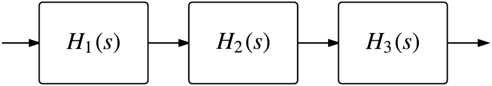
Pole-Zero representation

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_M)}$$

- Here, the transfer function is decomposed into a product of simpler transfer functions.
- The poles and zeros are real, or when they are complex, they occur in conjugate pairs (why?).
- So, in effect any $H(s)$ of practical interest can be split into first and second order systems,

$$G_1(s) = \frac{s+a}{s+b} \qquad G_2(s) = \frac{a_2s^2 + a_1s + a_0}{b_2s^2 + b_1s + b_0}$$

Cascading of simpler systems (1^{st} and 2^{nd} systems)



Pole-Zero representation

$$H(s) = K\frac{(s - z_1)(s - z_2)\cdots(s - z_M)}{(s - p_1)(s - p_2)\cdots(s - p_M)}$$

- The number of poles represent the number of memory elements in the system.
- The poles can tell us about the stability of the system. Let $H(s) = \frac{1}{s+a}$. Answer the following quesitons about this system:
  1. Where is the pole of the system?
  2. If the system is causal, what is the ROC? What is the ROC when the system is anti-causal?
  3. For a causal system, for what values of a is the system stable?

  Answering, these questions will help you understand the how the poles of a transfer function immediately tell us about the stability of a system.

Partial Fraction Representation

$$H(s) = R(s) + \sum_{i=1}^{N} \sum_{r=1}^{j_i} \frac{A_i}{(s - p_i)^r}$$

Here, $A_i, p_i \in \mathbb{C}$. In order to have real coefficients for realizing a system, the abov equation can be re-written as the following,

$$H(s) = R(s) + \sum_{i=1}^{L} \sum_{r=1}^{j_i} \frac{A_i}{(s - p_i)^r} + \sum_{i=1}^{M} \sum_{r=1}^{k_i} \frac{B_i s + C_i}{(s^2 + b_i s + c_i)^r}$$

where, $A_i, B_i, C_i, p_i, b_i, c_i \in \mathbb{R}$, and $b_i^2 - 4c_i < 0$.
For a proper rational function, where the order of the numerator is less than the order of the denominator, $R(s) = 0$.

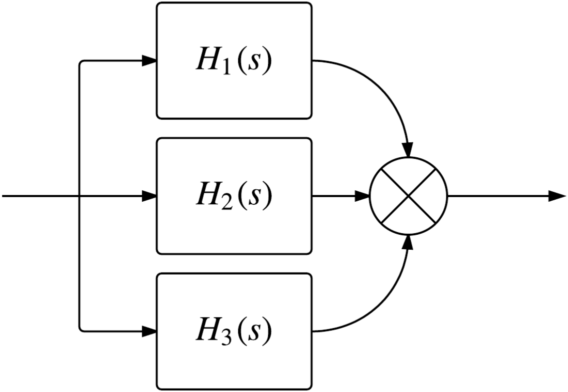
How does one find out the A_is, B_is and C_is?

Examples: Find the partial fraction representation for: (a) $\frac{s+3}{s^2+3s+5}$; (b) $\frac{s^3+5s^2+10s+3}{s^2+3s+5}$; and (c) $\frac{1}{s^3+s^2+2s+1}$.

Partial Fraction Representation

- Using a partial fraction representation one can implement a transfer fucntion as a parallel combination of simpler (1^{st} and 2^{nd} systems).

Parallel implementation of a system.



Frequency Response

The frequency response of a LTI system can be obtained from its transfer function.

$$H(\omega) = H(s)\Big|_{s=j\omega}$$

For a causal and stable system, the poles of the system must lie in the left half of the $j\omega$ axis. This means that the ROC includes the $j\omega$ axis, and thus implies that the $H(j\omega)$ exists.
Example:

$$H(s) = \frac{1}{s+a} \implies H(\omega) = \frac{1}{j\omega + a}$$

$$H(s) = \frac{s+b}{s^2+cs+d} \implies H(\omega) = \frac{j\omega + b}{d - \omega^2 + j\omega d}$$

Geometric estimation of frequency response.

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

$$\Longrightarrow \quad H(\omega) = K\frac{(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_M)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_N)}$$

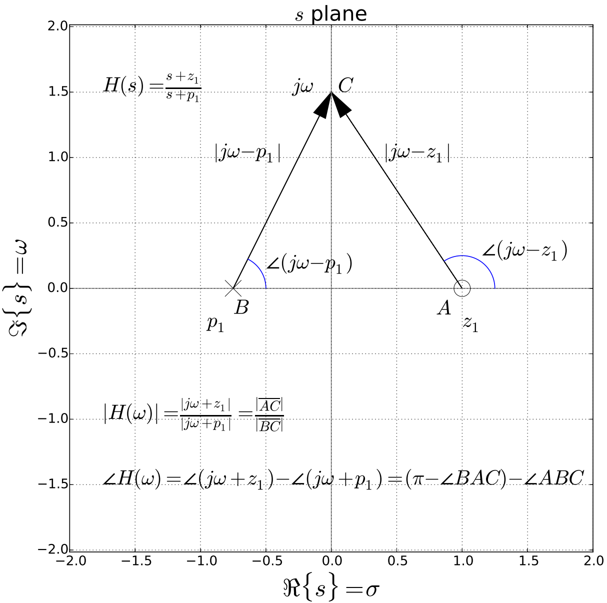
Here, the magnitude and phase responses can be estimated as the product and the sum of the contributions from the poles and the zeros.

$$\text{Magnitude Response} = |H(\omega)| = K\frac{\Pi_{i=1}^{M}|j\omega - z_i|}{\Pi_{i=1}^{N}|j\omega - p_i|}$$

$$\text{Phase Response} = \angle H(\omega) = \sum_{i=1}^{M}\angle(j\omega - z_i) - \sum_{i=1}^{N}\angle(j\omega - p_i)$$

Geometric estimation of frequency response.

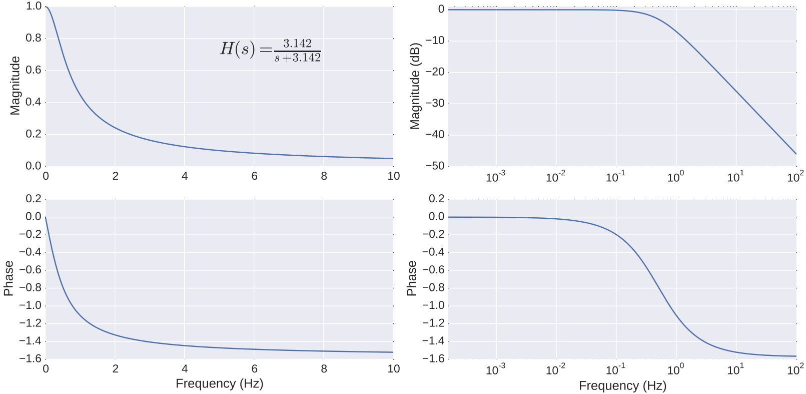
The following figure demonstrates how to determine the frequency response geometrically.



Frequency response of 1^{st} order system

Consider a 1^{st} order system, $H(s) = \frac{p}{s+p}$

The plot of the magnitude and phase responses are shown in the following figure. **Verify using the geometric method that these responses are correct.**



The right column shown the magnitude and phase responses in with a logarithmic x axis. Note that in the top-right plot the magnitude is presented in *decibels*, i.e. $20 \log |H(\omega)|$.

Frequency response of 2^{nd} order system

Consider a 2^{nd} order system, $H(s) = \frac{a_0}{s^2+a_1 s+a_0}$. This transfer function can be characterized in three different ways:

1. By the parameters, a_0 and a_1.
2. By the real and imaginary values of the poles, $\Re\{p_1\}$, $\Re\{p_2\}$, $\Im\{p_1\}$, and $\Im\{p_2\}$. $H(s) = \frac{a_0}{(s-p_1)(s-p_2)}$
3. By the resonant frequency ω_n and the quality factor Q. $H(s) = \frac{\omega_n^2}{s^2+(\omega_n/Q)s+\omega_n^2}$

$Q = \left|\frac{H(\omega_n)}{H(0)}\right|$ is the ratio of the magnitude at $\omega = \omega_n$ with respect to that of $\omega = 0$. It is also approximately equal to the ratio of the resonant frequency and the 3dB bandwidth around the resonant frequency $Q = \frac{\omega_n}{\Delta\omega_{3dB}}$. And also,

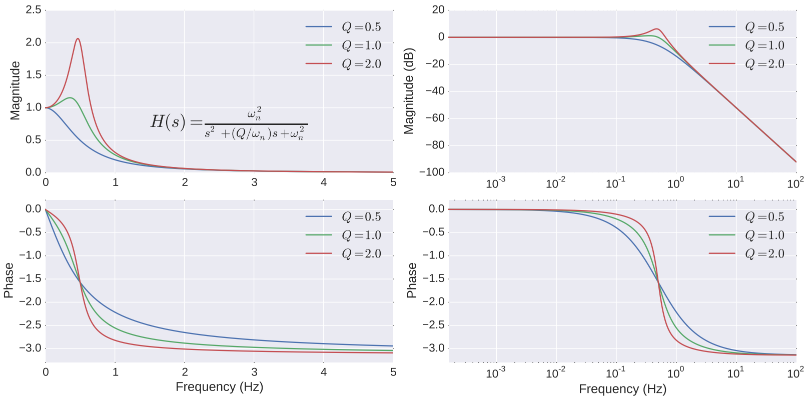
$$\frac{\Im\{p_1\}}{\Re\{p_1\}} = \sqrt{4Q^2 - 1}$$

What is the locus of all pole locations on the s plane with the same Q factor?

Frequency response of 2^{nd} order system

Consider a 2^{nd} order system, $H(s) = \frac{\omega_n^2}{s^2 + (\omega_n/Q)s + \omega_n^2}$

The plot of the magnitude and phase responses are shown in the following figure. **Verify using the geometric method that these responses are correct.**



Bode plots

- Bode plots provide a manual method to quickly sketch approximate magnitude and phase responses.
- Bode plots are asymptotic approximations of the actual magnitude and phase responses.
- Both the magnitude and phase plots are plotted on a logarithmic frequency axis.
- Magnitude is represented in *decibels*. The use of the decibel scale, will make the product of the different zeros and poles into a summation.

$$\text{Magnitude Response} = |H(\omega)| = K \frac{\Pi_{i=1}^{M} |j\omega - z_i|}{\Pi_{i=1}^{N} |j\omega - p_i|}$$

$$\text{Magnitude Response (in dB)} = 20 \log |H(\omega)|$$

$$= 20 \log K + \sum_{i=1}^{M} 20 \log |j\omega - z_i| - \sum_{i=1}^{N} 20 \log |j\omega - p_i|$$

Bode plots: Magnitude plot

- To understand how to plot the magnitude response, we will look at two cases: (a) a system with a single real pole; and (b) a system with a single real zero.
- System with a single real pole.

$$H(\omega) = \frac{\omega_0}{j\omega + \omega_0} \implies |H(\omega)|_{dB} = 20\log\left|\frac{\omega_0}{j\omega + \omega_0}\right|$$

$$|H(\omega)|_{dB} = -20\log\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx \begin{cases} 0 & \omega \ll \omega_0 \\ -20\log\left(\frac{\omega}{\omega_0}\right) & \omega \gg \omega_0 \end{cases}$$

This implies that for frequencies well below ω_0, the magnitude is close to 0dB.
When the frequency is well above ω_0, the magnitude decreases linearly as a log of the frequency. A 10 fold increase in $\frac{\omega}{\omega_0}$, will lead to -20dB reduction in the magnitude.

Bode plots: Magnitude plot

- System with a single real zero.

$$H(\omega) = \frac{j\omega + \omega_0}{\omega_0} \implies |H(\omega)|_{dB} = 20\log\left|\frac{j\omega + \omega_0}{\omega_0}\right|$$

$$|H(\omega)|_{dB} = 20\log\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx \begin{cases} 0 & \omega \ll \omega_0 \\ 20\log\left(\frac{\omega}{\omega_0}\right) & \omega \gg \omega_0 \end{cases}$$

This implies that for frequencies well below ω_0, the magnitude is close to 0dB.
When the frequency is well above ω_0, the magnitude decreases linearly as a log of the frequency. A 10 fold increase in $\frac{\omega}{\omega_0}$, will lead to +20dB increase in the magnitude.

Bode plots: Phase plot

We can do a similar approximation for the phase plot.

- System with a single real pole.

$$H(\omega) = \frac{\omega_0}{j\omega + \omega_0} \implies \angle H(\omega) = -\arctan\left(\frac{\omega}{\omega_0}\right)$$

$$\angle H(\omega) \approx \begin{cases} 0 & \omega < 0.1\omega_0 \\ -\frac{\pi}{4} & \omega = \omega_0 \\ -\frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

- System with a single real zero.

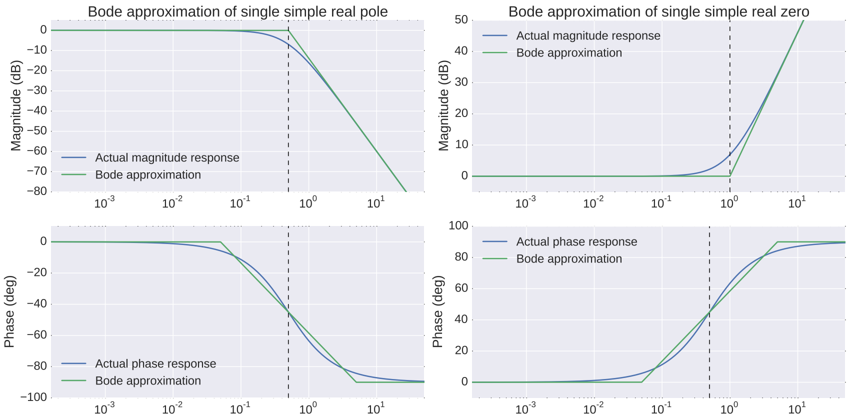
$$H(\omega) = \frac{j\omega + \omega_0}{\omega_0} \implies \angle H(\omega) = \arctan\left(\frac{\omega}{\omega_0}\right)$$

$$\angle H(\omega) \approx \begin{cases} 0 & \omega < 0.1\omega_0 \\ \frac{\pi}{4} & \omega = \omega_0 \\ \frac{\pi}{2} & \omega > 10\omega_0 \end{cases}$$

Between $0.1\omega_0$ and $10\omega_0$, we use a linear approximation as a function of the log of the frequency.

Bode plots of first order pole and zero

Frequency response and Bode approximations of systems with a first order pole and first order zero.



Bode plots: Magnitude plot of 2^{nd} order system

System with 2^{nd} order poles

$$H(\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j\omega\frac{\omega_n}{Q}} \implies |H(\omega)|_{dB} = 20\log\left|\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + j\frac{\omega/\omega_n}{Q}}\right|$$

$$|H(\omega)|_{dB} = -20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{\omega/\omega_n}{Q}\right)^2}$$

$$|H(\omega)|_{dB} \approx \begin{cases} 0 & \omega \ll \omega_0 \\ -20\log\left(\frac{1}{Q}\right) & \omega = \omega_0 \\ -40\log\left(\frac{\omega}{\omega_0}\right) & \omega \gg \omega_0 \end{cases}$$

The Bode approximation works well for small and large ω, but close to ω_0, the approximation can be poor depending on the value of Q. For the case of 2^{nd} order zeros, the signs of the magnitude response will need to reversed.

Bode plots: Phase plot of 2^{nd} order system

System with 2^{nd} order poles

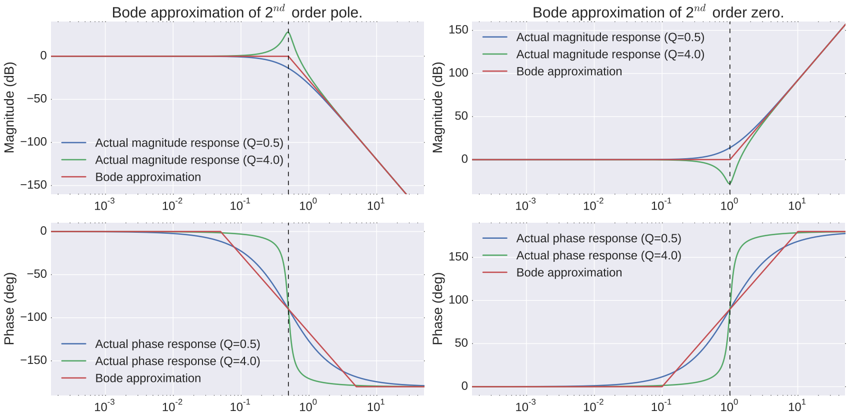
$$H(\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j\omega\frac{\omega_n}{Q}} \implies \angle H(\omega) = -\arctan\left(\frac{\omega/\omega_n Q}{1 - (\omega/\omega_n)^2}\right)$$

$$\angle H(\omega) \approx \begin{cases} 0 & \omega \ll 0.1\omega_0 \\ -\frac{\pi}{2} & \omega = \omega_0 \\ -\pi & \omega \gg 10\omega_0 \end{cases}$$

For the case of 2^{nd} order zeros, the signs of the magnitude response will need to reversed.

Bode plots of second order poles and zeros

Frequency response and Bode approximations of systems with a 2^{nd} order poles and zeros.



Bode plot example

Consider the following transfer function,

$$H(s) = \frac{(s+100\pi)(s+2000\pi)(s+10000\pi)}{(s+20\pi)(s+1000\pi)(s+2000\pi)(s+15000\pi)}$$