

# Introduction to Digital Signal Processing

## Fourier Representation of Continuous-Time Signals

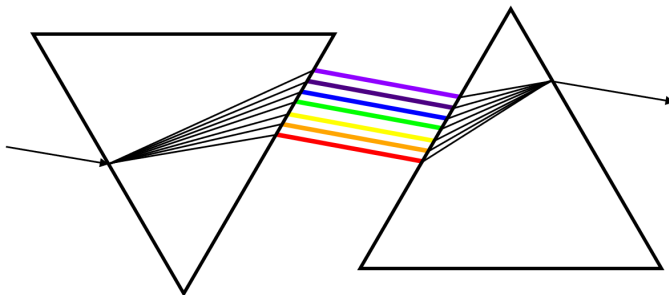
Sivakumar Balasubramanian

Department of Bioengineering  
Christian Medical College, Bagayam  
Vellore 632002

## Representation of signals as a linear combination of other signals

- ▶ Decomposition of an object into smaller/simpler components is a useful operation.
- ▶ Decomposing  $x[n]$  as a  $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$  was useful in understanding input-output relationship of LTI systems.
- ▶ Infinitely many ways of decomposing a signal.
- ▶ Decomposing into complex exponential signals is one of the oldest, most common, and very useful approaches → **Fourier analysis**.

## An example from Optics



- ▶ Sunlight (white light) can be decomposed into different colors using a prism.
- ▶ Individual colors can be combined to produce white light back.
- ▶ Using a filter between the two prisms will allow us to mix the individual colors in different combinations.

# Spectral analysis of signals

- ▶ Signals can be decomposed into different sinusoidal components.
- ▶ Each sinusoidal component is index or parametrized by its frequency.
- ▶ Let the set of all sinusoidal signals be  $S_{ct} = \{s_{\omega}(t) \mid \omega \in \mathbb{R}\}$  for continuous-time signals, or  $S_{dt} = \{s_{\Omega}[n] \mid \Omega \in (-\pi, \pi]\}$  for discrete-time signals.

$$x(t) = \int_{-\infty}^{\infty} X(\omega) s_{\omega}(t) d\omega \qquad y[n] = \int_{-\pi}^{\pi} Y(\Omega) s_{\Omega}[n] d\Omega$$

$X(\omega)$  is the “amount” of  $s_{\omega}(t)$  present in  $x(t)$ .

$Y(\Omega)$  is the “amount” of  $s_{\Omega}[n]$  present in  $y[n]$ .

## Fourier Series: Linear combination of periodic signals

- ▶ Consider the following set of sinusoidal signals  $\{\sin(k \cdot \omega_0 t) \mid k \in \mathbb{Z}_{>0}\}$ .
- ▶  $\omega_0 = 2\pi f_0 = 2\pi \frac{1}{T_0}$  is the fundamental angular frequency of the sinusoidal signal  $\sin(\omega_0 t)$ .
- ▶ Any signal of the following form will be periodic.

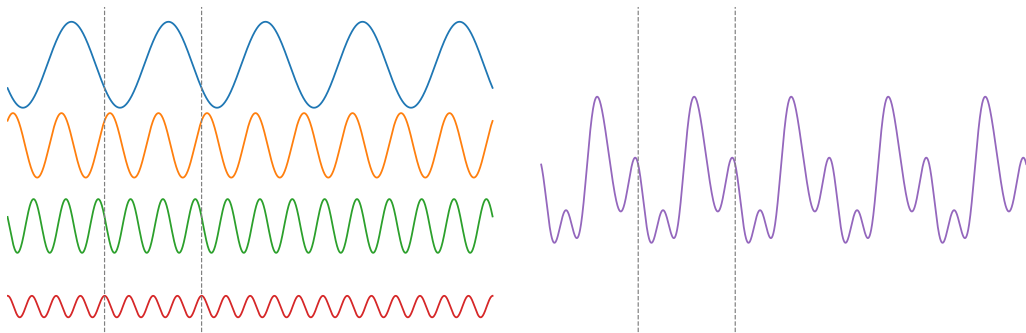
$$x(t) = r_0 + \sum_{k=1}^{\infty} r_k \sin(k \cdot \omega_0 t + \phi_k), \quad 0 \leq r_k \in \mathbb{R}, \quad \phi_k \in (-\pi, \pi]$$

What is the fundamental period of the signal?

## Fourier Series: Linear combination of periodic signals

- We can generate new periodic signals (of fundamental frequency  $f_0$ ) by mixing sinusoidal signals with fundamental frequencies that are integer multiples of  $f_0$ .

$$x(t) = r_0 + \sum_{k=1}^{\infty} r_k \sin(k \cdot \omega_0 t + \phi_k), \quad 0 \leq r_k \in \mathbb{R}, \quad \phi_k \in (-\pi, \pi]$$



# Fourier Series

$$\begin{aligned}x(t) &= r_0 + \sum_{k=1}^{\infty} r_k \sin(k \cdot \omega_0 t + \phi_k) \\&= r_0 + \sum_{k=1}^{\infty} a_k \sin(k \cdot \omega_0 t) + b_k \cos(k \cdot \omega_0 t) \\&= \sum_{k=0}^{\infty} (a_k \sin(k \cdot \omega_0 t) + b_k \cos(k \cdot \omega_0 t))\end{aligned}$$

Expressing  $\cos(\cdot)$  and  $\sin(\cdot)$  in terms of complex exponentials, and grouping terms together,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

# Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

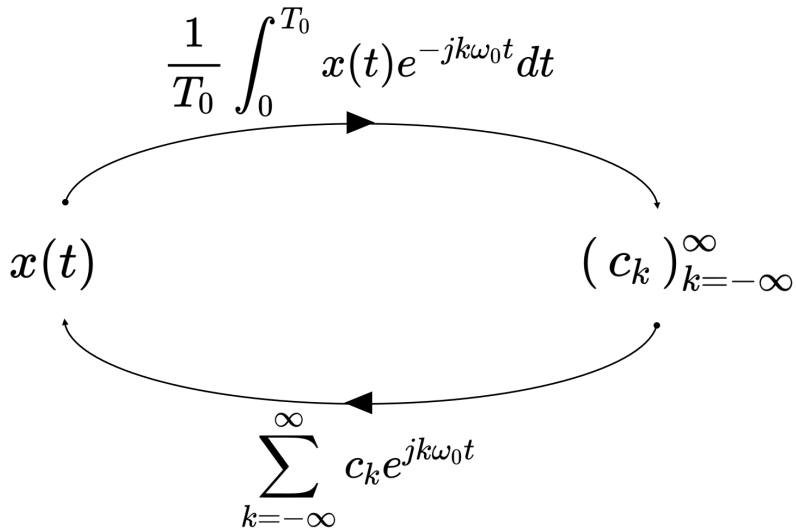
Knowing  $f_0$ , we can compute the signal  $x(t)$  from the list of numbers  $(c_k)_{k=-\infty}^{\infty}$ .

We can compute  $c_k$  as the following,

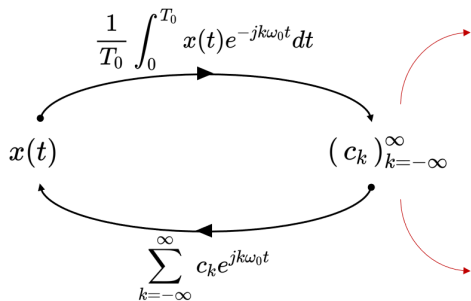
$$c_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k f_0 t} dt$$



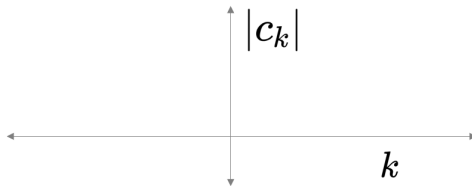
# Fourier Series



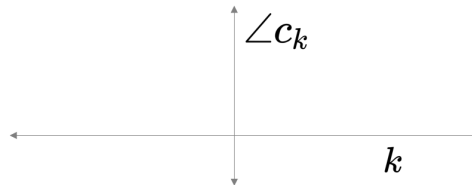
# Fourier Series



Magnitude Spectrum

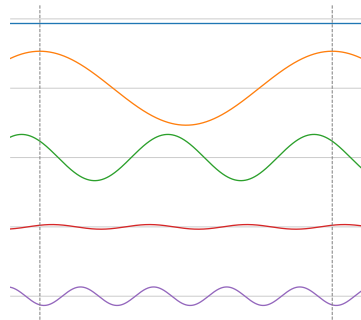
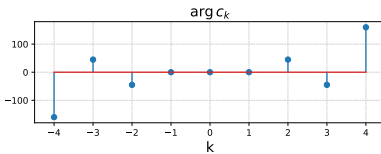
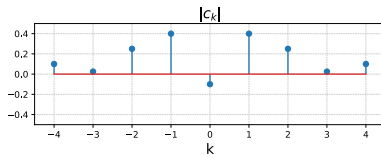
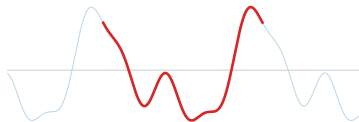


Phase Spectrum



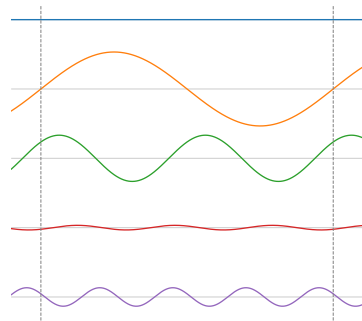
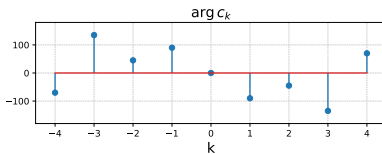
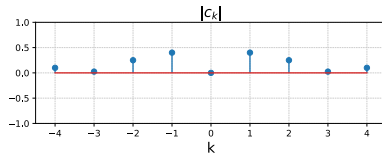
# Fourier Series

$$x(t) = -0.1 + 0.8 \cos(2\pi t) + 0.5 \cos\left(4\pi t + \frac{\pi}{4}\right) + 0.05 \cos\left(6\pi t - \frac{\pi}{4}\right) + 0.2 \cos\left(8\pi t + \frac{8\pi}{9}\right)$$



# Fourier Series

$$x(t) = 0.8 \sin(2\pi t) + 0.5 \sin\left(4\pi t + \frac{\pi}{4}\right) + 0.05 \sin\left(6\pi t - \frac{\pi}{4}\right) + 0.2 \sin\left(8\pi t + \frac{8\pi}{9}\right)$$



## Does any periodic signal have a Fourier series representation?

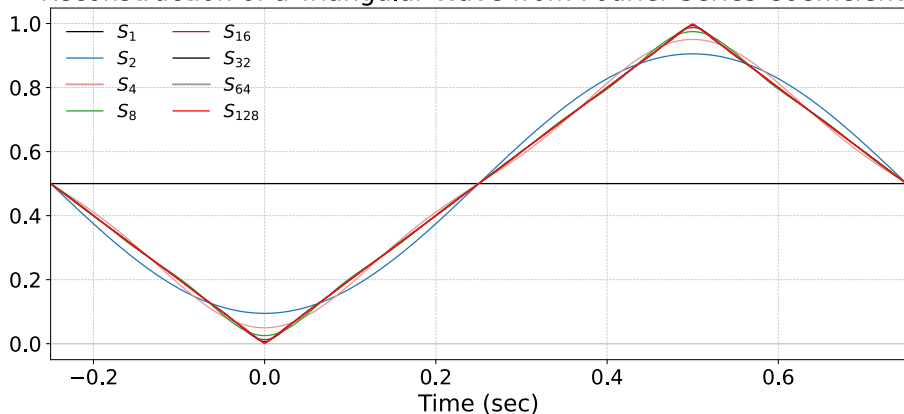
- ▶ If  $x(t)$  is absolutely integrable over a single cycle, then the Fourier series coefficients exist.
- ▶ Any continuous periodic function will have a Fourier series representation.
- ▶ When  $x(t)$  is continuous and finite, then reconstructed signal  $\sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t}$  will be equal to  $x(t)$  pointwise.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t} \quad \forall t$$

## Does any periodic signal have a Fourier series representation?

$$x(t) = \begin{cases} t, & 0 \leq t < 0.5 \\ 1 - t, & 0.5 \leq t < 1 \end{cases} \longrightarrow c_k = \begin{cases} \frac{1}{4}, & k = 0 \\ \frac{4}{k^2 \omega_0^2} \sin^2 \left( \frac{k \omega_0}{4} \right) e^{-j \frac{k \omega_0}{2}}, & k \neq 0 \end{cases}$$

Reconstruction of a Triangular Wave from Fourier Series Coefficients



## Does any periodic signal have a Fourier series representation?

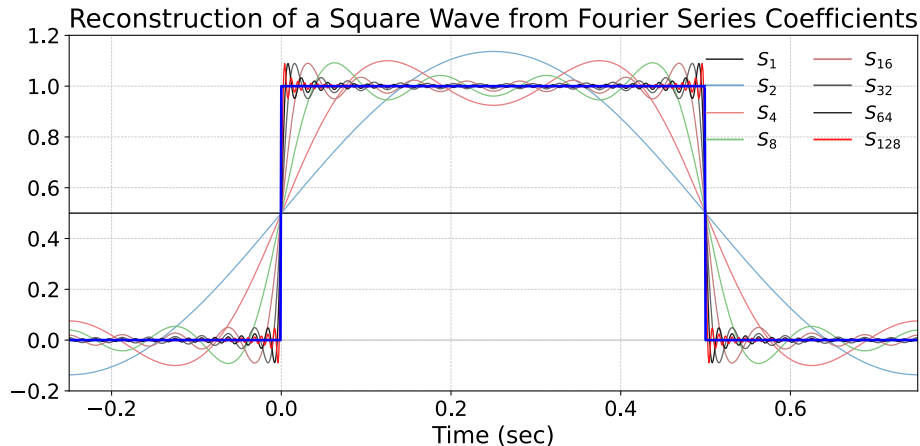
- If  $x(t)$  is finite but discontinuous  $\longrightarrow$  No pointwise equality. Only means squared convergence is possible.

$$\lim_{N \rightarrow \infty} \int_0^{T_0} \left| x(t) - \sum_{k=-N}^N c_k e^{j2\pi k f_0 t} \right|^2 dt = 0$$

This means that the reconstructed signal  $\sum_{k=-N}^N c_k e^{j2\pi k f_0 t}$  need not be equal to the signal  $x(t)$  at a discrete set of points, i.e. at the points where there is a discontinuity.

## Does any periodic signal have a Fourier series representation?

$$x(t) = \begin{cases} 1, & 0 \leq t < 0.5 \\ 0, & 0.5 \leq t < 1 \end{cases} \rightarrow c_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{2}{k\omega_0} \sin\left(\frac{k\omega_0}{4}\right) e^{-j\frac{k\omega_0}{4}}, & k \neq 0 \end{cases}$$





## Dirichlet conditions for Fourier series

The *Dirichlet conditions* guarantee that the  $c_k$  exists, and  $\sum_{k=-N}^N c_k e^{j2\pi k f_0 t}$  is equal to  $x(t)$  except at time points where there is a discontinuity.

At a discontinuity,  $\sum_{k=-N}^N c_k e^{j2\pi k f_0 t}$  converges to the midpoint of the discontinuity.

The *Dirichlet conditions* are that a single cycle of  $x(t)$ :

1. has a finite number of discontinuities.
2. has a finite number of maxima and minima.
3. is absolutely integrable.  $\int_0^{T_0} |x(t)| dt < \infty$

# Power Spectral Density of Periodic Signals

## Some definitions:

- ▶ Instantaneous power of a signal  $x(t) \triangleq |x(t)|^2$
- ▶ Total energy of a signal  $x(t)$  in a time interval  $[T_1, T_2] \triangleq \int_{T_1}^{T_2} |x(t)|^2 dt$
- ▶ Average power over a time interval  $[T_1, T_2] \triangleq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |x(t)|^2 dt$
- ▶ **Energy signal:** Signals with a finite total energy and zero average power over their entire duration.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt = 0$$

- ▶ **Power signal:** Signals with a finite average power, and infinite energy.

$$\lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \infty \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt < \infty$$

# Power Spectral Density of Periodic Signals

## Parseval's Identity.

Let  $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ , then

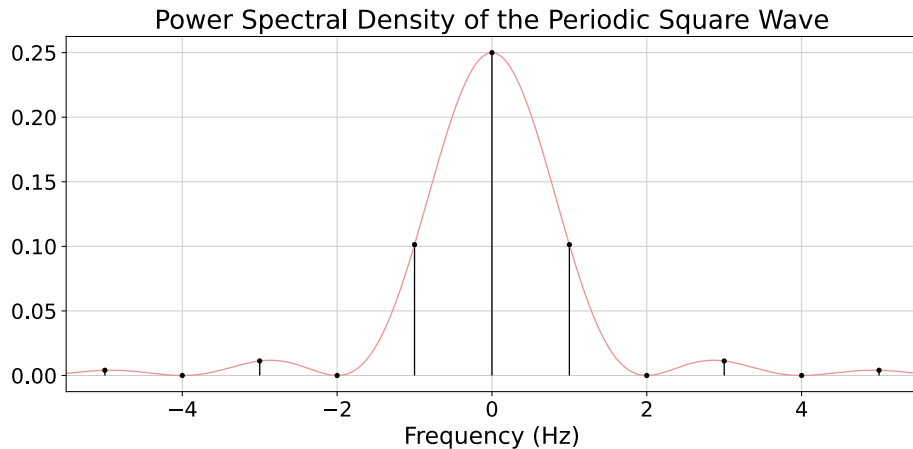
$$P_x = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Fourier series representation preserves the average power of the periodic signal  $x(t)$ .

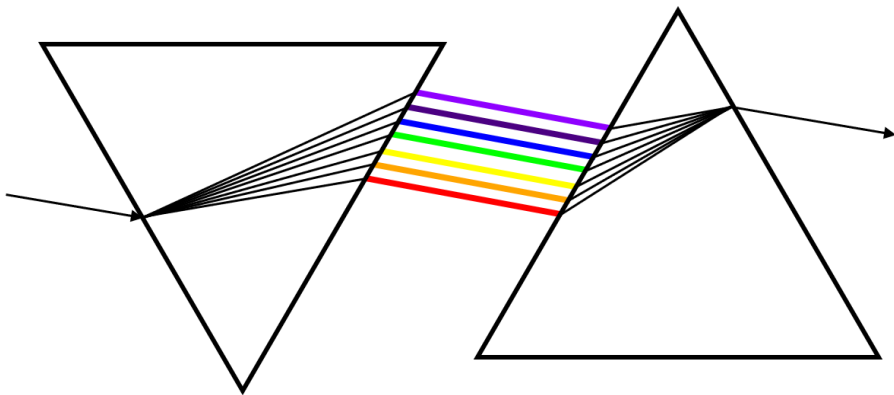
$|c_k|^2$  is the power in ofn the  $k^{th}$  harmonic.

$|c_k|^2$  as a function of  $k$  is the **Power Spectral Density** of  $x(t)$ .

# Power Spectral Density of Periodic Signals



# Power Spectral Density of Periodic Signals



## Fourier representation of aperiodic signals

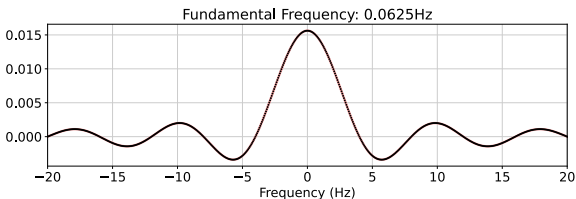
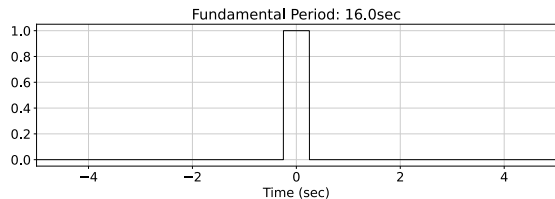
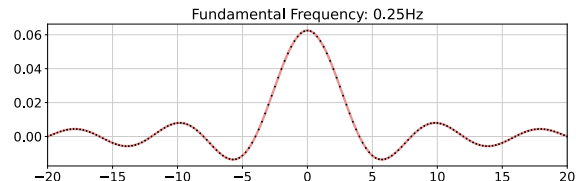
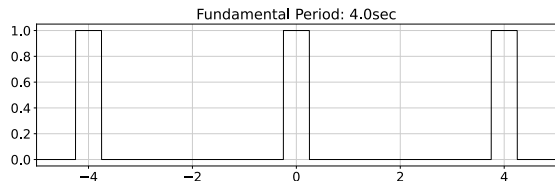
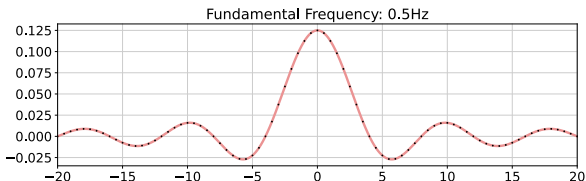
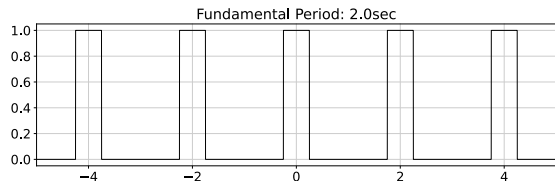
We can approach this problem starting from the Fourier series.

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \tau < |t| \leq \frac{T_0}{2} \end{cases}, \text{ where, } 0 < \tau < \frac{T_0}{2}$$



$$c_k = \frac{\tau}{T_0} \frac{\sin(\pi k f_0 \tau)}{\pi k f_0 \tau}, \quad k = 0, \pm 1, \pm 2, \dots$$

# Fourier representation of aperiodic signals



## Fourier representation of aperiodic signals: Fourier Transform

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \tau < |t| \leq \frac{T_0}{2} \end{cases}, \text{ where, } 0 < \tau < \frac{T_0}{2}$$



$$c_k = \frac{\tau}{T_0} \frac{\sin\left(\frac{k\omega_0\tau}{2}\right)}{\frac{k\omega_0\tau}{2}}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$T_0 \rightarrow \infty \quad \Rightarrow \quad \omega_0 \rightarrow 0 \quad \Rightarrow \quad \{k\omega_0\}_{k=-\infty}^{\infty} \rightarrow \omega \in \mathbb{R} \quad \Rightarrow \quad c_k \rightarrow X(\omega)$$

$$c_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt \quad \longrightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

This is the **Fourier transform**.



## Fourier representation of aperiodic signals

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < |t| \end{cases} \longrightarrow X(\omega) = \tau \cdot \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} = \tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

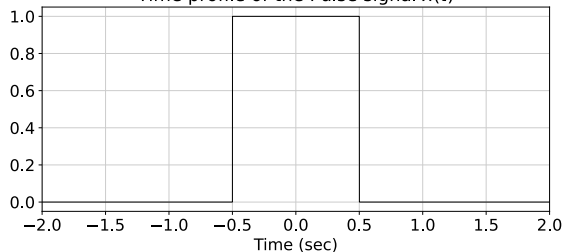
We can reconstruct the time-domain signal from the  $X(\omega)$ ,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

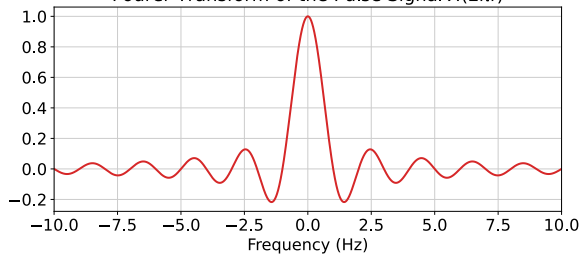
This is the **Inverse Fourier Transform**.

# Fourier Transform

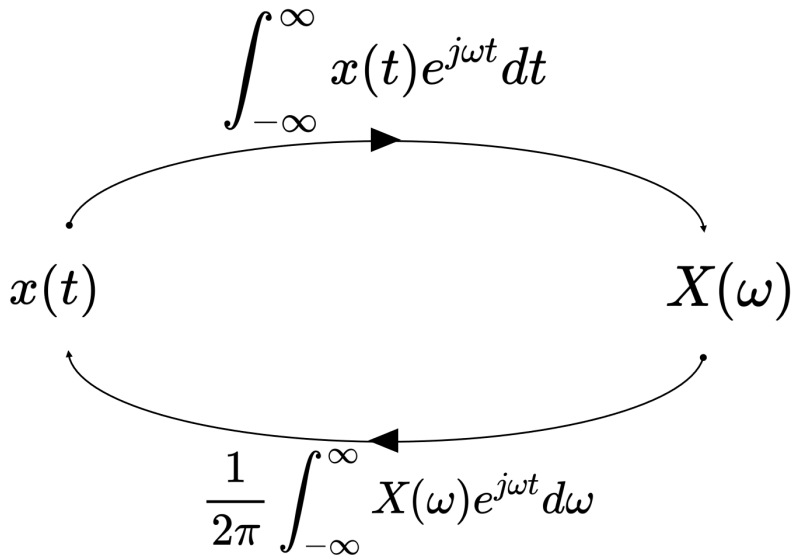
Time profile of the Pulse signal  $x(t)$



Fourier Transform of the Pulse Signal  $X(2\pi f)$



# Fourier Transform



## Dirichlet Conditions for the Fourier Transform

The *Dirichlet conditions* for the existence of the Fourier transform are that  $x(t)$ :

1. has a finite number of discontinuities.
2. has a finite number of maxima and minima.
3. is absolutely integrable.  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ .

This ensures that  $X(\omega)$  is finite and continuous.

We can still have Fourier transform for signal that are not absolutely integrable, but square integrable, i.e.  $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ .

**Example:**  $x(t) = \omega_0 \text{sinc}(\omega_0 t)$  is not absolutely integrable, but  $X(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}$ .

## Parseval's identity for aperiodic signals

Energy of an aperiodic signal  $x(t)$ :

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Parseval's identity:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$S_{xx}(\omega) = \frac{1}{2\pi} |X(\omega)|^2$  is the distribution of signal energy over frequency: **Energy density spectrum**.

# Properties of Fourier transform

- ▶ **Linearity:**  $\alpha x(t) + \beta y(t) \xleftrightarrow{\text{FT}} \alpha X(\omega) + \beta Y(\omega)$
- ▶ **Shift in time:**  $x(t - t_0) \xleftrightarrow{\text{FT}} e^{-j\omega t_0} X(\omega)$
- ▶ **Shift in frequency:**  $x(t)e^{j\omega_0 t} \xleftrightarrow{\text{FT}} X(\omega - \omega_0)$
- ▶ **Time and frequency scaling:**  $x(\alpha t) \xleftrightarrow{\text{FT}} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \alpha > 0$
- ▶ **Convolution in time:**  $x(t) * y(t) \xleftrightarrow{\text{FT}} X(\omega)Y(\omega)$