

# Introduction to Digital Signal Processing

## Digital Filters

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## LTI systems can be designed shape frequency spectrum

Let  $H$  be an LTI system,

$$e^{j\Omega n} \mapsto |H(\Omega)| e^{j\Omega n + \arg H(\Omega)}$$

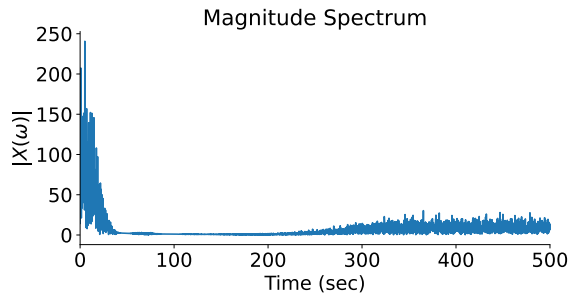
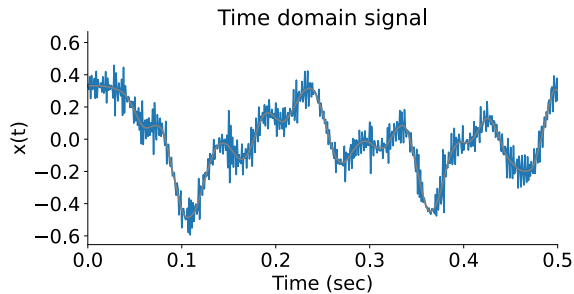
The amount of amplitude and phase modification of the input  $e^{j\Omega n}$  is determined by the magnitude and phase of the value of the transfer function  $H(\Omega)$ .

The frequency response a discrete-time LTI system can be obtained from its impulse response or the difference equation describing the system,

$$H(\Omega) = \sum_n h[n] e^{-j\Omega n} = \left. \frac{\sum_{k=1}^M b_k z^{-k}}{\sum_{l=1}^N a_l z^{-k}} \right|_{z=e^{j\Omega}}$$

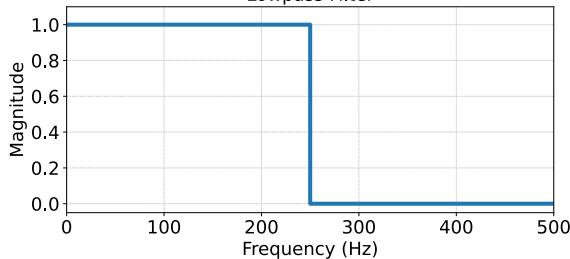
If we have a desired frequency response  $H_d(\Omega)$ , how do we choose the impulse response or the coefficients of the LTI system such that its frequency response  $H(\Omega)$  is as close to  $H_d(\Omega)$  as possible.

## Need for frequency selective filters

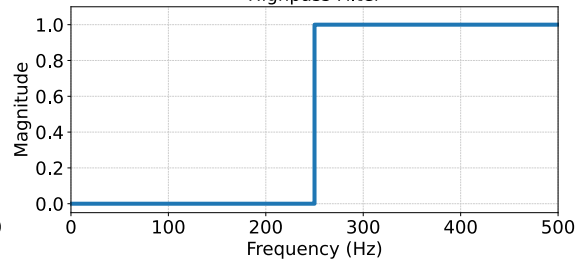


# Ideal Filters

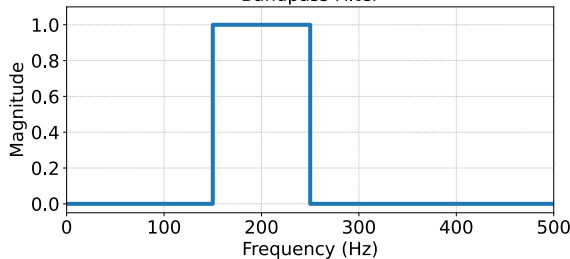
### Lowpass Filter



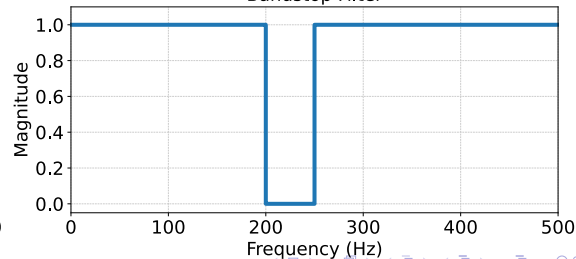
### Highpass Filter



### Bandpass Filter



### Bandstop Filter



## Characteristics of Ideal Filter

Consider the ideal lowpass filter,

$$H(\Omega) = \begin{cases} C \cdot e^{-j\omega n_0}, & |\Omega| \leq \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$$

$$\implies Y(\Omega) = X(\Omega) H(\Omega) = CX(\Omega) e^{-j\Omega n_0}$$

- ▶  $C \implies$  Constant scaling of amplitude in the passband.
- ▶  $e^{j\Omega n_0} \implies$  Linear phase, which a constant time delay for all frequency components.

But ideal filters are problematic!

## Problems with ideal

Ideal filters are not physically realizable.

Consider the ideal lowpass filter,  $H(\Omega) = \begin{cases} 1, & |\Omega| \leq \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$ . The inverse DTFT of  $H(\Omega)$  will give us the following impulse response for the system,

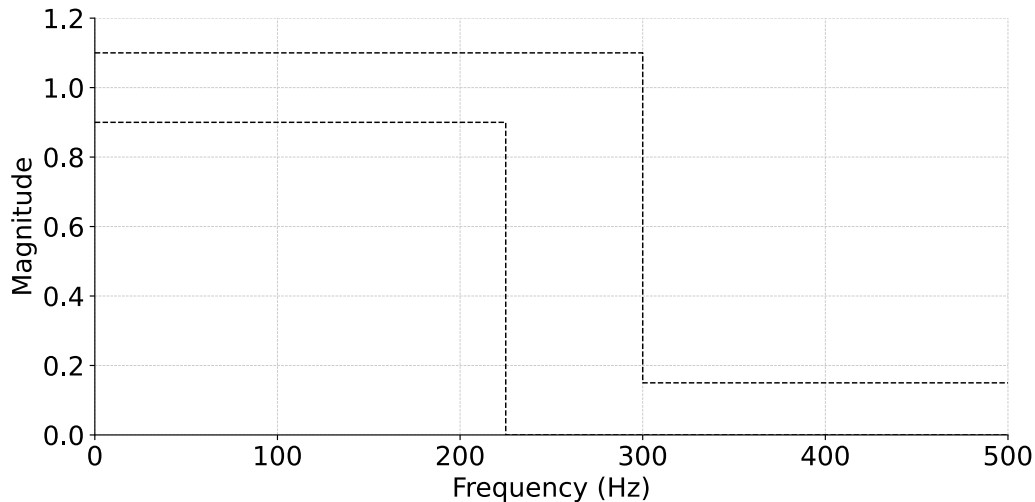
$$h[n] = 2\Omega_c \frac{\sin \Omega_c n}{\Omega_c n}$$

This is non-causal, with  $h[n]$  extending upto  $-\infty$ .

Physically realizable filters cannot have:

- ▶ Flat frequency response over a continuous interval.
- ▶ Cannot have step transitions.

## Real Filter Specifications



## Real Filter Specifications

$$\delta_1, \delta_2, f_p, f_s \longrightarrow H(\Omega) \longrightarrow \left. \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{l=1}^N a_l z^{-l}} \right|_{z=e^{j\Omega}}$$

Parameters to be chosen for a filter:  $N$ ,  $M$ ,  $(a_i)_{i=1}^N$ , and  $(b_i)_{i=0}^M$ .

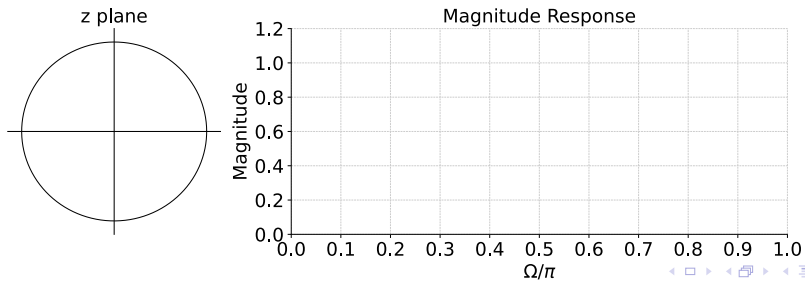
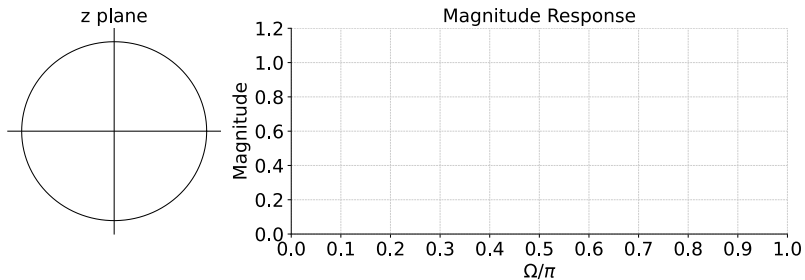
$$H(z) = b_0 \frac{\prod_{k=1}^M (1 - z_k z^{-1})}{\prod_{k=1}^N (1 - p_k z^{-1})}$$

- ▶  $(a_i)_{i=1}^N$  determine the **poles** of the transfer function  $(p_i)_{i=1}^N \longrightarrow$  Can be used to **emphasize certain frequencies**.
- ▶  $(b_i)_{i=0}^M$  determine the **zeros** of the transfer function  $(z_i)_{i=1}^M \longrightarrow$  Can be used to **attenuate certain frequencies**.

Appropriate placement of the poles and zeros will allow us to obtain a frequency response that satisfies the given filter specifications.



# Pole-Zero Placement and Frequency Response



## Pole-Zero Placement and Frequency Response

Consider the two pole system,

$$H(z) = \frac{b_0}{1 - 2pz^{-1} + p^2z^{-2}}$$

Determine  $b_0$  and  $p$  such that the frequency response,

$$H(\Omega = 0) = 1 \quad \text{and} \quad |H(\Omega = \frac{\pi}{2})| = \frac{1}{\sqrt{2}}$$



## Pole-Zero Placement and Frequency Response

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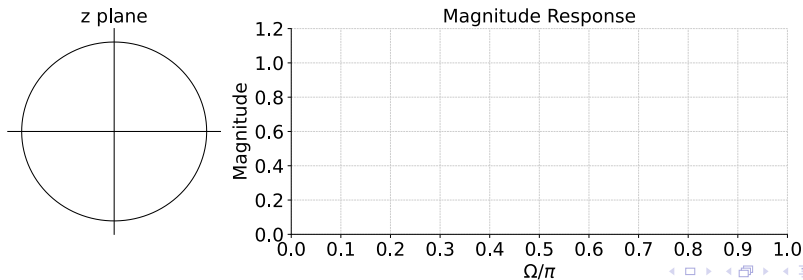
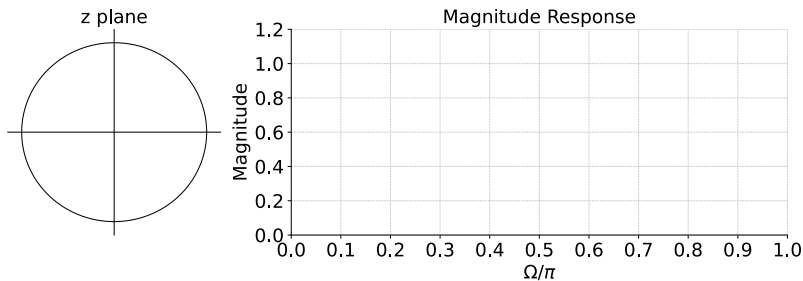
$$H(z) = \frac{b_0}{1 - 2pz^{-1} + p^2z^{-2}}$$

Determine  $b_0$  and  $p$  such that the frequency response,

$$H(\Omega = 0) = 1 \quad \text{and} \quad |H(\Omega = \frac{\pi}{4})| = \frac{1}{\sqrt{2}}$$

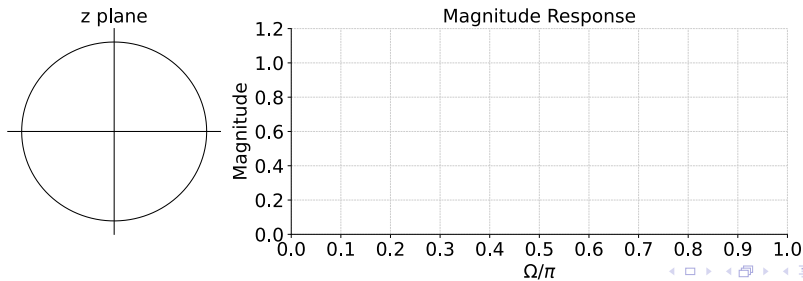
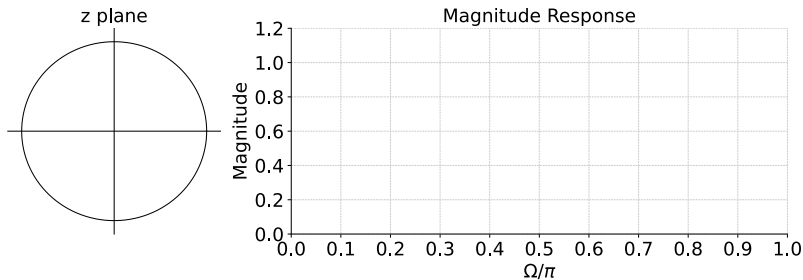


# Pole-Zero Placement and Frequency Response





# Pole-Zero Placement and Frequency Response





## Pole-Zero Placement and Frequency Response

$$H(z) = b_0 \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - p z^{-1})(1 - p^* z^{-1})}$$

Determine  $b_0$ ,  $z_1$ ,  $z_2$ , and  $p$  to design a bandpass filter such that,  $H(\Omega = 0) = 0$  and  $H(\Omega = \pi) = 0$ , the center of the passband is at  $\Omega = \frac{\pi}{2}$  and  $|H(\Omega = \frac{\pi}{4})| = \frac{1}{\sqrt{2}}$ .

# FIR and IIR Digital Filter

- ▶ Two types of filters:

- ▶ FIR filter:  $h[n]$  is of finite length.

$$h[n] = 0, \forall n \neq 0 \quad \text{and } n > N$$

- ▶ IIR filter:  $h[n]$  is of infinite length.

$$h[n] = 0, \forall n \neq 0$$

## Linear Phase FIR Filters

We will only deal with filter with real impulse response,

$$\implies H(\Omega) = H^*(-\Omega), \quad -\pi \leq \Omega < \pi \implies \begin{cases} |H(\Omega)| \text{ is an even function of } \Omega \\ \arg H(\Omega) \text{ is an odd function of } \Omega \end{cases}$$

We will only discuss the design of linear phase filters.

$$\arg H(\Omega) = \begin{cases} m \cdot \Omega, & H(\Omega) > 0 \\ m \cdot \Omega \pm \pi, & H(\Omega) < 0 \end{cases}$$

Any signal that is symmetric or anti-symmetric with response to some time point will have a linear phase response.

## Symmetric Impulse Response

**Symmetry impulse response** about  $n = 0$ .

$$\begin{aligned}h[n] = h[-n] &\implies H(\Omega) = 2 \sum_{n=1}^{\infty} h[n] \cos(\Omega n) \\&\implies \arg H(\Omega) = \begin{cases} 0, & H(\Omega) > 0 \\ \pi, & H(\Omega) < 0 \end{cases}\end{aligned}$$

**Anti-symmetry impulse response** about  $n = 0$ .

$$\begin{aligned}h[n] = -h[-n] &\implies H(\Omega) = -2j \sum_{n=1}^{\infty} h[n] \sin(\Omega n) \\&\implies \arg H(\Omega) = \begin{cases} -\frac{\pi}{2}, & \Im(H(\Omega)) > 0 \\ -\frac{3\pi}{2}, & \Im(H(\Omega)) < 0 \end{cases}\end{aligned}$$

## Symmetric, Causal, FIR Impulse Response

Let the length of the impulse response be  $M$  (odd). Then, a symmetric, causal impulse response is given by,

$$h[n] = h[M - n - 1] \implies H(\Omega) = H_r(\Omega) e^{j\Theta(\Omega)}$$

$$\Theta(\Omega) = \begin{cases} -\Omega \left( \frac{M-1}{2} \right), & H_r(\Omega) > 0 \\ -\Omega \left( \frac{M-1}{2} \right) + \pi, & H_r(\Omega) < 0 \end{cases}$$

## Anti-Symmetric, Causal, FIR Impulse Response

Let the length of the impulse response be  $M$  (odd). Then, an anti-symmetric, causal impulse response is given by,

$$h[n] = -h[M - n - 1] \implies H(\Omega) = H_r(\Omega) e^{j\Theta(\Omega)}$$

$$\Theta(\Omega) = \begin{cases} \frac{\pi}{2} - \Omega \left( \frac{M-1}{2} \right), & H_r(\Omega) > 0 \\ \frac{3\pi}{2} - \Omega \left( \frac{M-1}{2} \right), & H_r(\Omega) < 0 \end{cases}$$

## FIR Filter Design: Windowing

Window the impulse response of the desired frequency response.

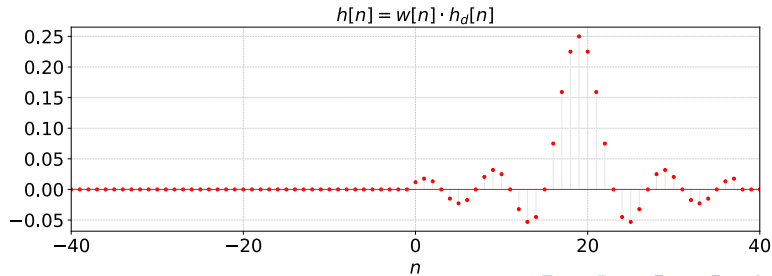
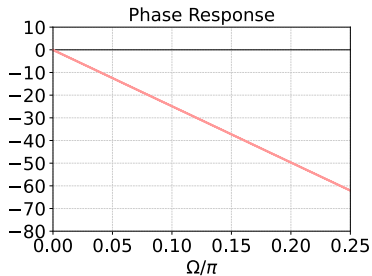
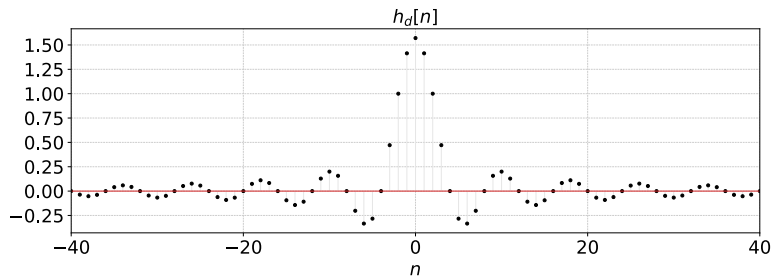
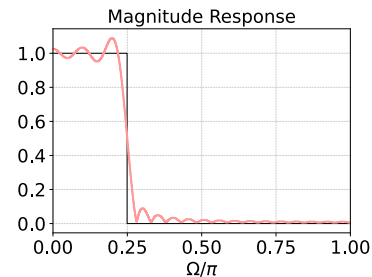
$$H_d(\Omega) \xrightarrow{\text{IDTFT}} h_d[n] \xrightarrow[w[n]]{\text{Window}} h[n] = w[n] \cdot h_d[n] \xrightarrow{\text{DTFT}} H(\Omega) \approx H_d(\Omega)$$

The resulting frequency response  $H(\Omega)$  is,

$$H(\Omega) = H_d(\Omega) * W(\Omega)$$

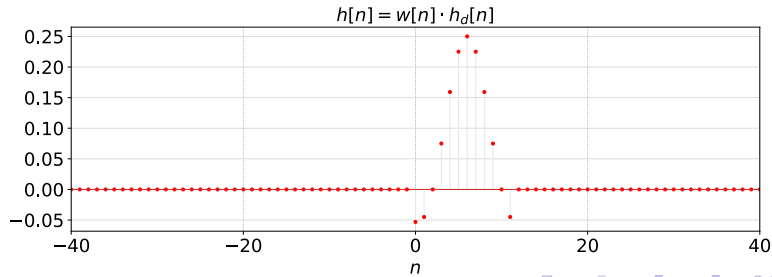
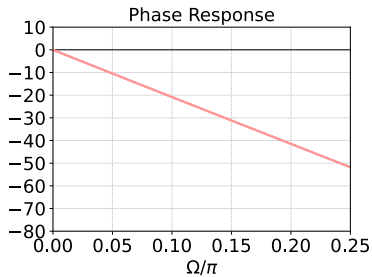
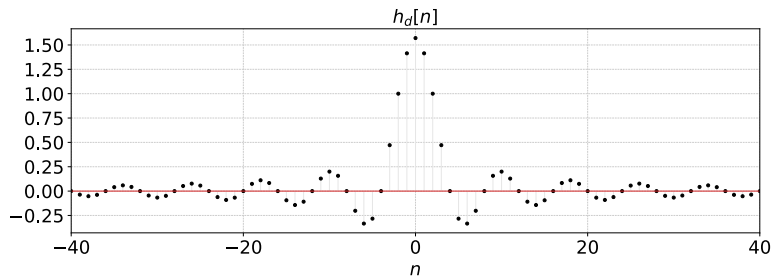
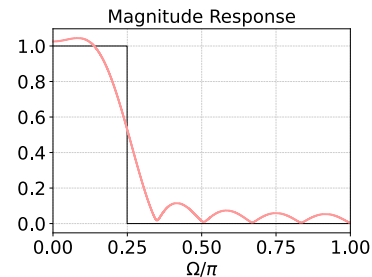
where,  $w[n] \xrightarrow{\text{DTFT}} W(\Omega)$ .

# FIR Filter Design: Windowing



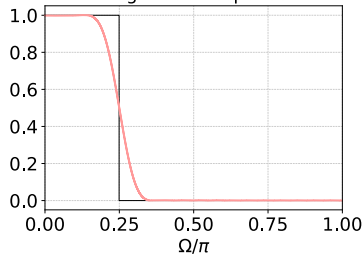


# FIR Filter Design: Windowing

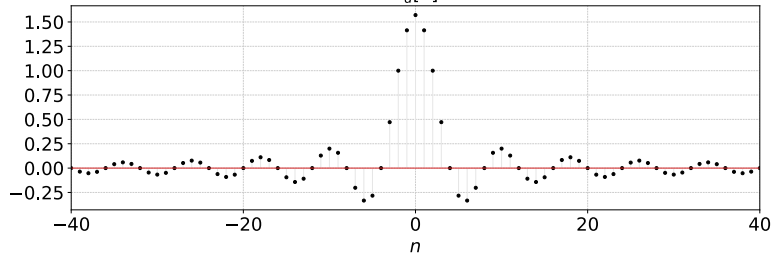


# FIR Filter Design: Windowing

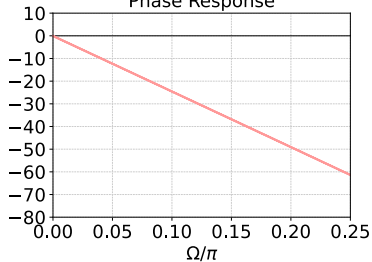
Magnitude Response



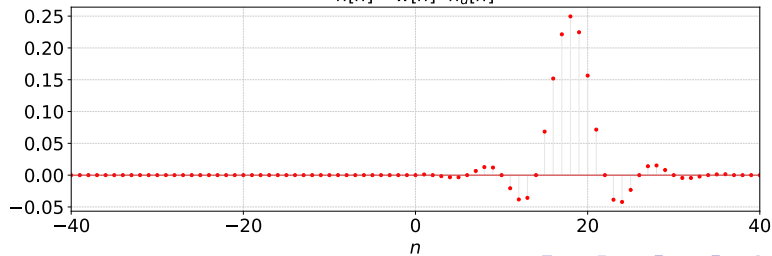
$h_d[n]$



Phase Response



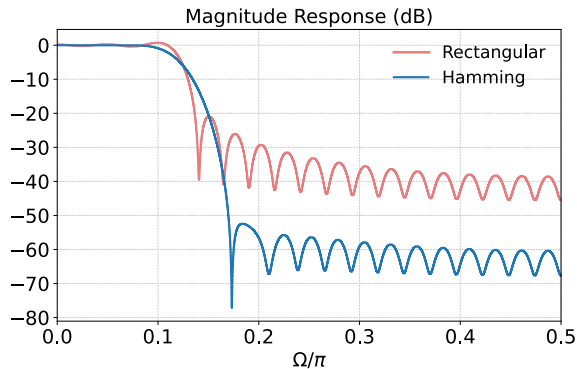
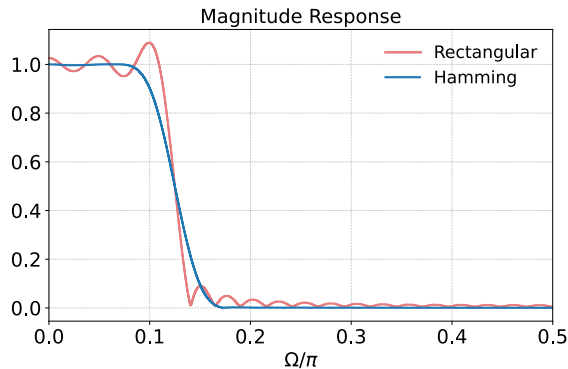
$h[n] = w[n] \cdot h_d[n]$



## FIR Filter Design: Windowing

$$H_d(\Omega) \xrightarrow{\text{IDTFT}} h_d[n] \xrightarrow[w[n]]{\text{Window}} h[n] = w[n] \cdot h_d[n] \longrightarrow b_i = h[i], \quad 0 \leq i \leq M$$

# FIR Filter Design: Windowing



Different windows trade-off between the width of the transition band and the level of passband attenuation, and nature of the ripple in the pass and stopbands.

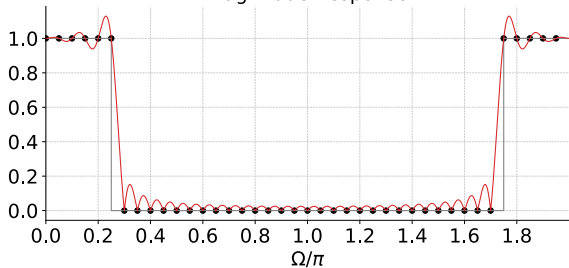
## FIR Filter Design: Frequency sampling

We can specify the desired frequency response  $H_d(\Omega)$  at a set of equally spaced frequencies,

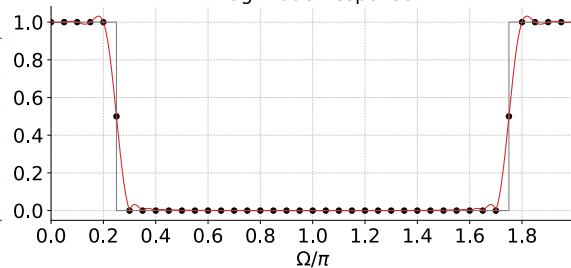
$$\Omega_k = \frac{2\pi k}{M}$$

# FIR Filter Design: Frequency sampling

Magnitude Response



Magnitude Response



# FIR Filter Design

Advantages of FIR filters:

- ▶ Always stable.  $\sum_n |h[n]| < \infty$
- ▶ Linear phase  $\implies$  No phase distortion.

Disadvantages of FIR filters:

- ▶ Same specs might require a longer filter.
- ▶ Might requires iterative numerical procedures for design.

## IIR Filter Design

We are interested in causal systems that have rational transfer functions,

$$H(z) = \frac{B(z)}{A(z)} \longrightarrow y[n] = \sum_{k=1}^M b_k \cdot x[n-k] - \sum_{l=1}^N a_l \cdot y[n-l]$$

Most popular approach is to design an analog filter and then translate it into a corresponding digital filter.

Digital filter specs  $\longrightarrow$  Analog filter specs  $\longrightarrow$  Analog filter  $H_a(s)$

Analog filter  $H_a(s)$   $\xrightarrow[\text{transformation}]{\text{Bilinear}}$  Digital filter  $H(z)$

**Bilinear transformation:** Substitute  $s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$ .

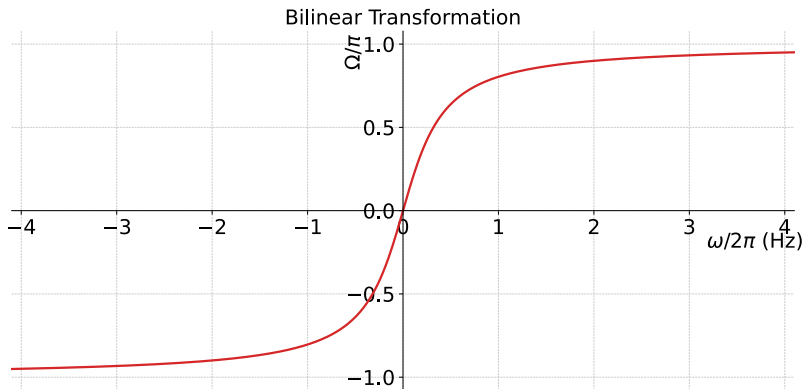
$$H(z) = H_a(s) \bigg|_{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$



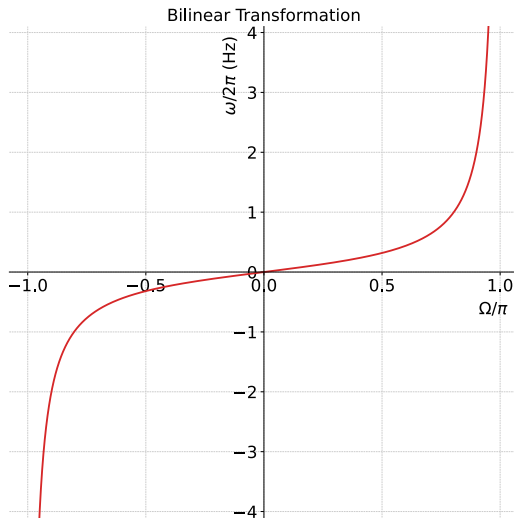
## IIR Filter Design

Analog to digital Frequency mapping done by the bilinear transformation.

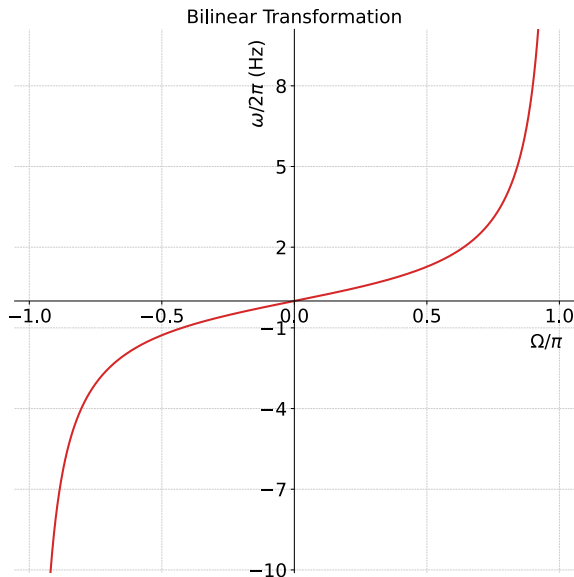
$$\omega = \frac{2}{T} \tan\left(\frac{\Omega}{2}\right) \implies \Omega = 2 \tan^{-1}\left(\frac{\omega T}{2}\right)$$



# IIR Filter Design



# IIR Filter Design



# IIR Filter Design

Advantages of IIR filters:

- ▶ Requires lower number of parameters than FIR.
- ▶ Design using analog filter design tools.

Disadvantages of IIR filters:

- ▶ Can be unstable.
- ▶ More sensitive to round-off errors.