Linear Systems: Vectors Assignment

Vectors

1. Given the matrices $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & -1 & -1 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -3 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -0 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Evaluate the following products.

   (a) $\mathbf{AB}$ (b) $\mathbf{A}^2\mathbf{B}$ (c) $\mathbf{CB}^T\mathbf{A}$ (d) $\mathbf{C}^3$ (e) $\mathbf{ABC}$

2. **Computational cost of different operations.** What is computational cost of the following matrix operations? Computational cost refers to the number of arithmetic operations required to carry out a particular matrix operation. Computational cost is a measure of the efficiency of an algorithm. For example, the consider the operation of vector addition, $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. This requires n addition/subtraction operations and zero multiplication/division operations.

   (a) Matrix multiplication: $\mathbf{AB}$, where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$
   (b) Inner product: $\mathbf{u}^T\mathbf{v}$
   (c) Gaussian Elimination for $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$.
   (d) Back substitution
   (e) Gauss-Jordan method to obtain the row echelon form: $\mathbf{A} \longrightarrow \mathbf{E}$.
   (f) Matrix inversion using the Gauss-Jordan method: $\left[\mathbf{A}|\mathbf{I}\right] \longrightarrow \left[\mathbf{I}|\mathbf{A}^{-1}\right]$

   Report the counts for the addition/subtraction and multiplication/division operations separately.

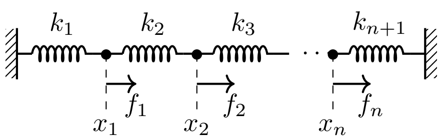
3. Prove $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

4. Consider the following matrix,

$$\mathbf{A} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

   Find out the expression for $\mathbf{A}_n = \mathbf{A}^n$. What is $\mathbf{A}_\infty = \lim_{n\to\infty} \mathbf{A}^n$?

5. Derive force and displacement relationship for a series of $n+1$ springs (with spring constants k_i) connected in a line. There are n nodes, with f_i and x_i representing the force applied and resulting displacement at the i^{th} node.



   (a) Represent the relationship in the following form,

$$\mathbf{f} = \mathbf{Kx}; \ \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}; \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

   (b) What kind of a pattern does $\mathbf{K}$ have?
   (c) Consider a specific case where $n = 4$ and $k = 1.5N.m^{-1}$. What should be forces applied at the four nodes in order to displace the spring $\mathbf{x} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \\ 0 \end{bmatrix} m$.

6. Prove that a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ can always be written as a sum a symmetric matrix $\mathbf{S}$ and a skew-symmetric matrix $\mathbf{A}$.

$$\mathbf{M} = \mathbf{S} + \mathbf{A}, \ \ \mathbf{S}^T = \mathbf{S} \ \text{ and } \ \mathbf{A}^T = -\mathbf{A}$$

   Does this property also hold for a complex matrix $\mathbf{M} \in \mathbb{C}^{n \times n}$?

7. The trace of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is defined as, $trace\,(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$. Prove the following,

   (a) $trace\,(\mathbf{A})$ is a linear function of $\mathbf{A}$.
   (b) $trace\,(\mathbf{AB}) = trace\,(\mathbf{BA})$
   (c) $trace\left(\mathbf{A}^T\mathbf{A}\right) = 0 \implies \mathbf{A} = 0$

8. Prove that the rank of an outer product $\mathbf{xy}^T$ is 1, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$.

9. Is there a relationship between the space of solutions to the following two equations?

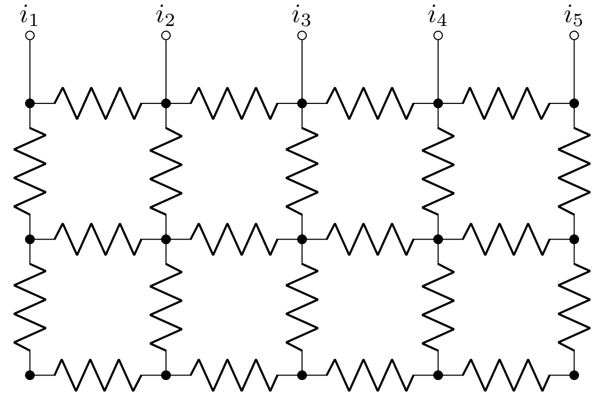
$$\mathbf{y}^T\mathbf{A} = \mathbf{c}^T \quad \text{and} \quad \mathbf{Ax} = \mathbf{b}$$

   If so, how are they related?

10. Consider an upper triangular and lower triangular matrices $\mathbf{U}$ and $\mathbf{L}$, respectively.

   (a) Is the product of two upper triangular matrices $\mathbf{U}_1\mathbf{U}_2$ upper triangular?
   (b) Is the product of two lower triangular matrices $\mathbf{L}_1\mathbf{L}_2$ upper triangular?
   (c) What is the $trace\,(\mathbf{LU})$?

11. Consider the following electrical circuit with rectangular grid of resistors R. The input to this grid is a set of current injected at the top node as shown in the figure, such that $\sum_{k=1}^{5} i_k = 0$.



Express the relationship between the voltages at the different nodes (represented by • in the figure) and the net current flowing in/out of the node in the following form, $\mathbf{Gv} = \mathbf{i}$. Where, $\mathbf{G}$ is the conductance matrix, $\mathbf{v}$ is the vector of node voltages, and $\mathbf{i}$ is the vector representing the net current flow in/out of the different node.

12. Consider the square full rank matrix $\mathbf{A}$, and let the $\mathbf{x}_i$ be then solution to the equation $\mathbf{A}\mathbf{x}_i = \mathbf{e}_i$. Show that the matrix $\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n\end{bmatrix}$ is the inverse of $\mathbf{A}$.

13. Consider the system of equation, $\mathbf{Ax} = \mathbf{b}$, such that a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. Are the following statements true? Explain your answer.
    (a) $rank\left(\mathbf{A}\right) \leq \min\left(m, n\right)$
    (b) The system is consistent if $rank\left(\mathbf{A}\right) = m$.
    (c) The system has a unique solution if $rank\left(\mathbf{A}\right) = n$.

14. Consider a linear function $f : V \to W$, where $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$. If V is a subspace of $\mathbb{R}^n$ then prove that W is a subspace of $\mathbb{R}^m$.

15. For a $n \times n$ square matrix $\mathbf{A}$, prove that if $\mathbf{AX} = \mathbf{I}$, then $\mathbf{XA} = \mathbf{I}$ and $\mathbf{X} = \mathbf{A}^{-1}$.

16. If two systems of linear equations are consistent, with augmented matices $[\mathbf{A}|\mathbf{b}]$ and $[\mathbf{A}|\mathbf{c}]$. Is $[\mathbf{A}|\mathbf{b}+\mathbf{c}]$ consistent?

17. Prove the following for the non-singular square matrices $\mathbf{A}$ and $\mathbf{B}$:
    (a) $\mathbf{AB}$ is non-singular.
    (b) $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$.
    (c) $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
    (d) $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$

18. Derive the inverse of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

19. Consider the following upper-triangular matrix,
$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$
where, $u_{ii} \neq 0,\ 1 \leq i \leq n$. Do the columns of this matrix form a linearly independent set? Explain your answer.

20. Verify that $\mathbf{A}$ and $\mathbf{B}$ are inverses of each other,
    (a) $\mathbf{A} = \mathbf{I} - \mathbf{uv}^T$ and $\mathbf{B} = \mathbf{I} + \mathbf{uv}^T / \left(1 - \mathbf{v}^T\mathbf{u}\right)$
    (b) $\mathbf{A} = \mathbf{C} - \mathbf{uv}^T$ and $\mathbf{B} = \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{uv}^T\mathbf{C}^{-1} / \left(1 - \mathbf{v}^T\mathbf{C}^{-1}\mathbf{u}\right)$
    (c) $\mathbf{A} = \mathbf{I} - \mathbf{UV}$ and $\mathbf{B} = \mathbf{I}_n + \mathbf{U}\left(\mathbf{I}_m - \mathbf{VU}\right)^{-1}\mathbf{V}$
    (d) $\mathbf{A} = \mathbf{C} - \mathbf{UD}^{-1}\mathbf{V}$ and $\mathbf{B} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{D} - \mathbf{VA}^{-1}\mathbf{U}\right)^{-1}\mathbf{VA}^{-1}$

    where, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{U} \in \mathbb{R}^{n \times m}$, $\mathbf{V} \in \mathbb{R}^{m \times n}$ and $\mathbf{D} \in \mathbb{R}^{m \times m}$.

21. Consider the matrices $\mathbf{A} \in \mathbb{R}^{m \times m}$, $\mathbf{B} \in \mathbb{R}^{n \times n}$ and $\mathbf{C} \in \mathbb{R}^{m \times n}$. Verify the following,
    (a) $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}$

22. **Two point boundary problem.** $\mathbf{Ax} = \mathbf{b}$ is often encountered in many practical applications. One such application is the numerical solution of differential equations of the following form,
$$\sum_{i=0}^{M} a_i\left(x\right) y^{(i)}\left(x\right) = f\left(x\right)$$
where, $x \in [a, b]$ and $y\left(a\right) = \alpha, y\left(b\right) = \beta$.

    Numerical methods are often employed for obtaining an approximate estimate of $y\left(x\right)$ at discrete points in the interval $[a, b]$. The interval is divided into subintervals of width Δx. The derivate of $y\left(x\right)$ at the different nodes (points between two subintervals) can be approximated as the following,
$$\begin{aligned} y'\left(x_i\right) &= \frac{y\left(x_i + \Delta x\right) - y\left(x_i - \Delta x\right)}{\Delta x} \\ y''\left(x_i\right) &= \frac{y\left(x_i + \Delta x\right) + 2y\left(x_i\right) - y\left(x_i - \Delta x\right)}{\Delta x^2} \end{aligned}$$
where, $x_i = a + i\Delta x,\ 0 \leq i \leq N+1$, and $b - a = \left(N+1\right)\Delta x$. Addition and subtracting the above two equations and neglecting terms involving higher orders of Δx, we get the following approximations for the derivatives of $y\left(x\right)$ at x_i.

    Replacing the derivatives of $y\left(x\right)$ by the above approximations and evaluating the equation at the different nodes x_is, we arrive a set of N linear equations with N unknowns $y\left(x_1\right), y\left(x_2\right), \dots y\left(x_N\right)$.

    Using this approach, compute an approximate solution for $y\left(x\right)$ for the following differential equations over the interval $x \in [0, 1]$.
    (a) $y''\left(x\right) = -x$
    (b) $y''\left(x\right) + y'\left(x\right) = x$

Solve these equations for different values of Δx, and compare the resulting approximate solution for $y(x)$ with the exact solution. Present your results as a plot the solution $y(x_i)$ versus x_i.

Comment on the dependence of the solution (x) on Δx. What is the best value for Δx to use in solving these equations?

23. **Ill-conditioned systems.** A system $\mathbf{Ax} = \mathbf{b}$ is said to be ill-conditioned when small changes in the components of $\mathbf{A}$ or $\mathbf{b}$ can produce large changes in the solution $\mathbf{x}$. Consider the following system,

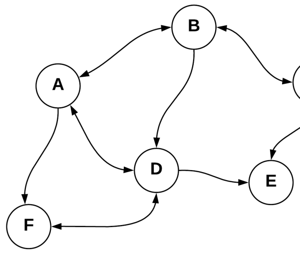
$$\begin{aligned} x - y &= 100 \\ 10 + (9 + \Delta)\, y &= 0 \end{aligned}$$

Find the solutions of the system for different values of $\Delta = -2, -1, 0, 1, 2$. How do the solutions change with Δ. Now consider the following system,

$$\begin{aligned} x - y &= 100 \\ 10 - (9 + \Delta)\, y &= 0 \end{aligned}$$

The second system is an example of an ill-conditioned system. What can you say about the geometries of these two systems?

24. **Connectivity matrices.** Another common application of matrices is in graph theory. A graph is a set of vertices or nodes connected by edges, as show in the following figure. A-F are the nodes of the graph, and the lines with the arrows are the edges that convey information about the connections or relationships between the nodes.



The above graph can be thought as a representation of different places in a city (represented by the nodes), and the lines with the arrows represent the roads connecting these different places. A line with two arrows allow two-way traffic, while line with single arrow only allow one way traffic. The connectivity between the different places can be summarized though the connectivity matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$, where n is the number of nodes in the graph. The elements of this connectivity matrix represents whether or not there is a direct path between two places.

$$c_{ij} = \begin{cases} 1 & \text{there is a direct road between places } i \,\&\, j. \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal element of $\mathbf{C}$ are zero, $c_{ii} = 0$.

Write down the connectivity matrix $\mathbf{C}$ for the graph shown above. How can we use the matrix $\mathbf{C}$ to answer the following questions? Explain exact matrix operation you would perform to answer these questions (Hint: Consider higher power of $\mathbf{C}$).

(a) Is there a path between two places i and j that goes via one other place? For example, we can go from A to D via B.

(b) How many paths are there between places i and j that goes via three other places?