Introduction to Signal Processing

Lecture 4: **Continuous-time Linear Time Invariant Systems**

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Operations on signals

Operations on the dependent variable

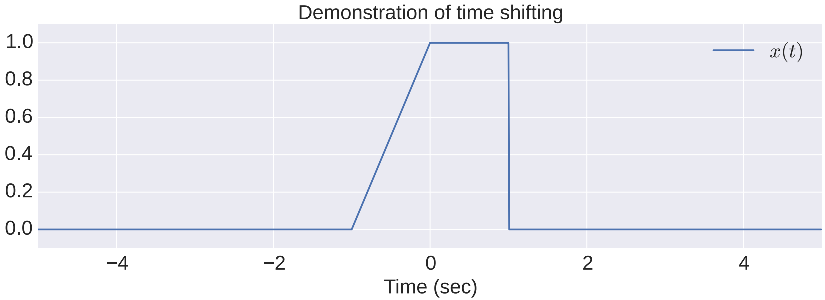
- **Scaling**: $y(t) = ax(t)$
- **Addition**: $y(t) = x_1(t) + x_2(t)$
- **Differentiation**: $y(t) = \frac{d}{dt}x(t)$
- **Integration**: $y(t) = \int_{-\infty}^{t} x(t)dt$

Operations on the independent variable

- **Time shifting**: $y(t) = x(t - \tau), \tau \in \mathbb{R}$
- **Time Scaling**: $y(t) = x(at)$

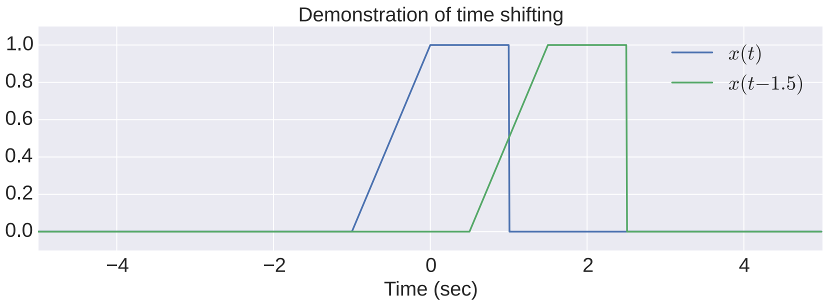
Operation on the independent variable: **Time shifting**

Consider $x(t)$ shown below,



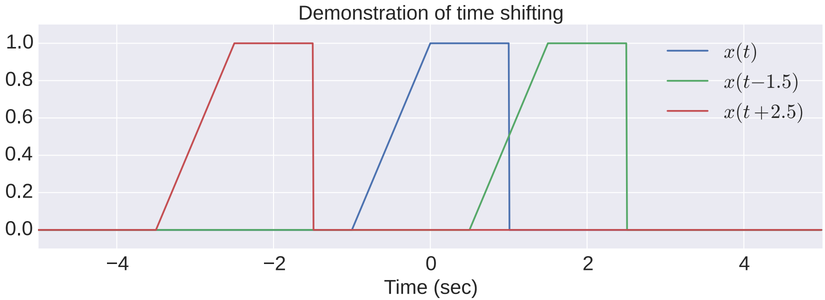
What does $x(t-1.5)$ look like?

Operation on the independent variable: **Time shifting**



What does $x(t+2.5)$ look like?

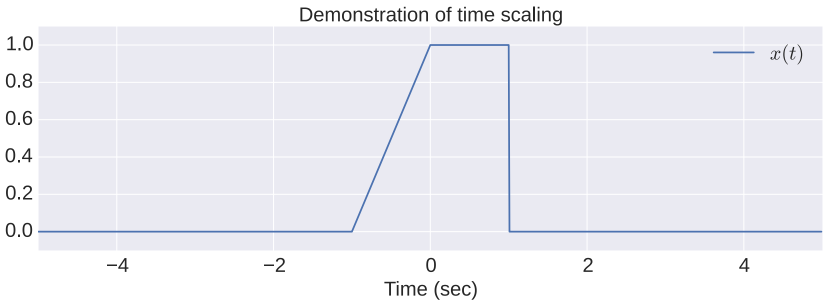
Operation on the independent variable: **Time shifting**



Effect of τ on time shifting operation,

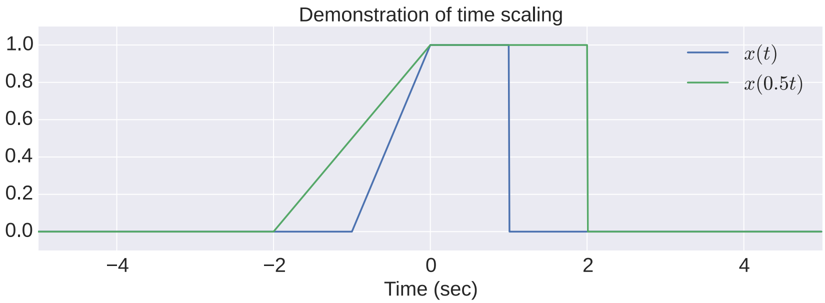
$$x(t-\tau) \longrightarrow \begin{cases} \tau > 0 & \text{Delays signal} \\ \tau < 0 & \text{Forwards signal} \end{cases}$$

Operation on the independent variable: **Time scaling**



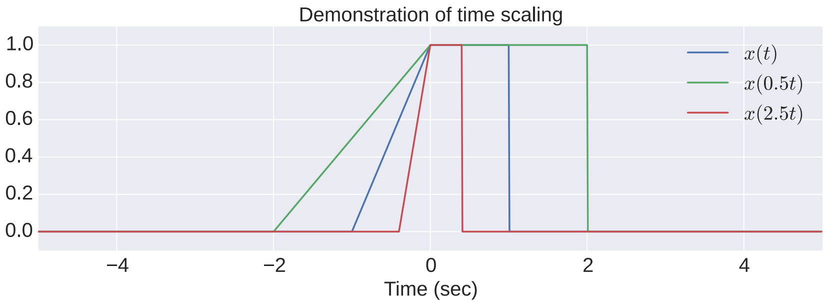
What does $x(0.5t)$ look like?

Operation on the independent variable: **Time scaling**



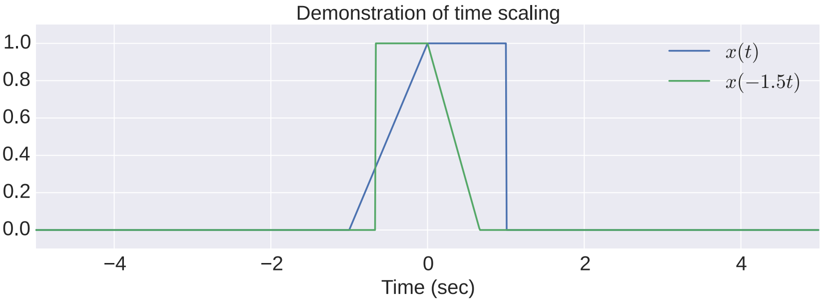
What does $x(2.5t)$ look like?

Operation on the independent variable: **Time scaling**



What does $x(-1.5t)$ look like?

Operation on the independent variable: **Time scaling**



Effect of a on time scaling operation,

$$x(at) \longrightarrow \begin{cases} 0 < a < 1 & \text{Exapands signal} \\ 1 < a < \infty & \text{Shrinks signal} \\ -1 < a < 0 & \text{Inverts and expands signal} \\ -\infty < a < -1 & \text{Inverts and shrinks signal} \end{cases}$$

Continous-time LTI Systems

Remember the definitions of **linearity** and **time invariance**?

$$x_i(t) \mapsto y_i(t) \implies \sum a_i x_i(t - \tau_i) \mapsto a_i \sum y_i(t - \tau_i)$$

Why are we interested in LTI systems?

- A reasonable approximation of real world systems.
- Well developed theory and tools for analysis and synthesis

Characterization of continous-time LTI Systems

- The analysis and synthesis of continuous time LTI system can be done either in the **time domain** or the **frequency domain**.
- Four ways to look at the time domain characteristics:
    - Impulse response
    - Differential equations
    - State space representation
    - Block diagram representation

Impulse Response of a LTI system

- We know everything about a system H, if we the output of the system $y(t)$ for any arbitrary input $x(t)$.
- A brute force method is to do so will be to tabulate all possible inputs and outputs!
- LTI systems allow a much more abbreviated representation.
- Let us assume that we have a set of inputs $x_i(t)$ with known outputs for a given LTI system H, such that

$$H\{x_i(t)\} = y_i(t)$$

- This implies that if there is an input $x(t)$ that is a linear combination of time shifted versions on $x_i(t)$, then

$$H\{x(t)\} = H\left\{\sum a_i x_i(t-\tau_i)\right\} = \sum a_i H\left\{x_i(t-\tau_i)\right\}$$

$$H\{x(t)\} = \sum a_i y_i(t)$$

Impulse Response of a LTI system

- Can we choose the signal $x_i(t)$ such that any arbitrary singal $x(t)$ can be represented as a linear combination of time shifted versions of $x_i(t)$?

Impulse Response of a LTI system

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- $\delta(t)$ acts as a value selector.

$$\int_{-\infty}^{\infty} f(\tau)\delta(\tau)d\tau = f(0)$$

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- What do we get here? $\int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau =?$

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Note: $f(\bullet)$ is assumed to be continuous at t.

Impulse Response of a LTI system

- The impulse response $h(t)$ of a system is the output of the system when $\delta(t)$ is applied as an input.
- We know that, $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)H\left\{\delta(t-\tau)\right\}d\tau$$

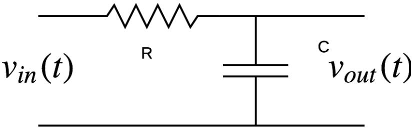
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = h(t) * x(t)$$

This is the *convolution integral* .

Impulse Response of a LTI system

- Properties of the convolution integral
  1. *Commutativity:*
  $$h(t) * x(t) = x(t) * h(t)$$
  2. *Associativity:*
  $$g(t) * (h(t) * x(t)) = (g(t) * h(t)) * x(t)$$
  3. *Connection with the inner product:*
  $$h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \langle x(t), h^*(t-\tau)\rangle$$

Impulse response of a simple RC circuit



Here, the output $v_{out}(t)$ is given by,

$$v_{out}(t) = e^{-t/RC}\left(\frac{1}{RC}\int_0^{\infty} e^{\tau/RC} v_{in}(\tau)d\tau\right) + v_{out}(0)\, e^{-t/RC}$$

Consider the input $x_T(t) = \begin{cases} \frac{1}{T} & 0 \leq t \leq T \\ 0 & \text{Otherwise} \end{cases}$.

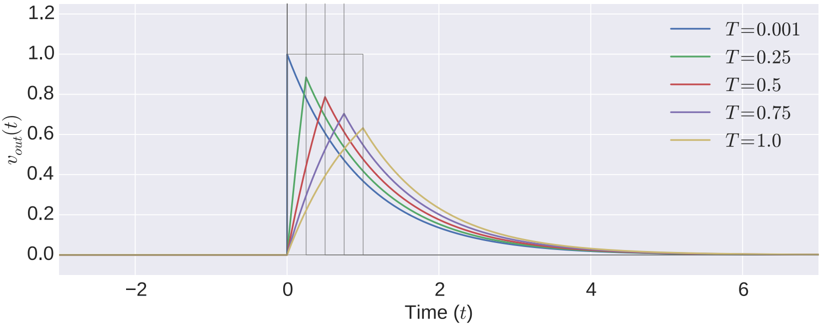
$$\delta(t) = \lim_{T \to 0} x_T(t)$$

Output of the system for $v_{in}(t) = x_T(t)$ will tend toward $h(t)$, when $T \to 0$.

Impulse response of a simple RC circuit

$$v_{out}(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{T}\left(1 - e^{-t/RC}\right) & 0 \leq t \leq T \\ \frac{1}{T}\left(1 - e^{-T/RC}\right) e^{-(t-T)/RC} & t > T \end{cases}$$

Verify that when $T \to 0$, $v_{out}(t) = \frac{1}{RC} e^{-t/RC}, \; t \geq 0$



Mechanics of the convolution integral

Consider the signals,

$$x(t) = \begin{cases} 1 & 0 \geq t \geq T1 \\ 0 & \text{Otherwise} \end{cases} \quad y(t) = \begin{cases} 1 & 0 \geq t \geq T2 \\ 0 & \text{Otherwise} \end{cases}$$

How does the convolution integral do?

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau$$

Mechanics of the convolution integral

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How does the convolution integral do?

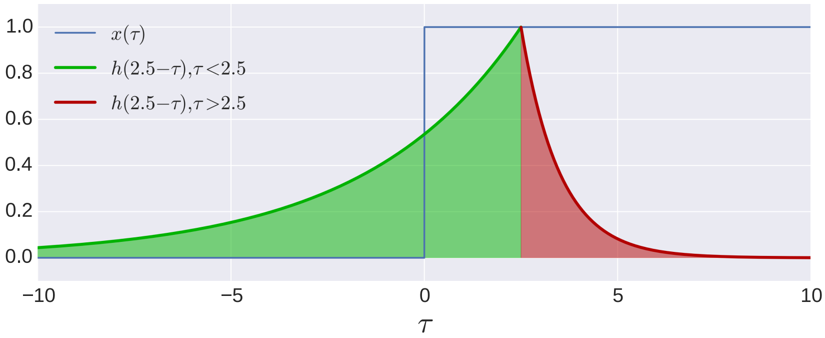
$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

$x(t) * y(t)$ is the integral of the function $x(\tau)y(t-\tau)$.

Note: The integration is with respect to τ, and t is a constant as far as the integral is concerned.

What is $x(t) * y(t)$?

Impulse response act like a weighting function



Here, $h(t) = \begin{cases} e^{-t} & t < 0 \\ e^{-0.25t} & t \geq 0 \end{cases}$ and $x(t) = u(t)$.

$$\begin{cases} h(t), \forall t < 0 & \text{Weightage for the future} \\ h(0) & \text{Weightage for the present} \\ h(t), \forall t > 0 & \text{Weightage for the past} \end{cases}$$

Impulse response and LTI system properties

- **(BIBO) Stability:** Let $|x(t)| < M_x, \ \forall t$, then H is BIBO stable, iff $|y(t)| < \infty$, i.e.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

The **impulse response must be absolutely integrable**.

- **Causality:** $h(t) = 0, \ \forall t < 0$.
- **Memoryless:** $h(t) = k\delta(t)$.

Differential Equations

Continuous-time LTI systems are often described by linear constant differential equations,

$$\sum_{i=0}^{N-1} a_i \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{M-1} b_j \frac{d^j}{dt^j} x(t)$$

The solution to this equation would provide the output for any given input, provided the appropriate initial conditions are available.

$$y(t) = y_p(t) + y_h(t)$$

where, $y_p(t)$ is the particular solution, and $y_h(t)$ is the homogenous solution.

State space representation of LTI systems

State space representation is a powerful and very useful tool in modern control theory.

Converts a N^{th} order differential equation into N 1^{st} order coupled differential equations.

$$\sum_{i=0}^{N-1} a_i \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{M-1} b_j \frac{d^j}{dt^j} x(t) \longrightarrow \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

Here, $\mathbf{x}(t)$ is called the *state* of the system, and $\mathbf{u}(t)$ is the *input*, and $\mathbf{y}(t)$ is the *output* of the system.

State space representation of LTI systems

Why care about this representation?

- Gives insight into the internal behavior of the system.
- Allows one to take the initial conditions of a system into account.
- Allows handling of "single input single output" (SISO) and "multi-input and multi-output" (MIMO) under a single framework.

Note: The state of the system $\mathbf{x}(t)$ is not unique, and there are infinitely many choices.

$$\tilde{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t) \implies \begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\tilde{\mathbf{x}}(t) + \mathbf{T}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{T}^{-1}\tilde{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

where, $\mathbf{T}$ is invertible.

State space representation of LTI systems

What is the state-space representation of the following?

$$\frac{d}{dt}y(t) + ay(t) = kx(t)$$

$$a_2\frac{d^2}{dt^2}y(t) + a_1\frac{d}{dt}y(t) + a_2y(t) = bx(t)$$

Block diagram Representation of LTI systems

Given a pictorial representation of the internal structure of a given system.

Basic blocks required for block diagram representation:

- **Addition:** $y(t) = \sum_i x_i(t)$
- **Scalar multiplication:** $y(t) = cx(t)$
- **Integration:** $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

This requires us to convert the differential equation to an integral equation.

$$F(t) = \int_{-\infty}^{t} f(\tau)d\tau, \quad f(t) = \frac{d}{dt}F(t)$$

What is the block diagram representation of the following system?

$$\frac{d}{dt}y\left(t\right) + ky\left(t\right) = x\left(t\right)$$