Linear Systems

Stability

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Internal stability

- There are two types of stability one can associate with a system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ – **Internal stability** and **Input-Output stability**.
- **Internal stability**: Deals with the stability of the zero-input response of the system states, i.e. $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$.
- An *equilibrium point* $\mathbf{x}_e$ of this system is defined as a point in the state space where, $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}_e) = \mathbf{0}$, i.e. if the system starts in this state, it stays in that state for all time.
- In the case of linear systems, we have $\mathbf{A}\mathbf{x}_e = \mathbf{0}$. The nullspace of $\mathbf{A}$ is the set of all equilibrium points of the linear system.

Internal stability

Find the equilibrium points for the following systems with $\mathbf{f}\left(\mathbf{x}\left(t\right)\right)$: (a) $\begin{bmatrix} x_2 \\ \sin x_1 \end{bmatrix}$; (b) $\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{bmatrix}$; (c) $\begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$; and (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Internal stability

- Definition of stability in the Lyapunov sense for linear systems:
  - The zero-input response of a linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is *stable or marginally stable* if every finite initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)$ $\forall t \geq 0$.
  $$\|\mathbf{x}(t)\| \leq d, \quad \forall t \geq 0$$
  - The zero-input response is *asymtotically stable* if everyf initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)$ that coverges to 0 as $t \to \infty$.
  $$\|\mathbf{x}(t)\| \leq d \text{ and } \lim_{t \to \infty} \|\mathbf{x}(t)\| = 0$$

Internal stability

- The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is marginally stable if and only if all eigenvales of $\mathbf{A}$ have either zero or negative real parts, and the eigenvalues with zero real parts have the same algebraic and geometric multiplicity.
- The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable if and only if all eigenvales of $\mathbf{A}$ have negative real parts.

Internal stability

- Consider the solution, $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-)\,,\;\; t \geq 0$, and $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$.

$$\|\mathbf{x}(t)\| = \left\|e^{t\mathbf{A}}\mathbf{x}(0^-)\right\| \leq \left\|e^{t\mathbf{J}}\right\| \left\|\mathbf{x}(0^-)\right\|$$

- When $\mathbf{A}$ is diagonalizable (λ_i are the eigenvalues of $\mathbf{A}$),
  - $\|\mathbf{x}(t)\| \leq e^{\sigma t}\|\mathbf{x}(0^-)\|$, where $\sigma = \max_i \Re\{\lambda_i\}$.
  - When $\sigma = 0$, $\|\mathbf{x}(t)\|$ is bounded $\forall t \geq 0$.
  - When $\sigma < 0$, $\lim_{t\to\infty}\|\mathbf{x}(t)\| = 0$.

Internal stability

- When $\mathbf{A}$ is not diagonalizable, then $\mathbf{J}$ is block diagonal.
  - Consider the i^{th} Jordan block, $\mathbf{J}_i = \lambda_i \mathbf{I} + \mathbf{N}$, Thus, $e^{t\mathbf{J}_i} = e^{\lambda_i t \mathbf{I}} e^{t\mathbf{N}} \implies \|\mathbf{x}(t)\| \leq e^{\sigma_i t} \left\| e^{t\mathbf{N}} \right\| \|\mathbf{x}(0^-)\|$
  - When $\sigma_i = 0$, $\left\| e^{t\mathbf{N}} \right\|$ grows with time, and thus $\mathbf{x}(t)$ is not bounded.
  - When $\sigma_i < 0$, the $e^{\sigma_i t}$ term does not allow $\mathbf{x}(t)$ to grow.

Internal stability

Comment of the stability: (a) $\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$; (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$; and (d) $\begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Internal stability – Discrete-time LTI systems

When $\mathbf{A}$ is diagonalizable (λ_i are the eigenvalues of $\mathbf{A}$),

- $\|\mathbf{x}[k]\| \leq |\lambda|^k \|\mathbf{x}[0]\|$, where $\lambda = \max_i |\lambda_i|$.
- When $|\lambda| = 1$, $\|\mathbf{x}[k]\|$ is bounded $\forall k > 0$.
- When $|\lambda| < 1$, $\lim_{k \to \infty} \|\mathbf{x}[k]\| = 0$.

Internal stability – Discrete-time LTI systems

When $\mathbf{A}$ is not diagonalizable, then $\mathbf{J}$ is block diagonal.

- Consider the i^{th} Jordan block, $\mathbf{J}_i^k = (\lambda_i \mathbf{I} + \mathbf{N})^k = \sum_{l=0}^{k} \frac{k!}{(k-l)!l!} \lambda_i^l \mathbf{N}^{k-l}$
- When $|\lambda_i| = 1$, $\left\|\mathbf{J}_i^k\right\|$ grows with time, and thus $\mathbf{x}[k]$ is not bounded.
- When $|\lambda_i| < 1$, the λ_i^l term does not allow $\mathbf{x}[k]$ to grow.

Input-Output stability

- Input-output stability or external stability deals with the forced response of a system, assuming the system is relaxed.
- Input-output stability is also known as BIBO (bounded input, bounded output) stability, i.e. a bounded input $\mathbf{u}(t)$ applied to the system produces a bounded output $\mathbf{y}(t)$.

Input-Output stability

- A single input, single output (SISO) LTI system with impulse response $h(t)$ is BIBO stable, if and only if
$$\int_0^\infty |h(t)|\, dt < \infty$$
When $h(t)$ is not absolutely integrable, then we are not guaranteed that bounded inputs will produce bounded outputs.
- A SISO system with a rational transfer function $H(s)$ is BIBO stable if and only if all its poles lie in the left half of the s-plane.

$$H(s) = \frac{B(s)}{A(s)} \xrightarrow{\mathcal{L}^{-1}} h(t) \text{ contains } e^{p_i t}, te^{p_i t}, \ldots t^{m-1}e^{p_i t}$$

Input-Output stability

- In the case of a muti-input, multi-output (MIMO) LTI system, the impulse response and transfer function matrices are given by,

$$\mathbf{G}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{B} + \mathbf{D}\delta(t) \text{ and } \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- A MIMO system is BIBO stable, if and only if each element of the impulse response matrix $\mathbf{G}(t)$ is absolutely integrable.

$$\int_0^\infty |g_{ij}(t)|\, dt < \infty, \;\; \forall 1 \leq i, j \leq n$$

Input-Output stability

- A MIMO LTI system is BIBIO stable, if and only if the poles of each element of the transfer function matrix $H(s)$ lie in the left half of the s-plane.
  Even if we have eigenvalue that have positive real parts, the system migth still be BIBO stable because of pole-zero cancellations in the individual elements of $\mathbf{G}(s)$.

Is this system externally stable? $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & -2 \end{bmatrix}$. Is this system internally stable?

Input-Output stability (discrete-time system)

- A SISO discrete-time LTI system with impulse response $h\left[k\right]$ is BIBO stable, if and only if

$$\sum_{k=0}^{\infty} \left|h\left[k\right]\right| < \infty$$

- A SISO system with a rational transfer function $H\left(z\right)$ is BIBO stable if and only if all its poles lie within the unit circle $\left|z\right| = 1$.

$$H\left(z\right) = \frac{B\left(z\right)}{A\left(z\right)} \xrightarrow{\mathcal{L}^{-1}} h\left[k\right] \text{ contains } p_i^k, kp_i^k, \ldots k^{m-1}p_i^k$$

Input-Output stability (discrete-time system)

- A MIMO discret-time LTI system is BIBO stable, if and only if each element of the impulse response matrix $\mathbf{G}[k]$ is absolutely summable.

$$\sum_{k=0}^{\infty} |g_{ij}[k]| < \infty, \ \ \forall 1 \leq i, j \leq n$$

- A MIMO discrete-time LTI system is BIBO stable, if and only if the poles of each element of the transfer function matrix $H(z)$ lie in the unit circle.