Linear Systems: Eigenvalues and Eigenvectors Assignment

1. Explain why an eigenvector cannot be associated with two eigenvalues.

2. What are the eigenspaces associated with the diagonal matrix $\mathbf{D} = \text{diag}\left(d_1, d_2, \ldots d_n\right)$?

3. If a matrix $\mathbf{A}$ has zero as one of its eigenvalues, explain why $\mathbf{A}$ must be singular.

4. For a matrix $\mathbf{A}$ with eigenvalues $\{\lambda_i\}_{i=1}^n$, verify for the following matrices that $\Pi_{i=1}^n \lambda_i = \det(\mathbf{A})$ and $\sum_{i=1}^n \lambda_i = trace(\mathbf{A})$.

   (a) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   (d) $\frac{1}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$

5. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ be the eigenpairs of a matrix $\mathbf{A}$. Then prove that,

   (a) $\left\{\lambda_i^k, \mathbf{v}_i\right\}_{i=1}^n$ are the eigenpairs of $\mathbf{A}^k$.

   (b) $\{p(\lambda_i), \mathbf{v}_i\}_{i=1}^n$ are the eigenpairs of $p(\mathbf{A})$, where $p(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \ldots + \alpha_k\mathbf{A}^k$.

6. Prove that if $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ are the eigenpairs of a matrix $\mathbf{A}$, then the eigenpairs of $\mathbf{A}^k$ are $\left\{\lambda_i^k, \mathbf{v}_i\right\}_{i=1}^n$.

7. Consider the matrices $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Are the eigenvalues of $\mathbf{AB}$ equal the eigenvalues of $\mathbf{BA}$?

8. Consider the matrices $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{v}$ is an eigenvector $\mathbf{B}$, underwhat condition will $\mathbf{v}$ also be the eignevector of $\mathbf{AB}$. Under these conditions, what will be corresponding eigenvalue of $\mathbf{v}$? How do your answers change in the case of $\mathbf{BA}$?

9. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ are the eignepairs of a matrix $\mathbf{A}$. What are the eigenpairs of the folliowing?

   (a) $2\mathbf{A}$

   (b) $\mathbf{A} - 2\mathbf{I}$

   (c) $\mathbf{I} - \mathbf{A}$

10. Let $\mathbf{A} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$. What is the value of: (a) A^2 (b) A^{100} (c) A^{∞}?

11. Show that $\mathbf{u} \in \mathbb{R}^2$ is an eigenvector of $\mathbf{A} = \mathbf{u}\mathbf{v}^T$. What are the two eigenvalues of $\mathbf{A}$?

12. Consider two similar matrices $\mathbf{A}$ and $\mathbf{B}$. Prove that the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are the same. How are the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ related to each of other for a given eigenvalue?

13. Find the eigenvectors of the following permutation matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

14. **Left eigenvectors**: Consider a matrix $\mathbf{A}$ with eigenpairs $\{\lambda_1, \mathbf{v}_i\}_{i=1}^n$. The left eigenvectors of the matrix $\mathbf{A}$ are the vectors that satisfy the equation, $\mathbf{A}^T\mathbf{w} = \mu\mathbf{w}$ (or $\mathbf{w}^T\mathbf{A} = \mu\mathbf{w}^T$), and let $\{\mu_i, \mathbf{w}_i\}_{i=1}^n$ be the left eigenpairs of $\mathbf{A}$. Show the following,

    (a) The eigenvalues of both $\mathbf{A}$ and $\mathbf{A}^T$ are the same.

    (b) $\mathbf{v}_i^T\mathbf{w}_j = 0$. The eigenvector $\mathbf{v}_i$ corresponding to the eigenvalue λ_i and the left eigvenvector $\mathbf{w}_j$ corresponding to the eigenvalue λ_j are orthogonal, when $\lambda_i \neq \lambda_j$.

    (c) The matrix A can be expressed as a sum of rank-one matrices,

    $$\mathbf{A} = \lambda_1\mathbf{v}_1\mathbf{w}_1^T + \lambda_2\mathbf{v}_2\mathbf{w}_2^T + \ldots + \lambda_n\mathbf{v}_n\mathbf{w}_n^T$$

15. Prove that $\mathbf{AA}^T$ has real and positive eigenvalues, and that the eigenvectors corresponding to distinct eigenvalues of $\mathbf{AA}^T$ are orthogonal.

16. If $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ are the eigenpairs of a non-singular matrix $\mathbf{A}$, the prove that $\left\{\lambda_i^{-1}, \mathbf{v}_i\right\}_{i=1}^n$ are the eigenpairs of $\mathbf{A}^{-1}$.

17. A matrix $\mathbf{A}$ is called *nilpotent* if $\mathbf{A}^k = \mathbf{0}$ for some finite positive integer k. Prove that the $trace(\mathbf{A}) = 0$ for a nilpotent matrix $\mathbf{A}$. What are all the eigenvalues of such a matrix?