Introduction to Digital Signal Processing

Z-transform

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Z transform

- Exponential signals are *eigenfucntiopns* of LTI systems.

$$z^n \longrightarrow H\left(z\right)z^n$$

  $H(z)$ is the eigenvalue corresponding to the eigenfunction z^n.

- If $x[n] = \sum_k \alpha_k z_k^n$, then $y[n] = \sum_k \alpha_k H\left(z_k\right) z_k^n$.

$$\left(\alpha_k\right)_{k\in\mathbb{Z}} \longrightarrow \text{Representation of } x[n] \text{ using } z_k^n$$
$$\left(H\left(z_k\right)\alpha_k\right)_{k\in\mathbb{Z}} \longrightarrow \text{Representation of } x[n] \text{ using } z_k^n$$

- The z-transoform allows us to find the representation of any disctet-time signal $x[n]$ in terms of the set of complex exponentials $\{z^n\}_{z\in\mathbb{C}}$

z transform

The z-transform of a discrete time signal $x[n]$ is defined as the following power seris,

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \mathcal{Z}(x[n])$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

where, $z \in \mathbb{C}$.

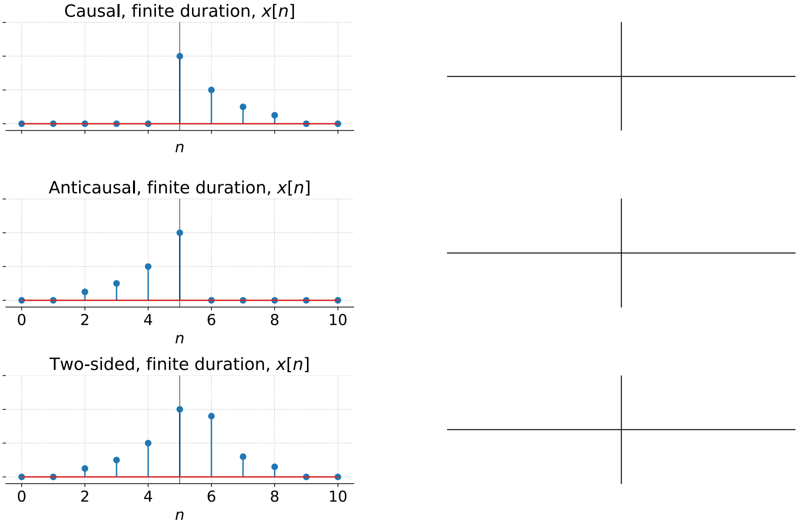
- The values of z for which the above summation covnerges is called the *region of conergence* of $X(z)$.

z transform

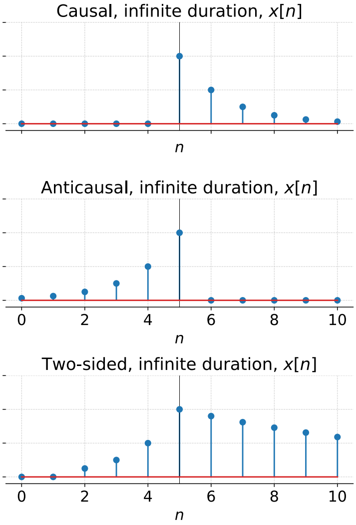
z-transform of some signals.

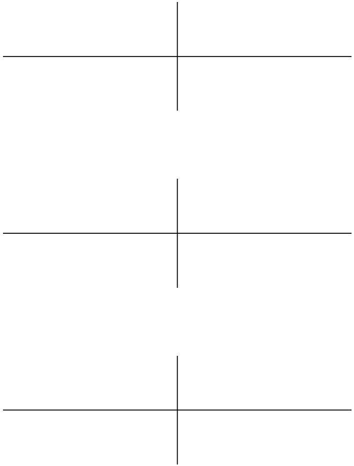
1. $\delta[n]$
2. $\delta[n-k]$
3. $\delta[n+k]$
4. $\sum_{k=0}^{5} \alpha_k \delta[n-k]$
5. $1[n]$
6. $a^k \cdot 1[n]$
7. $-a^k \cdot 1[-n-1]$

z-transform and ROCs



z-transform and ROCs





Properties of the z-transform

- Linearity
- Time-shifting
- Convolution in time
- Initital value theorem

Unilateral z-transform

z-transform of causal signals of the form $x[n] \cdot 1[n]$.

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

This is useful when analysing linear difference equations.
When the time domain signal $x[n]$ is delayed by a sample, such that the signal is $x[n-1] \cdot 1[n]$, then we have

$$x[n] \cdot 1[n] \overset{\mathcal{Z}_{ul}}{\longleftrightarrow} X(z) \implies x[n-1] \cdot 1[n] \overset{\mathcal{Z}_{ul}}{\longleftrightarrow} z^{-1} X(z) + x[-1]$$

Transfer function of an LTI system

The z-transform of the impulse response $h[n]$ is defined as the transfer function of a discrete-time LTI system.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

When the system is causal, then $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$.
The z-transforms of the input $x[n]$ and $y[n]$ are related to each other through the transfer function,

$$Y(z) = H(z) \cdot X(z)$$

Rational z-transforms

- In practice, we often come across rational polynomial of z.
- Consider a LTI system described by the following different equation,

$$y[n]+a_1y[n-1]+a_2y[n-2]+\cdot+a_Ny[n-N]=b_0x[n]+b_1x[n-1]+\cdots+b_Mx[n-M]$$

We are interested in sovling this equation from time $n=0$ for an input of the form $x[n]\cdot 1[n]$. Taking the unilateral z-transform on both sides,

$$\begin{aligned}
y[n] &\overset{\mathcal{Z}_{ul}}{\longleftrightarrow} Y(z)\\
y[n-1] &\overset{\mathcal{Z}_{ul}}{\longleftrightarrow} z^{-1}Y(z)+y[-1]\\
y[n-2] &\overset{\mathcal{Z}_{ul}}{\longleftrightarrow} z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\\
&\vdots\\
y[n-N] &\overset{\mathcal{Z}_{ul}}{\longleftrightarrow} z^{-N}Y(z)+z^{-(N-1)}y[-1]+z^{-(N-2)}y[-2]+\cdots+y[-N]
\end{aligned}$$