

Introduction to Digital Signal Processing

Fourier Representation of Discrete-Time Signals

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Discrete-Time Fourier Series

- ▶ Continuous-time complex sinusoids have frequencies $\omega \in (-\infty, \infty)$.
- ▶ Discrete-time sinusoids have frequencies $\Omega \in (-\pi, \pi]$
- ▶ A discrete-time periodic signal $x[n]$ with fundamental period N can be represented as a sum of discrete-time sinusoids,

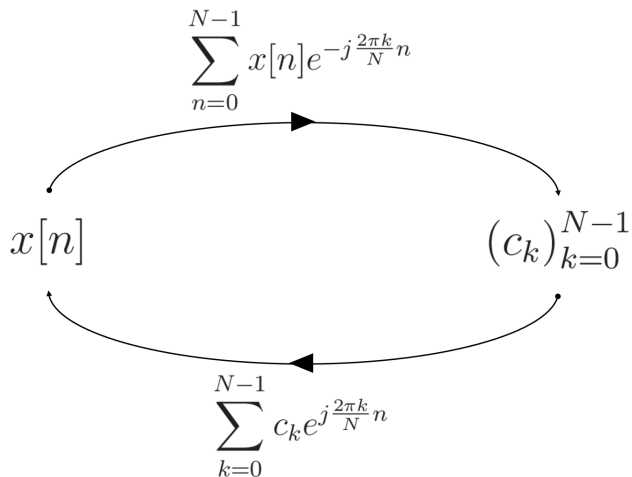
$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}$$

- ▶ Appropriate choice of c_k will let us represent $x[n]$ as a linear combination of $e^{j\frac{2\pi k}{N}n}$.
- ▶ The Fourier coefficient c_k can be determined by,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

Discrete-Time Fourier Series

- c_k is discrete like in the case of continuous-time Fourier series, and it is also periodic with fundamental period N , i.e. $c_k = c_{k+N}$.



Properties of Discrete-Time Fourier Series

- ▶ Fourier representation is discrete and periodic. (c_k is period with fundamental period N)
- ▶ When $N = 2M$ is even, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \leq l < \frac{N}{2}$$

- ▶ When $N = 2M + 1$ is odd, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \leq l < \frac{N-1}{2}$$

- ▶ Parseval's identity.

$$\frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

The distribution of $|c_k|^2$ as a function of $0 \leq k < N$ is the *power spectral density* of the periodic signal $x[n]$.

Discrete-Time Fourier Series

Find the DTFS of $x[n] = \begin{cases} 1, & 0 \leq n < M \\ 0, & M \leq n < N - 1 \end{cases}$ with fundamental period N .

Discrete-time Fourier Transform

- ▶ Similar to the continuous-time case, the Fourier representation of discrete-time aperiodic signals can be obtained as the limiting case of the periodic signals with increasing period N .
- ▶ The discrete-time Fourier transform (DTFT) of an aperiodic signal $x[n]$ with finite energy is given by,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, \quad -\pi \leq \Omega < \pi$$

- ▶ $X(\Omega)$ is continuous in Ω and periodic with period 2π .
- ▶ Inverse DTFT,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Discrete-time Fourier Transform

- ▶ DTFT exists only if $x[n]$ is absolutely summable.

$$\sum_n |x[n]| < \infty \implies |X(\Omega)| < \infty$$

- ▶ When $x[n]$ is only square summable, then DTFT converges to the true DTFT only in the mean squared sense.

E.g.,

$$x[n] = \begin{cases} \frac{\Omega_c}{n}, & n = 0 \\ \frac{\Omega_c}{n} \frac{\sin \Omega_c n}{\Omega_c n}, & n \neq 0 \end{cases} \longrightarrow X(\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$$

Properties of DTFT

- ▶ **Linearity:** $\alpha x[n] + \beta y[n] \xleftrightarrow{\text{DTFT}} \alpha X(\Omega) + \beta Y(\Omega)$
- ▶ **Shift in time:** $x[n - n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega)$
- ▶ **Shift in frequency:** $x[n]e^{j\Omega_0 n} \xleftrightarrow{\text{DTFT}} X(\Omega - \Omega_0)$
- ▶ **Symmetry in time:** $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$ is real.
- ▶ **Convolution in time:** $x[n] * y[n] \xleftrightarrow{\text{DTFT}} X(\Omega)Y(\Omega)$
- ▶ **Convolution in frequency:** $x[n] \cdot y[n] \xleftrightarrow{\text{DTFT}} X(\Omega) * Y(\Omega)$

Classification of signals based on the frequency spectrum

- ▶ We can also classify signals based on how their energy is distributed across frequency.
- ▶ **Low frequency signal.** Most of the energy is concentrated around 0Hz and frequencies around 0Hz.
- ▶ **High frequency signal.** Very less is concentrated around 0Hz, and most of the energy is in the higher frequencies all the way upto $\omega \rightarrow \infty$.
- ▶ **Bandpass signal.** Very less concentration around 0Hz and at high frequencies. Most of the energy is concentrated within a band of finite frequencies.
- ▶ **Bandlimited signal.** Signal energy is uniformly zero beyond a particular frequency, i.e. $|X(\omega)| = 0, \forall |\omega| > \omega_b$.

Bandwidth of a signal

- ▶ The band of frequencies over which most of the energy is distributed is the *bandwidth* of the signal.
- ▶ Several ways to define the bandwidth of a signal.
- ▶ **3 dB bandwidth.** Frequency range over which the spectral density is above a particular threshold.
Threshold value is often defined relative to the max. value of the spectral density.

Frequency-Domain and Time-Domain Properties

- ▶ **Continuous-Time, Periodic** \longrightarrow
- ▶ **Discrete-Time, Non-Periodic** \longrightarrow
- ▶ **Continuous-Frequency, Periodic** \longrightarrow
- ▶ **Discrete-Frequency, Non-Periodic** \longrightarrow