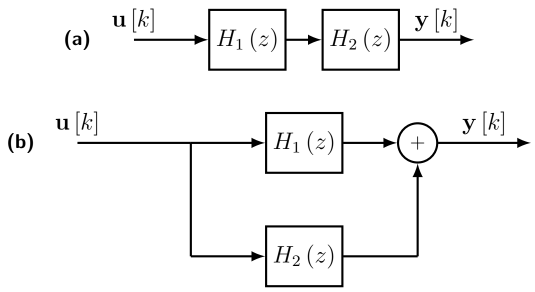
Linear Systems: LDS Solution, Stability, Controllability & Observability Assignment

1. Consider $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute $\mathbf{A}^{100}$ and $e^{t\mathbf{A}}$.

2. Assuming that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$. Prove that the eigenvalues of $e^{\mathbf{A}}$ are $e^{\lambda_1}, e^{\lambda_2}, \ldots e^{\lambda_n}$

3. Consider a linear time-variant discrete-time, $\mathbf{x}[k+1] = \mathbf{A}[k]\,\mathbf{x}[k] + \mathbf{B}[k]\,\mathbf{u}[k]$. Find the general expression for the zero-input solution, and the zero-state solution.

4. What are the conditions on the individual LTI discrete-time systems $H_1(z)$ and $H_2(z)$ for the following overall system to be: (a) internally stable; (b) externally stable; (c) controllable; and (d) observable

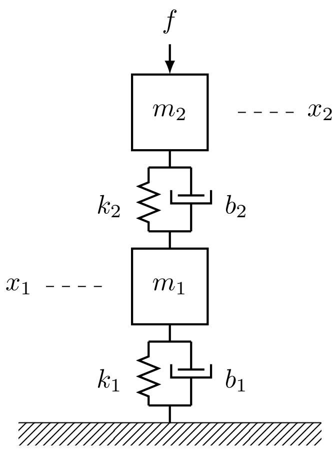


5. Consider a the LTI system, $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$. An experiment was carried out with this system, where the system was started at different initial conditions $\mathbf{x}_i(0^-)$, and the corresponding state trajectories were recorded to be $\mathbf{x}_i(t),\ \forall t \geq 0$. Assuming that the set of initial conditions$\{\mathbf{x}(0^-)\}_{i=1}^{n}$ are linearly independent, find the expression for the $e^{t\mathbf{A}}$.

6. Consider the following system, where the input is the force f applied to mass m_2, and the output of the system, positions of masses m_1 and m_2, are measured using a set of position sensors. Assume $m_1 = m_2 = 1Kg$, $k_1 = k_2 = 1Nm^{-1}$, and $b_1 = b_2 = 1Nm^{-1}s$.

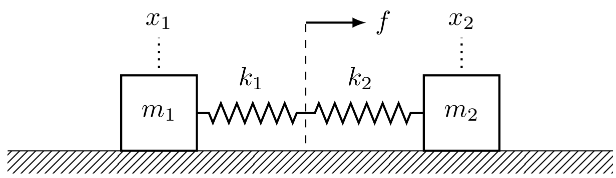
   (a) Is this system controllable? Is the system still controllable if the input f was acting on the mass m_1 instead of m_2?

   (b) Is this system observable? If instead of measuring both position x_1 and x_2, if the output of the system was either x_1 or x_2, is the system still observable?



Assume now that instead of f acting on m_2, it was acting on m_1, and the output of interest was the acceleration of the mass m_1. What would be correspondoing state and measurement equations in this case? Is this system still controllable and observable?

7. Write down the state and measurement equations of the following system, assuming the the output being measured are the positions of the masses m_1 and m_2. Assuming that $\frac{k_1}{m_1} = \frac{k_2}{m_2}$, is the system controllable? If $\frac{k_1}{m_1} \neq \frac{k_2}{m_2}$, is the system now controllable? Why are these two systems different in terms of controllability?



8. Consider the following system,

$$\mathbf{x}[n+1] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{u}[n]$$
$$\mathbf{y}[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}[n]$$

   Determine the conditions $a, b, c,$ and d for complete state controllability and complete observability.

9. Consider the following system,

$$\mathbf{x}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.16 & 1 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \mathbf{u}[n]$$

$$\mathbf{x}[0] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   Verify that this system is controllable. Determine the sequence of input signal $\mathbf{u}[0]$ and $\mathbf{u}[1]$ such that $\mathbf{x}[2] = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

10. Consider the following system,

$$
\begin{aligned}
\mathbf{x}[n+1] &= \begin{bmatrix} 0 & 1 \\ -0.16 & 1 \end{bmatrix} \mathbf{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}[n] \\
\mathbf{y}[n] &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}[n]
\end{aligned}
$$

Verify that this system is observable.