Introduction to Digital Signal Processing

Fourier Representation of Signals

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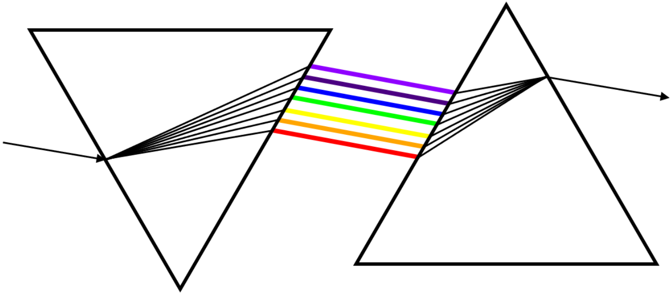
Representation of signals as a linear combination of other signals

- Decomposition of an object into smaller/simpler components is a useful operation.
- Decomposing $x[n]$ as a $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ was useful in understanding input-output relationship of LTI systems.
- Infinitely many ways of decomposing a signal.
- Decomposing into complex exponential signals is one of the oldest, most common, and very useful approaches $\longrightarrow$ **Fourier analysis**.

An example from Optics



- Sunlight (white light) can be decomposed into different colors using a prism.
- Individual colors can be cobmined to produce white light back.
- Using a filter between the two prisms will allow us to mix the individual colors in different combinations.

Spectral analysis of signals

- Signals can be decomposed into different sinusoidal components.
- Each sinusoidal component is index or parametrized by its frequency.
- Let the set of all sinusoidal signals be $S_{ct} = \{s_\omega(t) \mid \omega \in \mathbb{R}\}$ for continous-time signals, or $S_{dt} = \{s_\Omega[n] \mid \Omega \in (-\pi, \pi]\}$ for discrete-time singals.

$$x(t) = \int_{-\infty}^{\infty} X(\omega)\, s_\omega(t)\, d\omega \qquad y[n] = \int_{-\pi}^{\pi} Y(\Omega)\, s_\Omega[n]\, d\Omega$$

$X(\omega)$ is the "amount" of $s_\omega(t)$ present in $x(t)$.
$Y(\Omega)$ is the "amount" of $s_\Omega[n]$ present in $y[n]$.

Fourier Series: Linear combination of periodic signals

- Consider the following set of sinusoidal signals $\{\sin(k \cdot \omega_0 t) \mid k \in \mathbb{Z}_{>0}\}$.

- $\omega_0 = 2\pi f_0 = 2\pi \frac{1}{T_0}$ is the fundamental angular frequency of the sinusoidal signal $\sin(\omega_0 t)$.

- Any singal of the following signal will be periodic.

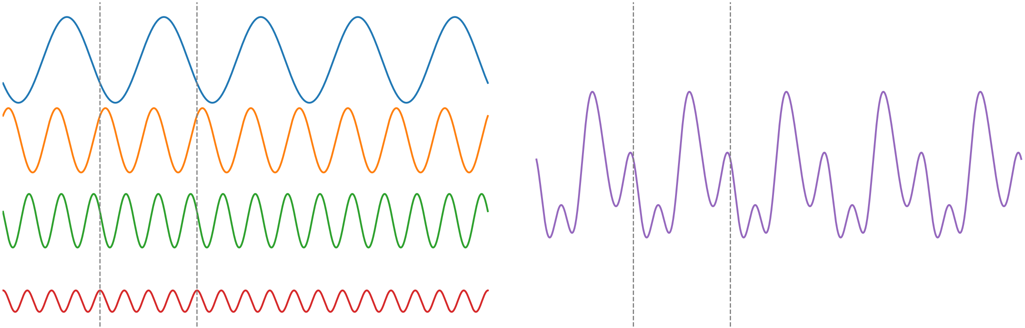
$$x(t) = r_0 + \sum_{k=1}^{\infty} r_k \sin(k \cdot \omega_0 t + \phi_k), \quad 0 \leq r_k \in \mathbb{R}, \ \phi_k \in (-\pi, \pi]$$

What is the fundamental period of the signal?

Fourier Series: Linear combination of periodic signals

- We can generate new periodic signals (of fundamental frequency f_0) by mixing sinusoidal signals with fundamental frequencies that are integer multiples of f_0.

$$x(t) = r_0 + \sum_{k=1}^{\infty} r_k \sin\left(k \cdot \omega_0 t + \phi_k\right), \quad 0 \leq r_k \in \mathbb{R}, \; \phi_k \in (-\pi, \pi]$$



Fourier Series

$$
\begin{aligned}
x(t) &= r_0 + \sum_{k=1}^{\infty} r_k \sin\left(k \cdot \omega_0 t + \phi_k\right) \\
&= r_0 + \sum_{k=1}^{\infty} a_k \sin\left(k \cdot \omega_0 t\right) + b_k \cos\left(k \cdot \omega_0 t\right) \\
&= \sum_{k=0}^{\infty} \left(a_k \sin\left(k \cdot \omega_0 t\right) + b_k \cos\left(k \cdot \omega_0 t\right)\right)
\end{aligned}
$$

Expressing $\cos(\cdot)$ and $\sin(\cdot)$ in terms of complex exponentials, and grouping terms together,

$$
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}
$$

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

Knowing f_0, we can compute the signal $x(t)$ from the list of numbers $(c_k)_{k=-\infty}^{\infty}$.

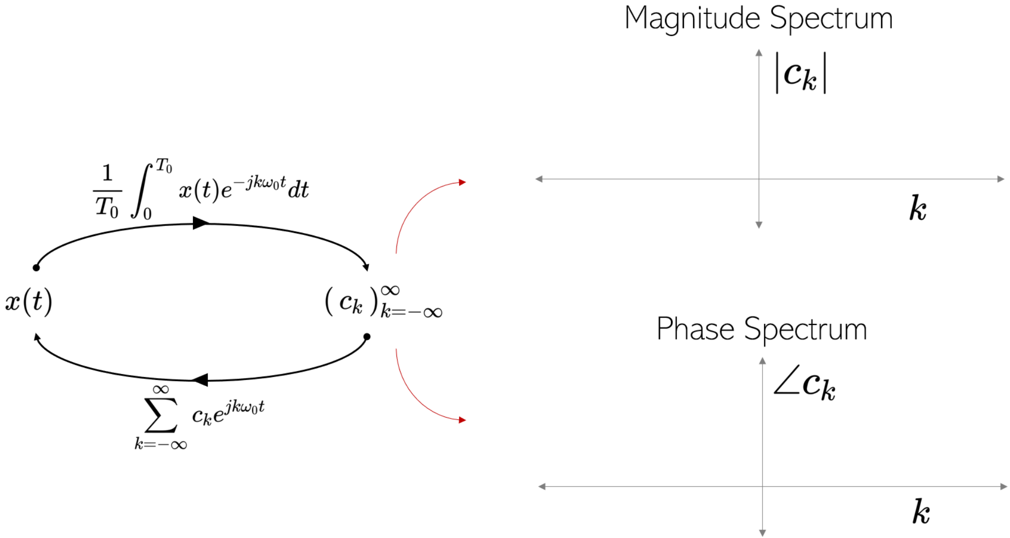
We can compute c_k as the following,

$$c_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t} dt$$

Fourier Series

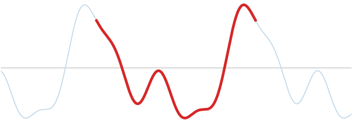
$$
\begin{tikzcd}
x(t) \arrow[rr, bend left, "\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt"] & & \left(c_k \right)_{k=-\infty}^{\infty} \arrow[ll, bend left, "\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}"]
\end{tikzcd}
$$

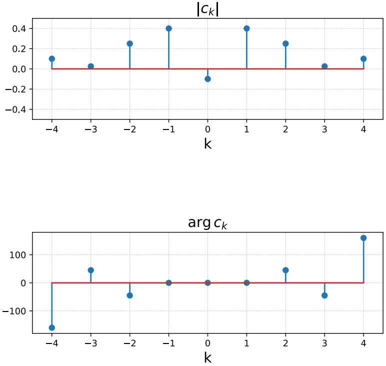
Fourier Series

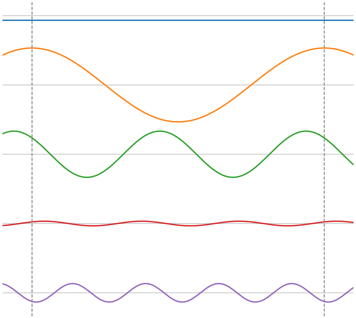


Fourier Series

$$x(t) = -0.1 + 0.8\cos(2\pi t) + 0.5\cos\left(4\pi t + \frac{\pi}{4}\right) + 0.05\cos\left(6\pi t - \frac{\pi}{4}\right) + 0.2\cos\left(8\pi t + \frac{8\pi}{9}\right)$$

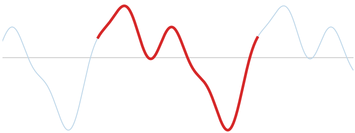


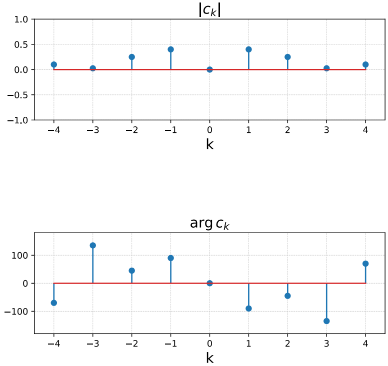


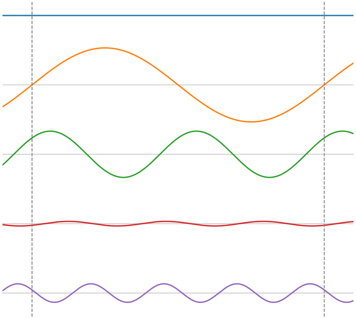


Fourier Series

$$x(t) = 0.8\sin\left(2\pi t\right) + 0.5\sin\left(4\pi t + \frac{\pi}{4}\right) + 0.05\sin\left(6\pi t - \frac{\pi}{4}\right) + 0.2\sin\left(8\pi t + \frac{8\pi}{9}\right)$$







Does any periodic signal have a Fourier series representation?

- If $x(t)$ is absolutely integrable over a single cycle, then the Fourier serious coefficients exist.

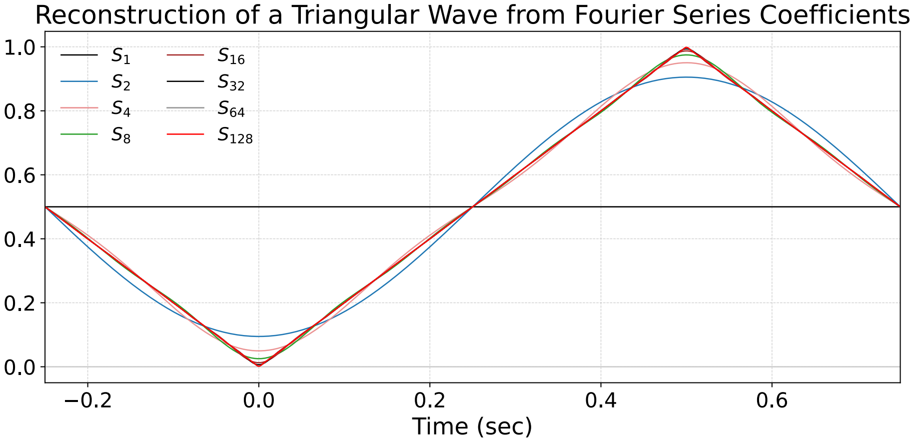
- Any continuous periodic function will have a Fourier series representation.

- When $x(t) =$ is continuous and finite, then reconstructed signal $\sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t}$ will be equal to $x(t)$ pointwise.

$$x\left(t\right) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t} \quad \forall t$$

Does any periodic signal have a Fourier series representation?

$$x(t) = \begin{cases} t, & 0 \leq t < 0.5 \\ 1 - t, & 0.5 \leq t < 1 \end{cases} \longrightarrow c_k = \begin{cases} \frac{1}{4}, & k = 0 \\ \frac{4}{k^2 \omega_0^2} \sin^2\left(\frac{k\omega_0}{4}\right) e^{-j\frac{k\omega_0}{2}}, & k \neq 0 \end{cases}$$



Does any periodic signal have a Fourier series representation?

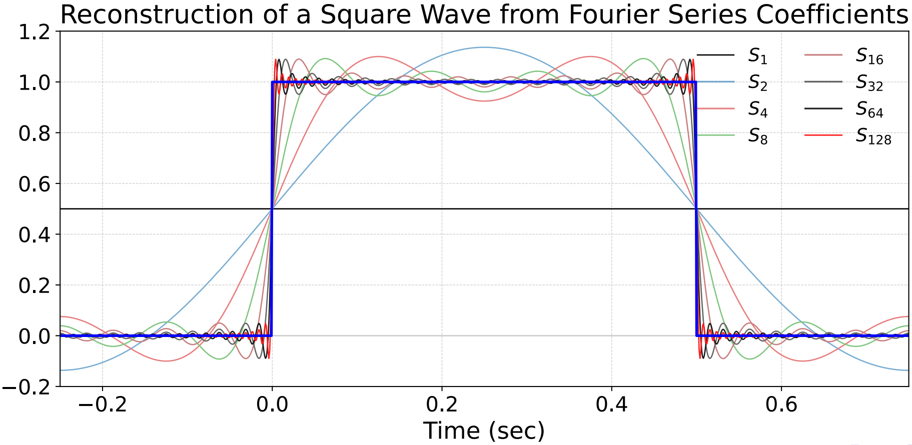
- If $x(t)$ is finite but discontinuous $\longrightarrow$ No pointwise equality. Only means squared convergence is possible.

$$\lim_{N \to \infty} \int_0^{T_0} \left| x(t) - \sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t} \right|^2 dt = 0$$

This means that the reconstructed signal $\sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t}$ need not be equal to the signal $x(t)$ at a discrete set of points, i.e. at the points where there is a discontinuity.

Does any periodic signal have a Fourier series representation?

$$x(t) = \begin{cases} 1, & 0 \leq t < 0.5 \\ 0, & 0.5 \leq t < 1 \end{cases} \longrightarrow c_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{2}{k\omega_0} \sin\left(\frac{k\omega_0}{4}\right) e^{-j\frac{k\omega_0}{4}}, & k \neq 0 \end{cases}$$



Dirichlet conditions for Fourier series

The *Dirichlet conditions* guarantee that the c_k exists, and $\sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t}$ is equal to $x(t)$ except at time points where there is a discontinuity.

At a discontinuity, $\sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t}$ converges to the midpoint of the discontinuity.

The *Dirichlet conditions* are that a single cycle of $x(t)$:

1. has a finite number of disconuities.
2. has a finite number of maxima and minima.
3. is absolutely integrable. $\int_0^{T_0} |x(t)| dt < \infty$

Power Spectral Density of Periodic Signals

Some definitions:

- Instantaneous power of a signal $x(t) \triangleq |x(t)|^2$
- Total energy of a signal $x(t)$ in a time interval $[T_1, T_2] \triangleq \int_{T_1}^{T_2} |x(t)|^2 dt$
- Average power over a time interval $[T_1, T_2] \triangleq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |x(t)|^2 dt$
- **Energy signal**: Signals with a finite total energy and zero average power over their entire duration.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \quad \text{and} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = 0$$

- **Power signal**: Signals with a finite average power, and infinite energy.

$$\lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \infty \quad \text{and} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt < \infty$$

Power Spectral Density of Periodic Signals

Parseval's Identity.

Let $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, then

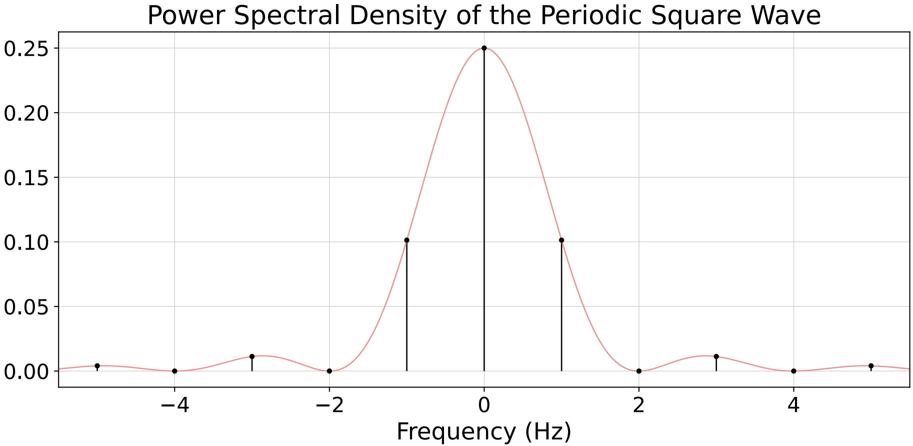
$$P_x = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Fourier series representation preserves the average power of the periodic signal $x(t)$.

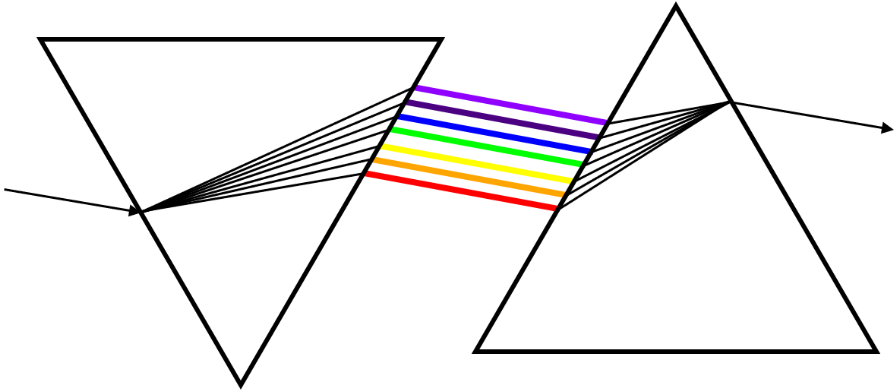
$|c_k|^2$ is the power in ofn the k^{th} harmonic.

$|c_k|^2$ as a function of k is the **Power Spectral Density** of $x(t)$.

Power Spectral Density of Periodic Signals



Power Spectral Density of Periodic Signals



Fourier representation of aperiodic signals

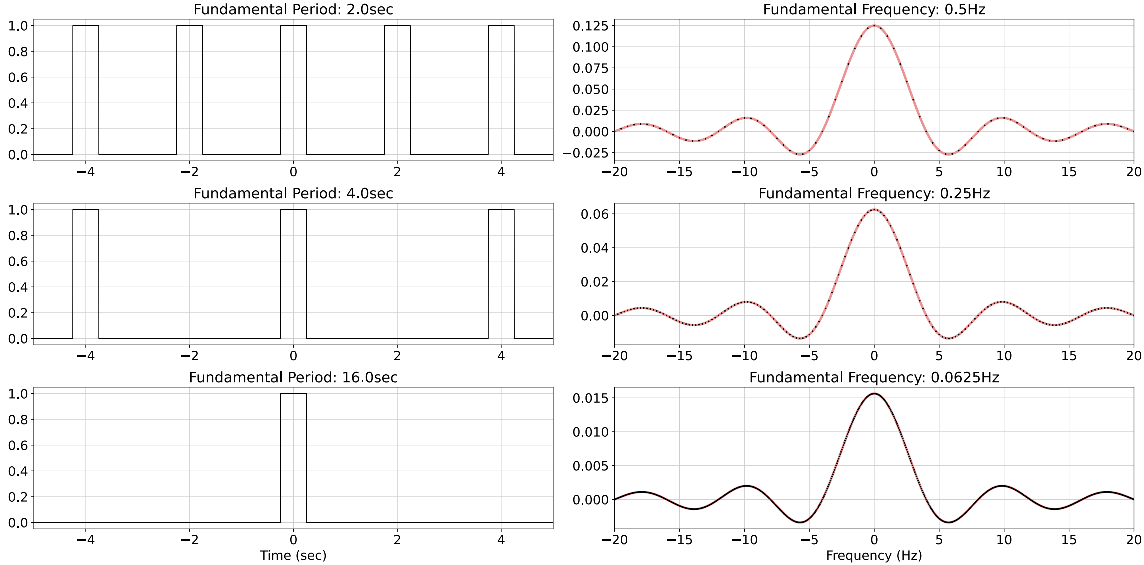
We can approach this problem starting from the Fourier series.

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \tau < |t| \leq \frac{T_0}{2} \end{cases}, \text{where, } 0 < \tau < \frac{T_0}{2}$$

$$\downarrow$$

$$c_k = \frac{\tau}{T_0} \frac{\sin\left(\pi k f_0 \tau\right)}{\pi k f_0 \tau}, \ k = 0, \pm 1, \pm 2, \ldots$$

Fourier representation of aperiodic signals



Fourier representation of aperiodic signals: Fourier Transform

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \tau < |t| \leq \frac{T_0}{2} \end{cases}, \text{where, } 0 < \tau < \frac{T_0}{2}$$

$$\downarrow$$

$$c_k = \frac{\tau}{T_0} \frac{\sin\left(\frac{k\omega_0\tau}{2}\right)}{\frac{k\omega_0\tau}{2}}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$T_0 \to \infty \quad \implies \quad \omega_0 \to 0 \quad \implies \quad \{k\omega_0\}_{k=-\infty}^{\infty} \to \omega \in \mathbb{R} \quad \implies \quad c_k \to X(\omega)$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t} dt \quad \longrightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

This is the **Fourier transform**.

Fourier representation of aperiodic signals

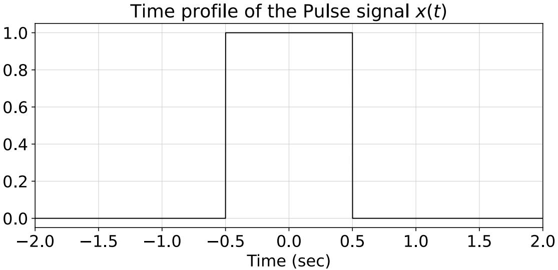
$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < |t| \end{cases} \quad \longrightarrow \quad X(\omega) = \tau \cdot \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} = \tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

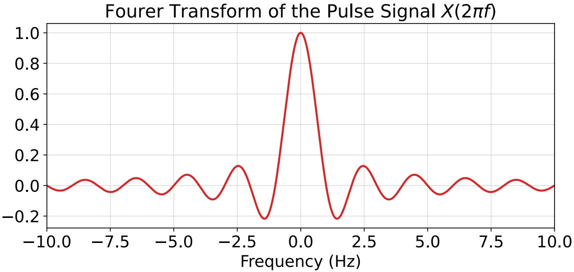
We can reconstruct the time-domain signal from the $X(\omega)$,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

This is the **Inverse Fourier Transform.**

Fourier Transform





Fourier Transform

$$
\begin{tikzcd}[column sep=huge]
x(t) \arrow[r, bend left=60, "\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt"] & X(\omega) \arrow[l, bend left=60, "\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega"]
\end{tikzcd}
$$

Dirichlet Conditions for the Fourier Transform

The *Dirichlet conditions* for the existence of the Fourier transform are that $x(t)$:

1. has a finite number of discontinuities.

2. has a finite number of maxima and minima.

3. is absolutely integrable. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.

This ensures that $X(\omega)$ is finite and continuous.

We can still have Fourier transform for signal that are not absolutely integrable, but square integrable, i.e. $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$.

Example: $x(t) = \omega_0 \text{sinc}(\omega_0 t)$ is not absolutely integrable, but $X(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}$.

Parseval's identity for aperiodic signals

Energy of an aperiodic signal $x(t)$:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Parseval's identity:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X\left(\omega\right)|^2 d\omega$$

$S_{xx}\left(\omega\right) = \frac{1}{2\pi}|X\left(\omega\right)|^2$ is the distribution of signal energey over frequency: **Energy density spectrum**.

Properties of Fourier transform

- **Linearity**: $\alpha x(t) + \beta y(t) \overset{\text{FT}}{\longleftrightarrow} \alpha X(\omega) + \beta Y(\omega)$
- **Shift in time**: $x(t - t_0) \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$
- **Shift in frequency**: $x(t)e^{j\omega_0 t} \overset{\text{FT}}{\longleftrightarrow} X(\omega - \omega_0)$
- **Time and frequency scaling**: $x(\alpha t) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \ \alpha > 0$
- **Convolution in time**: $x(t) * y(t) \overset{\text{FT}}{\longleftrightarrow} X(\omega)Y(\omega)$