Intro to DSP - Mathematical Preliminaries

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Introduction to Digital Signal Processing

Mathematical Preliminaries

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Sets

$S = \{\ ,\ ,\ ,\ \}$

- A set is a collection of distinct objects or elements.
- The definition of a set must make it clear to find out if an element belongs or does not belong to a set.
- Sets allow us to establish the universe of things that we are dealing with.
- Elements of a set are unique.
- Set are often represented by captial letters.

i/p → Program → o/p

$S = \{2, 4, 6, 8, 10\}$

$$A = \{1, 2, \pi, Orange\}$$

$$B = \{n \mid n \text{ is an even non-zero integer}\} = \{\ldots -4, -3, -2, -1, 1, 2, 3 \ldots\}$$

$= \{2, 2, 4, 6, 8, 10\}$

"such that"

3.41

Sets

Notations for some standard sets:

- Set of natural numbers.
$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$
counting
- Set of integers.
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$$
+ve
-ve
+4 -3
- Set of rational numbers.
$$\mathbb{Q} = \left\{ \frac{n}{m} \,\middle|\, n, m \in \mathbb{Z} \right\}$$
Fractions
4.35
- Set of real numbers. $\mathbb{R}$ → Real numbers →
- Set of complex numbers. $\mathbb{C}$

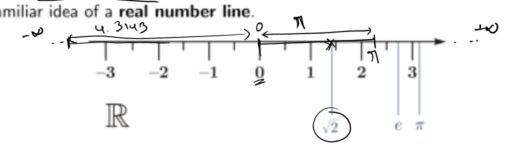
$-4.3143 \in \mathbb{R}$

$\pi \in \mathbb{R}$

Real numbers

The value of a continuous quantity, which can be represented as a distance on a line. This the familiar idea of a **real number line**.

$\in$ "belongs to"



What type of a number would we use for the following purposes?

1. The age of a person in years. $\mathbb{N}$
2. Cost of 3Kgs of banana (assuming we do not have fractions of a rupee). $\mathbb{N}$, $\mathbb{Q}$
3. Solution of the equation: $\boxed{x^2 = 2}$

$x^2 = 2 \quad x = \sqrt{2}$

Functions :

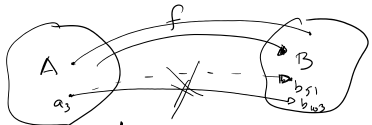
- A function is a relationship that associates elements from one set to exactly one element in another set.
- Let f be a function from set A to set B. We write, $f : A \mapsto B$.

$$\boxed{y = f(x)}, \text{ where, } x \in A, \ y \in B$$

$y = f(x)$

x

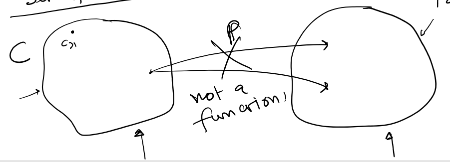
- Every element of A is mapped to an element in B.
- Every element of A is only mapped to one element in B.
- A is called the **domain** of f, and B is called the **range** of f.



$$\underset{B}{y} = f(\overset{A}{x})$$

Set of children

Set of ~~Mothers~~ Parents



$\boxed{m = M(c)}$

Moth

fath

$= P(c)$

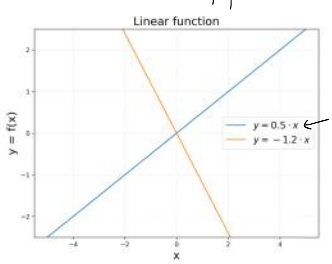
Functions

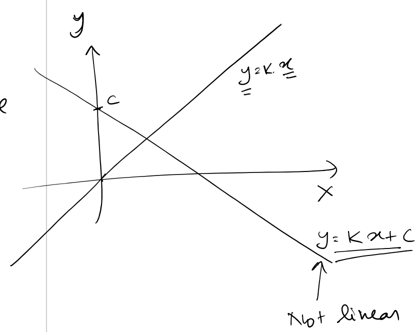
1. **Linear function** $f : \mathbb{R} \mapsto \mathbb{R}$

$$\boxed{y = f(x) = k \cdot x, \ k, x \in \mathbb{R}}$$

$y = k \cdot x$

slope of the line

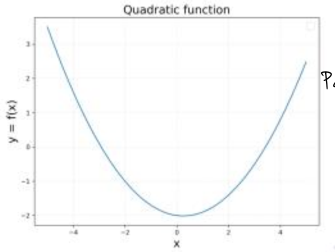




Functions

Quadratic function $f : \mathbb{R} \mapsto \mathbb{R}$

$$y = f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R}$$



Parabola

Functions

Exponential function $f : \mathbb{R} \mapsto \mathbb{R}$

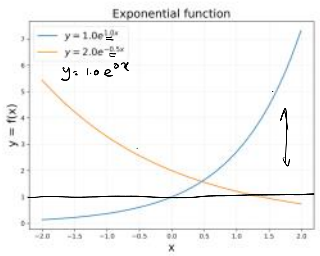
$$y = f(x) = ae^{kx}, \quad a, k, e \in \mathbb{R}$$

$a \cdot e^{kx}$

$a, e, k \in \mathbb{R}$

e → Mathematical constant → "Euler's number"

$= 2.718\ldots$



$y = a \cdot b^{kx}$

$= a\, e^{kx}$

$y = e^{x}$

$$\boxed{\frac{dy}{dx} = \frac{d}{dx} e^{x} = e^{x}}$$

Linear systems

$a\, e^{kx} = y$

$k = 0$

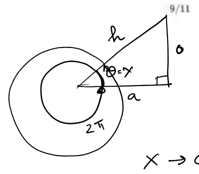
$y = a \cdot e^{0 \cdot x}$

$= a e^{0} = a$

Functions

Sine and Cosine function $f : \mathbb{R} \mapsto \mathbb{R}$

$$y_s = \sin(x) \quad \text{and} \quad y_c = \cos(x)$$

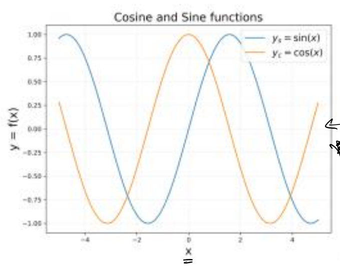


$\cos \overset{x}{\theta} = \frac{a}{h}$

$\sin \overset{x}{\theta} = \frac{o}{h}$

x → changes by 2π

$\sin(x) = \sin(x + 2\pi)$



Complex numbers ← Real number

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

imaginary

where, $i = \sqrt{-1} \implies i^2 = -1$.

- A complex number $x = a + ib$ consist of two components:
  - **real** part – a = Re(z)
  - **imaginary** part – b = Im(z)
- Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then,
- **Complex Addition**

$$z_3 = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

- **Complex Multiplication**

$$z_3 = z_1 \times z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_1b_2)$$

$$= (a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + ia_1b_2 + ia_2b_1 + i^2 b_1b_2$$

$$= a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$$

a,

$i = \sqrt{-1}$ $\quad i^2 = -1$

$x^2 = -2$

$x^2 = i^2 2$

$x = \pm\sqrt{i^2 2}$

$\boxed{x = \pm i\sqrt{2}}$ ← Pure imaginary number

$\mathbb{N}$ ←

$\mathbb{Z}$ $\quad x + 3 = 4 \Rightarrow x = 1$

n_1, n_2

$n_1 + n_2 = n_3$

$n_3 \geq n_1$

$n_3 \geq n_2$

$\boxed{x + 4 = 3 \rightarrow \text{no solution}}$

x can be an integer

$\boxed{x = -1} \rightarrow \mathbb{Z}$

$\boxed{4x + 1 = 3}$ → $\mathbb{Q}$

$\boxed{x^2 = 2}$ → $\mathbb{R}$

$x^2 = -2$ → $\mathbb{R}^2$

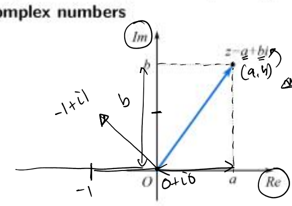
$ax^2 + bx + c = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← $b^2 - 4ac < 0$

$\mathbb{C}$

Complex numbers

- **Complex conjugate of a complex number** $\bar{z} = \overline{a + ib} = a - ib$
- **Length of a complex number** $|z|^2 = z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$
- **Geometry of complex numbers**



- **Euler forumla** $z = a + ib = re^{i\theta} = r\cos(\theta) + ir\sin(\theta) = |z|e^{i\arg(z)}$
  where, $r = |z| = \sqrt{a^2 + b^2}$, and $\theta = \arg(z) = \text{atan2}(b, a)$.

$z = a + ib = re^{i\theta}$

$= r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z = a + ib$

$\bar{z} = a - ib$

$-1 + i$

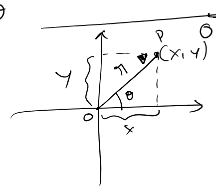
$z = a + ib \Leftarrow = re^{i\theta}$

Cartesian representation

$z \cdot \bar{z} = (a + ib)(a - ib)$

$= a^2 - iab + iab - i^2 b$

$= a^2 + b^2 = |z|^2 = z \cdot \bar{z}$



Cartesian representation (x, y)

Polar representation

(r, θ)

$e^{i\theta} = \cos\theta + i\sin\theta$

$0! = 1$

$n! = 1.2.3 \ldots n$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad x \in \mathbb{R}$$

$x = i\theta \in \mathbb{C}$

$$e^{i\theta} = 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$i^2 = -1$

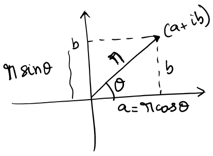
$i^3 = i^2 \cdot i = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

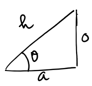


$r = \sqrt{a^2 + b^2}$

$r = |z|$

$\theta \rightarrow$ argument of z

$\theta = \text{atan2}(b, a)$



$\cos\theta = \dfrac{a}{h}$

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