Introduction to Signal Processing

Lecture 5: **Fourier Transform and its Properties**

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Reading material

- Continuous-time Fourier series and Fourier transform of Chapter 3 from [2].
- Chapter 4 from [3].
- **Extra reading material:** (Highly recommended) Sections 4.4.1-4.4.6, 4.5 from [5].

Eigenfunctions and Eigenvalues

- Convolution integral provides everything one needs to know about the input-output relationship of a LTI system.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- LTI systems have an interesting property, which comes across clearly through the convolution integral.

  When $x(t) = e^{st}$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$y(t) = e^{st}\left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)$$

Eigenfunctions and Eigenvalues

- The output of an LTI system to e^{st} is simply the product of e^{st} with scalar value $\lambda_s = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$.
- A function that satisfies this property called an **eigenfunction** of the system, i.e.

$$y(t) = H\left\{x(t)\right\} = \lambda x\left(t\right)$$

  where, λ is the **eigenvalue** corresponding to the input $x(t)$.
- Signals e^{st} are eigenfunctions of LTI systems, and their eigenvalue is given by,

$$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

- The eigenvalue is obtained from the impulse response.

What use is an eigenfunction?

- Eigenfunctions help replace the convolution operation to a multiplication process.
- If any signal $x(t)$ can be represented as a linear combination of eigenfunctions, then the output is the superposition of the scaled eigenfunctions.

$$x(t) = \sum_{i=0}^{N-1} a_i e^{s_i t} \mapsto y(t) = \sum_{i=0}^{N-1} a_i H(s_i) e^{s_i t}$$

where,

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

- When $s = j\omega$, then we get the **Fourier transform** of a signal $h(t)$.

$$H(\omega) \triangleq \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Fourier Transform

Definition:
The *Fourier transform* of a signal $x(t)$ is defined as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$X(\omega)$ exists if the above integral converges for all $\omega \in \mathbb{R}$. The *inverse Fourier transform* is given by,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$x(t)$ exists if the above integral converges for all $t \in \mathbb{R}$.
When both these exists, they are called the *Fourier transform pairs*,

$$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Fourier transform

- The Fourier transform definition is formally equivalent to the inner-product of finite energy signals (i.e. signals in $\mathcal{L}^2(\mathbb{R})$)

  $$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  Note: $e^{j\omega t}$ is not in $\mathcal{L}^2(\mathbb{R})$.
- Given that the Fourier transform and its inverse look very similar, the conditions for their existence are identical.
- For most practical signals we will encounter, we will not have to worry about the technical details of the existence of the Fourier transform and its inverse.

Existence of Fourier transform[1]

- When a signal $x(t)$ is absolutely integrable, i.e. $x(t) \in \mathcal{L}^1(\mathbb{R})$, then the Fourier transform $X(\omega)$ exists, and it is finite and continuous.
  You can understand this by analysing the following signal using the Fourier transform.

  $$x(t) = \begin{cases} \frac{1}{\sqrt{T}} & |t| \leq \frac{T}{2} \\ 0 & \text{Otherwise} \end{cases}$$

  Is $X(\omega)$ in $\mathcal{L}^1(\mathbb{R})$?
- Fourier transforms can exist even when a signal is not in $\mathcal{L}^1(\mathbb{R})$.
- Fourier transforms can be used even when a signal does not have finite energy and is not absolutely integrable, such as the Dirac delta function.

[1] This slide can be skipped

Properties of Fourier transform

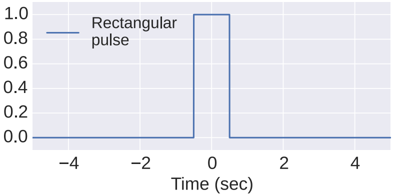
Here are some of the basic properties of Fourier transform. It is left as an exercise to prove these for yourself.

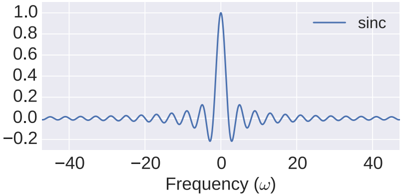
- **Linearity**: $\alpha x(t) + \beta y(t) \overset{\text{FT}}{\longleftrightarrow} \alpha X(\omega) + \beta Y(\omega)$
- **Shift in time**: $x(t - t_0) \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$
- **Shift in frequency**: $x(t)e^{j\omega_0 t} \overset{\text{FT}}{\longleftrightarrow} X(\omega - \omega_0)$
- **Time and frequency scaling**: $x(\alpha t) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \ \alpha > 0$
- **Differentiation in time**: $\frac{d^n x(t)}{dt^n} \overset{\text{FT}}{\longleftrightarrow} (j\omega)^n X(\omega)$
- **Integration in time**: $\int_{-\infty}^{t} x(\tau) d\tau \overset{\text{FT}}{\longleftrightarrow} \frac{1}{j\omega} X(\omega);\ X(0) = 0$
- **Convolution in time**: $x(t) * y(t) \overset{\text{FT}}{\longleftrightarrow} X(\omega)Y(\omega)$
- **Parseval's equality**: $\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$

There are several other properties, which are left as an exercise for the student to explore.

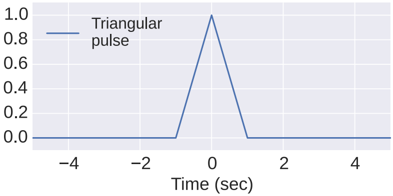
Some Fourier transform pairs

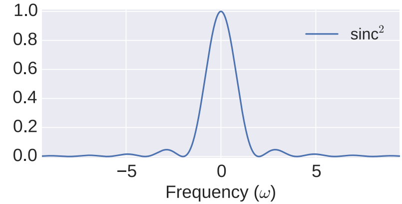
- Rectangular pulse $x(t) = \begin{cases} 1 & |t| < 0.5 \\ 0 & \text{Otherwise} \end{cases}$





- Triangular pulse $x(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{Otherwise} \end{cases}$





Some Fourier transform pairs

- DC signal, $x(t) = 1 \quad \overset{FT}{\longleftrightarrow} \quad 2\pi\delta(\omega)$
- Sinusoid, $x(t) = \sin \omega_0 t \quad \overset{FT}{\longleftrightarrow} \quad \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
- Decaying exponential, $x(t) = e^{-\alpha t}u(t), \alpha > 0 \quad \overset{FT}{\longleftrightarrow} \quad ?$
- Unit step, $x(t) = u(t) \quad \overset{FT}{\longleftrightarrow} \quad ?$
- Gaussian signal, $x(t) = e^{-\alpha t^2}, \alpha > 0 \quad \overset{FT}{\longleftrightarrow} \quad ?$

Frequency reseponse of LTI system

- The Fourier transform allows the representation of any aperiodic signal as a scaled superposition of sinusoids, as long as the Fourier transform representation exists.
- Once we know the Fourier representation of an input signal $x(t)$, we can use the eigenfunction property of a LTI system to estimate its output, i.e. scale each sinusoidal component (eigenfunction) by the appropriate eigenvalue and sum the scaled eigenfunctions to obtain the output.
- But, how does one find out the eigenvalues of a LTI system? *The impulse response*
- The eigenvalues are given by the Fourier transform of impulse response, which is called the **frequency response** of the LTI system.

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Frequency reseponse of LTI system

- $H(\omega)$ is the eigenvalue for the eigenfunction $e^{j\omega t}$.
- The output of the system to an input $x(t)$ is given by,

$$y(t) = \int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}d\omega$$

Can you explain what is going on here? Also, try to relate the convolution property of the Fourier transform with the above relation.

Frequency reseponse of LTI system

- $H(\omega)$ is a complex quantity with a real and imaginary part.

$$H(\omega) = H_r(\omega) + jH_i(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

  where, $|H(\omega)|$ is the *magnitude response*, and $\theta(\omega)$ is the *phase response*.

- The magnitude and phase response values are the amount of attenuation and the amount of phase shift introduced to an eigenfunction by an LTI system. i.e.

$$e^{j\omega t} \mapsto |H(\omega)|\, e^{j(\omega t + \theta(\omega))}$$

- This implies that LTI system can be used for implementing *frequency selective* filters, which manipulate the frequency content of any signal.

Properties of frequency response

Let $x(t) = x_r(t) + jx_i(t)$, and its Fourier transform is $X(\omega) = X_r(\omega) + jX_i(\omega)$.

Fourier transforms have the following properties:

- $x(t)$ is real $\implies X(-\omega) = X^*(\omega)$
- $x(t)$ is imaginary $\implies X(-\omega) = -X^*(\omega)$
- $x(t)$ is even $\implies X(\omega)$ is real
- $x(t)$ is odd $\implies X(\omega)$ is imaginary

When $x(t)$ is causal, such that $x(t) = 0,\ \forall t < 0$, then the real and imaginary components of the Fourier transform $X(\omega)$ are not independent of each other.

This implies that **one cannot have causal LTI system with arbitrary magnitude and phase responses**.

Fourier Series

The Fourier series representation allows periodic signals to be represented as a sum of sinusoids.

Definition The *Fourier series* representation of a periodic signal $x(t)$ with fundamental period T, is given by

$$x(t) = \sum_{k \in \mathbb{Z}} X_k e^{j(2\pi k/T)t}, \ t \in \left[\frac{-T}{2}, \frac{T}{2} \right)$$

where, X_k is the *Fourier series coefficient*

$$X_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j(2\pi k/T)t} dt, \ k \in \mathbb{Z}$$

X_k exists if the above integral defined and finite for all $k \in \mathbb{Z}$. When X_k exists, we have the *Fourier series pair*

$$x(t) \xleftrightarrow{FS} X_k$$

Fourier Series

- The Fourier series coefficient is related to the Fourier transform. Consider a periodic signal $x(t)$, and its Fourier series coefficient X_k.
  Let us now consider a signal $\hat{x}(t)$, such that

$$\hat{x}(t) = x(t), t \in [-T/2, T/2)$$

  In this case,

$$X_k = \frac{1}{T}\hat{X}\left(\frac{2\pi}{T}k\right), \quad k \in \mathbb{Z}$$

Existence of Fourier series[2]

- Existence of Fourier series coefficients are usually clearer than that of the Fourier transform, because the integral is over a finite interval.
- When a function is absolutely integrable in one period, then the Fourier serious coefficients exist. Any continuous periodic function will have Fourier series coefficients.
- Even when X_k exists, we are still left with the question of how well we can recover $x(t)$ from the X_ks.

  Consider $x(t)$, whose Fourier series coefficients are X_k. Now, considered the reconstructed signal, $\hat{x}(t)$:

$$\hat{x}(t) = \sum_{k \in \mathbb{Z}} X_k e^{j(2\pi k/T)t}$$

[2] This slide can be skipped

Existence of Fourier series[3]

- The reconstructed signal equals the original signal for every time point, i.e.

$$\hat{x}(t) = x(t), \ \ \forall t \in [-T/2, T/2), \iff \sum_{n\in\mathbb{Z}} |X_k| < \infty$$

- When $x(t)$ is finite but discontinuous, then we do not have pointwise equality. But the means squared error between $x(t)$ and $\hat{x}(t)$ is zero. $\int_{-T/2}^{T/2} |x(t) - \hat{x}(t)|^2 \, dt = 0$
- This can be understood with the following example. Determine the Fourier series coefficients for the following

$$x(t) = \begin{cases} 1 & t \in \left[-\frac{T}{4}, \frac{T}{4}\right] \\ 0 & t \in \left[-\frac{T}{2}, -\frac{T}{4}\right) \\ 0 & t \in \left(\frac{T}{4}, \frac{T}{2}\right) \end{cases}$$

What is value of the reconstructed signal $\hat{x}(t)$ at the discontinuities $t = \left\{-\frac{T}{4}, \frac{T}{4}\right\}$

[3] This slide can be skipped

Properties of Fourier series

The properties of the Fourier series are similar to that of the Fourier transform.

- **Linearity**: $\alpha x(t) + \beta y(t) \xleftrightarrow{\text{FT}} \alpha X_k + \beta Y_k$
- **Shift in time**: $x(t - t_0) \xleftrightarrow{\text{FT}} e^{-j(2\pi k/T)t_0} X_k$
- **Shift in frequency**: $x(t)e^{j(2\pi k_0/T)t} \xleftrightarrow{\text{FT}} X_{k-k_0}$

The other properties are left as an exercise for the student to explore.

Fourier series pairs

- Square wave, $x(t) = \begin{cases} 1 & t \in \left[-\frac{T}{4}, \frac{T}{4}\right] \\ 0 & t \in \left[-\frac{T}{2}, -\frac{T}{4}\right) \\ 0 & t \in \left(\frac{T}{4}, \frac{T}{2}\right) \end{cases} \quad \xleftrightarrow{FS} \quad ?$
- Triangular wave, $x(t) = \frac{1}{2} - |t|, t \in \left[\frac{-1}{2}, \frac{1}{2}\right) \quad \xleftrightarrow{FS} \quad ?$
- Dirac comb, $x(t) = s_T(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT) \quad \xleftrightarrow{FS} \quad ?$
- Sinusoid, $x(t) = \sin 0.75\pi t \quad \xleftrightarrow{FS} \quad ?$

Reconstruction of signals from finite number of Fourier coefficients

- When a period signal is bounded, we know that the Fourier series coefficients will exist, and the reconstructed signal from the Fourier series coefficient will be equal to the original signal in the mean square sense. i.e,

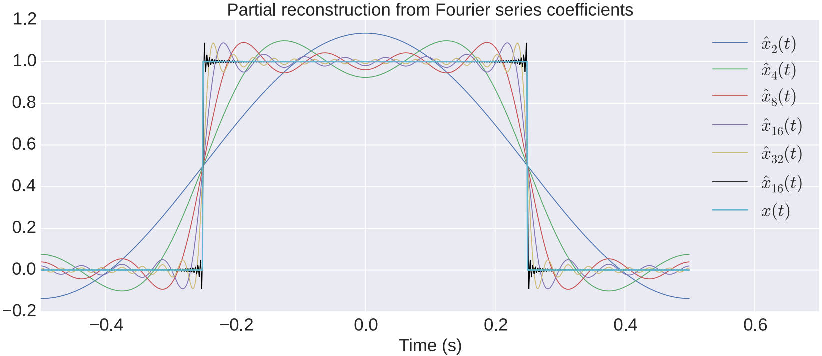
$$\int_{-T/2}^{T/2} \left| x(t) - \sum_{k \in \mathbb{Z}} X_k e^{j(2\pi k/T)t} \right|^2 dt = 0$$

- What happens if we use only a finite number of Fourier series coefficients?, i.e.

$$\hat{x}_N(t) = \sum_{k=-N}^{N} X_k e^{j(2\pi k/T)t}$$

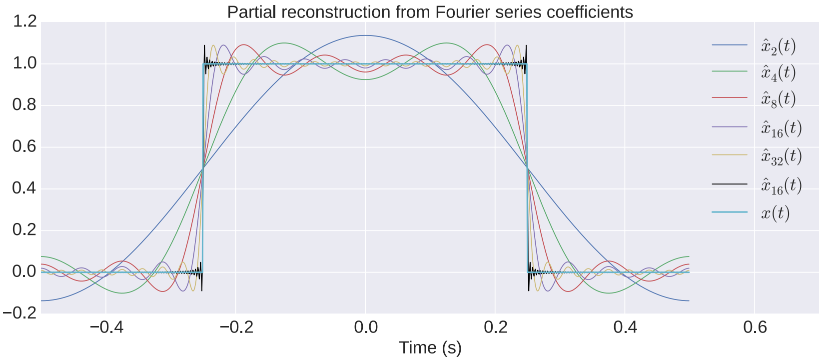
Reconstruction of signals from finite number of Fourier coefficients

Consider the following signal, $x(t) = \begin{cases} 1 & t \in [-0.25, 0.25] \\ 0 & t \in [-0.5, -0.25) \\ 0 & t \in (0.25, 0.5) \end{cases}$



What all do you notice in the above partial reconstructions?

Gibbs phenomenon



- As N increases, $\hat{x}(t) \to x(t)$ everywhere, except at points close to the discontinuities.
- $max_{t\in[-0.5,0.5)} |x(t) - \hat{x}(t)|$ does not approach zero for any finite N.
- This is the *Gibbs phenomenon*. When $x(t)$ is discontinuous, we only have convergence in the mean squared sense, i.e. $\int_{-T/2}^{T/2} |x(t) - \hat{x}(t)|^2 \, dt = 0$.

Fourier transform of periodic signals

- Periodic signals cannot have a Fourier transform in elementary sense, because periodic signals are not energy signals and they are not absolutely integrable. But using Dirac delta functions we can arrive at the Fourier transform on periodic signals.
- Consider a time-limited signal, $\hat{x}(t)$ that is restricted to $t \in [-T/2, T/2)$. Let $\hat{X}(\omega)$ be its Fourier transform.
- We can generate a periodic signal $x(t)$ from $\hat{x}(t)$, by the following,

$$x(t) = \sum_{n \in \mathbb{Z}} \hat{x}(t - nT) = s_T(t) * \hat{x}(t)$$

  Notice, that convolution between the Dirac delta comb and $\hat{x}(t)$ generates the periodic signal.

Fourier transform of periodic signals

$$x(t) = \sum_{n \in \mathbb{Z}} \hat{x}(t - nT) = s_T(t) * \hat{x}(t)$$

This implies that,

$$X(\omega) = S_T(\omega)\hat{X}(\omega)$$

Here (verify this),

$$S_T(\omega) = \sum_{n \in \mathbb{Z}} e^{-j\omega nT} = \frac{2\pi}{T} \sum_{k \in \mathbb{Z}} \delta\left(\omega - \frac{2\pi}{T}k\right) = \frac{2\pi}{T} s_{2\pi/T}(\omega)$$

$$X(\omega) = \frac{2\pi}{T} \sum_{k \in \mathbb{Z}} \hat{X}\left(\frac{2\pi}{T}k\right) \delta\left(\omega - \frac{2\pi}{T}k\right) = 2\pi \sum_{k \in \mathbb{Z}} X_k \delta\left(\omega - \frac{2\pi}{T}k\right)$$

Fourier transform of a periodic signal of period T is weighted Dirac comb with spacing $2\pi/T$.