Linear Systems

Linear Dynamical Systems: State Space View

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States of a system

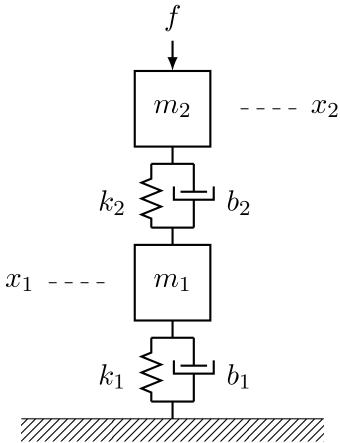
- A characteristic feature of most dynamical systems is their memory, i.e. the system's response (or output) depends on the present and past values of its input; We are only deal with causal systems here.
- If we get interested in a system at some arbitrary time t_0, we might not have a complete record of the past input to the system.
- The idea of a *state* deals with this problem.

States of a system

- **Defintion**: *The state* $\mathbf{x}(t_0)$ *of a system is the information at time* t_0, *which along with the input* $u(t)$, $\forall t \geq t_0$, *can be used to uniquely determine the system output* $y(t)$, $\forall t \geq t_0$.
- The state $\mathbf{x}(t_0)$ summarizes all the information ones needs to know about the system's past in order to predict its future.
- Examples of states of a system:
  - Position and velocity of a mass acted up on by a force.
  - Capacitor voltage and inductor current of a electrical network.
  - Initial conditions of a differential equation describing a system.

States of a System

States of a system



- In the system shown, the input $u(t)$ is the force $f(t)$ applied to m_2, and the output $y(t)$ is the position of m_2 $(x_2(t))$.
- $y(t)$ depends not only on $f(t)$, but also on: $\dot{x}_2(t)$, $x_1(t)$ and $\dot{x}_1(t)$.
- For the same input u, we can obtain different output y if the starting states are different. Thus, knowledge of the states are essential for correctly predicting the behavior of the system.
- In general, the dynamics of a system in terms of its states, input(s) and output(s) is mathematically represented as,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \rightarrow \textit{State Equation} \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \rightarrow \textit{Measurement Equation} \end{cases}$$

where, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^p$, and $\mathbf{y} \in \mathbb{R}^m$, and $t \in \mathbb{R}$ represents times.

States ofa system

States of a system

In general, the state and the input will determine the system's output.

$$\left.\begin{array}{l} \mathbf{x}(t_0) \\ u(t), \ \forall t \geq t_0 \end{array}\right\} \rightarrow y(t), \ \forall t \geq t_0$$

In the case of a linear system, if

$$\left.\begin{array}{l} \mathbf{x}_1(t_0) \\ u_1(t), \ \forall t \geq t_0 \end{array}\right\} \rightarrow y_1(t), \ \forall t \geq t_0 \ \text{ and } \ \left.\begin{array}{l} \mathbf{x}_2(t_0) \\ u_2(t), \ \forall t \geq t_0 \end{array}\right\} \rightarrow y_2(t), \ \forall t \geq t_0$$

$$\implies \left.\begin{array}{l} a_1\mathbf{x}_1(t_0) + a_2\mathbf{x}_2(t_0) \\ a_1u_1(t) + a_2u_2(t), \ \forall t \geq t_0 \end{array}\right\} \rightarrow a_1y_1(t) + a_2y_2(t), \ \forall t \geq t_0$$

States of a system

For a linear system, knowing the system output to the states and the input will allow us to know the complete output.

- **Zero State Response**: $\mathbf{x}(t_0) = \mathbf{0};\ u(t),\ t \geq t_0\} \rightarrow y_{zs}(t), \forall t \geq t_0$
- **Zero Input Response**: $\mathbf{x}(t_0);\ u(t) = 0,\ t \geq t_0\} \rightarrow y_{zi}(t), \forall t \geq t_0$

$y_{zs}(t) + y_{zi}(t)$ gives the complete response.

State space representation of linear systems

- In the case of a linear system, the equations representing the dynamics takes a simpler form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t)$$

where,

- $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$ is the *system* matrix.
- $\mathbf{B}(t) \in \mathbb{R}^{n \times p}$ is the *input* matrix.
- $\mathbf{C}(t) \in \mathbb{R}^{m \times n}$ is the *output* matrix.
- $\mathbf{D}(t) \in \mathbb{R}^{m \times p}$ is the *feedforward* matrix.

State space representation of linear systems

- In the case of time-invariant system, the matrices are constant.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- These two equations represent how the states and the measured outputs of the system are affected by the current states and inputs. The individual terms in these matrices indicate how a particular state/input affects another state/output.

State space representation of linear systems

Consider a LTI system represented by the following differential equation,

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = u(t)$$

We can obtain a state space representation of this differential equation by choosing two states, $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -a_2 x_1(t) - a_1 x_2(t) + u(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

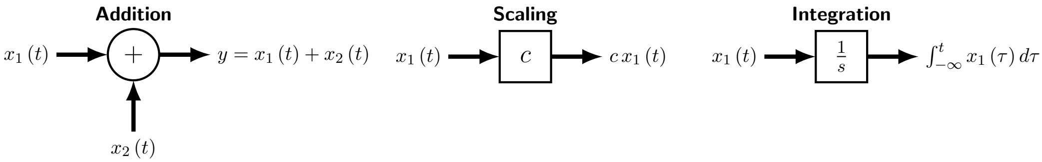
State space representation of linear systems

The choice of state for a system is not unique. If for a linear system, $\mathbf{x}(t)$ is a state, then so is $\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$, where $\mathbf{T}$ is invertible.

If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are the different matrices associated with a LTI system with state $\mathbf{x}(t)$. Derive the matrcies when $\mathbf{T}\mathbf{x}(t)$ is chosen as the state.

Block diagram representation of linear systems

- Pictorial representation of different componenets of a system and their inter-connections can provide insights into the behavior of the system.
- Helps breakdown a complex system into a set of simpler systems connected to each other.
- Linear systems in general can be built using three basic elements:



Represent the following linear differential equations using the three elementary block,

- $\dot{y}(t) + 0.1y(t) = u(t)$
- $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = u(t) - 2\ddot{u}(t)$

State space representation of discrete-time linear systems

- Discrete-time linear system,

$$\mathbf{x}[k+1] = \mathbf{A}[k]\,\mathbf{x}[k] + \mathbf{B}[k]\,\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}[k]\,\mathbf{x}[k] + \mathbf{D}[k]\,\mathbf{u}[k]$$

  where, $k \in \mathbb{Z}$ correspond to time index.

  - $\mathbf{A}[k] \in \mathbb{R}^{n \times n}$ is the *system* matrix.
  - $\mathbf{B}[k] \in \mathbb{R}^{n \times p}$ is the *input* matrix.
  - $\mathbf{C}[k] \in \mathbb{R}^{m \times n}$ is the *output* matrix.
  - $\mathbf{D}[k] \in \mathbb{R}^{m \times p}$ is the *feedforward* matrix.

- In the case of time-invariant system, the matrices are constant.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

State space representation of discrete-time linear systems

Consider a LTI system represented by the following differential equation,

$$y[k] + a_1 y[k-1] + a_2 y[k-2] = u[k]$$

We can obtain a state space representation of this difference equation by choosing two states, $x_1[k] = y[k-1]$ and $x_2[k] = y[k-2]$,

$$\mathbf{x}[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} y[k] \\ y[k-1] \end{bmatrix} = \begin{bmatrix} -a_1 x_1[k] - a_2 x_2[k] + u[k] \\ x_1[k] \end{bmatrix}$$

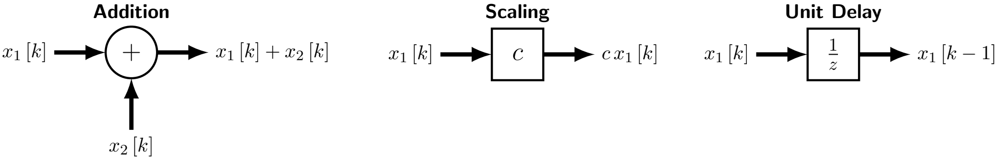
$$\mathbf{x}[k+1] = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}[k]$$

$$\mathbf{y}[k] = \begin{bmatrix} -a_1 & -a_2 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \end{bmatrix} \mathbf{u}[k]$$

The choice of state for a system is not unique. If for a linear system, $\mathbf{x}[k]$ is a state, then so is $\hat{\mathbf{x}}[k] = \mathbf{T}\mathbf{x}[k]$, where $\mathbf{T}$ is invertible.

Block diagram representation of discrete-time linear systems

- Discrete-time linear systems in general can be built using three basic elements:



Represent the following linear differential equations using the three elementary block,

- $y[k] + 0.1y[k] = u[k]$
- $y[k-2] + 2y[k-1] + 5y[k] = u[k] - 2u[k-2]$
- $y[k] = \frac{1}{5}\sum_{l=0}^{4} u[k-l]$

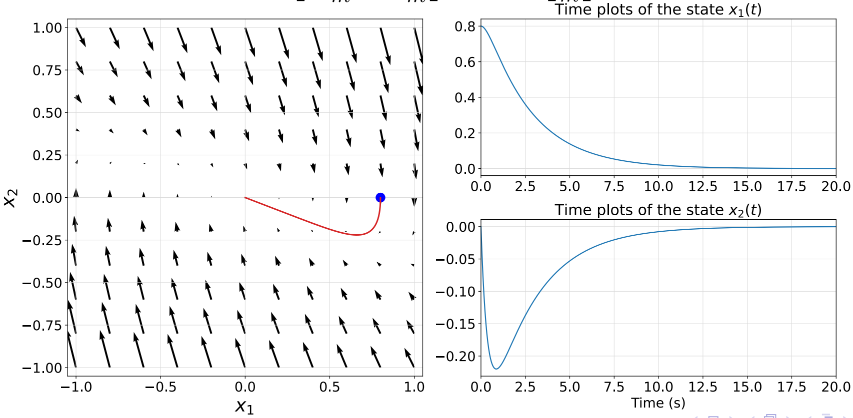
State space visualization

For systems with two states, we can visualize the state space trajectories of the system to gain better understanding of the system dynamics.

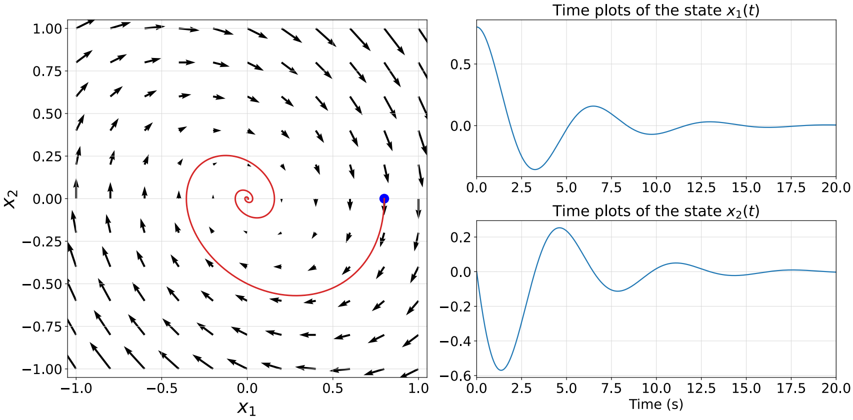
State space visualization

The state dynamics of a mass, spring and damper system is given by the following equation,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}(t)$$

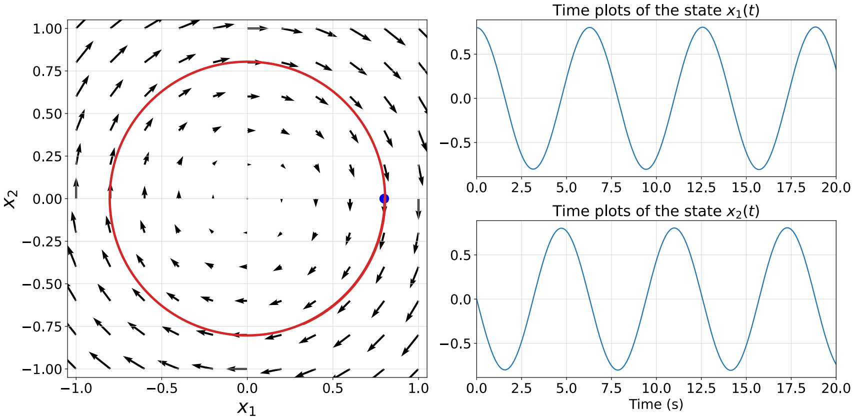


State space visualization



$$m = 1,\ b = 3,\ k = 1$$

State space visualization



$$m = 1,\ b = 3,\ k = 1$$