Introduction to Digital Signal Processing

Fourier Representation of Discrete-Time Signals

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Discrete-Time Fourier Series

- Continuous-time complex sinusoids have frequencies $\omega \in (-\infty, \infty)$.
- Discrete-time sinusoids have frequencies $\Omega \in (-\pi, \pi]$
- A discrete-time periodic signal $x[n]$ with fundamental period N can be represented as a sum of discrete-time sinusoids,

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}$$

- Appropriate choice of c_k will let us represent $x[n]$ as a linear combination of $e^{\frac{j2\pi k}{N}n}$.
- The Fourier coefficient c_k can be determined by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

Discrete-Time Fourier Series

- c_k is discrete like in the case of continuous-time Fourier series, and it is also periodic with fundamental period N, i.e. $c_k = c_{k+N}$.

$$\begin{tikzcd} x[n] \arrow[rr, bend left, "\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}"] & & \left(c_k\right)_{k=0}^{N-1} \arrow[ll, bend left, "\sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}"] \end{tikzcd}$$

Properties of Discrete-Time Fourier Series

- Fourier representation is discrete and periodic. (c_k is period with fundamental period N)
- When $N = 2M$ is even, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \leq l < \frac{N}{2}$$

- When $N = 2M + 1$ is odd, $0 < M \in \mathbb{Z}$.

$$c_{M+l} = c_{-M+l}, \quad 0 \leq l < \frac{N-1}{2}$$

- Parseval's identity.

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

The distribution of $|c_k|^2$ as a function of $0 \leq k < N$ is the *power spectral density* of the periodic signal $x[n]$.

Discrete-Time Fourier Series

Find the DTFS of $x[n] = \begin{cases} 1, & 0 \leq n < M \\ 0, & M \leq n < N - 1 \end{cases}$ with fundamental period N.

Discrete-time Fourier Transform

- Similar to the continus-time case, the Fourier representation of discrete -time aperiodic signals can be obtained as the limiting case of the a periodic signals with increasing period N.

- The discrete-time Fourier transform (DTFT) of an aperiodic signal $x[n]$ with finie energy is given by,
$$X\left(\Omega\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, \quad -\pi \leq \Omega < \pi$$

- $X\left(\Omega\right)$ is a continuous in Ω and periodic with period 2π.

- Inverse DTFT,
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(\Omega\right)e^{j\Omega n} d\Omega$$

Discrete-time Fourier Transform

- DTFT exists only if $x[n]$ is absolutely summable.

$$\sum_{n} |x[n]| < \infty \implies |X(\Omega)| < \infty$$

- When $x[n]$ is only square summable, then DTFT converges to the true DTFT only in the mean squared sense.
  E.g.,

$$x[n] = \begin{cases} \frac{\Omega_c}{n}, & n = 0 \\ \frac{\Omega_c}{n}\frac{\sin \Omega_c n}{\Omega_c n}, & n \neq 0 \end{cases} \longrightarrow X(\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$$

Properties of DTFT

- **Linearity**: $\alpha x[n] + \beta y[n] \xleftrightarrow{\text{DTFT}} \alpha X(\Omega) + \beta Y(\Omega)$

- **Shift in time**: $x[n - n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega)$

- **Shift in frequency**: $x[n]e^{j\Omega_0 n} \xleftrightarrow{\text{DTFT}} X(\Omega - \Omega_0)$

- **Time and frequency scaling**: $x(\alpha t) \xleftrightarrow{\text{FT}} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \ \alpha > 0$

- **Convolution in time**: $x[n] * y[n] \xleftrightarrow{\text{DTFT}} X(\Omega)Y(\Omega)$

Fourier Series

$$x\left(t\right) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

Knowing f_0, we can compute the signal $x\left(t\right)$ from the list of numbers $\left(c_k\right)_{k=-\infty}^{\infty}$.

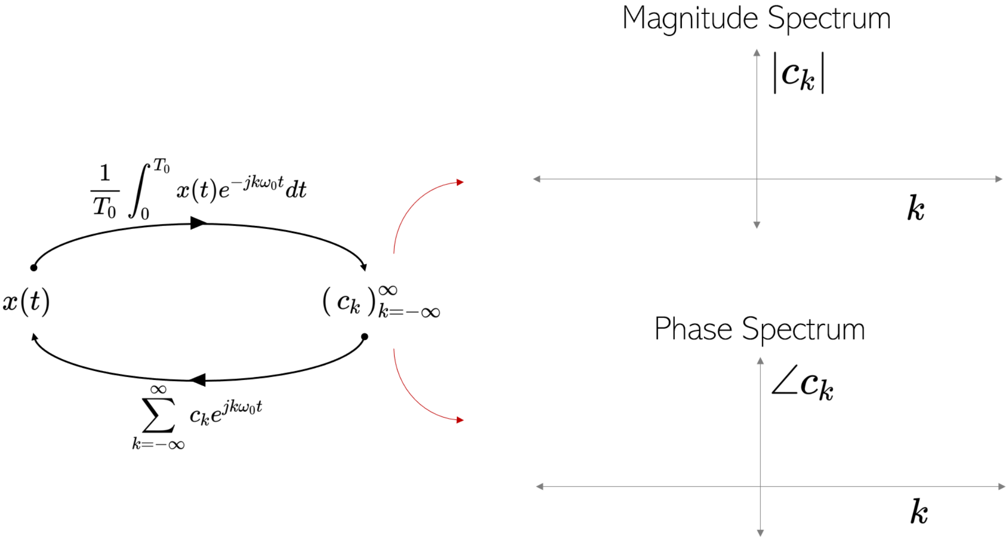
We can compute c_k as the following,

$$c_k = \frac{1}{T_0} \int_0^{T_0} x\left(t\right) e^{-j2\pi k f_0 t} dt$$

Fourier Series

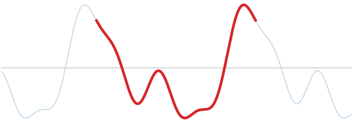
$$\begin{tikzcd}[column sep=huge] x(t) \arrow[r, bend left, "\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt"] & \left(c_k \right)_{k=-\infty}^{\infty} \arrow[l, bend left, "\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}"] \end{tikzcd}$$

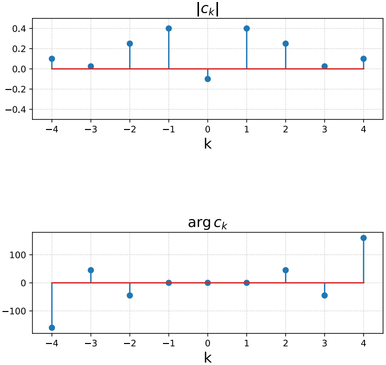
Fourier Series

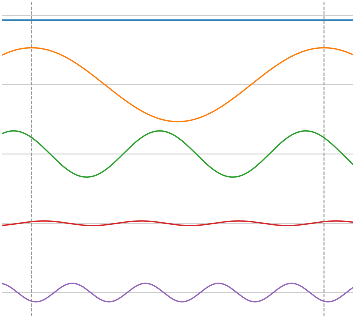


Fourier Series

$$x(t) = -0.1 + 0.8\cos\left(2\pi t\right) + 0.5\cos\left(4\pi t + \frac{\pi}{4}\right) + 0.05\cos\left(6\pi t - \frac{\pi}{4}\right) + 0.2\cos\left(8\pi t + \frac{8\pi}{9}\right)$$

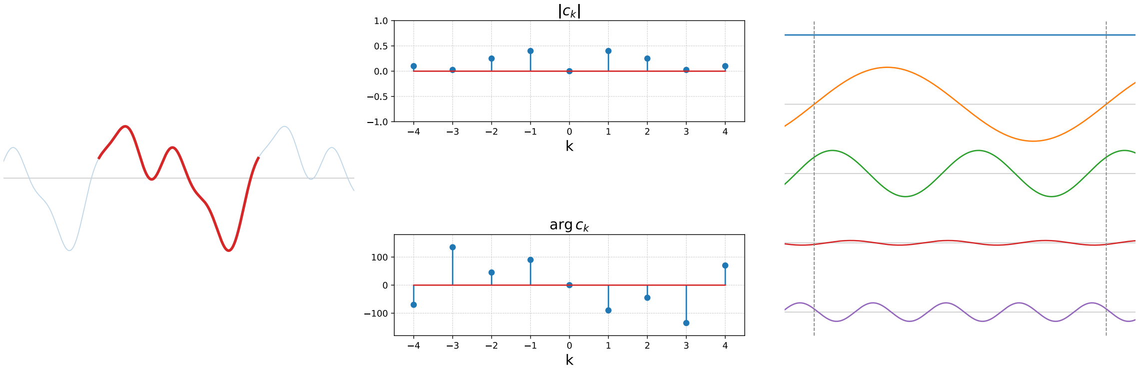






Fourier Series

$$x(t) = 0.8\sin\left(2\pi t\right) + 0.5\sin\left(4\pi t + \frac{\pi}{4}\right) + 0.05\sin\left(6\pi t - \frac{\pi}{4}\right) + 0.2\sin\left(8\pi t + \frac{8\pi}{9}\right)$$



Does any periodic signal have a Fourier series representation?

- If $x(t)$ is absolutely integrable over a single cycle, then the Fourier serious coefficients exist.

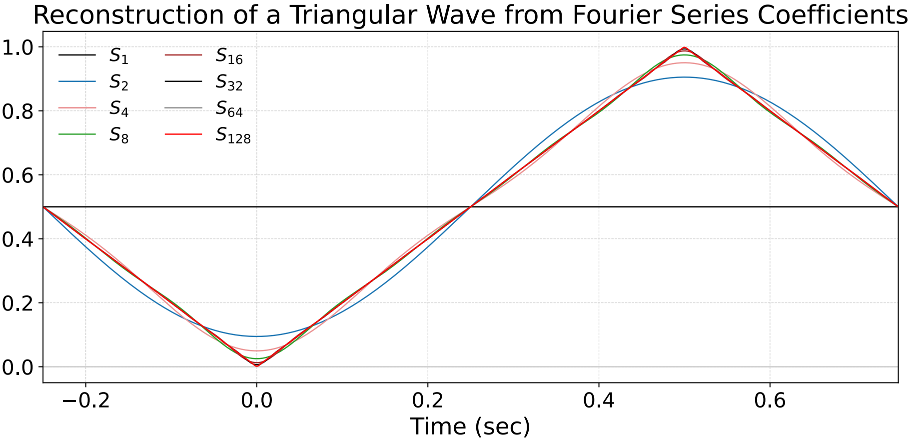
- Any continuous periodic function will have a Fourier series representation.

- When $x(t) =$ is continuous and finite, then reconstructed signal $\sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t}$ will be equal to $x(t)$ pointwise.

$$x\left(t\right) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t} \quad \forall t$$

Does any periodic signal have a Fourier series representation?

$$x(t) = \begin{cases} t, & 0 \leq t < 0.5 \\ 1 - t, & 0.5 \leq t < 1 \end{cases} \longrightarrow c_k = \begin{cases} \frac{1}{4}, & k = 0 \\ \frac{4}{k^2\omega_0^2} \sin^2\left(\frac{k\omega_0}{4}\right) e^{-j\frac{k\omega_0}{2}}, & k \neq 0 \end{cases}$$



Does any periodic signal have a Fourier series representation?

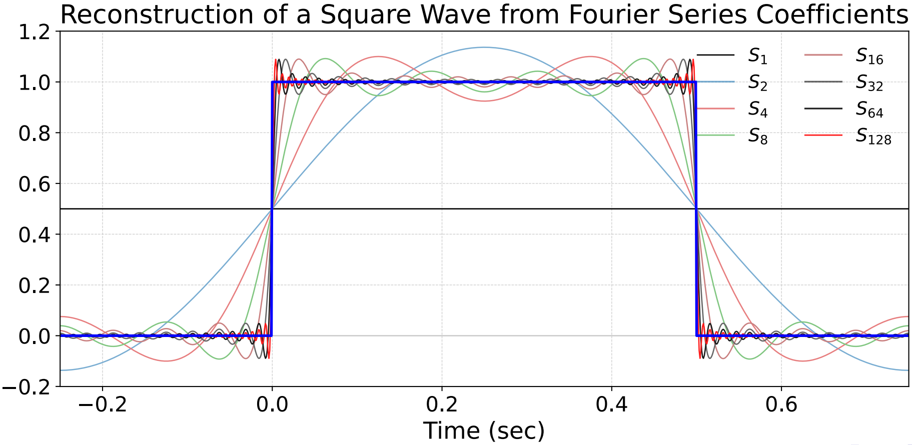
- If $x(t)$ is finite but discontinuous $\longrightarrow$ No pointwise equality. Only means squared convergence is possible.

$$\lim_{N\to\infty} \int_0^{T_0} \left| x(t) - \sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t} \right|^2 dt = 0$$

This means that the reconstructed signal $\sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t}$ need not be equal to the signal $x(t)$ at a discrete set of points, i.e. at the points where there is a discontinuity.

Does any periodic signal have a Fourier series representation?

$$x(t) = \begin{cases} 1, & 0 \leq t < 0.5 \\ 0, & 0.5 \leq t < 1 \end{cases} \longrightarrow c_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{2}{k\omega_0} \sin\left(\frac{k\omega_0}{4}\right) e^{-j\frac{k\omega_0}{4}}, & k \neq 0 \end{cases}$$



Dirichlet conditions for Fourier series

The *Dirichlet conditions* guarantee that the c_k exists, and $\sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t}$ is equal to $x(t)$ except at time points where there is a discontinuity.

At a discontinuity, $\sum_{k=-N}^{N} c_k e^{j2\pi k f_0 t}$ converges to the midpoint of the discontinuity.

The *Dirichlet conditions* are that a single cycle of $x(t)$:

1. has a finite number of disconuities.
2. has a finite number of maxima and minima.
3. is absolutely integrable. $\int_0^{T_0} |x(t)| dt < \infty$

Power Spectral Density of Periodic Signals

Some definitions:

- Instantaneous power of a signal $x(t) \triangleq |x(t)|^2$
- Total energy of a signal $x(t)$ in a time interval $[T_1, T_2] \triangleq \int_{T_1}^{T_2} |x(t)|^2 dt$
- Average power over a time interval $[T_1, T_2] \triangleq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} |x(t)|^2 dt$
- **Energy signal**: Signals with a finite total energy and zero average power over their entire duration.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \quad \text{and} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = 0$$

- **Power signal**: Signals with a finite average power, and infinite energy.

$$\lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \infty \quad \text{and} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt < \infty$$

Power Spectral Density of Periodic Signals

Parseval's Identity.

Let $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, then

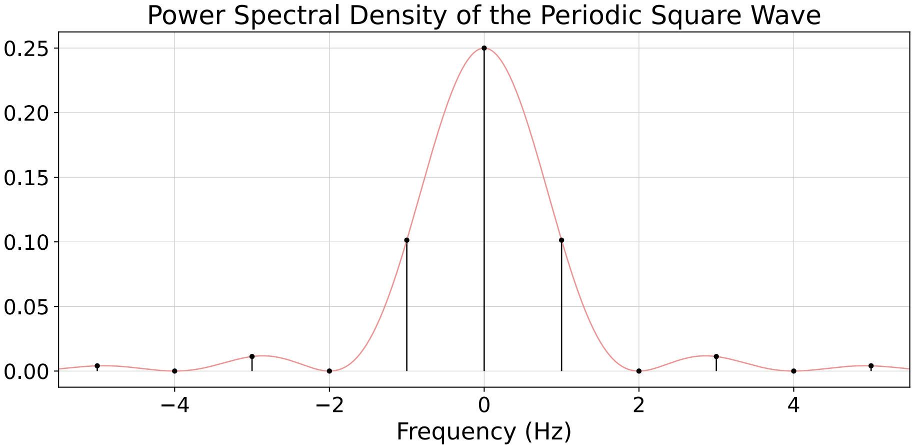
$$P_x = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Fourier series representation preserves the average power of the periodic signal $x(t)$.

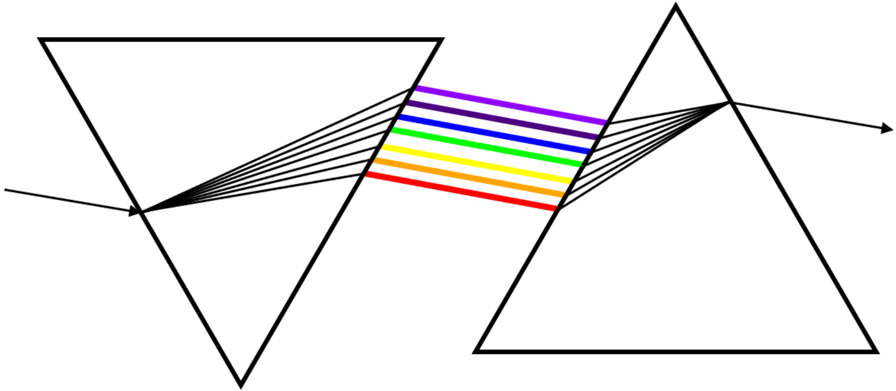
$|c_k|^2$ is the power in ofn the k^{th} harmonic.

$|c_k|^2$ as a function of k is the **Power Spectral Density** of $x(t)$.

Power Spectral Density of Periodic Signals



Power Spectral Density of Periodic Signals



Fourier representation of aperiodic signals

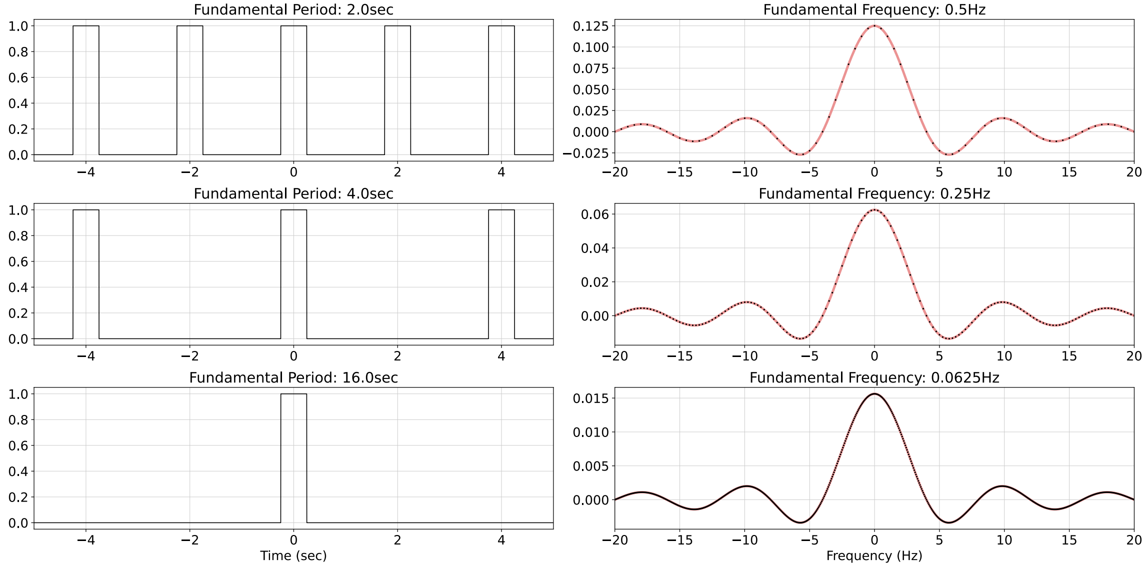
We can approach this problem starting from the Fourier series.

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \tau < |t| \leq \frac{T_0}{2} \end{cases}, \text{where, } 0 < \tau < \frac{T_0}{2}$$

$$\downarrow$$

$$c_k = \frac{\tau}{T_0} \frac{\sin\left(\pi k f_0 \tau\right)}{\pi k f_0 \tau}, \;\; k = 0, \pm 1, \pm 2, \ldots$$

Fourier representation of aperiodic signals



Fourier representation of aperiodic signals: Fourier Transform

$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \tau < |t| \leq \frac{T_0}{2} \end{cases}, \text{where, } 0 < \tau < \frac{T_0}{2}$$

$$\downarrow$$

$$c_k = \frac{\tau}{T_0} \frac{\sin\left(\frac{k\omega_0\tau}{2}\right)}{\frac{k\omega_0\tau}{2}}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$T_0 \to \infty \quad \implies \quad \omega_0 \to 0 \quad \implies \quad \{k\omega_0\}_{k=-\infty}^{\infty} \to \omega \in \mathbb{R} \quad \implies \quad c_k \to X(\omega)$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t} dt \quad \longrightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

This is the **Fourier transform**.

Fourier representation of aperiodic signals

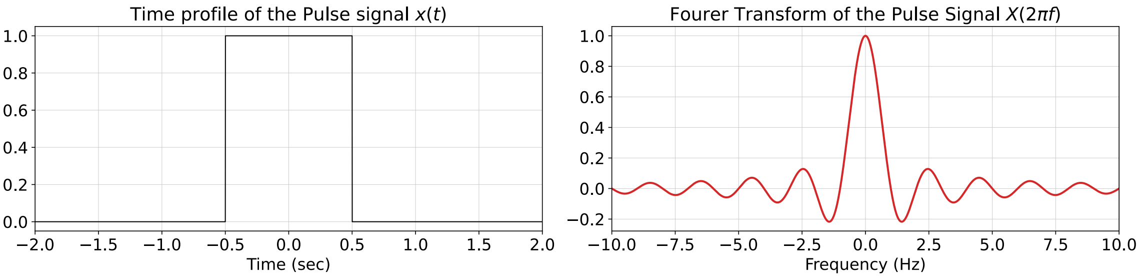
$$x(t) = \begin{cases} 1, & |t| \leq \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < |t| \end{cases} \quad \longrightarrow \quad X(\omega) = \tau \cdot \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} = \tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

We can reconstruct the time-domain signal from the $X(\omega)$,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

This is the **Inverse Fourier Transform.**

Fourier Transform



Fourier Transform

$$\begin{tikzcd}[column sep=huge] x(t) \arrow[r, bend left, "\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt"] & X(\omega) \arrow[l, bend left, "\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega"] \end{tikzcd}$$

Dirichlet Conditions for the Fourier Transform

The *Dirichlet conditions* for the existence of the Fourier transform are that $x(t)$:

1. has a finite number of discontinuities.
2. has a finite number of maxima and minima.
3. is absolutely integrable. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.

This ensures that $X(\omega)$ is finite and continuous.

We can still have Fourier transform for signal that are not absolutely integrable, but square integrable, i.e. $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$.

Example: $x(t) = \omega_0 \text{sinc}(\omega_0 t)$ is not absolutely integrable, but $X(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}$.

Parseval's identity for aperiodic signals

Energy of an aperiodic signal $x(t)$:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Parseval's identity:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$S_{xx}(\omega) = \frac{1}{2\pi} |X(\omega)|^2$ is the distribution of signal energey over frequency: **Energy density spectrum**.

Properties of Fourier transform

- **Linearity**: $\alpha x(t) + \beta y(t) \overset{\text{FT}}{\longleftrightarrow} \alpha X(\omega) + \beta Y(\omega)$
- **Shift in time**: $x(t - t_0) \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$
- **Shift in frequency**: $x(t)e^{j\omega_0 t} \overset{\text{FT}}{\longleftrightarrow} X(\omega - \omega_0)$
- **Time and frequency scaling**: $x(\alpha t) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right), \ \alpha > 0$
- **Convolution in time**: $x(t) * y(t) \overset{\text{FT}}{\longleftrightarrow} X(\omega)Y(\omega)$