Introduction to Signal Processing

Lecture 8: **Design and implementation of analog filters**

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Reading material

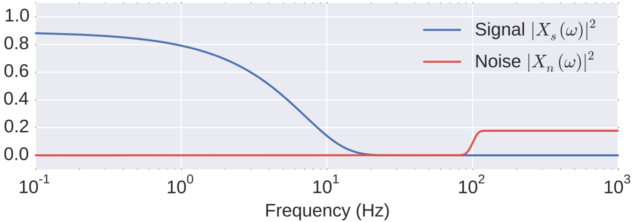
- Sections 8.1-8.7 from reference [2]

Filters

- A *filter* is any system that suppresses or removes undesired information from some desired information.
- In signal processing, we are mostly interested in selecting information that is restricted to a particular frequency band. The filters used for his purpose are *frequency selective filters*.
- All LTI system act as some sort of frequency selective filter, and will thus introduce some sort of amplitude and phase distortion (wanted or unwanted) to the input signal.
- The amount of distortion introduced is given by the frequency response of the LTI system.

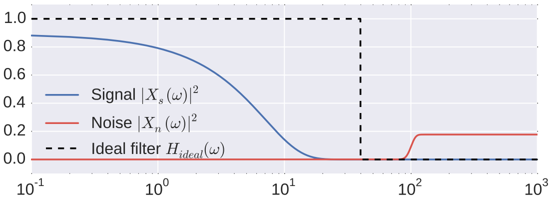
Ideal frequency selective filters

- Most often information of interest (signal) is confined to a particular frequency band , and is corrupted with unwanted information (noise) which occupies another frequency band.



- One could separate the signal and the noise in this case by passing this signal through an LTI system that has non-zero magnitude response for the band of frequencies occupied by the signal, and zero magnitude response in the noise frequency band,

Ideal frequency selective filters



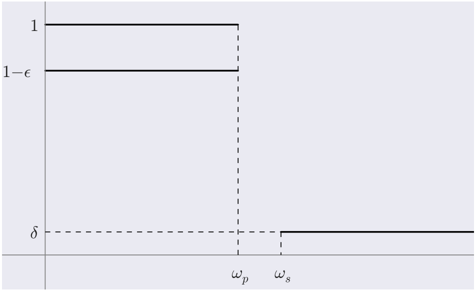
- Passing $X_s(j\omega) + X_n(j\omega)$ through $H_{ideal}\,(j\omega)$ will remove $X_n(j\omega)$ from the signal, and allow us to obtain $X_s(j\omega)$ undistorted.

$$H_{ideal}(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{Otherwise} \end{cases} \implies h_{ideal}(t) = \frac{\omega_c}{\pi}\frac{\sin\omega_c t}{\omega_c t}$$

- $H_{ideal}(j\omega)$ is a non-causal system and thus cannot be realized. This is true of any piecewise constant frequency response.
- However, for practical purpose $H_{idea}(j\omega)$ can be approximated by allowing deviations from the ideal response, which can lead to some distortion of $X_s(j\omega)$.

Real frequency selective filters

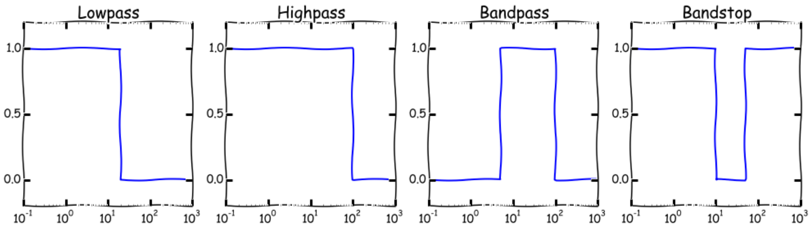
- Ideal frequency responses can be approximated by allowing deviations from the piecewise constant response, as depicted below.



$$\begin{cases} 1-\epsilon \le |H(j\omega)| \le 1, \ \ \forall|\omega| \le \omega_p & \text{Passband} \\ |H(j\omega)| \le \delta, \ \ \forall|\omega| \ge \omega_s & \text{Stopband} \\ \omega_p < |\omega| < \omega_p & \text{Transition band} \end{cases}$$

- ω_p, ω_s are the *passband and stopband cut-off frequencies*, and ϵ and δ are tolerance parameters.

Four types of frequency selective filters



Frequency selective filters can be broadly classified into four different types: *lowpass*, *highpass*, *bandpass* and *bandstop* filter. The above figure shows the ideal responses of these four different filter types.
We will discuss in details the design of lowpass filters, which can be easily transformed into the other filter types.

Real frequency selective filters

- With these specifications the design of the filter is carried out in terms of the frequency response, rather than the impulse response.
- The design of the filter then involves two steps:
  - **Approximation**: involves coming with a rational transfer function whose frequency response satisfy the specifications. The rational transfer function must be both *stable* and *causal*.
  - **Realization**: involves the physical realisation of the rational transfer function with physical elements (resistors, inductors and capacitors for electrical systems, or springs, dampers and masses for mechanical systems).

Approximation

- The purpose of this step is to approximate the ideal frequency response using a rational transfer function $H(s)$, that is both stable and causal.

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0}$$

- For $H(s)$ to be realizable:
  - All the coefficients $\{a_i\}_{i=0}^{N}$ and $\{b_i\}_{i=0}^{M}$ must be real.
  - $a_i > 0, \ \forall i \in [0, 1, \ldots, N]$
  - $N \geq M$
- The approximation step takes the filter specifications about a filter's magnitude response, and given use the coefficients $\{a_i\}_{i=0}^{N}$ and $\{b_i\}_{i=0}^{M}$, or the poles and zeros of the transfer function $H(s)$.

Simple cases of $H(s)$: *Bilinear* transfer function

Consider a $H(s)$ of the following form,

$$H(s) = \frac{b_1 s + b_0}{a_1 s + a_0}$$

This is called the *bilinear* transfer function.

For any given transfer function, we can talk about transmission zeros, i.e. frequencies at which the $H(j\omega) = 0$. Transmission zeros are different from that of zeros, because a transfer function with no zeros can have transmission zeros at ∞.

Consider the following special case,

$$H(s) = \frac{b_0}{a_1 s + a_0}$$

$H(s)$ has a *transmission* zero at ∞, as $\lim_{\omega \to \infty} H(j\omega) = 0$.

Simple cases of $H(s)$: *Bilinear* transfer function

Consider a $H(s)$ of the following form,

$$H(s) = \frac{b_1 s + b_0}{a_1 s + a_0}$$

Special cases

$$\begin{cases} \frac{b_0}{a_1 s + a_0} & \text{Poles} : -\frac{a_0}{a_1}; \ \ \text{Transmission Zeroes} : \infty \\ \frac{b_1 s}{a_1 s + a_0} & \text{Poles} : -\frac{a_0}{a_1}; \ \ \text{Transmission Zeroes} : 0 \end{cases}$$

Which of these behave likes a lowpass filter, and which one like a highpass filter?

How does $H(s) = \frac{b_1 s + b_0}{a_1 s + a_0}$ behave when $\frac{b_0}{a_0} > \frac{b_1}{a_1}$ and $\frac{b_0}{a_0} < \frac{b_1}{a_1}$?

Simple cases of $H(s)$: *Biquaratic* transfer function

Consider a $H(s)$ of the following form,

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

$H(s)$ has two poles (P) and two transmission zeros (TZ).

Special cases

$$\begin{cases} \frac{b_0}{a_2 s^2 + a_1 s + a_0} & \mathrm{P}: \frac{-a1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}; \quad \mathrm{TZ}: 2 \text{ at } \infty \\ \frac{b_2 s^2}{a_2 s^2 + a_1 s + a_0} & \mathrm{P}: \frac{-a1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}; \quad \mathrm{TZ}: 2 \text{ at } 0 \\ \frac{b_1 s}{a_2 s^2 + a_1 s + a_0} & \mathrm{P}: \frac{-a1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}; \quad \mathrm{TZ}: 1 \text{ at } 0, 1 \text{ at } \infty \\ \frac{b_2 s^2 + b_0}{a_2 s^2 + a_1 s + a_0} & \mathrm{P}: \frac{-a1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}; \quad \mathrm{TZ}: \pm\sqrt{\frac{b_0}{b_2}} \end{cases}$$

What type of filters do these three special cases correspond to?

Approximation

The purpose of the discussion on bilinear and biquadratic transfer functions was to give you an idea about the nature of the numerator and denominator polynomials of the transfer function for the different filter types.

Hopefully now by looking at a transfer function (and through knowledge of its poles and transmission zeros) you would be able to roughly say the type of filter represented by the transfer function.

We will now look at two different approximation procedures that are used to obtain rational transfer functions for the given filter specification:

- **Maximally flat** magnitude response
- **Equiripple** magnitude response

Butteworth filter

The squared magnitude response of a N^{th} order Butterworth filter is given by,

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2N}}$$

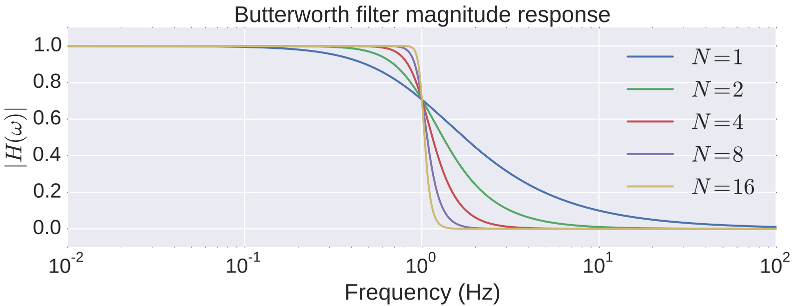
Has a maximally flat response, which formally means the following,

$$\frac{\partial^k}{\partial \omega^k} |H(j\omega)|^2 = 0, \;\; \forall k \in \{1, 2, 3, \dots, 2N-1\}$$

Butterworth filter: Properties

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2N}}$$

- It is an all pole filter. All of its transmission zeros are at infinity.
- $|H(0)| = 1$ for any given order N.
- $|H(\omega_0)| = \frac{1}{\sqrt{2}}$, which corresponds to the -3dB point.
- Has a roll-off of $-20N$dB in the stopband.



Butteworth filter: Properties

$$\left|H(j\omega)\right|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega) = H(s)H(-s)\Big|_{s=j\omega}$$

$$\implies H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j\omega_0}\right)^{2N}} = \frac{1}{1+(-1)^N\left(\frac{s}{\omega_0}\right)^{2N}}$$

$$B_N(s)B_N(-s) = 1+(-1)^N\left(\frac{s}{\omega_0}\right)^{2N}$$

$B_N(s)$ is the *Butterworth polynomial.*
Let us consider the following cases to understand where the poles of $H(s)H(-s)$ are,

$$N=1 \longrightarrow B_1(s)B_1(-s) = 1-\left(\frac{s}{\omega_0}\right)^2 = \left(1-\frac{s}{\omega_0}\right)\left(1+\frac{s}{\omega_0}\right)$$

The poles are at ω_0 and $-\omega_0$.

Butteworth filter: Properties

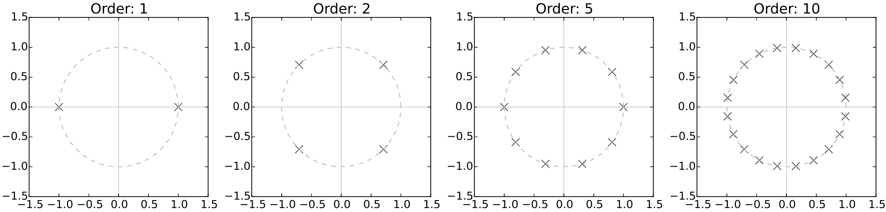
$$N = 2 \longrightarrow B_2(s)B_2(-s) = 1 + \left(\frac{s}{\omega_0}\right)^4 \quad \text{Poles: } \omega_0 e^{j(2k+1)\frac{\pi}{4}}, \ \ k = 0, 1, 2, 3$$

$$N = 3 \longrightarrow B_3(s)B_3(-s) = 1 - \left(\frac{s}{\omega_0}\right)^6 \quad \text{Poles: } \omega_0 e^{jk\frac{\pi}{3}}, \ \ k = 0, 1, \ldots 5$$

In general, the poles of an N^{th} order filter are located at,

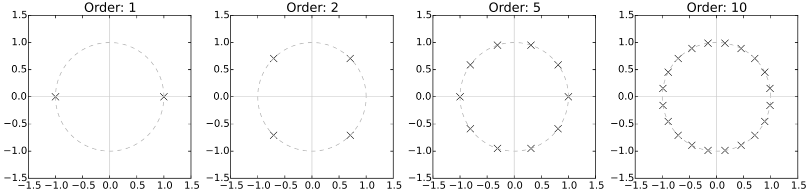
$$\begin{cases} N \text{ is odd} & \longrightarrow \omega_0 e^{jk\frac{\pi}{N}}, k = 0, 1, 2, \ldots 2N-1 \\ N \text{ is even} & \longrightarrow \omega_0 e^{j(2k+1)\frac{\pi}{2N}}, k = 0, 1, 2, \ldots 2N-1 \end{cases}$$

This means that the poles of $H(s)H(-s)$ are distributed on a circle of radius ω_0.



Butteworth filter: Properties

Poles of $H(s)H(-s)$ are distributed on a circle of radius ω_0.



A stable and causal filter can be implemented by choosing the poles on the left half of the s-plane, such that

$$H(s) = \begin{cases} N \text{ is even} & \longrightarrow \frac{1}{\Pi_k(s+2\omega_0\cos\phi_k s+\omega_0^2 p_i)} \\ N \text{ is odd} & \longrightarrow \frac{1}{\Pi_k(s+\omega_0)(s+2\omega_0\cos\phi_k s+\omega_0^2 p_i)} \end{cases}$$

Where, ϕ_k is angle of the complex conjugate poles. ϕ_k can be found using the following simple rule.

1. $\phi_1 = \frac{N+1}{2N}\pi$
2. ϕ_k and ϕ_{k+1} are separated by $\frac{\pi}{N}$.

Design of Butterworth filter

Filter design specifications: $\epsilon = 0.9, \delta = 0.1, \omega_p = 50\pi, \omega_s = 100\pi$. We need determine the values of Butterworth filter parameters N and ω_0 for the given specifications.

The Butterworth magnitude response is given by, $|H(j\omega)|_{dB} = 20\log\left[1 + \left(\frac{\omega}{\omega_0}\right)^{2N}\right]^{-\frac{1}{2}}$

$$\left|H(j\omega_p)\right|_{dB} = 20\log\epsilon = 20\log\left[1 + \left(\frac{\omega_p}{\omega_0}\right)^{2N}\right]^{-\frac{1}{2}} \implies \frac{1}{\epsilon^2} - 1 = \left(\frac{\omega_p}{\omega_0}\right)^{2N} \qquad (1)$$

$$\left|H(j\omega_s)\right|_{dB} = 20\log\delta = 20\log\left[1 + \left(\frac{\omega_s}{\omega_0}\right)^{2N}\right]^{-\frac{1}{2}} \implies \frac{1}{\delta^2} - 1 = \left(\frac{\omega_s}{\omega_0}\right)^{2N} \qquad (2)$$

$$\implies \frac{\left|H(j\omega_p)\right|_{dB}}{\left|H(j\omega_s)\right|_{dB}} = \frac{\frac{1}{\epsilon^2} - 1}{\frac{1}{\delta^2} - 1} = \left(\frac{\omega_p}{\omega_s}\right)^{2N} \implies N = \frac{1}{2}\frac{\log\left(\frac{\frac{1}{\epsilon^2}-1}{\frac{1}{\delta^2}-1}\right)}{\log\left(\frac{\omega_p}{\omega_s}\right)} \qquad (3)$$

The order N must be chosen as the lowest integer that is greater than the value obtained from Eq. 3. Once the N is determined, ω_0 can be determined using Eq. 1 or Eq. 2.

$$\omega_0 = \frac{\omega_p}{\left(\frac{1}{\epsilon^2} - 1\right)^{\frac{1}{2N}}} \text{ or } \omega_0 = \frac{\omega_s}{\left(\frac{1}{\delta^2} - 1\right)^{\frac{1}{2N}}}$$

For the given specifications,

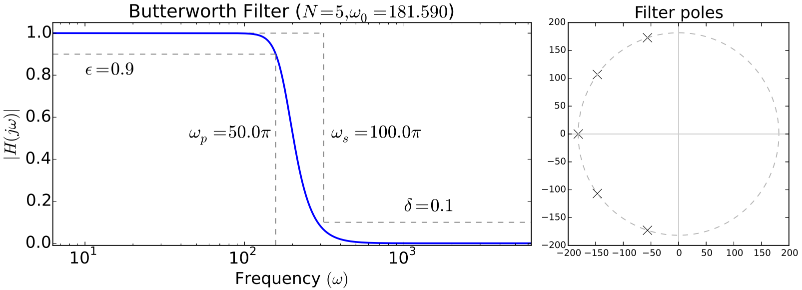
$$N = \frac{1}{2}\frac{\log\left(\frac{\frac{1}{0.9^2}-1}{\frac{1}{0.1^2}-1}\right)}{\log\left(\frac{10\pi}{15\pi}\right)} = 4.36 \implies N = 5$$

$$\omega_0 = \frac{10\pi}{\left(\frac{1}{0.9^2} - 1\right)^{\frac{1}{10}}} = 181.590$$

Design of Butterworth filter

Filter design specifications: $\epsilon = 0.9, \delta = 0.1, \omega_p = 50\pi, \omega_s = 100\pi$.
Butterworth filter parameters: $N = 5$, and $\omega_0 = 36.318$

$$H(s) = \frac{1}{(s + 181.59)\left(s^2 + 2(181.59)\cos\left(\frac{3\pi}{5}\right) + 181.59^2\right)\left(s^2 + 2(181.59)\cos\left(\frac{4\pi}{5}\right) + 181.59^2\right)}$$



Chebyshev filter

The Butterworth filter outperforms the filter specification in the passband, because of its monotonic response as a function of the frequency ω.
The Chebyshev filter (an example of a equiripple filter) makes use of the specifications to give a filter with a lower order, by allowing the mangitude response to have ripples in the passband.
The squared magnitude response of a N^{th} order Chebyshev filter is given by,

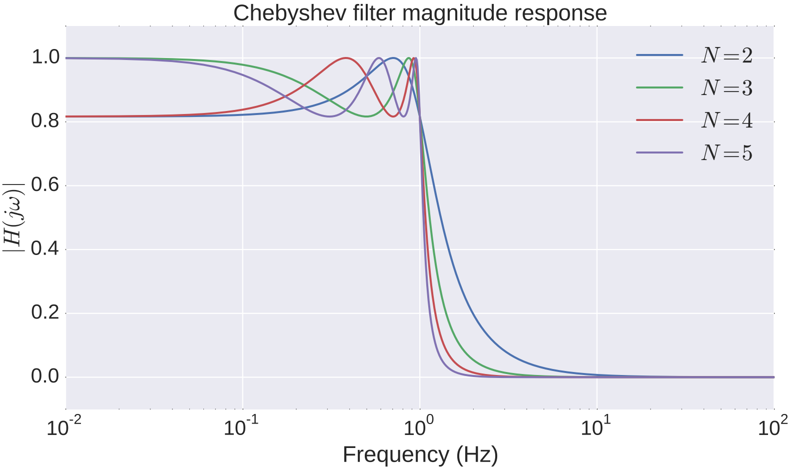
$$|H(j\omega)|^2 = \frac{1}{1+\gamma^2 C_N^2(\omega/\omega_0)}$$

where, N is the order of the filter, $C_N(\bullet)$ is the *Chebyshev polynomial*, and $1/\sqrt{1+\gamma^2}$ is the value of $|H(j\omega_0)|$.

$$C_N(\omega) = \begin{cases} \cos\left(N\cos^{-1}\omega\right) & |\omega| < 1 \\ \cosh\left(N\cosh^{-1}\omega\right) & |\omega| \geq 1 \end{cases}$$

Chebyshev filter

$$|H(j\omega)|^2 = \frac{1}{1 + \gamma^2 C_N^2(\omega/\omega_0)}$$



Design of Chebyshev filter: Properties

Filter design specifications: $\epsilon = 0.9, \delta = 0.1, \omega_p = 50\pi, \omega_s = 100\pi$.
We need determine the values of Chebyshev filter parameters N, γ and ω_0 for the given specifications.

The Chebyshev magnitude response is given by, $H(j\omega) = \left[1 + \gamma^2 C_N^2(\omega/\omega_0)\right]^{-\frac{1}{2}}$.
In the case of the Chebyshev filter, the value of ω_0 is equal to ω_p, which implies that,

$$\epsilon^2 = \frac{1}{1+\gamma^2} \implies \gamma = \sqrt{\frac{1}{\epsilon^2} - 1} \tag{4}$$

$$|H(j\omega_s)| = \delta = \left[1 + \gamma^2 C_N^2(\omega_s/\omega_0)\right]^{-\frac{1}{2}} = \left[1 + \gamma^2 \left(\cosh\left(N \cosh^{-1}(\omega_s/\omega_0)\right)\right)^2\right]^{-\frac{1}{2}} \tag{5}$$

$$N = \frac{\cosh^{-1}\sqrt{\frac{\frac{1}{\delta^2}-1}{\gamma^2}}}{\cosh^{-1}(\omega_s/\omega_0)} \tag{6}$$

The order N must be chosen as the lowest integer that is greater than the value obtained from Eq. 6.
For the given specifications,

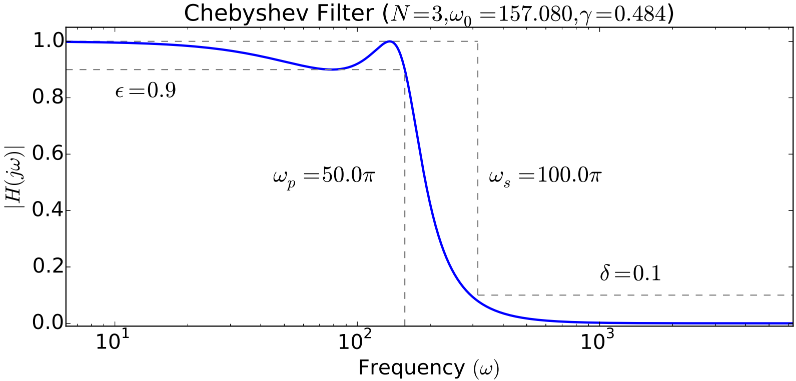
$$\gamma = \sqrt{\frac{1}{0.9^2} - 1} = 0.484 \quad \text{and} \quad \omega_0 = \omega_p = 50\pi$$

$$N = \frac{\cosh^{-1}\sqrt{\frac{\frac{1}{0.1^2}-1}{0.484^2}}}{\cosh^{-1}(100\pi/50\pi)} = 2.82 \implies N = 3$$

Design of Chebyshev filter

Filter design specifications: $\epsilon = 0.9, \delta = 0.1, \omega_p = 50\pi, \omega_s = 100\pi$.

Chebyshev filter parameters: $N = 3$, $\omega_0 = 157.080$ and $\gamma = 0.484$. Note that the Chebyshev filter has resulted in a lower order than the corresponding Butterworth filter for the same specifications.



Frequency transformations

- We have so far only focused on the design of lowpass filter. This is because, a lowpass filter can be converted to another type (such a highpass, bandpass or bandstop) through an appropriate frequency transformation.
- Given a lowpass transfer function $H_0(S)$, a frequency transformation procedure will replace the independent In a frequency transformation, the independent variable S is replaced by another variable $S = f(s)$, such that $H(s) = H_0(f(s))$ is the transformed filter (highpass, bandpass or bandstop) of interest.