Introduction to Digital Signal Processing

Digital Filters

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LTI systems can be designed shape frequency spectrum

Let H be an LTI system,

$$e^{j\Omega n} \mapsto |H(\Omega)| e^{j\Omega n + \arg H(\Omega)}$$

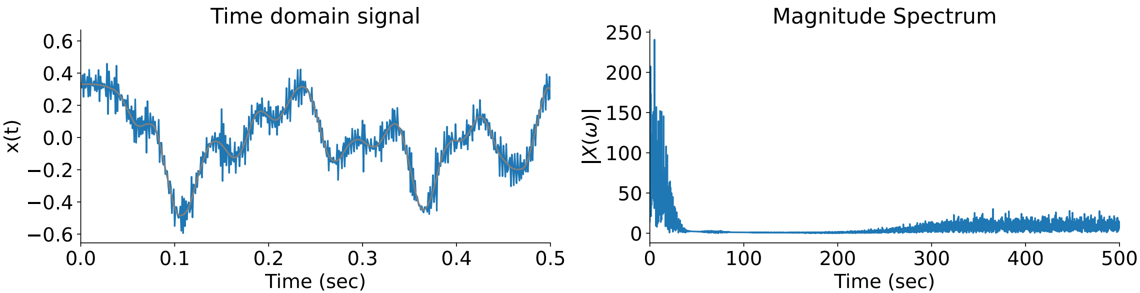
The amount of amplitude and phase modification of the input $e^{j\Omega n}$ is determined by the magntiude and phase of the value of the transfer function $H(\Omega)$.

The frequency response a discrete-time LTI system can be obtained from its impulse response or the difference equation describing the system,

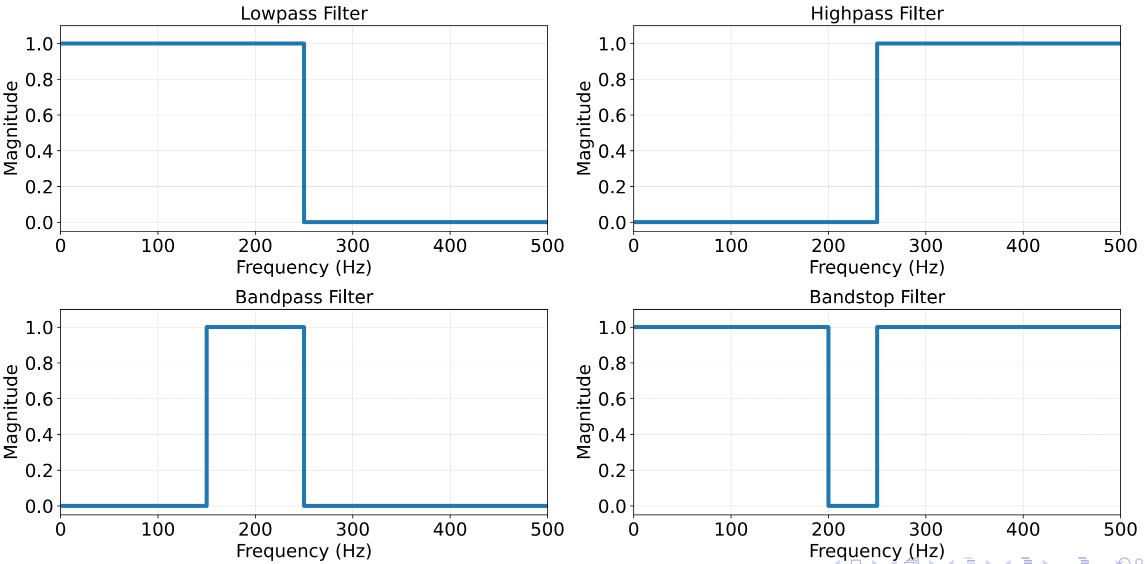
$$H(\Omega) = \sum_n h[n] e^{-j\Omega n} = \left. \frac{\sum_{k=1}^{M} b_i z^{-k}}{\sum_{l=1}^{N} a_l z^{-k}} \right|_{z=e^{j\Omega}}$$

If we have a desired frequency response $H_d(\Omega)$, how do we choose the impulse response or the coefficients of the LTI system such that its frequency response $H(\Omega)$ is as close to $H_d(\Omega)$ as possible.

Need for frequency selective filters



Ideal Filters



Characteristics of Ideal Filter

Consider the ideal lowpass filter,

$$H(\Omega) = \begin{cases} C \cdot e^{-j\omega n_0}, & |\Omega| \leq \Omega_c \\ 0, & \Omega_c <> |\Omega| \leq \pi \end{cases}$$

$$\implies Y(\Omega) = X(\Omega) H(\Omega) = CX(\Omega) e^{-j\Omega n_0}$$

- $C \implies$ Constant scaling of amplitude in the passband.
- $e^{j\Omega n_0} \implies$ Linear phase, which a constant time delay for all frequency components.

But ideal filters are problematic!

Problems with ideal

Ideal filters are not physically realizable.

Consider the ideal lowpass filter, $H\left(\Omega\right) = \begin{cases} 1, & |\Omega| \leq \Omega_c \\ 0, & \Omega_c |\Omega| \leq \pi \end{cases}$. The inverse DTFT of $H\left(\Omega\right)$ will give us the following impulse response for the system,

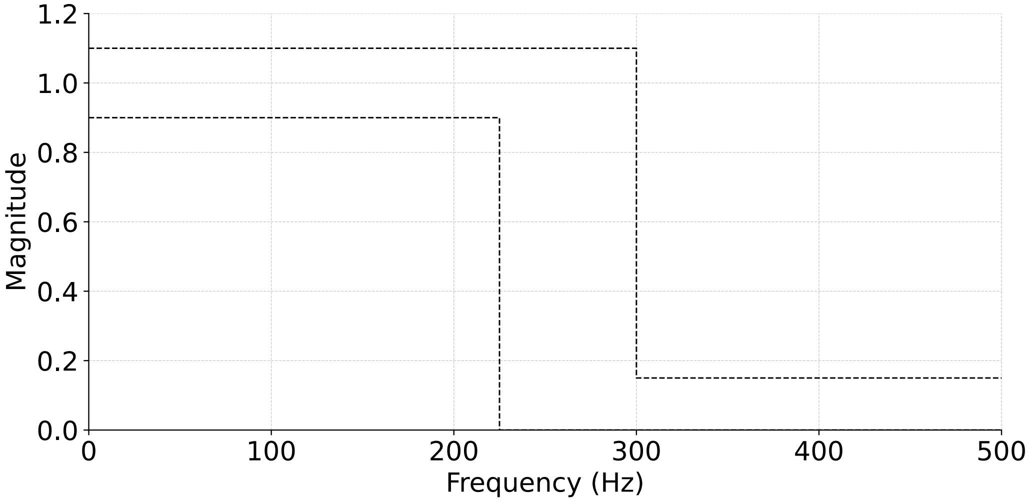
$$h[n] = 2\Omega_c \frac{\sin \Omega_c n}{\Omega_c n}$$

This is non-causal, with $h[n]$ extending upto $-\infty$.

Physically realizable filters cannot have:

- Flat frequecny response over a continuous interval.
- Cannot have step transitions.

Real Filter Specifications



Real Filter Specifications

$$\delta_1, \ \delta_2, \ f_p, \ f_s \longrightarrow H(\Omega) \longrightarrow \left. \frac{\sum_{k=0}^{M} b_i z^{-k}}{1 + \sum_{l=1}^{N} a_l z^{-k}} \right|_{z=e^{j\Omega}}$$

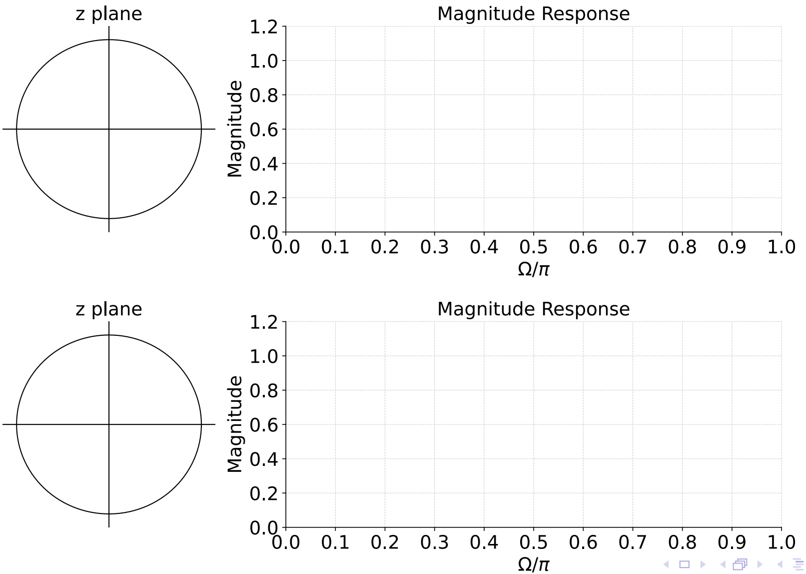
Parameters to be chosen for a filter: N, M, $(a_i)_{i=1}^{N}$, and $(b_i)_{i=0}^{M}$.

$$H(z) = b_0 \frac{\prod_{k=1}^{M} \left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - p_k z^{-1}\right)}$$

- $(a_i)_{i=1}^{N}$ determine the **poles** of the transfer function $(p_i)_{i=1}^{N} \longrightarrow$ Can be used to **emphasize certain frequencies**.
- $(b_i)_{i=0}^{M}$ determine the **zeros** of the transfer function $(z_i)_{i=1}^{M} \longrightarrow$ Can be used to **attenuate certain frequencies**.

Appropriate placement of the poles and zeros will allow us to obtain a frequency response that satisfies the given filter specifications.

Pole-Zero Placement and Frequency Response



Pole-Zero Placement and Frequency Response

Consider the two pole system,

$$H(z) = \frac{b_0}{1 - 2pz^{-1} - p^2 z^{-2}}$$

Determine b_0 and p such that the frequency response,

$$H(\Omega = 0) = 1 \quad \text{and} \quad \left|H(\Omega = \frac{\pi}{2})\right| = \frac{1}{\sqrt{2}}$$

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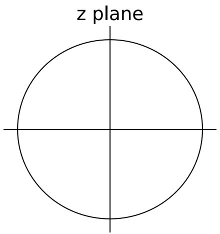
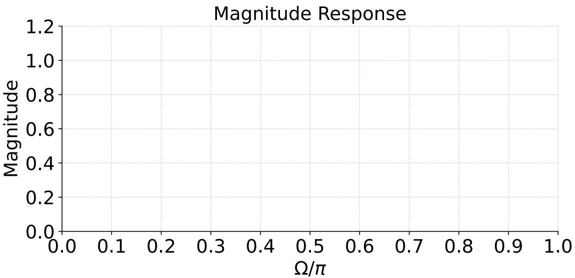
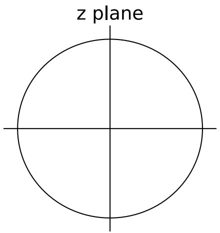
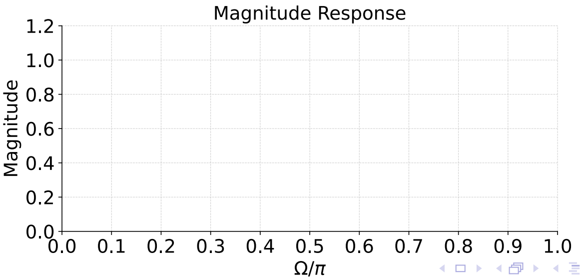
Pole-Zero Placement and Frequency Response

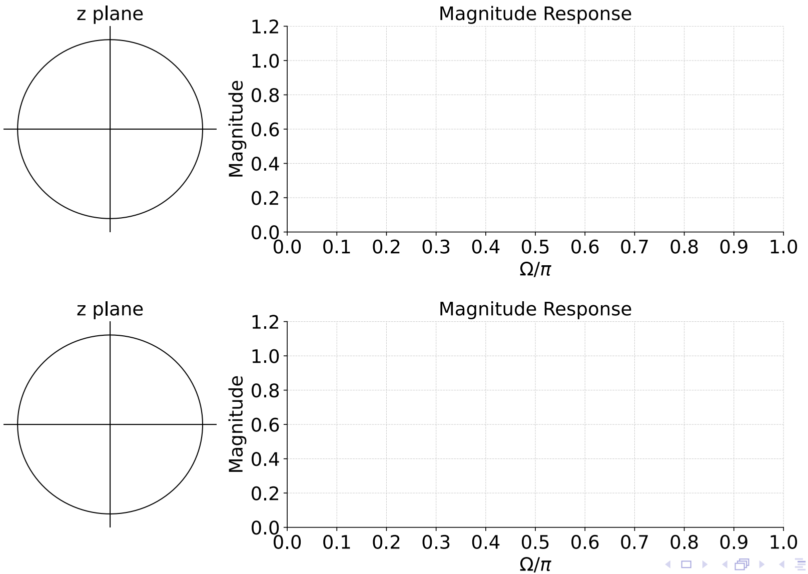








Pole-Zero Placement and Frequency Response



Pole-Zero Placement and Frequency Response

$$H(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)}{\left(1 - p z^{-1}\right)\left(1 - p^* z^{-1}\right)}$$

Determine b_0, z_1, z_2, and p to desgin a bandpass filter such that, $H\left(\Omega = 0\right) = 0$ and $H\left(\Omega = \pi\right) = 0$, the center of the passband in at $\Omega = \frac{\pi}{4}$ and $\left|H\left(\Omega = \frac{\pi}{2}\right)\right| = \frac{1}{\sqrt{2}}$.

FIR and IIR Digital Filter

- Two types of filters:
  - FIR filter: $h[n]$ is of finite length.

$$h[n] = 0, \ \forall n \neq 0 \quad \text{and } n > N$$

  - IIR filter: $h[n]$ is of infinite length.

$$h[n] = 0, \ \forall n \neq 0$$

Linear Phase FIR Filters

We will only deal with filter with real impulse response,

$$\implies H(\Omega) = H^*(-\Omega), \quad -\pi \leq \Omega < \pi \implies \begin{cases} |H(\Omega)| \text{ is an even function of } \Omega \\ \arg H(\Omega) \text{ is an off function of } \Omega \end{cases}$$

We will only discuss the design of linear phase filters.

$$\arg H(\Omega) = \begin{cases} m \cdot \Omega, & H(\Omega) > 0 \\ m \cdot \Omega + \pi, & H(\Omega) < 0 \end{cases}$$

Any signal that is symmetric or anti-symmetric with response to some time point will have a linear phase response.

Symmetric Impulse Response

Symmetry impulse response about $n = 0$.

$$h[n] = h[-n] \implies H(\Omega) = 2\sum_{n=1}^{\infty} h[n]\cos(\Omega n)$$

$$\implies \arg H(\Omega) = \begin{cases} 0, & H(\Omega) > 0 \\ \pi, & H(\Omega) < 0 \end{cases}$$

Anti-symmetry impulse response about $n = 0$.

$$h[n] = -h[-n] \implies H(\Omega) = -2j\sum_{n=1}^{\infty} h[n]\sin(\Omega n)$$

$$\implies \arg H(\Omega) = \begin{cases} -\frac{\pi}{2}, & \Im(H(\Omega)) > 0 \\ -\frac{3\pi}{2}, & \Im(H(\Omega)) < 0 \end{cases}$$

Symmetric, Causal, FIR Impulse Response

Let the length of the impulse response be M (odd). Then, a symmetric, causal impulse response is given by,

$$h[n] = h[M-n-1] \implies H(\Omega) = H_r(\Omega)\, e^{j\Theta(\Omega)}$$

$$\Theta(\Omega) = \begin{cases} -\Omega\left(\frac{M-1}{2}\right), & H_r(\Omega) > 0 \\ -\Omega\left(\frac{M-1}{2}\right) + \pi, & H_r(\Omega) < 0 \end{cases}$$

Anti-Symmetric, Causal, FIR Impulse Response

Let the length of the impulse response be M (odd). Then, an anti-symmetric, causal impulse response is given by,

$$h[n] = -h[M-n-1] \implies H(\Omega) = H_r(\Omega) e^{j\Theta(\Omega)}$$

$$\Theta(\Omega) = \begin{cases} \frac{\pi}{2} - \Omega\left(\frac{M-1}{2}\right), & H_r(\Omega) > 0 \\ \frac{3\pi}{2} - \Omega\left(\frac{M-1}{2}\right) + \pi, & H_r(\Omega) < 0 \end{cases}$$

FIR Filter Design: Windowing

Window the impulse response of the desired frequency response.

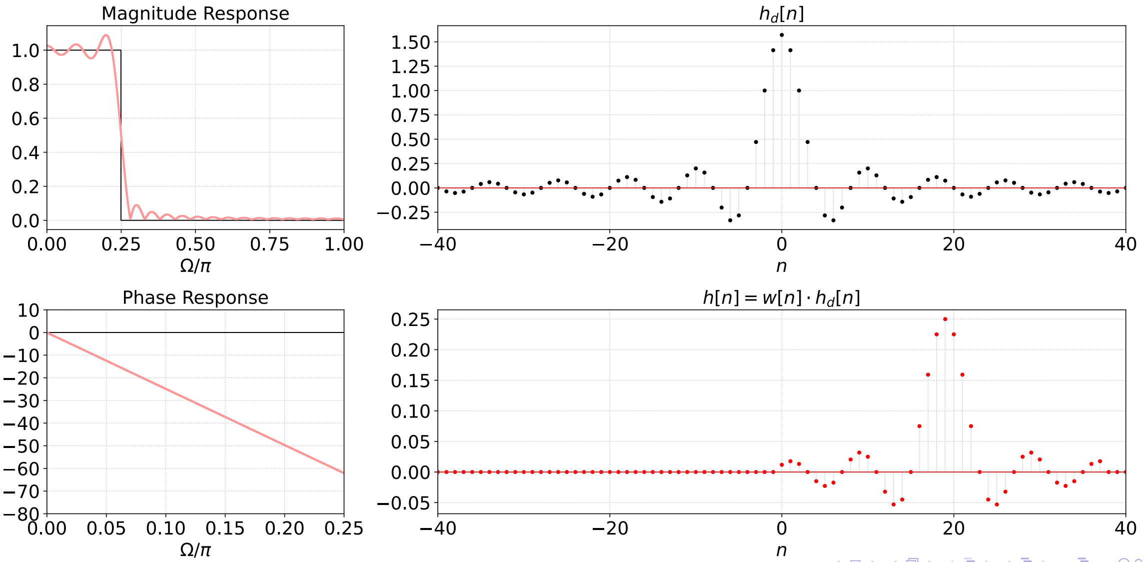
$$H_d(\Omega) \xrightarrow{\text{IDTFT}} h_d[n] \xrightarrow[w[n]]{\text{Window}} h[n] = w[n] \cdot h_d[n] \xrightarrow{\text{DTFT}} H(\Omega) \approx H_d(\Omega)$$

The resulting frequency response $H(\Omega)$ is,

$$H(\Omega) = H_d(\Omega) * W(\Omega)$$

where, $w[n] \xrightarrow{\text{DTFT}} W(\Omega)$.

FIR Filter Design: Windowing



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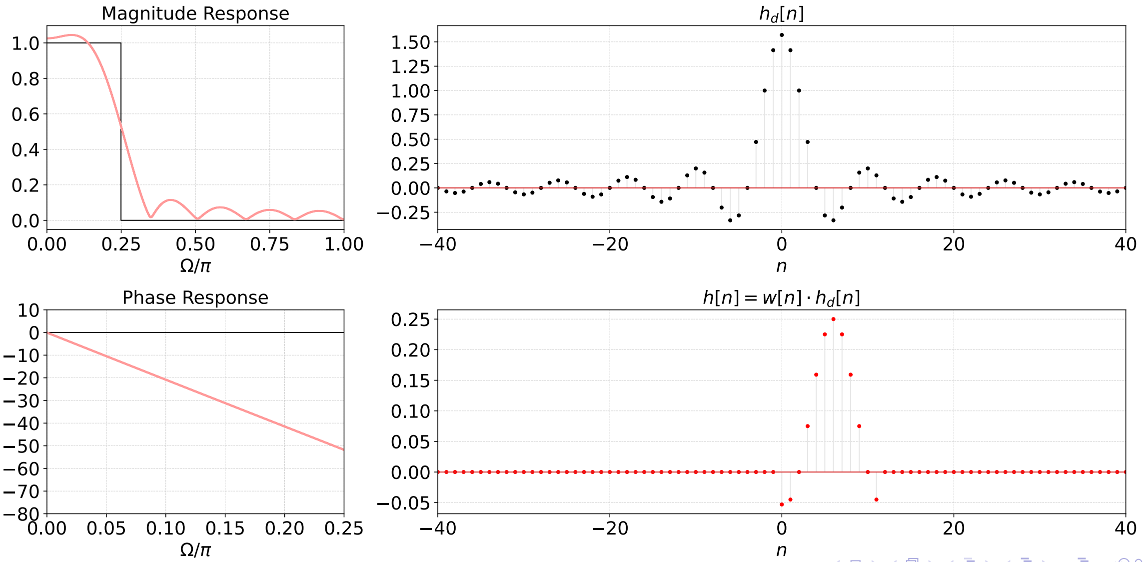
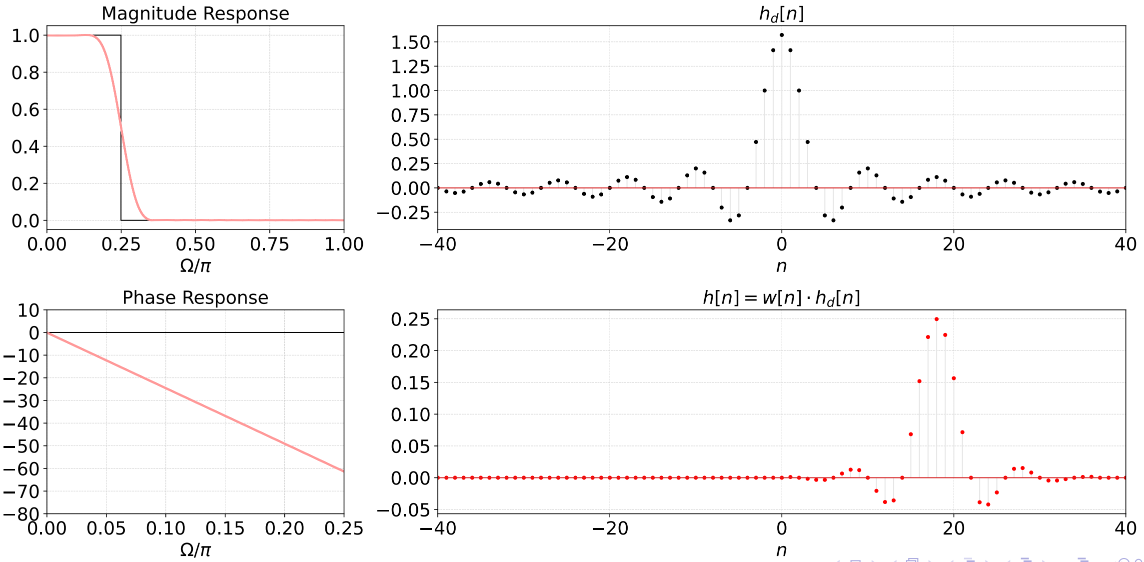


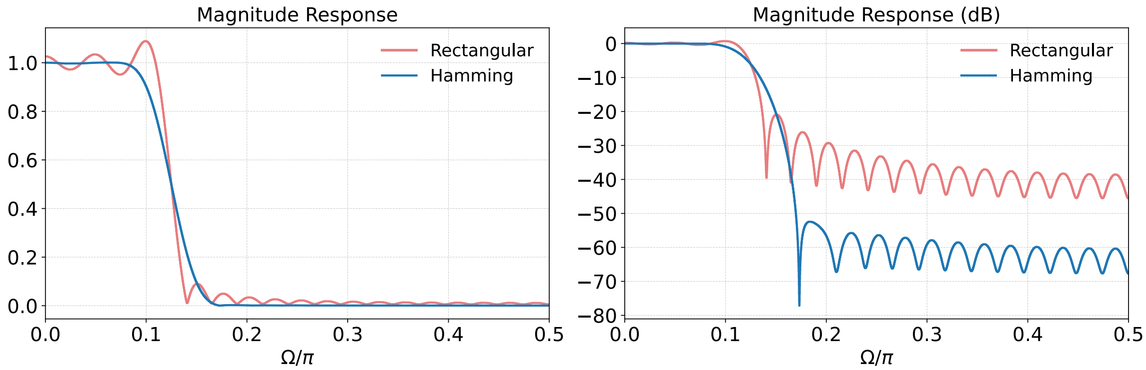
FIR Filter Design: Windowing



FIR Filter Design: Windowing

$$H_d(\Omega) \xrightarrow{\text{IDTFT}} h_d[n] \xrightarrow[w[n]]{\text{Window}} h[n] = w[n] \cdot h_d[n] \longrightarrow b_i = h[i],\ 0 \leq i \leq M$$

FIR Filter Design: Windowing



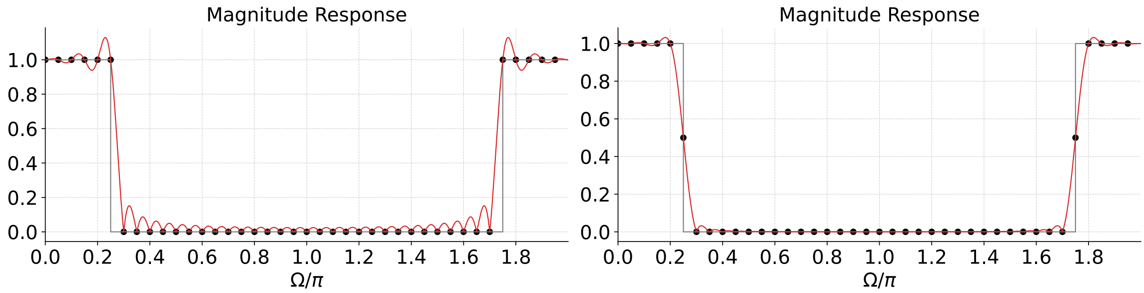
Different windows trade-off between the width of the transition band and the level of passband attentuation, and nature of the ripple in the pass and stopbands.

FIR Filter Design: Frequency sampling

We can specify the desired frequency response $H_d\left(\Omega\right)$ at a set of equally spaced frequenceies,

$$\Omega_k = \frac{2\pi k}{M}$$

FIR Filter Design: Frequency sampling



FIR Filter Design

$$H_d\left(\Omega\right) \xrightarrow{\text{IDTFT}} h_d[n] \xrightarrow[w[n]]{\text{Window}} h[n] = w[n] \cdot h_d[n] \longrightarrow b_i = h[i],\ 0 \leq i \leq M$$

FIR Filter Design

Advantages of FIR filters:

- Always stable. $\sum_n |h[n]| < \infty$
- Linear phase $\implies$ No phase distortion.

Disadvantages of FIR filters:

- Same specs might require longer filter.
- Might requires iterative numerical procedures for design.

IIR Filter Design

We are intersted in causal systems that have rational transfer functions,

$$H(z) = \frac{B(z)}{A(z)} \longrightarrow y[n] = \sum_{k=1}^{M} b_k \cdot x[n-k] - \sum_{l=1}^{N} a_l \cdot y[n-l]$$

Most popular approach is to design a analog filter and then translate it into a corresponding digital filter.

$$\text{Digital filter specs} \longrightarrow \text{Analog filter specs} \longrightarrow \text{Analog filter } H_a(s)$$

$$\text{Analog filter } H_a(s) \xrightarrow[\text{transformation}]{\text{Bilinear}} \text{Digital filter } H(z)$$

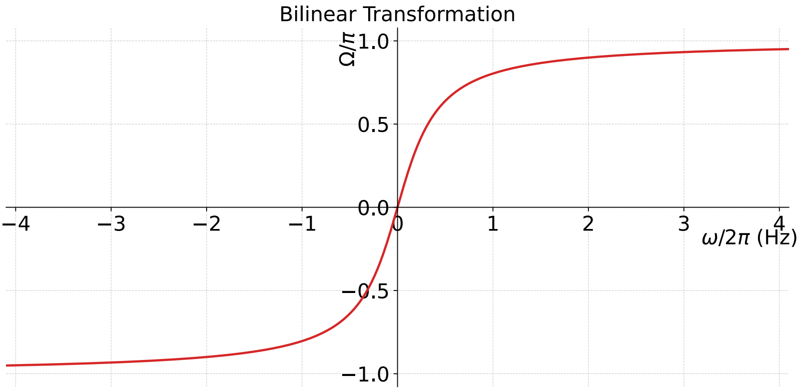
Bilinear transformation: Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$.

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

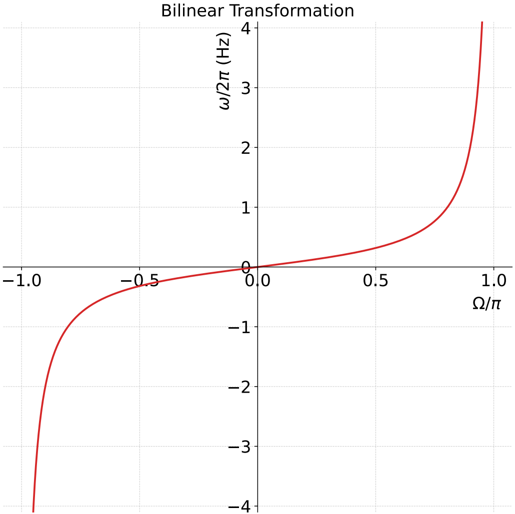
IIR Filter Design

Analog to digital Frequency mapping done by the bilinear transformation.

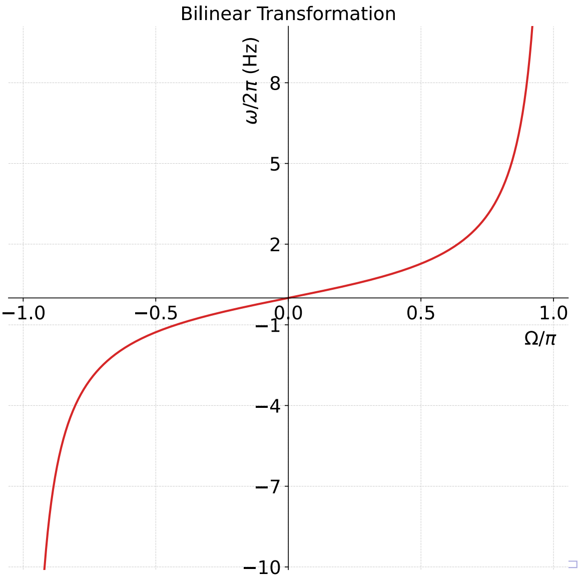
$$\omega = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right) \implies \Omega = 2\tan^{-1}\left(\frac{\omega T}{2}\right)$$



IIR Filter Design



IIR Filter Design



IIR Filter Design

Advantages of IIR filters:

- Requires lower number of parameters than FIR.
- Design using analog filter design tools.

Disadvantages of IIR filters:

- Can be unstable.
- More sensitive to round-off errors.