# Linear Systems

## Linear Dynamical Systems: State Space View

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# States of a system

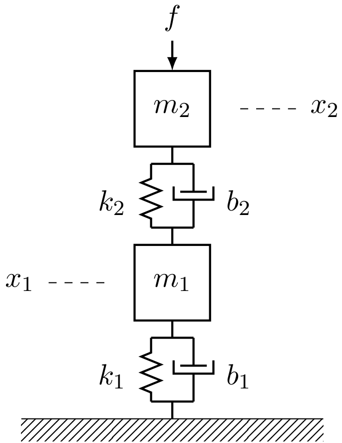
- A characteristic feature of most dynamical systems is their memory, i.e. the system's response (or output) depends on the present and past values of its input; We are only dealing with causal systems here.
- If we get interested in a system at some arbitrary time $t_0$, we might not have a complete record of the past input to the system.
- The idea of a *state* deals with this problem.

# States of a system

- **Defintion**: *The state* $\mathbf{x}(t_0)$ *of a system is the information at time* $t_0$, *which along with the input* $u(t)\,,\ \forall t \geq t_0$, *can be used to uniquely determine the system output* $y(t)\,,\forall t \geq t_0$.
- The state $\mathbf{x}(t_0)$ summarizes all the information ones needs to know about the system's past in order to predict its future.
- Examples of states of a system:
  - Position and velocity of a mass acted up on by a force.
  - Capacitor voltage and inductor current of a electrical network.
  - Initial conditions of a differential equation describing a system.

# States of a System

# States of a system




- In the system shown, the input $u(t)$ is the force $f(t)$ applied to $m_2$, and the output $y(t)$ is the position of $m_2$ $(x_2(t))$.
- $y(t)$ depends not only on $f(t)$, but also on: $\dot{x}_2(t)$, $x_1(t)$ and $\dot{x}_1(t)$.
- For the same input $u$, we can obtain different output $y$ if the starting states are different. Thus, knowledge of the states are essential for correctly predicting the behavior of the system.
- In general, the dynamics of a system in terms of its states, input(s) and output(s) is mathematically represented as,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \rightarrow \textit{State Equation} \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \rightarrow \textit{Measurement Equation} \end{cases}$$

where, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^p$, and $\mathbf{y} \in \mathbb{R}^m$, and $t \in \mathbb{R}$ represents times.

# States of a system

# States of a system

In general, the state and the input will determine the system's output.

$$\left.\begin{array}{l}\mathbf{x}\left(t_0\right) \\ u\left(t\right), \ \forall t \geq t_0\end{array}\right\} \rightarrow y\left(t\right), \ \forall t \geq t_0$$

In the case of a linear system, if

$$\left.\begin{array}{l}\mathbf{x}_1\left(t_0\right) \\ u_1\left(t\right), \quad \forall t \geq t_0\end{array}\right\} \rightarrow y_1\left(t\right), \ \forall t \geq t_0 \quad \text{and} \quad \left.\begin{array}{l}\mathbf{x}_2\left(t_0\right) \\ u_2\left(t\right), \ \forall t \geq t_0\end{array}\right\} \rightarrow y_2\left(t\right), \ \forall t \geq t_0$$

$$\implies \left.\begin{array}{l}a_1\mathbf{x}_1\left(t_0\right) + a_2\mathbf{x}_2\left(t_0\right) \\ a_1u_1\left(t\right) + a_2u_2\left(t\right), \ \forall t \geq t_0\end{array}\right\} \rightarrow a_1y_1\left(t\right) + a_2y_2\left(t\right), \ \forall t \geq t_0$$

# States of a system

For a linear system, knowing the system output to the states and the input will allow us to know the complete output.

- **Zero State Response**: $\mathbf{x}(t_0) = \mathbf{0};\ u(t), t \geq t_0\} \rightarrow y_{zs}(t), \forall t \geq t_0$
- **Zero Input Response**: $\mathbf{x}(t_0);\ u(t) = 0, t \geq t_0\} \rightarrow y_{zi}(t), \forall t \geq t_0$

$y_{zs}(t) + y_{zi}(t)$ gives the complete response.

# State space representation of linear systems

- In the case of a linear system, the equations representing the dynamics takes a simpler form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t)$$

where,

- $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$ is the *system* matrix.
- $\mathbf{B}(t) \in \mathbb{R}^{n \times p}$ is the *input* matrix.
- $\mathbf{C}(t) \in \mathbb{R}^{m \times n}$ is the *output* matrix.
- $\mathbf{D}(t) \in \mathbb{R}^{m \times p}$ is the *feedforward*matrix.

## State space representation of linear systems

- In the case of time-invariant system, the matrices are constant.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- These two equations represent how the states and the measured outputs of the system are affected by the current states and inputs. The individual terms in these matrices indicate how a particular state/input affects another state/output.

# State space representation of linear systems

Consider a LTI system represented by the following differential equation,

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = u(t)$$

We can obtain a state space representation of this differential equation by choosing two states, $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -a_2 x_1(t) - a_1 x_2(t) + u(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

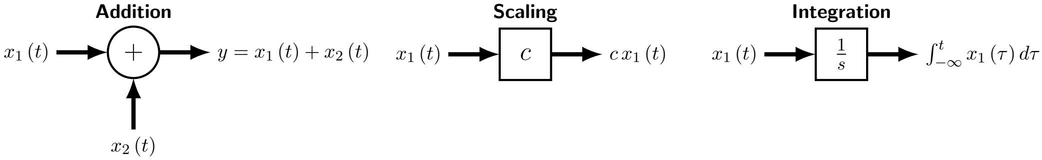
# State space representation of linear systems

The choice of state for a system is not unique. If for a linear system, $\mathbf{x}(t)$ is a state, then so is $\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$, where $\mathbf{T}$ is invertible.

If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are the different matrices associated with a LTI system with state $\mathbf{x}(t)$. Derive the matrcies when $\mathbf{T}\mathbf{x}(t)$ is chosen as the state.

# Block diagram representation of linear systems

- Pictorial representation of different componenets of a system and their inter-connections can provide insights into the behavior of the system.
- Helps breakdown a complex system into a set of simpler systems connected to each other.
- Linear systems in general can be built using three basic elements:




Represent the following linear differential equations using the three elementary block,

- $\dot{y}(t) + 0.1y(t) = u(t)$
- $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = u(t) - 2\ddot{u}(t)$

# Block diagram representation of linear systems

# State space representation of discrete-time linear systems

- Discrete-time linear system,

$$\mathbf{x}[k+1] = \mathbf{A}[k]\,\mathbf{x}[k] + \mathbf{B}[k]\,\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}[k]\,\mathbf{x}[k] + \mathbf{D}[k]\,\mathbf{u}[k]$$

  where, $k \in \mathbb{Z}$ correspond to time index.
  - $\mathbf{A}[k] \in \mathbb{R}^{n\times n}$ is the *system* matrix.
  - $\mathbf{B}[k] \in \mathbb{R}^{n\times p}$ is the *input* matrix.
  - $\mathbf{C}[k] \in \mathbb{R}^{m\times n}$ is the *output* matrix.
  - $\mathbf{D}[k] \in \mathbb{R}^{m\times p}$ is the *feedforward* matrix.
- In the case of time-invariant system, the matrices are constant.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

# State space representation of discrete-time linear systems

Consider a LTI system represented by the following differential equation,

$$y[k] + a_1 y[k-1] + a_2 y[k-2] = u[k]$$

We can obtain a state space representation of this difference equation by choosing two states, $x_1[k] = y[k-1]$ and $x_2[k] = y[k-2]$,

$$\mathbf{x}[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} y[k] \\ y[k-1] \end{bmatrix} = \begin{bmatrix} -a_1 x_1[k] - a_2 x_2[k] + u[k] \\ x_1[k] \end{bmatrix}$$

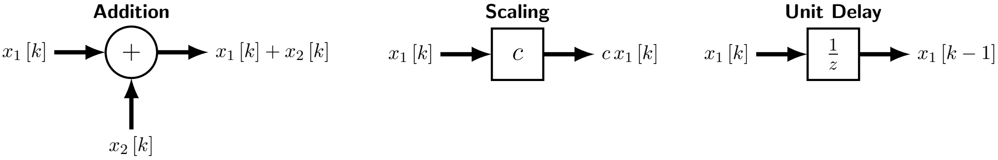
$$\mathbf{x}[k+1] = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}[k]$$

$$\mathbf{y}[k] = \begin{bmatrix} -a_1 & -a_2 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \end{bmatrix} \mathbf{u}[k]$$

The choice of state for a system is not unique. If for a linear system, $\mathbf{x}[k]$ is a state, then so is $\hat{\mathbf{x}}[k] = \mathbf{T}\mathbf{x}[k]$, where $\mathbf{T}$ is invertible.

# Block diagram representation of discrete-time linear systems

- Discrete-time linear systems in general can be built using three basic elements:




Represent the following linear differential equations using the three elementary block,

- $y[k] + 0.1y[k] = u[k]$
- $y[k-2] + 2y[k-1] + 5y[k] = u[k] - 2u[k-2]$
- $y[k] = \frac{1}{5}\sum_{l=0}^{4} u[k-l]$

# Block diagram representation of linear systems

# State space visualization

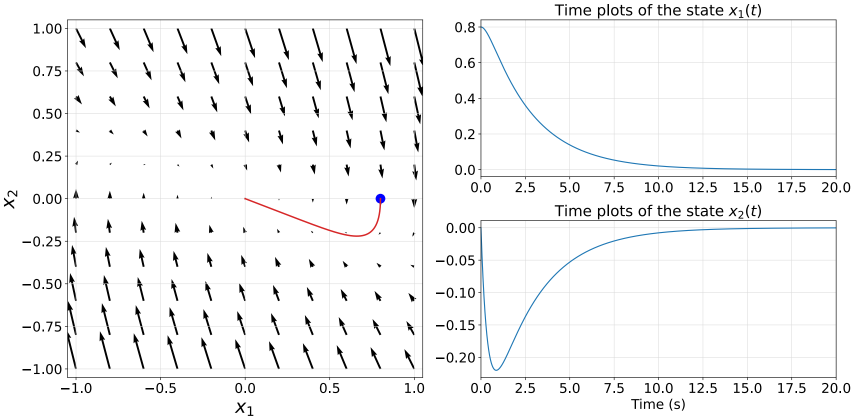
For systems with two states, we can visualize the state space trajectories of the system to gain better understanding of the system dynamics.

# State space visualization

The state dynamics of a mass, spring and damper system is given by the following equation,

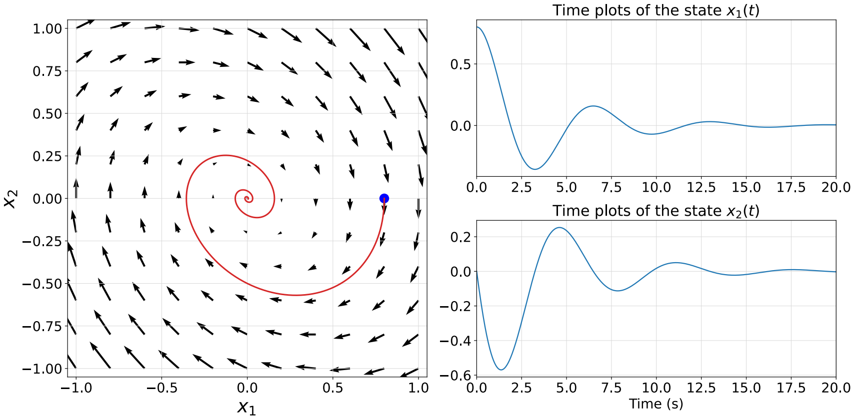
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}(t)$$

# State space visualization



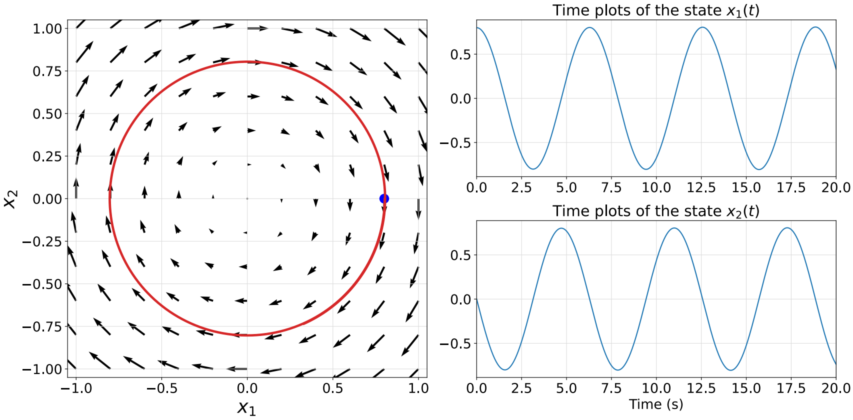

$$m = 1,\ b = 3,\ k = 1$$

# State space visualization




$$m = 1, \; b = 0.5, \; k = 1$$

# State space visualization




$$m = 1,\ b = 0,\ k = 1$$