Linear Systems: Orthogonality Assignment

1. Consider an orthonormal set of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_r\}, \ \mathbf{v}_i \in \mathbb{R}^n \ \forall i \in \{1, 2, \dots r\}$. If there is a vector $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v}_i^T \mathbf{w} = 0 \ \forall i \in \{1, 2, \dots r\}$. Prove that $\mathbf{w} \notin span\,(V)$.

2. Consider the following set of vectors in $\mathbb{R}^4$.
$$V = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} \right\}$$
Find the set of all vectors that are orthogonal to V?

3. For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, prove that $C(\mathbf{A}) \perp N\left(\mathbf{A}^T\right)$ and $C\left(\mathbf{A}^T\right) \perp N(\mathbf{A})$.

4. If the columns of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are orthonormal, prove that $\mathbf{A}^{-1} = \mathbf{A}^T$. What is $\mathbf{A}^T\mathbf{A}$ when $\mathbf{A}$ is rectangular ($\mathbf{A} \in \mathbb{R}^{m \times n}$) with orthonormal columns?

5. What will happen when the Gram-Schmidt procedure is applied to: (a) orthonormal set of vectors; and (b) orthogonal set of vectors? If the set of vectors are columns of a matrix $\mathbf{A}$, then what are the corresponding $\mathbf{Q}$ and $\mathbf{R}$ matrices for the orthonormal and orthogonal cases?

6. Consider the linear map, $\mathbf{y} = \mathbf{A}\mathbf{x}$, such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let us assume that $\mathbf{A}$ is full rank. What conditions must $\mathbf{A}$ satisfy for the following statements to be true,
   (a) $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, for all $\mathbf{x}, \mathbf{y}$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$.
   (b) $\mathbf{y}_1^T\mathbf{y}_2 = \mathbf{x}_1^T\mathbf{x}_2$, for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ such that $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1$ and $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2$.

   Note: A linear map $\mathbf{A}$ with the aforementioned properties preserves lengths and angle between vectors. Such maps are encountered in rigid body mechanics.

7. Prove that the rank of an orthogonal projection matrix $\mathbf{P}_S = \mathbf{U}\mathbf{U}^T$ onto a subspace $\mathcal{S}$ is equal to the dim $\mathcal{S}$, where the columns of $\mathbf{U}$ form an orthonormal basis of $\mathcal{S}$.

8. If the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ represent a basis for the subspace $\mathcal{S} \subset \mathbb{R}^m$. Find the orthogonal projection matrix $\mathbf{P}_\mathcal{S}$ onto the subspace $\mathcal{S}$. Hint: Gram-Schmidt orthogonalization.

9. Consider two orthogornal matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Is the $\mathbf{Q}_2^T\mathbf{Q}_1$ an orthogonal matrix? If yes, prove that it is so, else provide a counter-example showing $\mathbf{Q}_2^T\mathbf{Q}_1$ is not orthogonal.

10. Let $\mathbf{P}_\mathcal{S}$ represent an orthogonal projection matrix onto to the subspace $\mathcal{S} \subset \mathbb{R}^n$. What can you say about the rank of the matrix $\mathbf{P}_\mathcal{S}$? Explain how you can obtain an orthonormal basis for $\mathcal{S}$ from $\mathbf{P}_\mathcal{S}$.

11. Consider a 1 dimensional subspace spanned by the vector $\mathbf{u} \in \mathbb{R}^n$. What kind of a geometric operation does the matrix $\mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}$ represent?

12. Prove that when a triangular matrix is orthogonal, it is diagonal.

13. If an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is to be partitioned such that, $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}$, then prove that $C(\mathbf{Q}_1) \perp C(\mathbf{Q}_2)$.

14. Find an orthonormal basis for the subspace spanned by $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$.