Introduction to Digital Signal Processing

Linear Time-Invariant Systems

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Input-Output Relationship of System

Input-Output Relationship of Linear System

Linearity:

$$x_i[n] \mapsto y_i[n] \implies \sum_i \alpha_i x_i[n] \mapsto \sum_i \alpha_i y_i[n]$$

Input-Output Relationship of Time-Invariant System

Time-invariance:

$$x_i[n] \mapsto y_i[n] \implies x_i[n-k] \mapsto y_i[n-k]$$

Linear Time Invariant (LTI) System

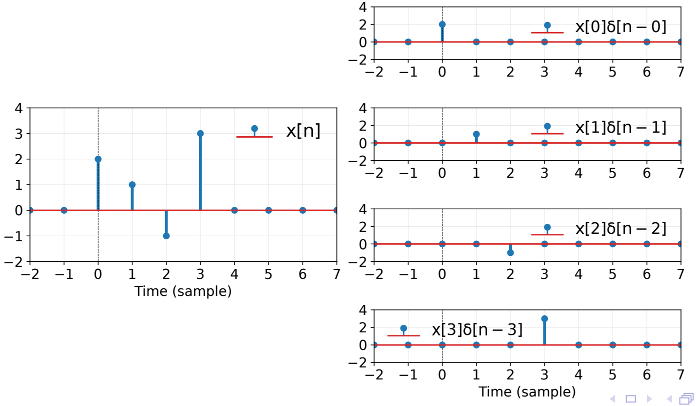
Input-Output Relationship

$$x_i[n] \mapsto y_i[n] \implies \sum_i \alpha_i x_i[n-k_i] \mapsto \sum_i \alpha_i y_i[n-k_i]$$

Importance of the Impulse Signal

Any signal $x[n]$ can be represented as a linear combinration of time-shifted impulse signals.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



Impulse Response of an LTI System

Impulse Response: The response of an LTI system to an impulse input.

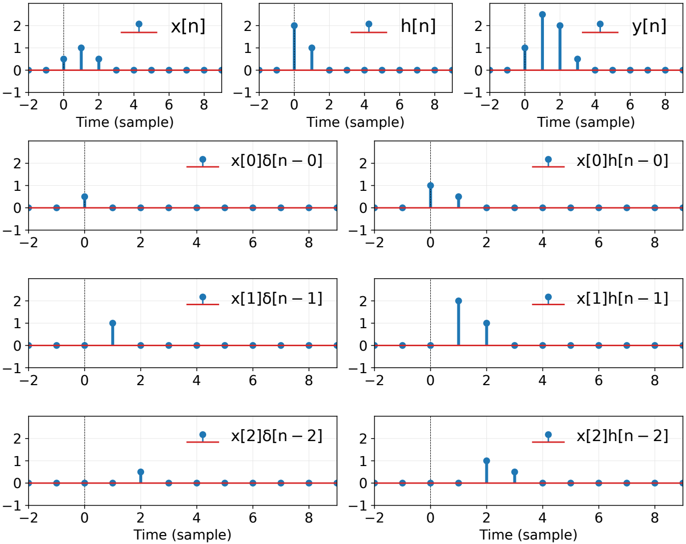
$$h[n] = \mathcal{H}\left(\delta[n]\right)$$

If we know this, then we know the following for an LTI system:

$$\delta[n] \mapsto h[n] \implies \begin{cases} \delta[n-k] & \mapsto h[n-k] \\ \alpha_k \cdot \delta[n-k] & \mapsto \alpha_k \cdot h[n-k] \\ \sum_k \alpha_k \cdot \delta[n-k] & \mapsto \sum_k \alpha_k \cdot h[n-k] \end{cases}$$

$$x[n] = \sum_k x[k] \cdot \delta[n-k] \xrightarrow{\mathcal{H}} \sum_k x[k] \cdot h[n-k] = x[n] * h[n]$$

Output of an LTI System



Convolution Sum

$$y[n] = x[n] * h[n] = \sum_{k} x[k] \cdot h[n-k]$$

Alternative View of the Convolution Sum

$$y[n] = x[n] * h[n] = \sum_{k} x[k] \cdot h[n-k]$$

k	$\cdots$	-3	-2	-1	0	1	2	3	4	5	6	7	$\cdots$
$x[k]$	$\cdots$	0	0	0	0.5	1	0.5	0	0	0	0	$\cdots$	
$h[\quad -k]$	$\cdots$												$\cdots$
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What does the impulse response tell us?

$$
\begin{aligned}
y[n] = x[n] * h[n] &= \sum_k x[k] \cdot h[n-k] \\
&= h[n] * x[n] = \sum_k h[k] \cdot x[n-k] \\
&= \cdots + h[2] \cdot x[n-2] + h[1] \cdot x[n-1] \\
&\quad + h[0] \cdot x[n] \\
&\quad + h[-1] \cdot x[n+1] + h[-2] \cdot x[n+2] + \cdots
\end{aligned}
$$

Properties of convolution sum

- **Commutative**

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

- **Associative**

$$x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$$

- **Distributive**

$$x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$$

- **Multiplicative identity**

$$x_1[n] * \delta[n] = x_1[n]$$

Impulse response, causality, and stability

Let $h[n]$ be the impulse response of and LTI system $\mathcal{H}$.

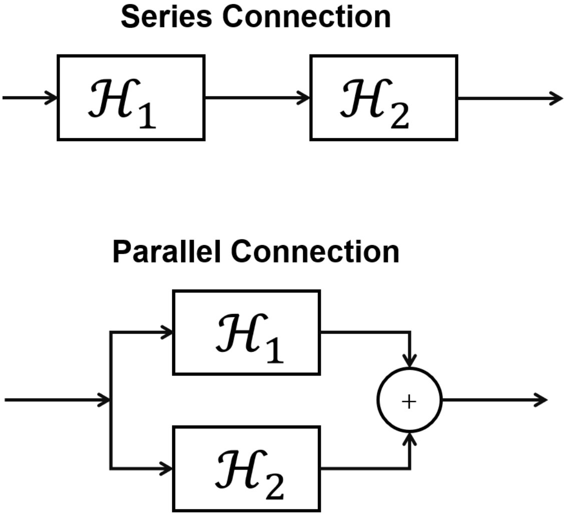
- **Causaility**. The LTI system $\mathcal{H}$ is causal, if and only if,

$$h[n] = 0, \ \forall n < 0$$

- **Stability**. The LTI system $\mathcal{H}$ is stable, if and only if,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Interconnection of LTI systems



Finite and Infinite Impulse Response system

Discrete-time LTI systems can be classified into two types based on the duration of the impulse response:

- **Finite Impulse Response (FIR) system**. $h[n]$ is non-zero only for a finite duration of time, and it is uniformly zero outside this finite duration.

$$h[n] = 0, \;\; n < M_1 \text{ and } n > M_2 \implies y[n] = \sum_{k=M_1}^{M_2} h[k]x[n-k]$$

Causality?

Stability?

How would be implement this system?

Finite and Infinite Impulse Response system

Discrte-time LTI system can be classified into two types of system based on the duration of the impulse response:

- **Infinite Impulse Response (IIR) system**. $h[n]$ is of infinitely long duration.,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Causality?

Stability?
How would be implement this system?

Linear, constant-coefficient difference equations

Consider the following system,

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

This system has a simple recursive form,

$$y[n] = y[n-1] + x[n]$$

Linear, constant-coefficient difference equations

A class of IIR systems can be expressed in recursive form through linear, constant coefficient difference equations,

$$y[n] = -\sum_{k=1}^{n} a_k \cdot y[n-k] + \sum_{k=0}^{m} b_k \cdot x[n-k]$$

This allows us to implement such a system.

Linear, constant-coefficient difference equations

A class of IIR systems can be expressed in recursive form through linear, constant coefficient difference equations,

$$y[n] = -\sum_{k=1}^{N} a_k \cdot y[n-k] + \sum_{k=0}^{M} b_k \cdot x[n-k]$$

This allows us to implement such a system.

Linear, constant-coefficient difference equations

Find the output of the system for $n \geq 0$ for the input $x[n] = u[n]$.

$$y[n] = a_1 \cdot y[n-1] + x[n]$$

Zero state and Zeo Input Responses of LTI system

$$y[n] = a_1 \cdot y[n-1] + x[n] \longrightarrow$$

Zero state response:

Zero state response:

Linearity of a general recursive LTI system

$$y[n] = -\sum_{k=1}^{N} a_k \cdot y[n-k] + \sum_{k=0}^{M} b_k \cdot x[n-k]$$

1. The total system response is the sum of the *zero state* response and *zero input* responses.

2. The *zero state* response satisfies the property of scaling and superposition.

3. The *zero input* response satisfies the property of scaling and superposition.

Solution of linear constant coefficient difference equations

$$y[n] = -\sum_{k=1}^{n} a_k \cdot y[n-k] + \sum_{k=0}^{m} b_k \cdot x[n-k]$$

The general solution is given by (assuming the input is applied at time $n = 0$),

$$y[n] = y_h[n] + y_p[n]$$

- $y_h[n]$ is the homogenous solution, i.e. the solution when the input is 0.
- $y_p[n]$ is the particular solution, i.e. the solution for the given input $x[n]$.

Homogenous Solution

This is the solution to the following equation,

$$y[n] + \sum_{k=1}^{N} a_k \cdot y[n-k] = 0$$

We are interested in the solution starting at $n = 0$ with the initial conditions $\{y[-1], y[-2], y[-3], \ldots, y[-N+1]\}$.

We first replace the term $y[n-k]$ to z^{n-k}, resulting in the following equation,

$$z^n + \sum_{k=1}^{N} a_k \cdot z^{n-k} = 0 \implies z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N = 0$$

Solution of linear constant coefficient difference equations

$$z^n + \sum_{k=1}^{N} a_k \cdot z^{n-k} = 0 \implies z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N = 0$$

Let $\{_i\}_{i=1}^{N}$ be the roots of the above polynominal equations, assuming there are N distinct roots. Then, the homogenous solution has the following form,

$$y_h[n] = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n + \ldots + C_N \lambda_N^n, \quad n \geq 0$$