

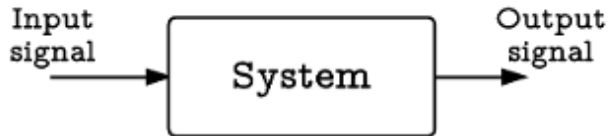
Introduction to Digital Signal Processing

Linear Time-Invariant Systems

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Input-Output Relationship of System



Input-Output Relationship of Linear System

Linearity:

$$x_i[n] \mapsto y_i[n] \implies \sum_i \alpha_i x_i[n] \mapsto \sum_i \alpha_i y_i[n]$$

Input-Output Relationship of Time-Invariant System

Time-invariance:

$$x_i[n] \mapsto y_i[n] \implies x_i[n - k] \mapsto y_i[n - k]$$

Linear Time Invariant (LTI) System

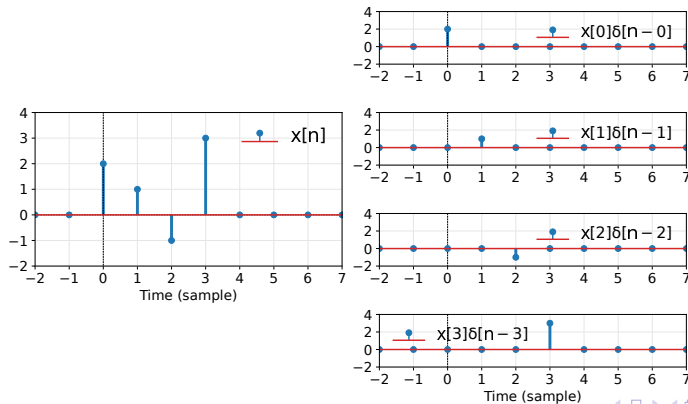
Input-Output Relationship

$$x_i[n] \mapsto y_i[n] \implies \sum_i \alpha_i x_i[n - k_i] \mapsto \sum_i \alpha_i y_i[n - k_i]$$

Importance of the Impulse Signal

Any signal $x[n]$ can be represented as a linear combination of time-shifted impulse signals.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$



Impulse Response of an LTI System

Impulse Response: The response of an LTI system to an impulse input.

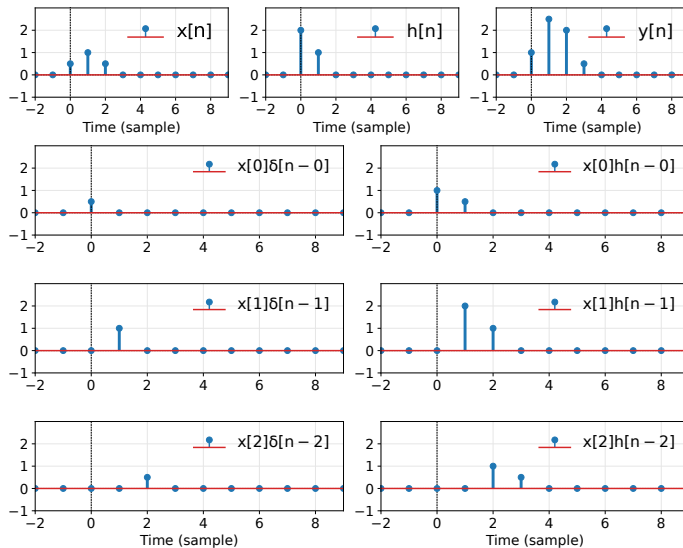
$$h[n] = \mathcal{H}(\delta[n])$$

If we know this, then we know the following for an LTI system:

$$\delta[n] \mapsto h[n] \implies \begin{cases} \delta[n-k] & \mapsto h[n-k] \\ \alpha_k \cdot \delta[n-k] & \mapsto \alpha_k \cdot h[n-k] \\ \sum_k \alpha_k \cdot \delta[n-k] & \mapsto \sum_k \alpha_k \cdot h[n-k] \end{cases}$$

$$x[n] = \sum_k x[k] \cdot \delta[n-k] \xrightarrow{\mathcal{H}} \sum_k x[k] \cdot h[n-k] = x[n] * h[n]$$

Output of an LTI System



Convolution Sum

$$y[n] = x[n] * h[n] = \sum_k x[k] \cdot h[n - k]$$

Alternative View of the Convolution Sum

$$y[n] = x[n] * h[n] = \sum_k x[k] \cdot h[n - k]$$

k	...	-3	-2	-1	0	1	2	3	4	5	6	7	...
$x[k]$...	0	0	0	0.5	1	0.5	0	0	0	0	...	
$h[-k]$
$h[-k]$
$h[-k]$
$h[-k]$
$h[-k]$
$h[-k]$
$h[-k]$

What does the impulse response tell us?

$$\begin{aligned}y[n] &= x[n] * h[n] = \sum_k x[k] \cdot h[n - k] \\&= h[n] * x[n] = \sum_k h[k] \cdot x[n - k] \\&= \cdots + h[2] \cdot x[n - 2] + h[1] \cdot x[n - 1] \\&\quad + h[0] \cdot x[n] \\&\quad + h[-1] \cdot x[n + 1] + h[-2] \cdot x[n + 2] + \cdots\end{aligned}$$

Properties of convolution sum

► Commutative

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

► Associative

$$x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$$

► Distributive

$$x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$$

► Multiplicative identity

$$x_1[n] * \delta[n] = x_1[n]$$

Impulse response, causality, and stability

Let $h[n]$ be the impulse response of an LTI system \mathcal{H} .

- **Causality.** The LTI system \mathcal{H} is causal, if and only if,

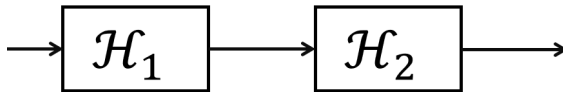
$$h[n] = 0, \quad \forall n < 0$$

- **Stability.** The LTI system \mathcal{H} is stable, if and only if,

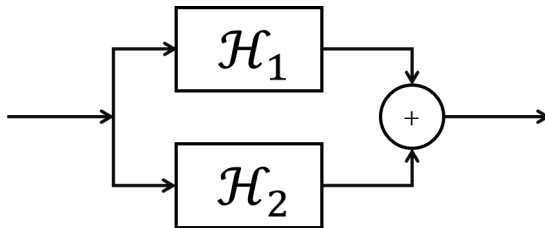
$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Interconnection of LTI systems

Series Connection



Parallel Connection



Finite and Infinite Impulse Response system

Discrete-time LTI systems can be classified into two types based on the duration of the impulse response:

- ▶ **Finite Impulse Response (FIR) system.** $h[n]$ is non-zero only for a finite duration of time, and it is uniformly zero outside this finite duration.

$$h[n] = 0, \quad n < M_1 \text{ and } n > M_2 \implies y[n] = \sum_{k=M_1}^{M_2} h[k]x[n-k]$$

Causality?

Stability?

How would be implement this system?

Finite and Infinite Impulse Response system

Discrete-time LTI system can be classified into two types of system based on the duration of the impulse response:

- ▶ **Infinite Impulse Response (IIR) system.** $h[n]$ is of infinitely long duration.,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Causality?

Stability?

How would be implement this system?

Linear, constant-coefficient difference equations

Consider the following system,

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

This system has a simple recursive form,

$$y[n] = y[n-1] + x[n]$$

Linear, constant-coefficient difference equations

A class of IIR systems can be expressed in recursive form through linear, constant coefficient difference equations,

$$y[n] = - \sum_{k=1}^n a_k \cdot y[n-k] + \sum_{k=0}^m b_k \cdot x[n-k]$$

This allows us to implement such a system.

Linear, constant-coefficient difference equations

A class of IIR systems can be expressed in recursive form through linear, constant coefficient difference equations,

$$y[n] = - \sum_{k=1}^N a_k \cdot y[n-k] + \sum_{k=0}^M b_k \cdot x[n-k]$$

This allows us to implement such a system.

Linear, constant-coefficient difference equations

Find the output of the system for $n \geq 0$ for the input $x[n] = u[n]$.

$$y[n] = a_1 \cdot y[n - 1] + x[n]$$

Zero state and Zero Input Responses of LTI system

$$y[n] = a_1 \cdot y[n-1] + x[n] \longrightarrow$$

Zero state response:

Zero state response:

Linearity of a general recursive LTI system

$$y[n] = - \sum_{k=1}^N a_k \cdot y[n - k] + \sum_{k=0}^M b_k \cdot x[n - k]$$

1. The total system response is the sum of the *zero state* response and *zero input* responses.
2. The *zero state* response satisfies the property of scaling and superposition.
3. The *zero input* response satisfies the property of scaling and superposition.

Solution of linear constant coefficient difference equations

$$y[n] = - \sum_{k=1}^n a_k \cdot y[n-k] + \sum_{k=0}^m b_k \cdot x[n-k]$$

The general solution is given by (assuming the input is applied at time $n = 0$),

$$y[n] = y_h[n] + y_p[n]$$

- ▶ $y_h[n]$ is the homogenous solution, i.e. the solution when the input is 0.
- ▶ $y_p[n]$ is the particular solution, i.e. the solution for the given input $x[n]$.

Homogenous Solution

This is the solution to the following equation,

$$y[n] + \sum_{k=1}^N a_k \cdot y[n-k] = 0$$

We are interested in the solution starting at $n = 0$ with the initial conditions $\{y[-1], y[-2], y[-3], \dots, y[-N+1]\}$.

We first replace the term $y[n-k]$ to z^{n-k} , resulting in the following equation,

$$z^n + \sum_{k=1}^N a_k \cdot z^{n-k} = 0 \implies z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N = 0$$

Homogenous Solution

$$z^n + \sum_{k=1}^N a_k \cdot z^{n-k} = 0 \implies z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N = 0$$

Let $\{\lambda_i\}_{i=1}^N$ be the roots of the above polynomial equations, assuming there are N distinct roots. Then, the homogenous solution has the following form,

$$y_h[n] = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n + \dots + C_N \lambda_N^n, \quad n \geq 0$$

The initial conditions can be used to solve for the constants C_i .

Homogenous Solution

$$z^n + \sum_{k=1}^N a_k \cdot z^{n-k} = 0 \implies z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N = 0$$

Let λ_1 be repeated m times, then the homogenous solution has the following form,

$$y_h[n] = \sum_{i=0}^{m-1} C_i n^i \lambda_1^n + \sum_{i=m+1}^N C_i \lambda_i^n, \quad n \geq 0$$

Particular Solution

The particular solution $y_p[n]$ satisfies the following equation,

$$y_p[n] + \sum_{k=1}^N a_k \cdot y_p[n-k] = \sum_{k=0}^m b_k \cdot x[n-k]$$

Based on the type of the input, we assume a particular form for the particular solution and solve for the constants in this solution to obtain the exact $y_p[n]$.

Input Signal	Particular Solution
A	K
$u[n]$	$K \cdot u[n]$
α^n	$K \cdot \alpha^n$
n^p	$\sum_{i=0}^p K_i n^i$
$\cos \Omega n, \sin \Omega n$	$K_1 \cos \Omega n + K_2 \sin \Omega n$

Example

Find the output of the LTI system described by the following equation,.

$$y[n] + 0.5y[n - 1] = x[n]; \quad x[n] = u[n], y[-1] = -2$$

Example

Example

Find the output of the LTI system described by the following equation,.

$$y[n] + y[n-1] - 2y[n-2] = x[n]; \quad x[n] = 0.1^n, y[-1] = 1, y[-2] = -1$$

Example

Impulse response of a Linear Time-invariant Recursive system

$$y[n] = - \sum_{k=1}^N a_k \cdot y[n-k] + \sum_{k=0}^M b_k \cdot x[n-k]$$

Impulse response is the *zero state* response for the input $\delta[n]$.

Find the impulse response of the following system,

$$y[n] + 0.5y[n-1] = x[n] + 2x[n-1]$$

$$y[n] + y[n-1] - 2y[n-2] = x[n] - x[n-1]$$

Impulse response of a Linear Time-invariant Recursive system

$$y[n] = - \sum_{k=1}^N a_k \cdot y[n-k] + \sum_{k=0}^M b_k \cdot x[n-k]$$

The impulse response has the general form,

$$h[n] = \sum_{i=1}^N C_i \lambda_i^n$$

We solve for C_i assuming zero initial conditions, $y[-1] = y[-2] = \dots = y[-N+1] = 0$.

The impulse response will be equal to the homogenous solution from $n = M+1$, with the initial conditions $y[n-M], y[n-M-1], y[n-M-2], \dots, y[n-M-N+1]$.

Exponential inputs to an LTI system

Consider the following LTI system,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If we apply an input $x[n] = z^n$, then we get

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) z^n = H(z) z^n$$