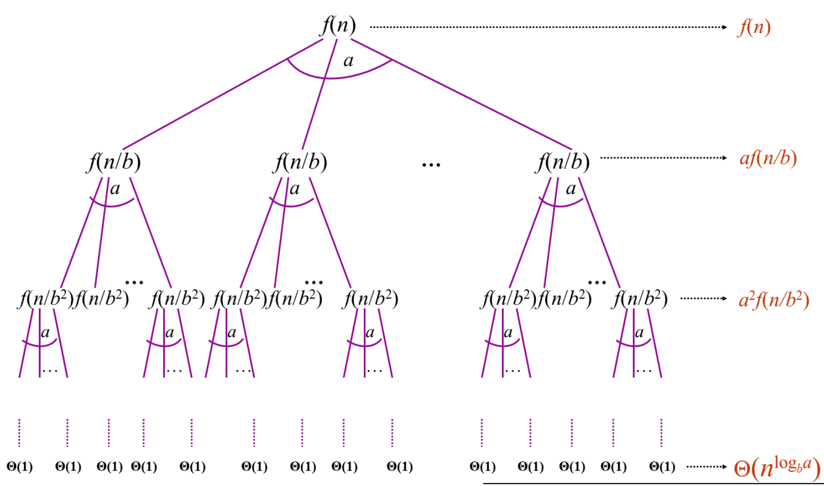
Your Exercise Answers:

Case1:
$T(n) = 2T(n/2) + 1 \sim O(n)$
$T(n) = 4T(n/2) + 1 \sim O(n^2)$
$T(n) = 4T(n/2) + n \sim O(n^2)$
$T(n) = 8T(n/2) + n^2 \sim O(n^3)$
$T(n) = 16T(n/2) + n^2 \sim O(n^4)$

Case3:
$T(n) = T(n/2) + n \sim O(n)$
$T(n) = 2T(n/2) + n^2 \sim O(n^2)$
$T(n) = 2T(n/2) + n^2 \log n \sim O(n^2 \log n)$
$T(n) = 4T(n/2) + n^3 \log^2 n \sim O(n^3 \log^2 n)$
$T(n) = 2T(n/2) + n^2 / \log n \sim O(n^2)$

Case2:
$T(n) = T(n/2) + 1 \sim O(\log n)$
$T(n) = 2T(n/2) + n \sim O(n \log n)$
$T(n) = 2T(n/2) + n \log n \sim O(n \log^2 n)$
$T(n) = 4T(n/2) + n^2 \sim O(n^2 \log n)$
$T(n) = 4T(n/2) + n^2 \log^2 n \sim O(n2 \log^3 n)$
$T(n) = 2T(n/2) + n / \log n \sim O(n \log\log n)$
$T(n) = 2T(n/2) + n / \log^2 n \sim O(n)$

Dividing Function Recursion Tree View:



Total: $T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$

Root Function:

$\mathbf{T(n) = T(\sqrt{n}) + 1}$

$T(2^m) = T(2^{m/2}) + 1$
$T(2^m) = S(m)$, $m = \log n$, $n = 2^m$
$S(m) = S(m/2) + 1$
$\log_2 1 = 0$, $k = 0$, $p = 0$, case 2-1,
$m = \log n$,
$O(\log \mathbf{m}) \rightarrow O(\log\log n)$

$x = \log_b a$ is the exponent for $a = b^x$.

Natural log: $\ln a = \log_e a$
Binary log: $\lg a = \log_2 a$

$\lg^2 a = (\lg a)^2$
$\lg \lg a = \lg (\lg a)$

$$a = b^{\log_b a}$$
$$\log_c(ab) = \log_c a + \log_c b$$
$$\log_b a^n = n \log_b a$$
$$\log_b a = \frac{\log_c a}{\log_c b}$$
$$\log_b(1/a) = -\log_b a$$
$$\log_b a = \frac{1}{\log_a b}$$
$$a^{\log_b c} = c^{\log_b a}$$

Complexity Comparison:

Substitute n with log (asymptotical comparison)

(1) **2^n : n^2**

nlog2 → n; $\log n^2$ → 2logn
$n = 2^{10}$: 20
$2^n > n^2$

(2) **3^n : 2^n**

nlog3 : nlog2
$3^n > 2^n$

(3) **n : $(\log n)^{100}$**

logn : 100 loglogn

$n = 2^{128}$

$n = 2^{1024}$

(4) **$n^{\log n}$: nlogn**

logn * logn : logn + loglogn

$n = 2^{1024}$

1024 * 1024 > 1024 + 10

(5) **$\sqrt{\log n}$: loglogn**

1/2**loglogn** : **logloglogn**

$n = 2^{1024}$

5 > 3.5

(6) $\mathbf{n^{\sqrt{n}}}$ **:** $\mathbf{n^{logn}}$

$\sqrt{n}$logn : logn*logn

$\sqrt{n}$: logn

1/2**logn** : **loglogn**

n= 2^{1024}

516 > 10

(7) $\mathbf{2^n}$ **:** $\mathbf{n^{3/2}}$ **: nlogn :** $\mathbf{n^{logn}}$

n : 3/2logn : logn + loglogn : logn * logn

logn*logn > logn + loglogn ←(4)

n > 3/2logn
n > logn + loglogn
n > logn * logn

logn * logn > 3/2logn

3/2logn > logn + loglogn

$\mathbf{2^n}$ **>** $\mathbf{n^{logn}}$ **>** $\mathbf{n^{3/2}}$ **> nlogn**

(8) f(n) = n^3 0 < n < 10,000
= n^2 n >= 10,000

g(n) = n 0 < n < 100
= n^3 n >= 100

	0 - 99	100 – 9,999	10,000
f(n)	n^3	n^3	n^2
g(n)	n	n^3	$\mathbf{n^3}$

(9) $\mathbf{n^{logn}}$ **:** $\mathbf{2^{\sqrt{n}}}$

logn * logn : $\sqrt{n}$
n= 2^{1024}
1024 * 1024 < 2^{516}

or substitute another log:

$\log^2 n$: $n^{1/2}$
2**loglogn** < 1/2**logn**

(10) $\mathbf{n^{\sqrt{n}}}$ **:** $\mathbf{2^{logn}}$

$\sqrt{n}$logn : logn*log2

$\sqrt{n}$ logn: logn

$\sqrt{n} > 1$

(11) **2^n : 2^{2n}**

nlog2 : 2nlog2
1 < 2
(don't forget we have applied log here)
$2^n < 2^{2n}$

(12) **n^2logn : n(logn)^{10}**

2logn + loglogn : **logn** + 10loglogn

n= 2^{1024}

n^2logn > n(logn)^{10}

(13) **$3n^{\sqrt{n}}$: $2^{\sqrt{n}\,\text{logn}}$**
$3n^{\sqrt{n}}$: $2^{\text{logn}\sqrt{n}}$
$3n^{\sqrt{n}}$: $n^{\sqrt{n}\ \ \text{log2}}$
$3n^{\sqrt{n}}$: $n^{\sqrt{n}}$
The same order (asymptotically equal) but the left is bigger $\Theta(n^{\sqrt{n}})$

(a)
$5n^2 + 100n$ **$3n^2 + 2$**

The same order (asymptotically equal) but the left is bigger $\Theta(n^2)$

(b)
$\log_3(n^2)$ **$\log_2(n^3)$**

$\log_b a = \log_c a / \log_c b$; A = 2lg$n$ / lg3, B = 3lg$n$, A/B =2/(3lg3)

$n^{\lg 4}$ **$3^{\lg n}$**

$\lg^2 n$ **$n^{1/2}$**

Q. True or False? Check which one is bigger (big O), the same (big Θ)

1. $n^3 = O(n^3)$ True
2. $n^2 = \Theta(n^3)$ False
3. $(n + k)^m = \Theta(n^m)$ True
4. $2^{n+1} = O(2^n)$ True
5. $2^{2n} = O(2^n)$, $4^n > 2^n$ False
6. $\sqrt{\text{logn}} = O(\text{loglogn})$ False
7. $n^{\text{logn}} = O(2^n)$ True

For each of the following pairs of functions, either f(n) is O(g(n)), f(n) is Ω(g(n)), or f(n) = Θ(g(n)). Determine which relationship is correct.

1. $f(n) = \log n^2$; $g(n) = \log n + 5$, $f(n) = \Theta(\log n) = \Theta(g(n))$
2. $f(n) = n$; $g(n) = \log n^2$, $f(n) = \Omega(\log n) = \Omega(g(n))$
3. $f(n) = \log \log n$; $g(n) = \log n$, $f(n) = O(\log n) = O(g(n))$
4. $f(n) = n$; $g(n) = \log^2 n$, $f(n) = \Omega(\log n) = \Omega(g(n))$
5. $f(n) = n \log n + n$; $g(n) = \log n$, $f(n) = \Omega(\log n) = \Omega(g(n))$
6. $f(n) = 10$; $g(n) = \log 10$, $f(n) = \Theta(1) = \Theta(g(n))$
7. $f(n) = 2^n$; $g(n) = 10n^2$, $f(n) = \Omega(n^2) = \Omega(g(n))$
8. $f(n) = 2^n$; $g(n) = 3^n$, $f(n) = O(3^n) = O(g(n))$

Go back to asymptotic notation:

$1 < \log n < \sqrt{n} < n < n\log n < n^2 < n^3 < \ldots < 2^n < 3^n \ldots < n^n$

(1)

$f(n) = 2n^2 + 3n + 1 < 7n^2$ $O(n^2)$, $g(n) = 7n^2$

$f(n) = 2n^2 + 3n + 1 > n^2$ $\Omega(n^2)$, $g(n) = n^2$

Both exist so $\Theta(n^2)$

(2)

$f(n) = n^2 \log n + n > n^2 \log n$ $\Omega(n^2 \log n)$

$f(n) = n^2 \log n + n < 10n^2 \log n$ $O(n^2 \log n)$

Both exist so $\Theta(n^2 \log n)$

(3)

$f(n) = n!$

$n! = n * (n-1) * (n-2) * \ldots *2*1 <= n *n*n*n\ldots *n$

$1*1*1*..*1 =< n * (n-1) * (n-2) * \ldots *2*1 <= n *n*n*n\ldots *n$

$1 =< n! <= n^n$

$\Omega(1)$, $O(n^n)$ no Θ

(4)

$f(n) = \log n!$

$\log(1*1*1*..*1) =< \log(n * (n-1) * (n-2) * \ldots *2*1) <= \log(n *n*n*n\ldots *n)$

$1 =< \log n! <= \log n^n = n\log n$

$\Omega(1)$, $O(n\log n)$ no Θ

Properties

- *Theorem:*

  $f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n))$ and $f = \Omega(g(n))$
- Transitivity:
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for O and Ω
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for O and Ω
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$