PRODUCTION PLANNING

Integer Programming

Abstract

Create a production plan for next six months to minimize costs and avoid stockouts using integer programming.

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Problem Statement

In the next six months a company must, in each month, operate either a normal shift or an extended shift (if it produces at all). A normal shift costs 100,000 per month and can produce up to 5,000 units per month. An extended shift costs 180,000 per month and can produce up to 7,500 units per month.

It is estimated that changing from a normal shift in one month to an extended shift in the next month costs an extra £15,000. No extra cost is incurred in changing from an extended shift in one month to a normal shift in the next month.

Carrying cost is estimated to be 2 per unit per month (based on the stock held at the end of each month) and the initial stock is 3,000 units (produced by a normal shift). At the end of month 6, at least 2000 units should be in stock. The demand for the company's product is shown below:

Month	1	2	3	4	5	6
Demand	6,000	6,500	7,500	7,000	6,000	6,000

If the company produces anything in a particular month it must produce at least 2,000 units. If the company wants a production plan for the next six months that avoids stockouts, formulate their problem as an integer program.

Hint: first formulate the problem allowing non-linear constraints and then attempt to make all the constraints linear.

Solution:

Preliminary Analysis:

The following points are noted from above problem:

- The company operates either a normal shift or an extended shift if it produces at all. So, in any month only one shift will be there. The company can also decide to stop production if it wants for any month.
- Cost of production when the company operates in normal shift = £100,000 per month
- Maximum stock that can be produced in a month when the company operates in normal shift = 5,000 units per month
- Cost of production when the company operates in extended shift = £180,000 per month
- Maximum stock that can be produced in a month when the company operates in extended shift = 7,500 units per month
- Extra costs for switching from normal to extended shift = £15,000
- Extra costs for switching from extended to normal shift = £0
- Carrying cost for stock units = £2 per unit per month
- Initial stock available with the company = 3,000 units

- Initial stock available has been produced by operating in a normal shift.
- The company wants at least 2000 units of stock left at the end of month 6.
- The company wants a production plan for the next six months that avoids stockouts.
- The estimated demand for the company's product in each month for the next 6 months is given:

Month	1	2	3	4	5	6
Demand	6,000	6,500	7,500	7,000	6,000	6,000

Solving the Problem

Identify Decision Variables

Decision Variables
Let Xi be the binary variable for i'th month representing whether the monthly production is operated as a normal shift Hence Xi = 1 if production is operated as a normal shift in the month i. = 0 Otherwise Where i = 1,2,3,4,5,6
Let Yi be the binary variable for i'th month representing whether the monthly production is operated as an extended shift Hence Yi = 1 if production is operated as an extended shift in the month i. = 0 Otherwise Where i = 1,2,3,4,5,6
Let Zi be the binary variable for i'th month representing whether there is a switch in mode of operations from normal shift in (i-1) month to extended shift in month i. Hence Zi = 1 if production mode of operation is switched from a normal shift in (i-1) month to an extended shift in the month i. = 0 Otherwise

Where i = 1,2,3,4,5,6
Let Pi be the variable representing total number of units produced in a month i irrespective of the shift. Where i = 1,2,3,4,5,6
Let Di be the variable representing demand in number of units in a month i. Where i = 1,2,3,4,5,6 This demand is provided in the problem as a table.
Let Ii be the variable representing total number of units of stock left at the end of month i after meeting the projected demand. This is the closing inventory for month i. Where i = 1,2,3,4,5,6
Let Ki be the binary variable for i'th month representing whether there is any production in the month i irrespective of shift operated. Hence Ki = 1 if production is there in the month i. = 0 Otherwise Where i = 1,2,3,4,5,6

Write Objective Function

We need to minimize the total costs for the firm in order to increase their profits.

There are few costs involved to the firm every month:

Fixed Cost involved in producing the required stock. It can be either 100000 if stock is produced in normal shift or 180000 if stock is produced in extended shift. As per given inputs, stock is produced in only one of the operating modes. Hence these are mutually exclusive events.

Variable costs involve extra cost of 15000 if in case there is a switch in the operating mode of production from normal to extended shift. And also, the cost involved in storing the stock for next month. As per given inputs this cost will be 2 per unit per month.

So, the total cost for the firm in month i will be

100000 Xi + 180000 Yi + 15000 Zi + 2 Ii

So, the total cost involved in the next 6 months will be sum of the costs in the 6months period.

That is SUM (100000 Xi + 180000 Yi + 15000 Zi + 2 Ii)

Where i = 1,2,3,4,5,6

So, the objective function will be as follows:

Minimize total cost = 100000 X1 + 180000 Y1 + 15000 Z1 + 2 I1 +
100000 X2 + 180000 Y2 + 15000 Z2 + 2 I2 +
100000 X3 + 180000 Y3 + 15000 Z3 + 2 I3 +
100000 X4 + 180000 Y4 + 15000 Z4 + 2 I4 +
100000 X5 + 180000 Y5 + 15000 Z5 + 2 I5 +
100000 X6 + 180000 Y6 + 15000 Z6 + 2 I6

Write Constraints

The above objective has to be achieved subject to the following constraints:

Shift Constraints:

As per given inputs, in the next six months the company must, in each month, operate either a normal shift or an extended shift. So, stock can be produced in only one of the operating modes. Hence these are mutually exclusive events.

So, when stock is produced in a normal shift in month i:

Xi will be equal to 1 and Yi will be 0.

Similarly, when stock is produced in an extended shift in month i:

Xi will be equal to 0 and Yi will be 1.

Hence maximum value of the sum of these two variables is 1 as both the variables cannot be 1 at the same time. If one of them is 1 the other will be 0 when there is production.

In the case when there is no production, both Xi and Yi will be zero as there won’t be any shift operated in that time.

Therefore, the following equations can be written that satisfy the above constraints.

$X1 + Y1 <= 1$

$X2 + Y2 <= 1$

$X3 + Y3 <= 1$

$X4 + Y4 <= 1$

$X5 + Y5 <= 1$

$X6 + Y6 <= 1$

Inventory Constraints:

As per given inputs, 3,000 units of stock produced by a normal shift is available with the company as Initial stock. The same can be written as below equation:

$I0 = 3000$ (From given inputs)

It is also given that at the end of month 6, at least 2000 units should be in stock. The same can be written as below equation:

$I6 >= 2000$

The company wants a production plan for the next six months that avoids stockouts. Stockout can be understood like a situation where there is no stock available with the company to meet that particular month's demand. Hence the closing stock will be positive number if there are extra units left after meeting the month's demand. The closing stock will be equal to zero if the sum of produced units and carried stock is equal to the demand in that particular month. This number will be negative if sum of produced stock and carried stock is less than the demand in a particular month, which is the stockout scenario. Hence this closing inventory should be always greater than or equal to zero. The same can be written as below equations:

$I1 >= 0$

$I2 >= 0$

$I3 >= 0$

$I4 >= 0$

$I5 >= 0$

$I6 >= 0$

Closing stock left with the company at the end of every month is the sum of opening stock and stock produced in that month less demand for stock in that month.

Hence the following equation can be written:

Closing stock at the end of month i = Opening stock for month i + Production in the month i - Demand met during the month i

$=> I_i = I_{i-1} + P_i - D_i$

Where i = 1,2,3,4,5,6

Di values can be taken from the below table which has been provided as input in the problem.

Month (i)	1	2	3	4	5	6
Demand (Di)	6,000	6,500	7,500	7,000	6,000	6,000

Formulating constraints for closing inventory in i'th month as below:

$I1 = I0 + P1 - D1 => I1 = 3000 + P1 - 6000 => I1 = P1 -3000$

$I2 = I1 + P2 - D2 => I2 = I1 + P2 - 6500$

$I3 = I2 + P3 - D3 => I3 = I2 + P3 - 7500$

$I4 = I3 + P4 - D4 => I4 = I3 + P4 - 7000$

$I5 = I4 + P5 - D5 => I5 = I4 + P5 - 6000$

$I6 = I5 + P6 - D6 => I6 = I5 + P6 - 6000$

The above equations are derived into below format for easy representation in solver.

$P1 - I1 = 3000$

$P2 + I1 - I2 = 6500$

$P3 + I2 - I3 = 7500$

$P4 + I3 - I4 = 7000$

$P5 + I4 - I5 = 6000$

$P6 + I5 - I6 = 6000$

Production Constraints:

As per given inputs the company can produce up to 5,000 units per month when operating in a normal shift and can produce up to 7,500 units per month when operating in an extended shift.

So, in a particular month of i, the maximum stock that can be produced will be less than or equal to 5000 or 7500 when operating in normal shift or extended shift respectively. As per the given inputs company will not operate both the shifts in same month. Hence these shift operations are mutually exclusive events. So, the maximum production in i'th month can be expressed as below equation:

$P_i <= 5000 X_i + 7500 Y_i$

Where i = 1,2,3,4,5,6

So, the production constraints for the next 6 months based on above information will be as follows:

$P1 <= 5000\ X1 + 7500\ Y1$

$P2 <= 5000\ X2 + 7500\ Y2$

$P3 <= 5000\ X3 + 7500\ Y3$

$P4 <= 5000\ X4 + 7500\ Y4$

$P5 <= 5000\ X5 + 7500\ Y5$

$P6 <= 5000\ X6 + 7500\ Y6$

Also, it has been mentioned that, if the company produces anything in a particular month it must produce at least 2,000 units. That means that either the company will not produce any thing in a month or it produces at least 2000 units of stock. So, this is an Either-Or case where either Pi = 0 or Pi >= 2000

Either Pi = 0
Or 2000 – Pi <= 0

Therefore, for this we can have
Pi <= M(Ki)
2000 - Pi <= M(1 – Ki)

Here M is a very large value and Ki is a binary variable representing if production is there or not in i'th month. The maximum stock that can be produced in any month irrespective of shift is 7500 as per given inputs. So, we take the M value as 7500 in this case.

$P_i <= 7500(K_i)$ $2000 - P_i <= 7500(1 - K_i)$ That implies $P_i - 7500 K_i >= -5500$ Where i = 1,2,3,4,5,6

So, the production constraints for the next 6 months based on above either-or equations will be as follows:

$P_1 <= 7500 K_1$ $P_2 <= 7500 K_2$ $P_3 <= 7500 K_3$ $P_4 <= 7500 K_4$ $P_5 <= 7500 K_5$ $P_6 <= 7500 K_6$ $P_1 - 7500 K_1 >= -5500$ $P_2 - 7500 K_2 >= -5500$ $P_3 - 7500 K_3 >= -5500$ $P_4 - 7500 K_4 >= -5500$ $P_5 - 7500 K_5 >= -5500$ $P_6 - 7500 K_6 >= -5500$

Shift Change Constraints:

As per given inputs there will be extra cost 15000 when the operation mode changes from a normal shift to extended shift. As per our definition of Z_i value is considered as 1 if production mode of operation is switched from a normal shift in (i-1) month to an extended shift in the month i.

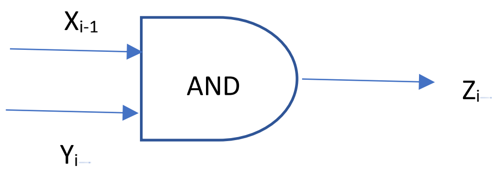
Hence $Z_i = 1$ When $X_{i-1} = 1$ and $Y_i = 1$

When we observe the values of X_{i-1}, Y_i and Z_i it will be like below:

X_{i-1}	Y_i	Z_i
0	0	0
0	1	0
1	0	0
1	1	1

This is AND gate where:

- Inputs are X_{i-1} and Y_i
- Output is Z_i



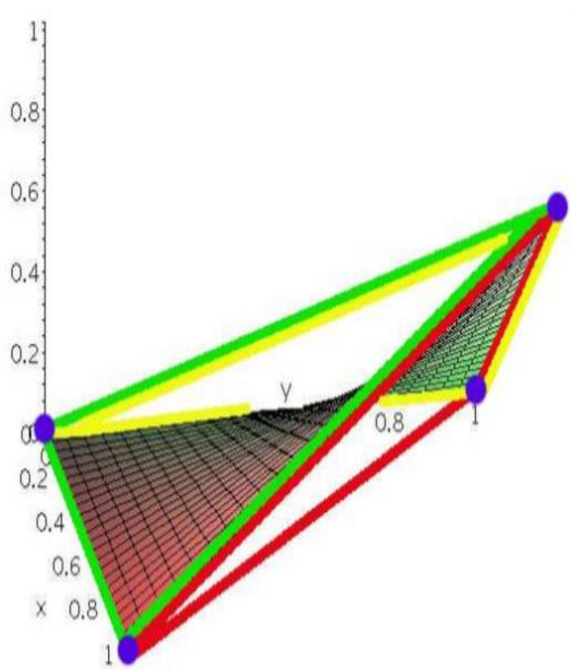
The logical expression for AND gate with two inputs can be represented as

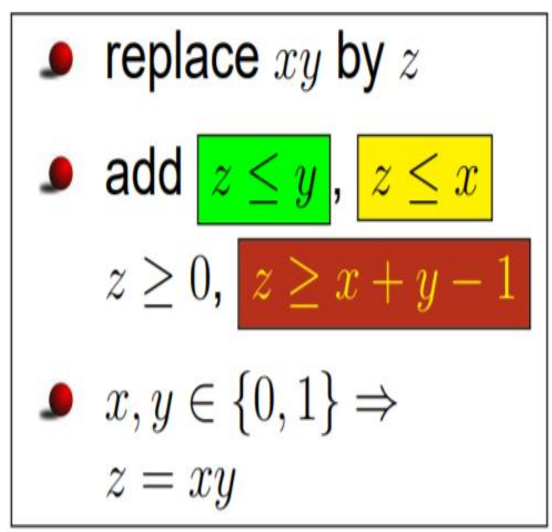
Output = Input 1 * Input 2

So, the below equation can be formulated that satisfies above conditions.

$Z_i = (X_{i-1}) * (Y_i)$
Where i = 1,2,3,4,5,6

But the above equation is a non-linear equation. So as per the given hint in the problem we will make this constraint in to linear by using standard trick. Hence the equivalent linear equations for the above non-linear constraint will be as follows:





$Z_i <= X_{i-1}$

$Z_i <= Y_i$

$Z_i >= X_{i-1} + Y_i - 1$

Where i = 1,2,3,4,5,6

The first two inequality equations ensure Z will be 0 if either X or Y is 0 and last one will take Z to 1 if X and Y are set as 1. So, the shift changes constraints for the next 6 months based on above equations will be as follows:

Z1 <= X0

Z1 <= Y1

Z1 >= X0 + Y1 – 1

Z2 <= X1

Z2 <= Y2

Z2 >= X1 + Y2 – 1

Z3 <= X2

Z3 <= Y3

Z3 >= X2 + Y3 – 1

Z4 <= X3

Z4 <= Y4

Z4 >= X3 + Y4 – 1

Z5 <= X4

Z5 <= Y5

Z5 >= X4 + Y5 – 1

Z6 <= X5

Z6 <= Y6

Z6 >= X5 + Y6 – 1

Solve the problem

By feeding the above equations in to Excel sheet and solving them, we can find out optimal values for decision variables and objective function.

The optimal values from solver are mentioned in the table below for reference.

	X (Stock produced in Normal Shift)	Y (Stock produced in Extended Shift)	Z (Change from Normal to Extended Shift)	K (Stock produced for the month or not)	P (Number of units produced in the month)	I (Number of units left after meeting the demand)
Month 1	1	0	0	1	4500	1500
Month 2	1	0	0	1	5000	0
Month 3	0	1	1	1	7500	0
Month 4	0	1	0	1	7000	0
Month 5	0	1	0	1	6500	500
Month 6	0	1	0	1	7500	2000
Minimum Cost for production in next 6 months	£ 943000					

By observing the values, we can understand the optimal production and operating mode required in each month.

Proposed Production Plan for next 6 months

By observing the values in above table, the following production plan can be suggested for the next six months in order to meet all the company's requirements.

There is no need to stop production in the next 6 months.

Month 1: Production can be operated in a normal shift. 4500 units must be produced in order to meet the month's demand.

Month 2: Production can be operated in a normal shift. 5000 units must be produced in order to meet the month's demand.

Month 3: Change from normal to extended shift is required in this month. Production must be operated in an extended shift. 7500 units must be produced in order to meet the month's demand. The maximum production capacity must be utilized in this month in order to meet the demands.

Month 4: Production must be operated in an extended shift. Continue with extended shift. 7000 units must be produced in order to meet the month's demand.

Month 5: Production must be operated in an extended shift. Continue with extended shift. 6500 units must be produced in order to meet the month's demand.

Month 6: Production must be operated in an extended shift. Continue with extended shift. 7500 units must be produced in order to meet the month's demand. The maximum production capacity must be utilized in this month in order to meet the company's requirements. Desired closing stock of 2000 units is possible at the end of this month by following the proposed production plan.

The above plan can be followed to meet the company's requirements with minimum cost of £ 943000 in the next 6 months.