Discrete signal processing on graphs: Sampling theory

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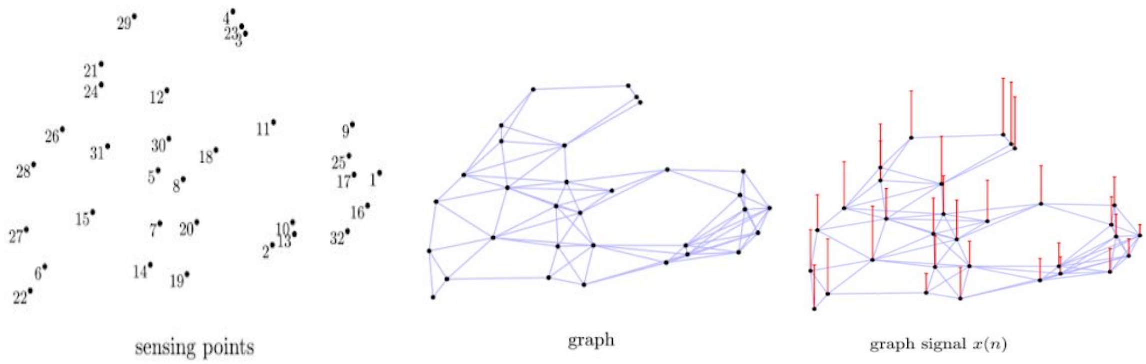
1 Abstract

This project proposes sampling theory for graph signals that are supported on directed or undirected graphs. With the proposed sampling theory, perfect recovery is possible for the bandlimited graph signals from its sampled signal. Designing the sampling operator using optimal sampling operator and random sampling operator for different graphs. We will show how to handle full band graph signals by using graph filter banks.

2 Introduction

2.1 Graph signal processing

In numerous practical cases the signal domain is not a set of equidistant instants in time or a set of points in space on a regular grid. The data sensing domain is kind of irregular and in some cases they are not related to time or space like many social media networks. Graph signal processing deals with the signals whose domain, defined by a graph is irregular. Graph signal processing is a useful tool for representing analyzing, and processing the signal lying on a graph. Spectral decomposition and sampling on graphs helps in understanding the variations in the graphs and compressing the large data.



2.1.1 Few concepts in graph signal processing

Graph: A graph consists of a set of vertices and edges. The set of vertices are used to define the graph signal, and edges describe the connections between those vertices.

Graph shift: The connections between the nodes can be represented using a weighted adjacency matrix.

Graph Signal: A graph signal is defined as the map on graph nodes that assigns the signal coefficients to the nodes in the graph. $x = [x_0\ x_1\ x_2 \ldots x_n]^T$.

2.1.2 Graph fourier transform:

The graph fourier transform can be calculated using the graph laplacian matrix or adjacency matrix. The connections in the graphs can be represented using the adjacency matrix. So we can decompose the graph shift A based on eigenvalues and vectors. The set of the adjacency matrix eigenvalues is called the graph adjacency spectrum.

$A = V \Lambda V^{-1}$, where V consists of eigenvectors of A and Λ is a diagonal matrix with diagonal elements has the eigenvalues of A.

Graph fourier transform :

$$\hat{x} = V^{-1} x.$$

Inverse graph fourier transform:

$$x = V\hat{x}$$

2.2 Sampling on Graphs and properties of sampled graph signal:

Background:

In the previous works, sampling on graphs is based on spectral graph theory. They defined the bandwidth based on the value of graph frequencies. They used graph laplacian matrices for doing sampling and finding the graph fourier transform.

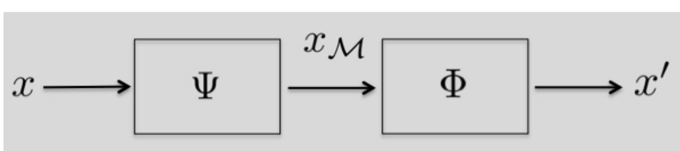
The disadvantages of this technique is that every has its own graph frequencies and so it is hard to define the general cutoff frequency. Also for generating the cutoff frequency we need to compute all graph frequencies which is computationally inefficient for large graphs.

Solution:

Here we define a graph signal is bandlimited if the graph fourier transform coefficients are zero after a finite coefficients. $x\hat{}$ = 0 for all k $\geq K$ where K $\in$ {0,1,..N-1}.

The smallest such K is the bandwidth of the graph signal x and if no such k exists then it is called full-band graph signal. The advantage of this is that we are considering the discrete nature of the graph frequencies and gives us a good intuition of understanding the graph signals. By defining the bandwidth of graph signals as above establishes a strong connection to linear algebra.

2.2.1 Sampling & Interpolation:



Let's say we have a graph signal 'x' with graph shift A (N x N matrix) and we want to sample *M* coefficients. Let the sampled signal is x_M which is obtained after sampling. Let ***M*** = *(*M_0*,* M_1*, ...* M_{M-1}*)* is the sequences of sampled indices. Once we have the sequence the sampling operator is defined as follows to obtain the sampled signal.

$$\Psi_{i,j} = \begin{cases} 1, & j = \mathcal{M}_i; \\ 0, & \text{otherwise}, \end{cases}$$

is an M x N matrix.

The interpolating operator is used to reconstruct original signal back, say Φ is an N x M matrix, then

Sampling: $x_M = \Psi x$ Interpolation: $x' = \Phi x_M = \Phi\Psi\, x$.

When $x' = x$, we will say that perfect recovery is obtained and it is possible when $\Phi\Psi$ is an identity matrix. This is not possible in general that the rank($\Phi\Psi$) $\leq$ M < N but for the bandlimited graph signals we can get $\Phi\Psi$ is an identity matrix by choosing the appropriate interpolation operator.

The interpolating operator is $\Phi = V_{(k)} (\Psi V_{(k)})^{-1}$

where $V_{(k)}$ is the first k columns of the eigenvector matrix.

By choosing k linearly independent rows in $V_{(k)}$ we can have $\Phi\Psi$ is an identity matrix and it is clear that any arbitrary sampling operator will not give perfect recovery only those sampling operators which gives perfect recovery are known as qualified sampling operator.

2.2.2 Sampled signal & Property of a sampled graph signal:

Let x_M is the sampled signal of x and the graph shift A_M associated with it can be obtained from the graph shift A of signal x as

$$\mathrm{A}_{\mathcal{M}} = \mathrm{U}^{-1}\, \Lambda_{(K)}\, \mathrm{U} \in \mathbb{C}^{K \times K}$$

where $U = (\Psi V_{(k)})^{-1}$

Graph fourier transform of x_M , $\hat{x}_M = Ux_M$ IGFT: $x_M = U^{-1}\hat{x}_M$.

The property of a sampled signal is that it preserves the first order difference between the sampled signal and the original signal. I.e.,

$$x_{\mathcal{M}} - \mathrm{A}_{\mathcal{M}}\, x_{\mathcal{M}} \;=\; \Psi\,(x - \mathrm{A}\, x)$$

3 Different strategies for sampling on graphs

We now know that only a qualified sampling operator leads to perfect recovery for bandlimited graph signals.For finding an optimal sampling operator we will select k linearly independent rows from $V_{(k)}$, when multiple such sets of k exist, we will choose the set noise k which helps in minimizing the noise. For some graphs random sampling also leads to perfect recovery.

3.1 Experimentally Designed Sampling

We will design an optimal sampling operator on any graph which is kind of robust to noise.

3.1.1 Optimal Sampling operator:

We know that at least one set of K linearly-independent rows in $V_{(K)}$ always exists. When we have multiple choices of K linearly-independent rows, we aim to find the optimal one to minimize the effect of noise.

Let the sampled signal is $x_{\mathcal{M}} = \Psi x + e,$ where Ψ is a qualified sampling operator and e is noise.

Then recovered signal x'_e

$$x'_e = \Phi x_{\mathcal{M}} = \Phi \Psi x + \Phi e = x + \Phi e.$$

To bound the noise, we have

$$\begin{aligned} \left\| x' - x \right\|_2 &= \left\| \Phi e \right\|_2 = \left\| \mathrm{V}_{(K)} \, \mathrm{U} \, e \right\|_2 \\ &\leq \left\| \mathrm{V}_{(K)} \right\|_2 \left\| \mathrm{U} \right\|_2 \left\| e \right\|_2 , \end{aligned}$$

Here e and $v_{(k)}$ are already fixed.

The only parameter we can minimize is U = Ψ $V_{(k)}$ the best choice comes from the U with the smallest spectral norm. This is equivalent to maximizing the smallest singular value of Ψ $V_{(k)}$,

$$\Psi^{opt} = \arg\max_{\Psi} \sigma_{\min}(\Psi \, \mathrm{V}_{(K)}),$$

where σ_{min} denotes the smallest singular value. Note that M is the sampling sequence, indicating which rows to select, and $(V_{(K)})_M$ denotes the sampled rows from $V_{(K)}$. When increasing the number of samples, the smallest singular value of Ψ $V_{(K)}$ grows, and thus, redundant samples make the algorithm robust to noise.

3.2 Random Sampling operator

Random sampling operator means that we can select any k rows in $V_{(k)}$ and do the sampling and interpolation on the graph signal. For some specific graphs, a random sampling operator leads to perfect recovery with high probabilities.

3.2.1 Frames with Maximal robustness to erasures:

A frame $\{f_0, f_1, \ldots f_{N-1}\}$ is a generating system for C^k with N>=k, when there exist two constants a > 0 and b < infinity, such that for all x belongs C^N.

$$a\,\|x\|^2 \le \sum_k |f_k^T x|^2 \le b\,\|x\|^2\,.$$

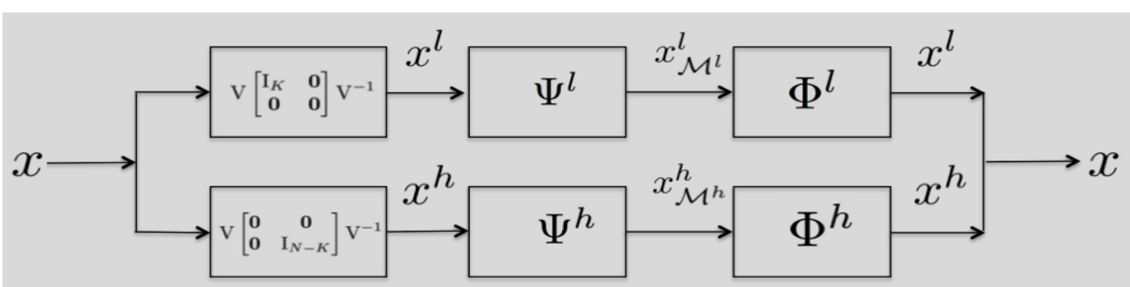
The frame is said to be maximally robust to erasures when every K x K submatrix is invertible. Few ex: polynomial transformation matrix and circulant adjacency matrix. When the inverse graph fourier transform matrix V is maximally robust to erasures then any sampling operator that samples at least K signal coefficients guarantees perfect recovery. The circulant matrix is a polynomial transform matrix .Since the graph Fourier transform matrix of a circulant graph is the discrete Fourier transform matrix, we can perfectly recover a circulant-graph signal with bandwidth K by sampling any M >=K signal coefficients.

3.2.2 Erdos-Renyi graph

This graph is constructed by connecting the nodes randomly, where every node has a probability p to connect to every other node independently. For this types of graph the probability that the spectral norm of the difference between the U^*U^T and the identity matrix is always <= ½ with very high probability. This shows that singular values of $\Psi V_{(k)}$ are bounded with high probability which means there is a high chance to get a full rank for $\Psi V_{(k)}$ matrix. So, perfect recovery is possible with high probabilities.

3.3 Graph filter Banks

Graph filter banks are very useful in sampling the full band graph signals. Here we basically divide the big graph into smaller ones so that the smaller ones become bandlimited signals. Then we can do sampling on the smaller signals individually, and finally, we can combine them to obtain our original signal back. Here once we have smaller graph signals, we will try to use a family of qualified sampling operators on the smaller graphs.



By downsampling a graph by 2, one can set the bandwidth to a half of the number of nodes, that is, K = N/2, and use it to obtain an optimal sampling operator. Perfect recovery is achieved when graph signals are bandlimited.

Let x be a full-band graph signal, which, without loss of generality, we can express without loss of generality as the addition of two bandlimited signals supported on the same graph, that is,

$x = x^l + x^h$ where $x^l = P^l x$ and $x^h = P^h x$

$$P^l = V \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix} V^{-1}, \quad P^h = = V \begin{bmatrix} 0 & 0 \\ 0 & I_{N-K} \end{bmatrix} V^{-1}.$$

We see that x^l contains the first K frequencies, contains the other N −K frequencies, and each is bandlimited. We do sampling and interpolation for and in two channels, respectively. By dividing and doing the sampling and interpolation on individual signals, and finally combining the resulting signals we can reconstruct our original signal back. This idea is very helpful in dealing with huge graphs which makes computations easier.

4 Simulations

4.1 Sampling on graphs

We designed an algorithm for doing sampling on graph signals by taking an adjacency matrix and the graph signal as inputs and gives the minimum number of samples needed to get perfect reconstruction and the corresponding sampling operator and interpolating operator. This algorithm tries to find the optimal sampling operator corresponding to the graph and do the sampling.

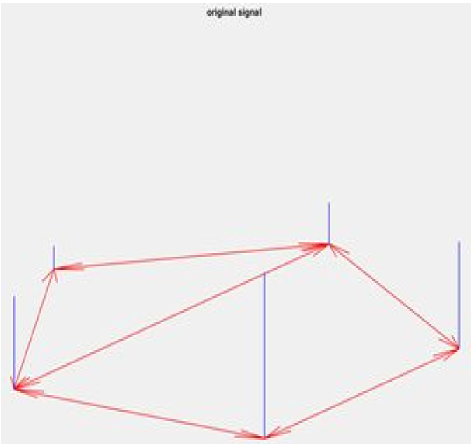
EX: Adjacency matrix taken
[[0,2/5,2/5,0,1/5];[2/3,0,1/3,0,0];[1/2,1/4,0,1/4,0];[0,0,1/2,0,1/2];[1/2,0,0,1/2,0]]

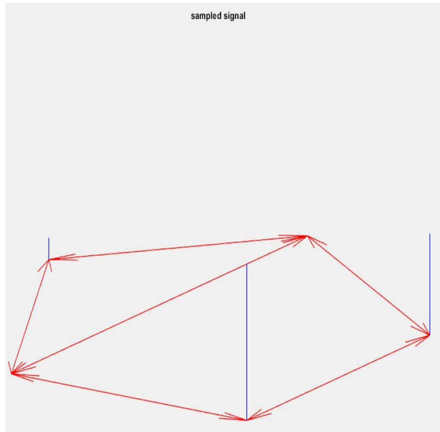
Graph signal - x = [8.4153;4.7259;18.9719;33.7691;21.8021] is bandlimited with bandwidth = 3

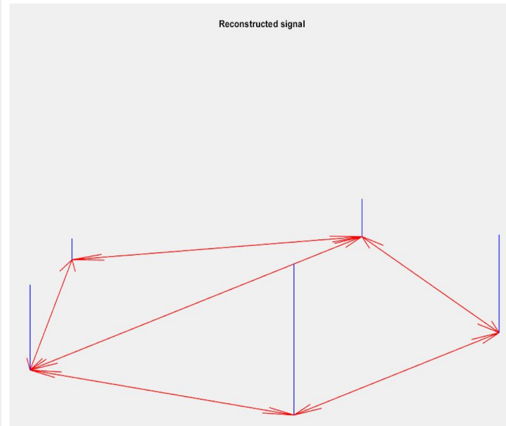
Original signal

Sampled signal

Reconstructed signal







4.2 Optimal sampling operator Vs Some qualified sampling operator in presence of noise

Algorithm 1 Optimal Sampling Operator via Greedy Algorithm

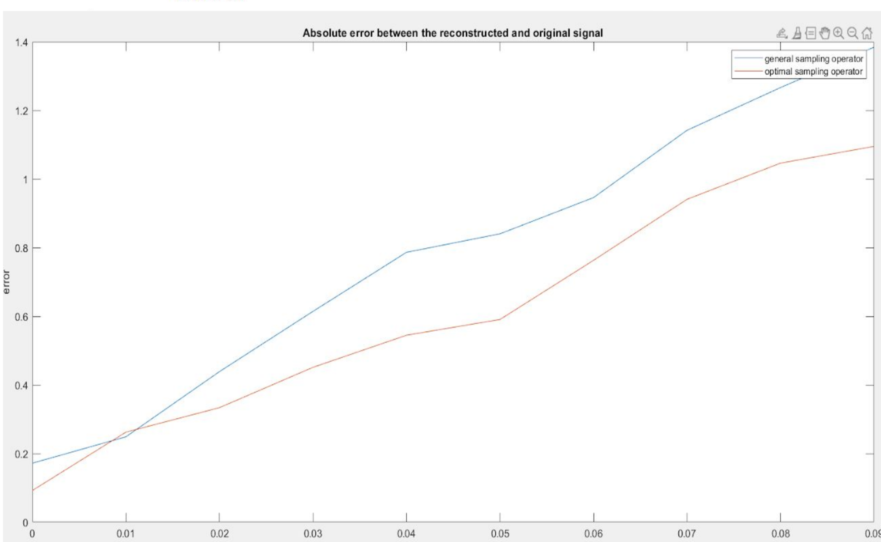
Input $V_{(K)}$ the first K columns of V
M the number of samples

Output $\mathcal{M}$ sampling set

Function

while $|\mathcal{M}| < M$
$m = \arg\max_i \sigma_{\min}\left((V_{(K)})_{\mathcal{M}+\{i\}}\right)$
$\mathcal{M} \leftarrow \mathcal{M} + \{m\}$
end
return $\mathcal{M}$

This is the pseudo algorithm for designing the optimal sampling operator.

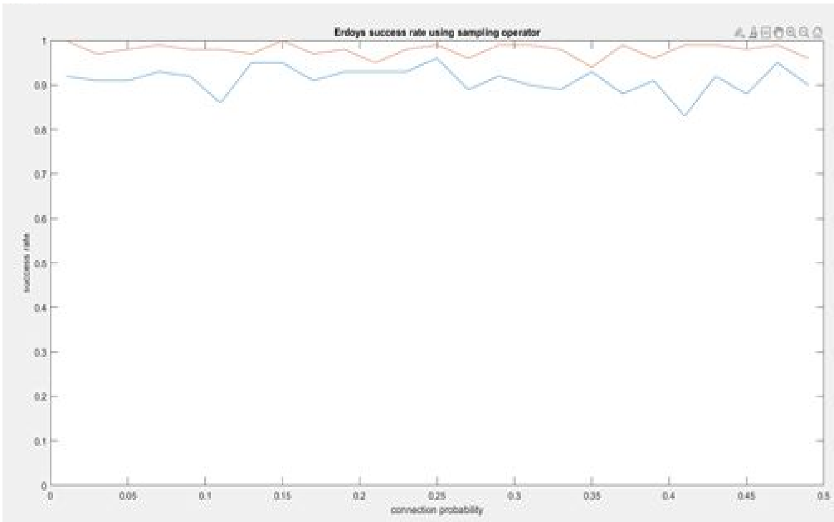


This is the plot when we sample a given signal of N = 5 and bandwidth 3 by introducing some random noise.

From the plot we can clearly see that the optimal sampling operator is more robust to noise compared to the some qualified sampling operator. If we simulate multiple graphs we can observe that optimal sampling is more robust to noise then random sampling.

4.3 Random Sampling operator

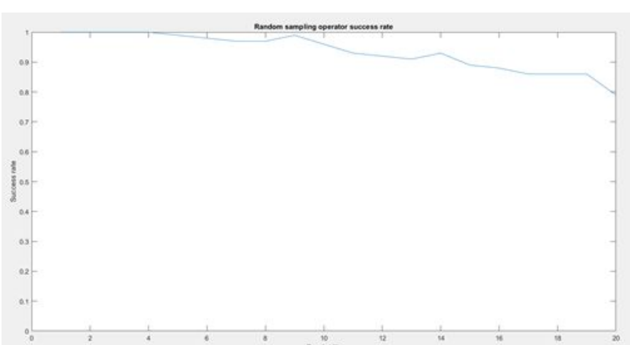
Here we will do sampling for erdos-renyi graphs by using a random sampling operator.



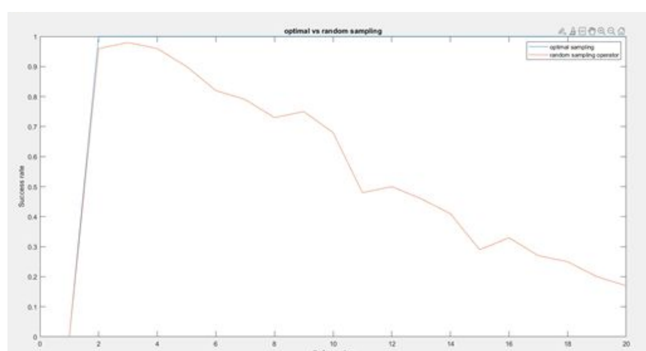
Here we took N=100 (Blue line in graph) and N = 500 (orange) and generated some random graphs. Now we randomly sampled any 5 rows and check whether the obtained 5x5 matrix is a full rank matrix or not. If it is fully ranked then we can recover our signal perfectly. We did it for different connection probabilities of erdos-renyi and found the success rate by doing 100 random tests for each connection probability. We can see that the success rate is above 90% and as connection probability increases is increasing slightly. So, we can say that perfect recovery for erdos-renyi type has high success rates by random sampling.

4.4 Success rates of optimal and random sampling operator

Random sampling success rate (a)



Optimal vs random sampling operator (b)



In the (a) figure we chose a number of vertices N = 1000 and generated some random adjacency matrix. We now start randomly sampling bandwidth B rows from the $V_{(B)}$ matrix and check whether the obtained BxB is a full rank matrix or not. If it is full ranked perfect recovery is possible. We tested 100 times for each bandwidth B and measured the success rate. We can observe that the success rate is decreasing as B increases because when we take more samples it is more probable to get previously sampled rows.

In the (b) figure we chose N = 100 and generated an adjacency matrix. We also created a bandlimited signal of bandwidth 2. We now started taking more and more samples during the sampling and try to do sampling and interpolation using random and optimal sampling operators. We see that the random sampling operator fails as we take more and more samples but optimal sampling still gives perfect recovery because the interpolation assumed is always a qualified sampling operator which is not the case for random sampling operator.

5 Conclusion:

From this project we learnt a few concepts of graph signal processing. We learnt about graph signals and computing graph fourier transform in different ways. We came to know that classical discrete signal processing is the special case of graph signal processing. We learnt a way of sampling on graphs. We also came to know the usefulness of compressed sensing and graph filter banks in graph signal processing.

We learnt sampling on graphs using the MATLAB and how to visualise the graph signals. Perfect reconstruction for graph signals is possible for bandlimited signals with fewer samples by choosing an optimal sampling operator. We observed a random sampling operator gives perfect recovery for specific graphs with high probabilities. We observed that the optimal sampling operator is more suitable for perfect recovery in any case and it is robust to noise compared to some other qualified sampling operator.

6 References:

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