1. Read Section 2.8. State that you've read Section 2.8 or part of Section 2.8 and give yourself a score out of 5 based on how much you have read.

   I read Section 2.8; 5/5.

2. (2.8.1) Let $S^1 \subset \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$ be the standard circle. Let $D = \{(0,0,t) \mid -2 \leq t \leq 2\}$ and $S^2(2) = \left\{x \in \mathbb{R}^3 \mid \|x\| = 2\right\}$. Then $S^2(2) \cup D$ is a deformation retract of $X = \mathbb{R}^3 \setminus S^1$. The space X is h-equivalent to $S^2 \vee S^1$.

   Proof. We describe the deformation retraction via the following pictures:

   For each $\theta \in [0, 2\pi)$ we specify the deformation retract within the closed half plane H_θ^+ given in green above. Points on $S^2(2) \cup D$ remain fixed throughout the homotopy. For points outside of $S^2(2)$, we follow the standard deformation retract and draw a line from such points to the origin and drag them along via the straight line homotopy until they hit $S^2(2)$. For points inside the sphere minus the point p where S^1 intersected H_θ^+, draw a line from p towards such points and drag those points along the straight line homotopy until they hit either D or $S^2(2)$ and map them there.

   To see that X is h-equivalent to the wedge of S^2 and S^1 we use the fact that deformation retracts are h-equivalences and that composing h-equivalences are still h-equivalences. So follow the deformation retract outlined earlier by the homotopy which drags the top point of D to its bottom point, outlined in the below pictures: $\square$

3. (2.8.4) Let i_{0*} in (2.6.2) be an isomorphism. Then j_{1*} is an isomorphism. This statement is a general formal property of pushouts. If i_{0*} is surjective, then j_{1*} is surjective.

Proof. We prove the above statements for pushouts in general; that is, that pushouts of isomorphisms (epimorphisms) are isomorphisms (epimorphisms). We start with the following pushout:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "j_1"] \\
Y \arrow[r, "j_2"'] & P
\end{tikzcd}$$

For the first, observe that when g is an isomorphism the following diagram commutes, using the universal property of pushouts:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "j_1"] \arrow[ddr, bend left, "\mathrm{id}_X"] & \\
Y \arrow[r, "j_2"'] \arrow[drr, bend right, "f\circ g^{-1}"'] & P \arrow[dr, dashed, "u"] \arrow[ddr, bend left, "\mathrm{id}_P"] & \\
& & X \arrow[d, "j_1"] \\
& & P
\end{tikzcd}$$

It follows that j_1 has left and right inverses, so it is an isomorphism.

For the second, assume g is an epimorphism and let $h_1, h_2 \colon P \to P'$ be morphisms such that $h_1 j_1 = h_2 j_1$. Then $h_1 j_1 f = h_1 j_2 g = h_2 j_2 g = h_2 j_1 f$, and since g is an epimorphism, $h_1 j_2 = h_2 j_2$. Now apply the universal property of the pushout to obtain the diagram

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "j_1"] \arrow[dr, bend left, "hj_1=h_1j_1=h_2j_1"] & \\
Y \arrow[r, "j_2"'] \arrow[rr, bend right, "hj_2=h_1j_2=h_2j_2"'] & P \arrow[r, dashed, "h"] & P'
\end{tikzcd}$$

where h is unique so that $h_1 = h_2 = h$ as desired. It follows that j_1 is an epimorphism.

Since (2.6.2) is a pushout the above is true for j_{1*} whenever i_{0*} is an isomorphism (epimorphism). $\square$

4. (2.8.5, last statement) ... we obtain $\pi_1(P^2) \cong \mathbb{Z}/2$.

Proof. We use the following pushout diagram which tells us we can obtain P^2 from $S^1 \cong P^1$ by attaching a

2-cell:

$$\begin{tikzcd} S^1 \arrow[r, "\varphi"] \arrow[d, "j"] & P^1 \arrow[d, "J"] \\ D^2 \arrow[r, "\Phi"] & P^2 \end{tikzcd}$$

We use the Seifert-van Kampen theorem due to the discussion in (2.8.10), and so we obtain the following diagram:

$$\begin{tikzcd} \pi_1(S^1) \cong \mathbb{Z} \arrow[r, "\varphi_*"] \arrow[d, "j_*"] & \pi_1(P^1) \cong \mathbb{Z} \arrow[d, "J_*"] \\ \pi_1(D^2) \cong 1 \arrow[r, "\Phi_*"] & \pi_1(P^2) \end{tikzcd}$$

and so the fundamental group of P^2 is isomorphic to $\pi_1(P^1)/\langle\varphi\rangle$ where $\langle\varphi\rangle$ denotes the normal subgroup generated by the image of φ_*. In terms of generators and relations, we take the amalgamated free product of $\langle a\rangle = \pi_1(S)$ and 1 to obtain $\langle a \mid \varphi_*(a) = e\rangle$. We deduce $\varphi_*(a)$ by taking a generator of S^1 (the identity $e^{i\theta} \mapsto e^{i\theta}$, a loop) and seeing that its image under φ is $[\cos(\theta), \sin(\theta)]$, but to go back to S^1 (from which we obtained the fundamental group) we apply the specified homeomorphism (given in problem statement) to see that $[\cos(\theta), \sin(\theta)]$ maps to $e^{i(2\theta)}$. (If $\theta \in [0, \pi)$ then there is no need to choose a representative; otherwise take the representative to be $[\cos(\theta - \pi), \sin(\theta - \pi)]$ and map this to $e^{i(2[\theta-\pi])} = e^{i(2\theta)}$) It follows that in the fundamental groups, $a \mapsto a^2$ so that $\pi_1(P^2) = \langle a \mid a^2 = e\rangle$, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. □

5. (2.8.6) The Klein bottle K can be obtained from two Möbius bands M by an identification of their boundary curves with a homeomorphism, $K = M \cup_{\partial M} M$.

Apply the theorem of Seifert and van Kampen and obtain the presentation $\pi_1(K) = \langle a, b \mid a^2 = b^2\rangle$. The elements a^2, ab generate a free abelian subgroup of rank 2 and of index 2 in the fundamental group. The element a^2 generates the center of this group, it is represented by the central loop ∂M. The quotient by the center is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$.

Proof. To show that the fundamental group of the Klein bottle K is given by $G = \pi_1(K) = \langle a, b \mid a^2 = b^2\rangle$, we first view K as the pushout of two inclusions of ∂M into M:

$$\begin{tikzcd} \partial M \arrow[r, hook] \arrow[d, hook] & M \arrow[d] \\ M \arrow[r] & M \cup_{\partial M} M \end{tikzcd}$$

With ∂M homeomorphic to S^1 and M homotopic to S^1, we apply Seifert-van Kampen and obtain another pushout:

$$\begin{tikzcd}
\pi_1(\partial M) \cong \langle c \rangle \arrow[r] \arrow[d] & \pi_1(M) \cong \langle b \rangle \arrow[d] \\
\pi_1(M) \cong \langle a \rangle \arrow[r] & \pi_1(M \cup_{\partial M} M)
\end{tikzcd}$$

The generator c represents going along the boundary of M, which amounts to going around the center line of M twice. As a result the image of c under the induced maps from $\pi_1(\partial M)$ into $\pi_1(M)$ are a^2 and b^2. Hence the fundamental group of $K = M \cup_{\partial M} M$ is presented as $\langle a, b \mid a^2 b^{-2} = e \rangle$ as desired.

The subgroup H generated by a^2, ab is torsion free because the only relation imposed on these generators is that $a^2 = b^2$, which could not cause a finite word to be the neutral element. To show commutativity, it suffices to show it for the generators: $a^2 ab = aa^2 b = ab^2 b = abb^2 = aba^2$. It follows that H is free and abelian. Observe also that since H is generated by $a^2 = b^2$ and ab, it follows that H contains only the words of even length in G, and exactly those (if $g \in G$ has even length, then by inserting in an even number of symbols and using the above identities to collect g into a product of generators of H, we find that $g \in H$.) Then the remaining words of G of even length may be obtained by prepending a to elements of H. Hence H, aH are the only two cosets of H in G, so H has index 2 in G.

The center Z is given by elements which commute with every element of G, in particular with the generators of G. Observe that $ab \neq ba$, so that if an element p were to commute with a or b, it must not be a or b. But observe that $a^2 = b^2$ will commute with a and b. So an element p commutes with a and b if it is the product of finitely many a^2. If p is of odd length then at some point p will cease to commute with a and b. Hence $Z = \langle a^2 \rangle$. In terms of generators and relations, G/Z is given by $\langle a, b \mid a^2 = b^2 = e \rangle$, which is isomorphic to any presentation of $\mathbb{Z}/2 * \mathbb{Z}/2$. □

The space $M/\partial M$ is homeomorphic to the projective plane P^2. If we identify the central ∂M to a point, we obtain a map $q \colon K = M \cup_{\partial M} M \to P^2 \vee P^2$. The induced map on the fundamental group is the homomorphism onto $\mathbb{Z}/2 * \mathbb{Z}/2$.

Proof. Pictographically, we take M and identify its boundary to a point, and see that we obtain a sphere with antipodal points identified, which is the definition of P^2:

Then in K if we identify the central ∂M to a point, it is the same as taking two Möbius strips M and quotienting out by ∂M, and then taking their wedge at the point ∂M was identified with. Using the previous result, it follows that quotienting out the central ∂M from K yields a space homeomorphic to $P^2 \vee P^2$; thus there is a map $q\colon K = M \cup_{\partial M} M \to P^2 \vee P^2$ which is the quotient map for the above composed with the appropriate homeomorphism, still a quotient map of spaces (it is surjective). Thus by Seifert-van Kampen, it follows that the induced homomorphism of fundamental groups is also surjective, and concretely the effect is to quotient out by the center Z of G. So $q_*\colon G \to G/Z \cong \mathbb{Z}/2 * \mathbb{Z}/2$ is the (surjective) quotient map of groups. $\square$