Let R be a commutative ring with 1.

1. (DF7.2.3) Define the set $R[[x]]$ of *formal power series* in the indeterminate x with coefficients from R to be all formal infinite sums
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
Define addition and multiplication of power series in the same way as for power series with real or complex coefficients i.e., extend polynomial addition and multiplication to power series as though they were "polynomials of infinite degree". (The term "formal" is used here to indicate that convergence is not considered, so that formal power series need not represent functions on R.)

   (a) Prove that $R[[x]]$ is a commutative ring with 1.

   Proof. The set $R[[x]]$ is an abelian group under addition since R is an abelian group under addition. The zero element is the zero power series (so every coefficient is 0_R). The addition on $R[[x]]$ is given by
$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$
from which it is easy to see that the addition given is a binary operation on $R[[x]]$ (the set is closed under this operation), and the addition is commutative since R is an abelian group under addition:
$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n = \sum_{n=0}^{\infty} (b_n + a_n) x^n = \sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} a_n x^n.$$
The addition is also associative since R is an additive group:
$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n\right) + \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} (a_n + b_n) x^n + \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} [(a_n + b_n) + c_n] x^n \\ &= \sum_{n=0}^{\infty} [a_n + (b_n + c_n)] x^n \\ &= \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} (b_n + c_n) x^n \\ &= \sum_{n=0}^{\infty} a_n x^n + \left(\sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} c_n x^n\right).\end{aligned}$$
Additive inverses are taken by taking additive inverses in R; that is,
$$-\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} (-a_n) x^n.$$

It follows that $R[[x]]$ is an additive abelian group. We now check that the multiplication on $R[[x]]$ given by

$$\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

makes $R[[x]]$ into a commutative ring with 1. It is clear that the multiplication is a binary operation on $R[[x]]$ (closure), and that the identity element $1_{R[[x]]}$ is the power series $\sum_{n=0}^{\infty} d_n x^n$ where $d_0 = 1$ and $d_n = 0$ for all $n \geq 1$ (i.e. just the constant term 1). The multiplication is associative:

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n \right) \times \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \times \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \left(\sum_{k=0}^{i} a_k b_{i-k} \right) c_{n-i} \right] x^n,
\end{aligned}$$

and the expansion of $\left[\sum_{i=0}^{n} \left(\sum_{k=0}^{i} a_k b_{i-k} \right) c_{n-i} \right]$ (expanding into rows) is

$$\begin{aligned}
& a_0 b_0 c_n + \\
& a_0 b_1 c_{n-1} + a_1 b_0 c_{n-1} + \\
& a_0 b_2 c_{n-2} + a_1 b_1 c_{n-2} + a_2 b_0 c_{n-2} \\
& \qquad\qquad\qquad + \\
& \qquad\qquad\qquad \vdots \\
& \qquad\qquad\qquad + \\
& a_0 b_n c_0 + a_1 b_{n-1} c_0 + a_2 b_{n-2} c_0 + \cdots + a_n b_0 c_0.
\end{aligned}$$

By rewriting this sum we have $\left[\sum_{i=0}^{n} a_i \left(\sum_{k=0}^{n-i} b_k c_{n-i-k} \right) \right]$ (this sum is interpreted as the columns of the above summation; similar to a double counting method). It follows that

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n \right) \times \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \left(\sum_{k=0}^{i} a_k b_{i-k} \right) c_{n-i} \right] x^n \\
&= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} a_i \left(\sum_{k=0}^{n-i} b_k c_{n-i-k} \right) \right] x^n \\
&= \sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k c_{n-k} \right) x^n \\
&= \sum_{n=0}^{\infty} a_n x^n \times \left(\sum_{n=0}^{\infty} b_n x^n \times \sum_{n=0}^{\infty} c_n x^n \right)
\end{aligned}$$

as desired, so that multiplication of formal power series is associative. The multiplication is commutative:

$$\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n-k} a_k \right) x^n,$$

and

$$\sum_{k=0}^{n} b_{n-k} a_k = (b_n a_0 + b_{n-1} a_1 + \cdots + b_1 a_{n-1} + b_0 a_n) = \sum_{i=0}^{n} b_i a_{n-i}$$

so that

$$\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n-k} a_k \right) x^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} b_i a_{n-i} \right) x^n = \sum_{n=0}^{\infty} b_n x^n \times \sum_{n=0}^{\infty} a_n x^n$$

as desired. With commutativity, we only need to check that multiplication distributes from one direction (e.g. from the right):

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \right) \times \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty} (a_n + b_n) x^n \times \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (a_k + b_k) c_{n-k} \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k c_{n-k} + \sum_{k=0}^{n} b_k c_{n-k} \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k c_{n-k} \right) x^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k c_{n-k} \right) x^n \\
&= \left(\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} c_n x^n \right) + \left(\sum_{n=0}^{\infty} b_n x^n \times \sum_{n=0}^{\infty} c_n x^n \right)
\end{aligned}$$

It follows that $R[[x]]$ is a commutative ring with 1. □

(b) Show that $1 - x$ is a unit in $R[[x]]$ with inverse $1 + x + x^2 + \cdots$.

Proof. We use the distributivity and commutativity of the multiplication on $R[[x]]$:

$$\begin{aligned}
(1 - x) \times \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} x^n + (-x) \times \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-1) x^{n+1} \\
&= (1 + x + x^2 + \cdots) + (0 + (-1)x + (-1)x^2 + \cdots) = 1.
\end{aligned}$$

Thus $1 - x$ and $1 + x + x^2 + \cdots$ are inverses of each other; it follows that $1 - x$ is a unit in $R[[x]]$. □

(c) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R. Remark: There is probably a really slick way to define the inverse of a power series whose constant term is nonzero using an idea from part (b) but I do not know how to do that.

Proof. Let $\sum_{n=0}^{\infty} a_n x^n$ be an element of $R[[x]]$.

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$. Then there is an inverse element $\sum_{n=0}^{\infty} b_n x^n$ such that $\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n = 1$. But due to the definition of the multiplication on $R[[x]]$, the coefficient of the constant term x^0 is given by $a_0 b_0$, so that $a_0 b_0 = 1$. (All of the other coefficients vanish, but they do not matter here.) It follows that a_0 is a unit in R since there is an element $b_0 \in R$ such that $a_0 b_0 = 1$.

Conversely, for $\sum_{n=0}^{\infty} a_n x^n$, suppose that a_0 is a unit in R. Then we can define an element $\sum_{n=0}^{\infty} b_n x^n$ such that $\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n = 1$, by defining each b_n recursively.

Define b_0 to be a_0^{-1}, since a_0 is a unit in R. Then by the definition of the multiplication on $R[[x]]$, the constant term in $\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n$ is given by $a_0 b_0 = a_0 a_0^{-1} = 1$. We want to define the remaining b_n so that the other coefficients vanish. We can start with the coefficient of x^1, given by $(a_0 b_1 + a_1 b_0)$. This term needs to be equal to zero, but we are given a_0, b_0, a_1. Rearranging yields that $b_1 = -a_0^{-1}(a_1 b_0) = -b_0(a_1 b_0)$.

In a similar manner, the coefficient of the x^n term is given by $(a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)$, which we want to be equal to zero. Thus $b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0) = -b_0(a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)$.

So by defining $b_0 = a_0^{-1}$ and $b_n = -b_0(a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)$ for $n \geq 1$, we have a recursively defined the coefficients for the multiplivative inverse of $\sum_{n=0}^{\infty} a_n x^n$, written as $\sum_{n=0}^{\infty} b_n x^n$.

Hence $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R. □

2. (DF7.2.4) Prove that if R is an integral domain then the ring of formal power series $R[[x]]$ is also an integral domain.

Proof. We already saw above that $R[[x]]$ is a commutative ring with $1_{R[[x]]}$ when R is a commutative ring with 1. What remains to show that if R has no zero divisors, then $R[[x]]$ also has no zero divisors.

We show that the product of any two nonzero power series is nonzero. Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two nonzero power series; that is, at least one of the coefficients a_n, b_n are nonzero. Let a_i be the first nonzero coefficient in $\sum_{n=0}^{\infty} a_n x^n$ and let b_j be the first nonzero coefficient in $\sum_{n=0}^{\infty} b_n x^n$. Then in the product $\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n$, the coefficient of x^{i+j} is given by $(a_0 b_{i+j} + \cdots + a_i b_j + \cdots a_{i+j} b_0)$. By our choice of a_i, b_j, it follows that every term vanishes except for $a_i b_j$, since R is an integral domain. It follows that the coefficient of x^{i+j} in the product is nonzero. It did not matter what happened to any of the other terms in the product since we have shown that there is at least one nonzero term; hence the product $\sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty} b_n x^n$ does not vanish. Since $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} a_n x^n$ were arbitrary nonzero power series, it follows that there are no zero divisors in $R[[x]]$, meaning that $R[[x]]$ is an integral domain. □

3. (DF7.2.5) Let F be a field and define the ring $F((x))$ of *formal Laurent series* with coefficients from F by

$$F((x)) = \left\{ \sum_{n \geq N}^{\infty} a_n x^n \mid a_n \in F \text{ and } N \in \mathbb{Z} \right\}$$

(Every element of $F((x))$ is a power series in x plus a polynomial in $1/x$, i.e., each element of $F((x))$ has only a finite number of terms with negative powers of x.)

(a) Prove that $F((x))$ is a field.

Proof. Define the addition on this set by

$$\sum_{n\geq N}^{\infty} a_n x^n + \sum_{n\geq M}^{\infty} b_n x^n = \sum_{n\geq \min\{N,M\}} c_n x^n$$

where $c_n = a_n + b_n$ whenever $n \geq \max\{N, M\}$, and when $n < \max\{N, M\}$ let $c_n = \begin{cases} a_n & \text{if } N < M \\ b_n & \text{if } M < M \end{cases}$ (so we just add in extra zero terms wherever needed and add in the natural way). It is clear that this operation is a binary operation (closure) and the additive identity is the zero series (where every coefficient is zero).

The addition is associative since the addition in F is associative; it is similar to the proof from before in that we add three series in two different groupings; they become equal since the bracketing in each term of the resulting series can be adjusted to obtain equality.

The addition is also commutative since the addition in F is commutative as well. To see this we can add two Laurent series and simply interchange the order of addition in each term of the resulting series to obtain the addition of the series in the reverse order.

Additive inverses are also given similarly:

$$-\left(\sum_{n\geq N}^{\infty} a_n x^n\right) = \sum_{n\geq N}^{\infty} (-a_n) x^n.$$

It follows that $F((x))$ is an abelian group under addition.

The multiplication on $F((x))$ is given by

$$\sum_{n\geq N}^{\infty} a_n x^n \times \sum_{n\geq M}^{\infty} b_n x^n = \sum_{n\geq N+M}^{\infty} \left(\sum_{\substack{i,j\in\mathbb{Z} \\ i+j=n}} a_i b_j\right) x^n$$

(i.e., we multiply through in the natural way and cancel powers of x wherever possible, then combine like terms.) It is clear that the multiplicative identity on $F((x))$ is the series whose coefficients are zero except for the coefficient of the x^0 term, which we put to be 1_F (so 1_F viewed as a Laurent series). The multiplication is associative since F is a ring:

$$\begin{aligned}
\left(\sum_{n\geq N}^{\infty} a_n x^n \times \sum_{n\geq M}^{\infty} b_n x^n\right) \times \sum_{n\geq L}^{\infty} c_n x^n &= \sum_{n\geq N+M}^{\infty} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=n}} a_i b_j\right) x^n \times \sum_{n\geq L}^{\infty} c_n x^n \\
&= \sum_{n\geq (N+M)+L}^{\infty} \left[\sum_{\substack{\ell,m\in\mathbb{Z}\\ \ell+m=n}} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=\ell}} a_i b_j\right) c_m\right] x^n \\
&= \sum_{n\geq N+M+L}^{\infty} \left(\sum_{\substack{i,j,k\in\mathbb{Z}\\ i+j+k=n}} a_i b_j c_k\right) x^n \\
&= \sum_{n\geq N+(M+L)}^{\infty} \left[\sum_{\substack{\ell,m\in\mathbb{Z}\\ \ell+m=n}} a_\ell \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=m}} b_i c_j\right)\right] x^n \\
&= \sum_{n\geq N}^{\infty} a_n x^n \times \sum_{n\geq M+L}^{\infty} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=n}} b_i c_j\right) x^n \\
&= \sum_{n\geq N}^{\infty} a_n x^n \times \left(\sum_{n\geq M}^{\infty} b_n x^n \times \sum_{n\geq L}^{\infty} c_n x^n\right).
\end{aligned}$$

The multiplication is commutative since F is a commutative ring itself:

$$\begin{aligned}
\sum_{n\geq N}^{\infty} a_n x^n \times \sum_{n\geq M}^{\infty} b_n x^n &= \sum_{n\geq N+M}^{\infty} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=n}} a_i b_j\right) x^n \\
&= \sum_{n\geq N+M}^{\infty} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=n}} b_j a_i\right) x^n \\
&= \sum_{n\geq M}^{\infty} b_n x^n \times \sum_{n\geq N}^{\infty} a_n x^n.
\end{aligned}$$

Then we check distributivity from one direction (e.g. from the right) since we know the multiplication is commutative:

$$\left(\sum_{n\geq N}^{\infty} a_n x^n + \sum_{n\geq M}^{\infty} b_n x^n\right) \times \sum_{n\geq L}^{\infty} c_n x^n = \sum_{n\geq \min\{N,M\}+L} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=n}} d_i c_j\right) x^n,$$

where $d_i = a_i + b_i$ and we use the convention that if a_i (or b_i) is not defined (i.e. the index i is taken outside of the bounds for which a_i (or b_i) was originally defined), then a_i (or b_i) is zero. Continuing with this convention, we have that

$$\sum_{n\geq\min\{N,M\}+L}\left(\sum_{\substack{i,j\in\mathbb{Z}\\i+j=n}} d_ic_j\right)x^n = \sum_{n\geq\min\{N+L,M+L\}}\left[\left(\sum_{\substack{i,j\in\mathbb{Z}\\i+j=n}} a_ic_j\right)+\left(\sum_{\substack{i,j\in\mathbb{Z}\\i+j=n}} b_ic_j\right)\right]x^n$$
$$=\left(\sum_{n\geq N}^{\infty}a_nx^n\times\sum_{n\geq L}^{\infty}c_nx^n\right)+\left(\sum_{n\geq M}^{\infty}b_nx^n\times\sum_{n\geq L}^{\infty}c_nx^n\right)$$

as desired. Hence $F((x))$ is a commutative ring with 1, and we show that nonzero elements have multiplicative inverses.

Let $f=\sum_{n\geq N}^{\infty}a_nx^n$ be a nonzero element of $F((x))$, so that a_N is not zero, and hence is a unit in F since F is a field. Then write $f=x^N\sum_{n=0}^{\infty}a_{N+n}x^n$, and let $g=\sum_{n=0}^{\infty}a_{N+n}x^n$ (so that $f=x^Ng$). View g as a nonzero element of the ring of formal power series $F[[x]]$ (it is clear that $F[[x]]$ is a subring of $F((x))$), and construct its multiplicative inverse g^{-1} via the recursive algorithm given earlier, writing a_n as a_{N+n} in the algorithm (all of this is possible only because a_N is a unit in F).

It follows that f has a multiplicative inverse, given by $f^{-1}=x^{-N}g^{-1}$, since $f\times f^{-1}=x^Ng\times x^{-N}g^{-1}=x^Nx^{-N}gg^{-1}=1_F$. Since f was an arbitrary nonzero element, it follows that every nonzero element in $F((x))$ has a multiplicative inverse.

Hence $F((x))$ is a field. □

(b) Define the map

$$\nu\colon F((x))^\times\to\mathbb{Z}\quad\text{by}\quad\nu\left(\sum_{n\geq N}^{\infty}a_nx^n\right)=N$$

where a_N is the first nonzero coefficient of the series (i.e., N is the "order of zero or pole of the series at zero"). Prove that ν is a discrete valuation on $F((x))$ whose discrete valuation ring is $F[[x]]$, the ring of formal power series (cf. Exercise 26, Section 1).

Proof. Let $a=\sum_{n\geq N}^{\infty}a_nx^n$ and $b=\sum_{n\geq M}^{\infty}b_nx^n$ be arbitrary elements of $F((x))^\times$. Observe that

$$\sum_{n\geq N}^{\infty}a_nx^n\times\sum_{n\geq M}^{\infty}b_nx^n=\sum_{n\geq N+M}^{\infty}\left(\sum_{\substack{i,j\in\mathbb{Z}\\i+j=n}}a_ib_j\right)x^n,$$

and the coefficient of the x^{N+M} term is given by a_Nb_M, which is nonzero since a_N,b_M must be nonzero elements of F by assumption. Similarly,

$$\sum_{n\geq N}^{\infty}a_nx^n+\sum_{n\geq M}^{\infty}b_nx^n=\sum_{n\geq\min\{N,M\}}c_nx^n,$$

where $c_n=a_n+b_n$ (noting that if either of a_n or b_n are not defined for a particular n then take it to be zero). If the resulting sum is nonzero, then its first nonzero coefficient occurs at some index greater than or equal to $\min\{N,M\}$, because each coefficient is found termwise.

It follows that $\nu(ab) = N + M = \nu(a) + \nu(b)$ and $\nu(a+b) \geq \min\{N, M\} = \min\{\nu(a), \nu(b)\}$ whenever $a + b \neq 0$. Furthermore, for any integer L, we can take the Laurent series given by one term, x^L, and it follows that $\nu(x^L) = L$. Hence ν is surjective, and we have shown that ν is a discrete valuation on $F((x))$.

We show that $R = \{f \in F((x)) \mid \nu(f) \geq 0\} \cup \{0\}$ is equal to $F[[x]]$. It is clear that 0 is in both sets, and that nonzero elements of $F[[x]]$ are contained in R since a nonzero element of $F[[x]]$ only has terms with nonnegative powers of x, so that its first nonzero term has index greater than or equal to zero. Similarly, a nonzero element of R has the condition that its valuation is nonnegative, meaning that its first nonzero term has index greater than or equal to zero. This is exactly the same condition needed for a series to be an element of $F[[x]]$, so the reverse containment is also clear. Hence the discrete valuation ring of ν is $F[[x]]$. □

4. (DF7.5.5) If F is a field, prove that the field of fractions of $F[[x]]$ (the ring of formal power series in the indeterminate x with coefficients in F) is the ring $F((x))$ of formal Laurent series (cf. Exercises 3 and 5 of Section 2). Show the field of fractions of the power series $\mathbb{Z}[[x]]$ is properly contained in the field of Laurent series $\mathbb{Q}((x))$. [Consider the series for e^x.]

Proof. Given the field of fractions Q of $F[[x]]$ (which is a field since F is an integral domain and we take the denominators to be the nonzero elements of $F[[x]]$), recall that we can write every nonzero element r/d of Q as rd^{-1} for $r \in F[[x]]$ and $d \in F[[x]]^\times$. This is because every $d \in F[[x]]^\times$ viewed as an element of Q has a multiplicative inverse, given by d^{-1} viewed as an element of Q. ("Viewed as" means to send an element through the injective embedding map from $F[[x]]$ into Q.) It follows that elements of Q can be viewed as elements of $F((x))$ (as products of power series).

Then observe that every nonzero element of $F((x))$ can also be decomposed into a fraction of power series. Observe that $F[[x]]$ is a subset of $F((x))$, and $F[[x]]$ can be viewed as a subset of Q by sending an element $r \in F[[x]]$ to re/e for any $e \in F[[x]]^\times$. Take any nonzero Laurent series $f = \sum_{n\geq N}^{\infty} a_n x^n$ such that N is negative (so that f is not a power series), and write it as $x^N \sum_{n=0}^{\infty} a_{N+n} x^n$, and let $g = \sum_{n=0}^{\infty} a_{N+n} x^n$, so that $f = x^N g$. View g as a nonzero element of the ring of formal power series $F[[x]]$. Since N is negative, we can write f instead as g/x^{-N}. (This is really an injection from $F((x))$ to Q.) It follows that f can be written as a fraction of power series (as g/x^{-N}).

Since 0 is an element of both $F((x))$ and Q (0 viewed as the zero fraction in Q), it follows from the above decompositions that $F((x))$ is contained in Q, and Q is contained in $F((x))$. Hence $F((x))$ is the field of fractions of $F[[x]]$. (We also could have used the fact that Q is the unique smallest field containing $F[[x]]$ in place of saying Q was contained in $F((x))$ to arrive at the conclusion.)

To see that $\mathbb{Q}((x))$ has an element not contained in the field of fractions of $\mathbb{Z}[[x]]$, we consider the Taylor series for e^x viewed as a rational Laurent series:

$$e^x \coloneqq \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

What makes it impossible for this series to be an element of the field of fractions of $\mathbb{Z}[[x]]$ is that every term after the constant term in this series has a distinct denominator; that is, $n!$ is distinct for every $n \geq 1$. Elements of the field of fractions of $\mathbb{Z}[[x]]$ can be viewed as rd^{-1} where $r \in \mathbb{Z}[[x]]$ and $d \in \mathbb{Z}[[x]]^{\times}$, and if e^x is a fraction of integer power series, then we could multiply through by the power series in the denominator to obtain a power series with only integer coefficients.

If there exists a suitable power series $g = \sum_{n=0}^{\infty} b_n x^n$ with $b_n \in \mathbb{Z}$ such that g is the denominator of a supposed fractional form for e^x, we investigate the product $g \times e^x$:

The product is given by

$$b_0 + (b_0 + b_1)x + (b_0/2 + b_1 + b_2)x^2 + (b_0/6 + b_1/2 + b_2 + b_3)x^3 + \cdots (b_0/n! + b_1/(n-1)! + \cdots b_n/0!)x^n + \cdots .$$

We determine what the coefficients b_n need to be in order for the product to have a chance at being an integer power series. For example, in the expansion of the product, we have that the term $b_0/n!$ appears for each n, meaning that for a given n either $b_0/n!$ needs to be an integer or it must be cancelled out by perhaps another term occurring in forming whatever coefficient $b_0/n!$ appears to be a summand of. Both are impossible (even for b_i instead of b_0) since in the sequence $\{n!\}_{n=0}^{\infty}$ we can find n large enough so that *any* prime p can divide $n!$. That is, infinitely many distinct primes must divide each b_n, which is impossible since b_n is an integer for each n. Hence the series for e^x is an element of $\mathbb{Q}((x))$ but not the field of fractions of $\mathbb{Z}[[x]]$. $\square$