## Graded

1. If  $F: \mathcal{C} \to \text{Sets}$  is a representable functor, prove that the representing object is unique up to isomorphism. Suggested strategy: Yoneda

*Proof.* Let F be representable by two objects  $A, B \in \mathcal{C}$  with natural isomorphisms  $\alpha \colon h_A = \hom_{\mathcal{C}}(A, -) \to F$  and  $\beta \colon h_B = \hom_{\mathcal{C}}(B, -) \to F$ .

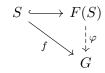
Then by Yoneda there is a bijection of the sets hom(B, A) and the set of natural transformations from  $h_A$  to  $h_B$  natural in A and  $h_B$ ; similarly, there is a bijection of the sets hom(A, B) and the set of natural transformations from  $h_B$  to  $h_A$  natural in B and  $h_A$ .

So from the above bijections obtain from  $\beta^{-1} \circ \alpha$  and  $\alpha^{-1} \circ \beta$  the morphisms  $f: B \to A$  and  $g: A \to B$  respectively. We must show that the compositions of f and g produce identities both ways.

I am not sure how to use the naturality conditions to obtain this from here.

5. (i) What is a free object on a set S in the category of Abelian groups?

We describe the free Abelian group F(S) on S to be the group of formal finite sums  $\mathbb{Z}[S] = \{\sum_{\text{finite}} c_s s \mid c_s \in \mathbb{Z}, s \in S\}$  with the group addition defined componentwise. This object satisfies the universal property of free objects in the category of Abelian groups. Given any group G and a set map from S into G, there is a unique map  $\varphi \colon F(S) \to G$  making the following diagram commute:



Define  $\varphi$  as the map which takes each  $s \in F(S)$  to f(s) and extend by linearity (so extended to a  $\mathbb{Z}$ -module homomorphism).

- (ii) What is a left adjoint to the forgetful functor from Abelian groups to sets?
  - The free functor F sending sets S to F(S) as above and sending set maps to their extensions as  $\mathbb{Z}$ -module maps. Let U be the forgetful functor. We show that for any set S and Abelian group G we have a natural isomorphism  $\operatorname{AbGrp}(F(S), G) \cong \operatorname{Set}(S, U(G))$ . Informally, we can see this since every  $\mathbb{Z}$ -module map is determined by its action on basis elements and vice versa.
- (iii) Show that the free Abelian group on S is not free in the category of groups.

The free Abelian group on S when S has two elements or more will not be free since the free group on S cannot be Abelian. The word  $aba^{-1}b^{-1}$   $(a, b \in S)$  is a nonidentity element of the free group on S while it is equal to the identity in the free Abelian group on S. A free Abelian group is a free group if and only if they are isomorphic in the category of groups; if the free group were isomorphic to the free Abelian group then the free group must be Abelian.

## **Additional Problems**

3. For a commutative ring R, define

$$E(R) = \{(x, y) \in R^2 \colon y^2 = x^3 - x\}.$$

- (i) Define E on morphisms to make it a functor from commutative rings to sets. For a commutative ring homomorphism  $\varphi \colon R \to S$ , define  $E(\varphi) \colon E(R) \to E(S)$  to be the map taking (x,y) to  $(\varphi(x), \varphi(y))$ . Note  $(\varphi(x), \varphi(y)) \in E(S)$  since  $\varphi$  is a ring homomorphism.
- (ii) Show that E is representable.

I feel like the object will be some kind of polynomial ring or ring of rational functions but I don't know how to do something like this without a field. Perhaps  $\mathbb{Z}$  or  $\mathbb{Q}$  are good candidates for the coefficients but I am not sure... Not complete.

The functor E is the set of points of an elliptic curve in algebraic geometry. The representing object is the coordinate ring of E (the ring of algebraic functions on E).

- 4. Let U be the forgetful functor from the category of fields to the category of integral domains.
  - (i) Show that sending an integral domain to its field of fractions is a functor F from the category of integral domains to fields.
  - (ii) Prove that F and U are adjoint functors. Which is the left one?

Not done.

## **Feedback**

- 1. None.
- 2. I am hoping that over the break I can reset and get back onto a regular schedule. I suspect I have burned out.