1. *Reduced homology.* Let X be a topological space with $x_0 \in X$. Define the *reduced singular chain group* by $\tilde{S}_n(X) = S_n(X, x_0)$ and *reduced singular homology* by $\tilde{H}_n(X) = H_n(X, x_0)$. Use the long exact sequence for the homology of a pair to show that $\tilde{H}_n(X) \cong H_n(X)$ for $n \geq 1$. The case $n = 1$ requires extra care. Use the maps $i\colon x_0 \to X$, $r\colon X \to x_0$ and the splitting lemma to show that $H_0(X) \cong \mathbb{Z} \oplus \tilde{H}_0(X)$.

   Proof. We know that the homology groups $H_i(x_0)$ are zero for nonzero i and is $\mathbb{Z}$ for $i = 0$. Then in the long exact sequence
   $$\cdots \xrightarrow{\partial} H_n(x_0) \to H_n(X) \to \tilde{H}_n(X) \xrightarrow{\partial} H_{n-1}(x_0) \to \cdots$$
   we have for $n > 1$ the exact sequence
   $$0 \to H_n(X) \to \tilde{H}_n(X) \xrightarrow{\partial} 0.$$
   But by exactness we must have that the middle arrow is both injective and surjective, so it is an isomorphism. For $n = 1$ we have the exact sequence
   $$0 \to H_1(X) \to \tilde{H}_1(X) \xrightarrow{\partial} H_0(x_0) \cong \mathbb{Z}.$$
   By exactness it is clear the middle arrow is injective, and it is surjective as it is a quotient map of groups. Hence the middle arrow is also an isomorphism.

   For $n = 0$ we obtain the short exact sequence:
   $$0 \to H_0(x_0) \cong \mathbb{Z} \to H_0(X) \to \tilde{H}_0(X) \to 0.$$
   The second arrow is injective because it is induced by the injective map $i\colon x_0 \to X$, which has left inverse $r\colon X \to x_0$. Thus $(\mathrm{id}_{x_0})* = (ri)_* = r_* i_*$ so indeed i_* is injective. But because r_* is a retract of i_* we have by the splitting lemma that $H_0(x_0)$ is a direct summand of $H_0(X)$, and since the third arrow is an isomorphism of the cokernel of the second arrow with $\tilde{H}_0(X)$, we must have that $H_0(X) \cong \mathbb{Z} \oplus \tilde{H}_0(X)$. □

2. *Long exact sequence for reduced homology of a pair.* Consider a pair (X, A) where A is nonempty. Define $\tilde{C}_n(X, A) = \tilde{C}_n(X)/\tilde{C}_n(A)$. Use the Nine Lemma to show that $\tilde{C}_n(X, A) \cong C_n(X, A)$. Use the short exact sequence of chain complexes $0 \to \tilde{C}_\bullet(A) \to \tilde{C}_\bullet(X) \to \tilde{C}_\bullet(X, A) \to 0$ and the above isomorphism to obtain the long exact reduced homology sequence of a pair.

Proof. In the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_n(A) \arrow[r] \arrow[d] & \tilde{C}_n(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \tilde{C}_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & \tilde{C}_n(X,A) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the columns are exact by definition of $C_n(X,A)$ and $\tilde{C}_n(X,A)$. The top two rows are exact as well since the third arrows of those rows are projection maps. The third arrow of the third row takes $x + C_n(A)$ to $[x] + \tilde{C}_n(A)$ where $[x] \in \tilde{C}_n(X)$, and it is well defined since for $a \in C_n(A)$, $[a] \in \tilde{C}_n(X)$ is either zero or in $\tilde{C}_n(A)$. It follows by the Nine Lemma that the third row is also exact. Then the sequence $0 \to C_n(X,A) \to \tilde{C}_n(X,A) \to 0$ is exact so that the middle arrow is an isomorphism (it is injective and surjective).

Then by the above isomorphism we obtain a short exact sequence of chain complexes $0 \to \tilde{C}_\bullet(A) \to \tilde{C}_\bullet(X) \to C_\bullet(X,A) \to 0$, which gives a long exact sequence as follows:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$$

□

3. *Homology of spheres.* Prove that $H_m(S^n) = \mathbb{Z}$ if $m = 0$ or n and is otherwise 0. That is, $\tilde{H}_m(S^n) = \mathbb{Z}$ if $m = n$ and is otherwise 0. Hint: Consider the pair (D^n, S^{n-1}) and the long exact sequence for reduced homology of a pair. Recall: using excision one may prove that $H_n(X,A) \cong \tilde{H}_n(X \cup CA)$ and if $A \subset X$ is a cofibration then $H_n(X,A) \cong \tilde{H}_n(X/A)$.

Proof. The inclusion $S^{n-1} \subset D^n$ is a cofibration. It follows that for all m, $H_m(X,A) \cong \tilde{H}_m(D^n/S^{n-1}) \cong \tilde{H}_m(S^n)$ Since disks are contractible, the reduced homology of a disk is all zero. Then in the long exact sequence for reduced homology of a pair we have

$$\cdots \to \tilde{H}_m(S^{n-1}) \to 0 \to \tilde{H}_m(S^n) \to \tilde{H}_{m-1}(S^{m-1}) \to 0 \to \cdots .$$

By exactness it follows that $\tilde{H}_m(S^n) \cong \tilde{H}_{m-1}(S^{n-1})$, and by induction we have for $n < m$ that $\tilde{H}_m(S^n) \cong \tilde{H}_{m-n}(S^0) \cong 0$, for $n = m$ that $\tilde{H}_m(S^n) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$, and for $n > m$ that $\tilde{H}_m(S^n) \cong \tilde{H}_0(S^{n-m}) \cong 0$ (as $H_0(S^{n-m}) \cong \mathbb{Z}$ since spheres are path connected). Thus the reduced homology $\tilde{H}_m(S^n)$ is $\mathbb{Z}$ if $n = m$ and is zero otherwise as desired. □

4. *The suspension isomorphism* Let X be a topological space. Prove that for all n, $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$. Hint: Consider the pair (CX, X).

Proof. Interpret ΣX as $CX \cup CX$ (with common subspace X; I think you meant free suspension in the problem statement). Then by excision we have $H_n(CX, X) \cong \tilde{H}_n(CX \cup CX) = \tilde{H}_n(\Sigma X)$ In the long exact reduced homology sequence of the pair (CX, X) we have

$$\cdots \to \tilde{H}_n(X) \to \tilde{H}_n(CX) \to \tilde{H}_n(\Sigma X) \xrightarrow{\partial} \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(CX) \to \cdots .$$

But CX is contractible so we have the exact sequence

$$\cdots \to 0 \to \tilde{H}_n(\Sigma X) \xrightarrow{\partial} \tilde{H}_{n-1}(X) \to 0 \to \cdots .$$

Thus by exactness the connecting homomorphism must always be an isomorphism, so $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$ as desired. □

5. *Homology of a bouquet of spheres.* Let A be some set. Then $\bigoplus_{\alpha\in A} \mathbb{Z}\alpha$ is the free Abelian group generated by the set A. Prove that $\tilde{H}_m(\bigvee_{\alpha\in A} S^n_\alpha) \cong \bigoplus_{\alpha\in A} \mathbb{Z}\alpha$ if $m = n$ and is 0 otherwise.

Proof. Let $X = \bigvee_{\alpha\in A} S^n_\alpha$. Let B be a neighborhood of the central point to which all the spheres were wedged together at; note that B is contractible. Let C be the all of the spheres minus a smaller neighborhood of the center point contained in B so that $B \cap C$ is nonzero and is homotopic to the disjoint union $\sqcup_{\alpha\in A} S^{n-1}_\alpha$. Note that C itself is homotopic to the disjoint union $\sqcup_{\alpha\in A} D^{n-1}_\alpha$, which is homotopic to the disjoint union of $|A|$ many points. Then consider the reduced Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_m(B \cap C) \to \tilde{H}_m(B) \oplus \tilde{H}_m(C) \to \tilde{H}_m(X) \to \tilde{H}_{m-1}(B \cap C) \to \tilde{H}_{m-1}(B) \oplus \tilde{H}_{m-1}(C) \to \cdots ,$$

which because disks are contractible and homology of a disjoint union is the direct sum of homologies, we obtain the following exact sequence:

$$\cdots \to \bigoplus_{\alpha\in A} \tilde{H}_m(S^{n-1}_\alpha) \to 0 \to \tilde{H}_m(X) \to \bigoplus_{\alpha\in A} \tilde{H}_{m-1}(S^{n-1}_\alpha) \to 0 \to \cdots$$

By exactness it follows that $\tilde{H}_m(X) \cong \bigoplus_{\alpha\in A} \tilde{H}_{m-1}(S^{n-1}_\alpha)$, but by a previous result we have that $H_{m-1}(S^{n-1}_\alpha) \cong \mathbb{Z}\alpha$ if $m = n$ but is 0 otherwise. Hence $\tilde{H}_m(X) \cong \bigoplus_{\alpha\in A} \mathbb{Z}\alpha$ if $m = n$ but is 0 otherwise. □

6. *Homology of a bouquet of spaces.* Assume that for all $\alpha \in A$, (X_α, x_α) is a pointed space which is a pair of spaces with the homotopy lifting property. Prove that for all n, $\tilde{H}_n(\bigvee_{\alpha\in A} X_\alpha) \cong \bigoplus_{\alpha\in A} \tilde{H}_n(X_\alpha)$. Hint: consider the pair $(\coprod_{\alpha\in A} X_a, \coprod_{\alpha\in A} x_a)$.

Proof. The coproduct of cofibrations is a cofibration. Then the pair $(X, x) = (\coprod_{\alpha\in A} X_a, \coprod_{\alpha\in A} x_a)$ has the homotopy lifting property for all spaces. Note that X/x is the wedge sum $\bigvee_{\alpha\in A} X_\alpha$. It follows that

$H_n(X, x) \cong \tilde{H}_n(X/x) = \tilde{H}_n(\bigvee_{\alpha \in A} X_\alpha)$. Then in the long exact reduced homology sequence of the pair (X, x) we have

$$\cdots \to 0 = \tilde{H}_n(\sqcup_{\alpha \in A} x_\alpha) \to \tilde{H}_n(\sqcup_{\alpha \in A} X_\alpha) \to \tilde{H}_n(\bigvee_{\alpha \in A} X_\alpha) \to 0 = \tilde{H}_{n-1}(\sqcup_{\alpha \in A} x_\alpha).$$

By exactness, for each n we must have $\bigoplus_{\alpha \in A} \tilde{H}_n(X_\alpha) \cong \tilde{H}_n(\sqcup_{\alpha \in A} X_\alpha) \cong \tilde{H}_n(\bigvee_{\alpha \in A} X_\alpha)$ as desired. $\square$