HOMEWORK 8

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Do one.

(i) Show, if p is a polynomial of degree at least two, then the sum of the residues of $\frac{1}{p}$ is zero.

(ii) Suppose $a > e$ and $n \in \mathbb{N}^+$. Show $e^z - az^n = 0$ has exactly n solutions in the unit disc $\mathbb{D} = \{|z| < 1\}$.

Proof (ii). Apply Rouché's theorem.

The functions f, g given by $f(z) = \exp(z) - az^n$ and $g(z) = az^n$ are meromorphic on $\mathbb{D}$. Furthermore, f has no poles in $\mathbb{D}$, and g has n zeroes but no poles in $\mathbb{D}$.

We have for $z \in S^1$, $|f(z) + g(z)| = |\exp(z)| = \exp(\mathrm{Re}(z)) \leq e$, and $a = |az^n| \leq |f(z)| + |g(z)|$. Since $a > e$, $|f(z) + g(z)| < |f(z)| + |g(z)|$ for all $z \in S^1$. By Rouché's theorem the number of zeroes of $f(z)$ is n as desired. $\square$