Graded

1. If $F\colon \mathcal{C} \to \text{Sets}$ is a representable functor, prove that the representing object is unique up to isomorphism. Suggested strategy: Yoneda

Proof. Let F be representable by two objects $A, B \in \mathcal{C}$ with natural isomorphisms $\alpha\colon h_A = \hom_{\mathcal{C}}(A, -) \to F$ and $\beta\colon h_B = \hom_{\mathcal{C}}(B, -) \to F$.

Then by Yoneda there is a bijection of the sets $\hom(B, A)$ and the set of natural transformations from h_A to h_B natural in A and h_B; similarly, there is a bijection of the sets $\hom(A, B)$ and the set of natural transformations from h_B to h_A natural in B and h_A.

So from the above bijections obtain from $\beta^{-1} \circ \alpha$ and $\alpha^{-1} \circ \beta$ the morphisms $f\colon B \to A$ and $g\colon A \to B$ respectively. We must show that the compositions of f and g produce identities both ways.

By a similar argument the identity natural transformations id_{h_A} and id_{h_B} are induced by the identity maps on A and B respectively. But $f \circ g$ and $g \circ f$ will also induce (also by post composition) the identity maps on the hom functors h_A and h_B respectively; by Yoneda there can only be one map for each situation which does this, the identity maps. It follows that f and g are isomorphisms so that the representing objects A and B are isomorphic. □

5. (i) What is a free object on a set S in the category of Abelian groups?

We describe the free Abelian group $F(S)$ on S to be the group of formal finite sums $\mathbb{Z}[S] = \{\sum_{\text{finite}} c_s s \mid c_s \in \mathbb{Z}, s \in S\}$ with the group addition defined componentwise. This object satisfies the universal property of free objects in the category of Abelian groups. Given any group G and a set map from S into G, there is a unique map $\varphi\colon F(S) \to G$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "f"'] & F(S) \arrow[d, dashed, "\varphi"] \\ & G \end{tikzcd}$$

Define φ as the map which takes each $s \in F(S)$ to $f(s)$ and extend by linearity (so extended to a $\mathbb{Z}$-module homomorphism).

(ii) What is a left adjoint to the forgetful functor from Abelian groups to sets?

The free functor F sending sets S to $F(S)$ as above and sending set maps to their extensions as $\mathbb{Z}$-module maps. Let U be the forgetful functor. We show that for any set S and Abelian group G we have a natural isomorphism $\text{AbGrp}(F(S), G) \cong \text{Set}(S, U(G))$. Informally, we can see this since every $\mathbb{Z}$-module map is determined by its action on basis elements and vice versa.

(iii) Show that the free Abelian group on S is not free in the category of groups.

The free Abelian group on S when S has two elements or more will not be free since the free group on S cannot be Abelian. The word $aba^{-1}b^{-1}$ $(a, b \in S)$ is a nonidentity element of the free group on S

while it is equal to the identity in the free Abelian group on S. A free Abelian group is a free group if and only if they are isomorphic in the category of groups; if the free group were isomorphic to the free Abelian group then the free group must be Abelian.

Additional Problems

3. For a commutative ring R, define

$$E(R) = \left\{(x, y) \in R^2 \colon y^2 = x^3 - x\right\}.$$

(i) Define E on morphisms to make it a functor from commutative rings to sets.

For a commutative ring homomorphism $\varphi\colon R \to S$, define $E(\varphi)\colon E(R) \to E(S)$ to be the map taking (x, y) to $(\varphi(x), \varphi(y))$. Note $(\varphi(x), \varphi(y)) \in E(S)$ since φ is a ring homomorphism.

(ii) Show that E is representable.

I feel like the object will be some kind of polynomial ring or ring of rational functions but I don't know how to do something like this without a field. Perhaps $\mathbb{Z}$ or $\mathbb{Q}$ are good candidates for the coefficients but I am not sure... Not complete.

The functor E is the set of points of an elliptic curve in algebraic geometry. The representing object is the coordinate ring of E (the ring of algebraic functions on E).

4. Let U be the forgetful functor from the category of fields to the category of integral domains.

(i) Show that sending an integral domain to its field of fractions is a functor F from the category of integral domains to fields.

Let R be an integral domain with 1 and let $\mathrm{Frac}(R)$ be its field of fractions. Given a homomorphism of integral domains $\phi\colon R \to S$, extend φ into a map $\Phi\colon \mathrm{Frac}\, R \to \mathrm{Frac}\, S$ by defining it in the natural way: Given an element $a/b \in \mathrm{Frac}\, R$, $\Phi(a/b) = \phi(a)/\phi(b)$ and Φ is well defined because ϕ is a ring homomorphism. The associativity axioms follow from the associativity of integral domain homomorphisms and we can use the identity maps on these rings to obtain the identity maps on the corresponding field of fractions.

(ii) Prove that F and U are adjoint functors. Which is the left one?

F is left adjoint to U. We check that $\mathrm{Field}(F(R), K) \cong \mathrm{Ring}(R, U(K))$: Every field homomorphism of an integral domain's field of fractions contains the data needed to define the corresponding ring homomorphism (look at the image of $a/1$ for each $a \in R$ and check that "restricting" to these elements gives an integral domain homomorphism) Obtain a field homomorphism from an integral domain homomorphism in the manner described above (extend to numerator and denominator). These assignments are natural.

Feedback

1. None.

2. I am hoping that over the break I can reset and get back onto a regular schedule. I suspect I have burned out.