1. (13.34) Prove that if (f_n) is a dominated sequence, then it is uniformly integrable. Give an example of a sequence (f_n) that converges in L^1 (and is thus uniformly integrable), but is not dominated.

Proof. Let (f_n) be a sequence of measurable functions dominated by some $g \in L^1$ so that $|f_n| \leq |g|$ for all n.

It follows that for all n,
$$\|f_n\|_1 = \int |f_n| \leq \int |g| = \|g\|_1,$$
so $(\|f_n\|_1)$ is bounded by $\|g\|_1$.

For any x, n, and $M > 0$ observe that if $|f_n(x)| \geq M$ then $|g(x)| \geq M$. Thus for any n, $M > 0$, $\{x\colon |f_n(x)| \geq M\} \subseteq \{x\colon |g(x)| \geq M\}$.

It follows that for any n and $M > 0$,
$$\int_{|f_n| \geq M} |f_n| \leq \int_{|f_n| \geq M} |g| \leq \int_{|g| \geq M} |g|,$$
and we can take M large enough (independently of n) so that $\int_{|g| \geq M} |g|$ is arbitrarily small by the Dominated Convergence Theorem. (The sequence $(\mathbf{1}_{|g| \geq M}|g|)$ for $M \in \mathbb{Z}_+$ converges to the zero function a.e. (since $|g| \in L^1$), and $\mathbf{1}_{|g| \geq M}|g| \leq |g|$ for each M.) It follows that $\sup\left\{\int_{|f_n| \geq M} |f_n|\right\}$ (bounded above by $\int_{|g| \geq M} |g|$) tends to zero as M tends to ∞.

For any n and integer $k > 0$, define g_k as $\sup\left\{\mathbf{1}_{|f_n| \leq 1/k}|f_n|\right\}$, and observe that for every k, $g_k \leq |g|$ since for any n, $\mathbf{1}_{|f_n| \leq 1/k}|f_n| \leq |g|$. But as k tends to ∞, g_k converges pointwise to the zero function so $\int g_k$ tends to zero as k tends to ∞ (DCT). Furthermore, we have $\int_{|f_n| \leq 1/k} |f_n| \leq \int g_k$ for every n, from which it follows that $\sup\left\{\int_{|f_n| \leq 1/k} |f_n|\right\}$ tends to zero as k tends to ∞.

It follows that (f_n) is uniformly integrable. $\square$

An example of a sequence converging in L^1 which is not dominated is the sequence $(n\mathbf{1}_{[0,1/n^2]})$ which converges to the zero function in L^1 (as $\int n\mathbf{1}_{[0,1/n^2]} = 1/n$). But this sequence could not be dominated by any L^1 function (the height of the box tends to ∞).

2. (13.37) Prove that if (f_n) is a dominated sequence, and (f_n) converges to f a.e., then (f_n) converges to f almost uniformly. (Hint: imitate the proof of Egorov's theorem.) (Thus for dominated sequences, a.e. and a.u. convergence are equivalent.)

Proof. We mimic the proof of Egorov's theorem almost everywhere.

Let (f_n) be a dominated sequence (by $g \in L^1$) converging to f almost everywhere. Without loss of generality modify each of the f_n on null sets so that (f_n) converges to f everywhere. For $N, k \geq 1$, let $E_{N,k} = \cup_{n=N}^{\infty} \{x\colon |f_n(x) - f(x)| \geq 1/k\}$. Observe that $E_{1,k}$ for fixed k has finite measure: for any n, if $|f_n(x) - f(x)| \geq 1/k$ then by the triangle inequality $2|g(x)| = |2g(x)| \geq 1/k$. Thus $E_{1,k} \subseteq \{x\colon |2g(x)| \geq 1/k\}$, and the latter has finite measure since $2g \in L^1$ (otherwise we arrive at a contradiction).

Let k be fixed. Then for each x there is an N such that $|f_n(x) - f(x)| < 1/k$ for all $n \geq N$. It follows that $\cap_{N=1}^{\infty} E_{N,k} = \emptyset$. The $E_{N,k}$ are decreasing in N and are contained in $E_{1,k}$ which has finite measure; by dominated convergence for sets, we have for fixed k the sequence $(\mu(E_{N,k}))_N$ tends to zero.

Let $\varepsilon > 0$ be given. For each k choose N_k such that $\mu(E_{N_k,k}) < \varepsilon 2^{-k}$. Let $E = \cup_{k=1}^{\infty} E_{N_k,k}$, so that $\mu(E) < \varepsilon$. We show that (f_n) converges uniformly to f on E^c. Let $\eta > 0$ be given and choose k such that $1/k < \eta$. Let $x \in E^c$ and take $n \geq N_k$; since $E^c \subseteq E^c_{N_k,k}$ we have $|f_n(x) = f(x)| < 1/k < \eta$. Since x was arbitrary in E^c we have that (f_n) converges to f uniformly on E^c, so (f_n) converges to f almost uniformly. □

3. (19.6) (Integral as the area under a graph). Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, and give $\mathbb{R}$ the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$ and Lebesgue measure m (restricted to $\mathscr{B}_{\mathbb{R}}$). An unsigned function $f\colon X \to [0, +\infty)$ is measurable if and only if the set

$$G_f \coloneqq \{(x,t) \in X \times \mathbb{R} \colon 0 \leq t \leq f(x)\}$$

is measurable. In this case,

$$(\mu \times m)(G_f) = \int_X f \, \mathrm{d}\mu \,.$$

Proof. Suppose that f is measurable. Then for every $n \in \mathbb{Z}_+$, $f + 1/n$ is measurable also. Fix n. There exists a sequence of increasing simple functions $(s_{k,n})_k$ converging pointwise to $f + 1/n$. Writing some $s_{k,n}$ as $\sum_{j=1}^{d} c_j \mathbf{1}_{E_j}$, observe that $G_{s_{k,n}} = \{(x,t) \in X \times \mathbb{R} \colon 0 \leq t \leq s_{k,n}(x)\} = \cup_{j=1}^{d} E_j \times [0, c_j]$. Hence for every k, $G_{s_{k,n}}$ is measurable so that $\cup_{k=1}^{\infty} G_{s_{k,n}}$ is measurable.

We have that $G_f \subseteq \cup_{k=1}^{\infty} G_{s_{k,n}} \subseteq G_{f+1/n}$ for any n since for any x, $(s_{k,n}(x))_k$ converges to $f(x) + 1/n$ (but $f(x) + 1/n$ need not be attained).

Then $\cap_{n=1}^{\infty} \cup_{k=1}^{\infty} G_{s_{k,n}}$ must be equal to G_f: We have one inclusion by the above. For the reverse inclusion, fix x. We have (x,t) in $\cap_{n=1}^{\infty} \cup_{k=1}^{\infty} G_{s_{k,n}}$ if one of $0 \leq t \leq f(x) + 1/n$ or $0 \leq t < f(x) + 1/n$ holds (the distinction of $<$ or $\leq$ depends on the n) for every n, which means that $0 \leq t \leq f(x)$. It follows that G_f is measurable (countable intersection of countable union of measurable sets).

Conversely, suppose G_f is measurable. Then by Theorem 15.7, we obtain a measurable $p\colon X \to [0, +\infty]$ given by $p(x) = m((G_f)_x) = \int_X \mathbf{1}_{(G_f)_x} \mathrm{d}m$. But $m((G_f)_x) = m([0, f(x)]) = f(x)$ so p agrees with f on X; it follows that f is measurable.

Using the first version of Tonelli's theorem we have

$$\begin{aligned}
(\mu \times m)(G_f) &= \int_{X \times \mathbb{R}} \mathbf{1}_{G_f} \, \mathrm{d}\mu \times m \\
&= \int_X \int_{\mathbb{R}} (\mathbf{1}_{G_f})_x(t) \, \mathrm{d}m(t) \, \mathrm{d}\mu(x) \\
&= \int_X m((G_f)_x) \, \mathrm{d}\mu(x) \\
&= \int_X f \mathrm{d}\mu
\end{aligned}$$

as desired. □