Do any THREE problems.

1. For a) implies b): for any $f \in L^1(\mu)$, we have $\int |gf| = \int |g||f| \leq M \int |f| < +\infty$ since $f \in L^1(\mu)$.

   The reverse implication I feel is shady, but here was what I had so far:

   We try a proof by contrapositive, so assume that there does not exist an M such that $|g| \leq M$ almost everywhere; that is, for every M the set $G_M = \{x \in X : |g(x)| \geq M\}$ has positive measure. Also by dominated convergence for sets we should have that as M goes to infinity, the measure of the sets G_M goes to zero (I am interpreting $g\colon X \to \mathbb{R}$ to mean that g does not take on $+\infty$), but I did not find this to be useful.

   I formed a measurable partition of X via the sets $E_n = \{x \in X : n \leq |g|(x) \leq n+1\} = G_n \cap G_{n+1}^c$, so that $X = \cup_{n=0}^{\infty} E_n$. Some of the E_n may have measure zero; more importantly, there must be an infinite number of the E_n with nonzero measure, since otherwise there would be N large enough with $|g| \leq N$. It follows that $\mu(X) = \sum_{n=0}^{\infty} \mu(E_n)$ is still an infinite sum.

   Observe that $\sum_{n=0}^{N-1} n\mathbf{1}_{E_n} + N\mathbf{1}_{G_N}$ for N finite is a simple function that is less than or equal to $|g|$. My idea was to define an $f \in L^1(\mu)$ such that the above simple function could be modified to become a simple function less than $|gf|$ whose integral could be made arbitrarily large as N tends to $+\infty$ (which means the integral of $|gf|$ is arbitrarily large and so gf is not in $L^1(\mu)$).

   A function I had in mind was $f\colon X \to \mathbb{R}$ such that f takes on the value $\frac{1}{n^2\mu(E_n)}$ on E_n whenever $\mu(E_n) \neq 0$ and $n \geq 1$ and zero everywhere else. Then the integral of $|f| = f$ would be equal to or at least bounded above by $\sum_{\substack{n\in\mathbb{Z}_+ \\ \mu(E_n)\neq 0}} \frac{1}{n^2\mu(E_n)}\mu(E_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$, so that $f \in L^1(\mu)$. But then $\sum_{n=1}^{N-1} \frac{1}{n^2\mu(E_n)} \cdot n\mathbf{1}_{E_n} + \frac{1}{N^2\mu(G_N)} \cdot N\mathbf{1}_{G_N}$ is a simple function less than or equal to $|gf|$ for all N, but its integral is bounded below by $\sum_{\substack{1\leq n\leq N \\ \mu(E_n)\neq 0}} \frac{1}{n}$. As N is made arbitrarily large, this sum should diverge (or I hope it should), so $\int |gf|$ must also diverge. This should prove the contrapositive, but somehow I feel like something is wrong.

3. We check first that the function of two variables $g(x,t) = (1/2h)f(x-t)\mathbf{1}_{[-h,h]}(t)$ is in $L^1(\mathbb{R}^2)$. This function and its absolute value should be measurable if $f(x-t)$ is measurable, but I am not sure how to show this directly. The function $f(x-t)$ should still be measurable, and we proceed by showing that $g(x,t)$ is absolutely integrable using Tonelli's theorem:

$$\begin{aligned}\int_{\mathbb{R}^2} |g(x,t)| \,\mathrm{d}A &= \frac{1}{2h}\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-t)|\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2h}\int_{\mathbb{R}} \mathbf{1}_{[-h,h]}(t)\int_{\mathbb{R}} |f(x-t)|\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2h}\int_{\mathbb{R}} \mathbf{1}_{[-h,h]}(t)\,\mathrm{d}t \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x = \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x < +\infty.\end{aligned}$$

Now apply Fubini's theorem to obtain the following chains of inequalities:

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \frac{1}{2h} f(x-t)\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}A\right| &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2h} f(x-t)\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}t\,\mathrm{d}x\right| \\
&\leq \int_{\mathbb{R}}\left|\int_{\mathbb{R}} \frac{1}{2h} f(x-t)\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}t\right|\mathrm{d}x \\
&= \int_{\mathbb{R}}\left|\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,\mathrm{d}t\right|\mathrm{d}x \\
&= \int_{\mathbb{R}} |f_h(x)|\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}}\left|\int_{\mathbb{R}} \frac{1}{2h} f(x-t)\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}t\right|\mathrm{d}x &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2h}|f(x-t)|\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2h}|f(x-t)|\mathbf{1}_{[-h,h]}(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x
\end{aligned}$$

by the above. It follows that $\int_{\mathbb{R}} |f_h(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x$.

4. In the background let $X = X_+ \cup X_-$ be the Hahn decomposition of X (so that in the Jordan decomposition of ν, ν_+ is a measure on X_+ and ν_- is a measure on X_- and they are mutually singular)

   (a) We show inequalities in both ways.

   Let E be measurable. Then $|\nu(E)| = |\nu_+(E) - \nu_-(E)| \leq |\nu_+(E)| + |\nu_-(E)| = \nu_+(E) + \nu_-(E) = |\nu|(E)$. Then for any finite partition $\{E_j\}_{j=1}^n$ of E, we have $\sum_{j=1}^n |\nu(E_j)| \leq \sum_{j=1}^n |\nu|(E_j) = |\nu|(E)$, where the last equality was due to additivity as the E_j form a partition of E. It follows that

$$|\nu|(E) \geq \sup\left\{\sum_{j=1}^n |\nu(E_j)| \colon E_1, \ldots, E_n \text{ form a partition for } E\right\}.$$

   For the other inequality we show that $|\nu|(E)$ is an element of the set we are taking the supremum over: Consider the partition of E into the sets $E \cap X_+$ and $E \cap X_-$ (since $X = X_+ \cup X_-$). Then since ν_+, ν_- are mutually singular measures, we have

$$\begin{aligned}
|\nu(E\cap X_+)| + |\nu(E\cap X_-)| &= |\nu_+(E\cap X_+)| + |\nu_-(E\cap X_-)| \\
&= \nu_+(E\cap X_+) + \nu_-(E\cap X_-) \\
&= \nu_+(E\cap X_+) + \nu_+(E\cap X_-) + \nu_-(E\cap X_-) + \nu_-(E\cap X_+) \\
&= \nu_+(E) + \nu_-(E) \\
&= |\nu|(E).
\end{aligned}$$

   Thus the reverse inequality is obtained, so we have equality.

(b) Define $|f|_{X_+}$ to be the function on X that is 0 on X_- and is $|f|$ on X_+; define $|f|_{X_-}$ similarly to be the function which is 0 on X_+ and is $|f|$ on X_-. It follows that $|f| = |f|_{X_+} + |f|_{X_-}$. Then we have

$$\left|\int f\,\mathrm{d}\nu\right| = \left|\int f\,\mathrm{d}\nu_+ - \int f\,\mathrm{d}\nu_-\right| \tag{1}$$

$$\leq \int |f|\,\mathrm{d}\nu_+ + \int |f|\,\mathrm{d}\nu_- \tag{2}$$

$$= \int |f|_{X_+}\,\mathrm{d}|\nu| + \int |f|_{X_-}\,\mathrm{d}|\nu| \tag{3}$$

$$= \int |f|_{X_+} + |f|_{X_-}\,\mathrm{d}|\nu| \tag{4}$$

$$= \int |f|\,\mathrm{d}|\nu|\,. \tag{5}$$

The equality in (3) is justified since $|\nu| = \nu_+ + \nu_-$, and any simple unsigned function less than or equal to $|f|_{X_+}$ (resp. $|f|_{X_-}$) must be zero on X_- (resp. X_+). Every simple function of this form may also be viewed as (by restriction) a simple function on X_+ (resp. X_-) less than or equal to $|f|_{X_+}$ (resp. $|f|_{X_-}$). Hence equality (3) follows.