HOMEWORK 7

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Let $\Omega_\pm$ denote the upper and lower half planes and identify $\mathbb{R}$ with the real axis in $\mathbb{C}$ and suppose $f : \overline{\Omega_+} \to \mathbb{C}$. Show, if

(i) f is continuous;
(ii) the restriction of f to Ω_+ is analytic; and
(iii) $f(\mathbb{R}) \subseteq \mathbb{R}$,

then $g : \overline{\Omega_-} \to \mathbb{C}$ defined by $g(z) = f(z^*)^*$ is continuous, analytic on Ω_- and agrees with f on $\mathbb{R}$. Finally, show the function $F : \mathbb{C} \to \mathbb{C}$ defined by

$$F(z) = \begin{cases} f(z) & z \in \overline{\Omega_+} \\ g(z) & z \in \overline{\Omega_-} \end{cases}$$

is an analytic extension of f to an entire function.

Proof. Complex conjugation is a continuous operation. Given $\varepsilon > 0$, if $|z - w| < \varepsilon$ then $|z^* - w^*| < \varepsilon$. Since the composition of continuous functions is continuous, $g = \cdot^* \circ f \circ \cdot^*$ is continuous.

We use Morera's theorem to show g is analytic on Ω_-. Given a triangle $T \subset \Omega_-$, we can find a triangle $T^* \subset \Omega_+$ which is just the pointwise image of the conjugation map on T. The orientation of the triangle will be reversed under conjugation. It follows that $\int_T g = -\int_{T^*} f^* = (\int_{T^*} f)^* = 0$ since f is analytic on Ω_+. Since T was arbitrary it follows g is analytic on Ω_-. Since $f(R) \subset R$, we have for real x that $g(x) = f(x^*)^* = f(x)^* = f(x)$, so g agrees with f on $\mathbb{R}$.

By the pasting lemma ($\mathbb{C} = \overline{\Omega_+} \cup \overline{\Omega_+}$ and f, g are continuous on $\overline{\Omega_+}, \overline{\Omega_-}$ respectively and agree on $\mathbb{R} = \overline{\Omega_+} \cap \overline{\Omega_-}$) F is continuous on $\mathbb{C}$.

We use Morera's theorem again to show F is analytic on $\mathbb{C}$. If T is any triangle contained in Ω_+ or Ω_-, then by analyticity of f or g an integral along the boundary of

T will vanish. So we consider the case when $T \cap \mathbb{R}$ is not trivial. If $T \cap \mathbb{R}$ is a single point the integral along the boundary of T vanishes. If $T \cap \mathbb{R}$ is an interval then the integral vanishes by continuity (literal sketch):

If $T \cap \mathbb{R}$ is a two point set then a similar continuity argument may be used to show the integral vanishes. Hence all integrals along triangles vanish, so F is entire. $\square$