1. (4.5.2) Let $S(k) = I^k/\partial I^k$. We have canonical homeomorphisms

$$\text{(a) } \Omega^k(Y) = F^0(S(k), Y) \cong F((I^k, \partial I^k), (Y, *)) \quad \text{and} \quad \text{(b) } \Omega^k\Omega^l(Y) \cong \Omega^{k+l}(Y).$$

Proof. Let p be the quotient map from I^k to $I^k/\partial I^k = S(k)$. Then the homeomorphism in (a) is given by the assignment $f\colon S(k) \to Y \mapsto f \circ p\colon I \to Y$. We show that the assignment is bijective, continuous, and open (briefly, as 4.5.6 is a similar problem and the work is more clear there):

For two distinct $f, g \in \Omega^k(Y)$, they differ at a point in the interior of I^k, so $f \circ p$ and $g \circ p$ differ. For some $w \in F((I^k, \partial I^k), (Y, *))$ consider $\overline{w}$ which is w on the interior of I^k, but takes on $*$ on the equivalence class for ∂I^k. This is continuous as w itself agrees on points on ∂I and $\overline{w} \circ p$ agrees with w as needed. Hence the assignment is bijective.

For continuity and openness we consider subbase elements. Let $\{w \in \Omega^k(Y) \mid w(K) \subset U\}$ be an element of the subbase for the subspace topology on ΩY. Then its preimage is given by $\{f \mid (f \circ p)(K) \subset U\}$, which is $\{f \mid f(p(K)) \subset U\}$. Since $p(K)$ is compact, the preimage is open so the assignment is continuous. Let $\{f \mid f(K) \subset U\}$ be a subbase element of $F((I^k, \partial I^k), (Y, *))$. Then the image under the assignment is $\{f \circ p \mid f(K) \subset U\} = \{f \circ p \mid (f \circ p)(p^{-1}(K)) \subset U\}$ and $p^{-1}(K)$ is compact, $p(p^{-1}(K)) = K$ since p is surjective. It follows that this assignment is open.

Hence the assignment is a homeomorphism as desired.

For (b) Since products of intervals are compact (locally compact), we can use Theorem 2.4.6 (Exponential law) to see that the adjunction map $\alpha\colon Y^{I^k \times I^l} \xrightarrow{\cong} (Y^{I^l})^{I^k}$ is a homeomorphism. Restricting to the subspaces which fix the image of the boundaries of these cubes to the basepoint of Y, we should also obtain a homeomorphism of subspaces $\Omega^k\Omega^l(Y) \cong \Omega^{k+l}(Y)$. In particular, the subspace $\Omega^{k+l}(Y)$ would be sent to the subspace of $(Y^{I^l})^{I^k}$ whose elements are maps which all send the boundaries of cubes to the basepoint of Y, which is $\Omega^k\Omega^l(Y)$. □

2. (4.5.3) The space $F((I, 0), (Y, *)) \subset Y^I$ is pointed contractible.

Proof. We show the identity map is homotopic to the constant map: For $x \in X = F((I, 0), (Y, *)) \subset Y^I$, define for $s \in I$, $sx\colon I \to Y$ by $sx(t) = x((1 - s)t)$ (it is clear each sx is continuous). Then the homotopy $H\colon X \times I \to X$ given by $H(x, s) = sx$ starts with $H(x, 0) = 0x = x$ and ends at $H(x, 1) = 1x = 1_*$, with 1_* being the path sending I to $* \in Y$ (since $x(0) = *$ for all $x \in X$). Thus X is contractible to its base point, the constant map sending I to $*$. □

3. (4.5.6) Verify the homeomorphism $F^0(I/\partial I, Y) \cong \Omega Y$.

Proof. The homeomorphism $F^0(I/\partial I, Y) \cong \Omega Y$ is given by the assignment $f\colon I/\partial I \to Y \mapsto f \circ p\colon I \to Y$, where f is a pointed continuous map taking ∂I to $* \in Y$.

We show that this assignment is bijective, continuous, and open. It is clear that the assignment is injective since if $f, g \in F^0(I/\partial I, Y)$ are distinct then $f \circ p$ and $g \circ p$ differ at some $t \in (0,1)$. Given some path $w \in \Omega Y$, define $\overline{w}\colon I/\partial I \to Y$ which agrees with w on $(0,1)$ and on ∂I is $w(0) = w(1) = *$. This is continuous since w agrees on ∂I, and $\overline{w} \circ p$ agrees with w on I as needed.

For continuity and openness it suffices to check on the subbase for the compact open topology. Let $\{w \in \Omega Y \mid w(K) \subset U\}$ be an element of the subbase for the subspace topology on ΩY. Then its preimage is given by $\{f \mid (f \circ p)(K) \subset U\}$, which is $\{f \mid f(p(K)) \subset U\}$. Since $p(K)$ is compact also, we have an open set in $F^0(I/\partial I, Y)$, so the assignment is continuous. Let $\{f \mid f(K) \subset U\}$ be a subbase element of $F^0(I/\partial I, Y)$. Then the image under the assignment is $\{f \circ p \mid f(K) \subset U\} = \big\{f \circ p \mid (f \circ p)(p^{-1}(K)) \subset U\big\}$ and $p^{-1}(K)$ is compact, $p(p^{-1}(K)) = K$ since p is surjective. It follows that this assignment is open.

Hence the assignment is a homeomorphism as desired. □

4. (4.6.1) Let the left square in the next diagram be a pushout with an embedding j and hence an embedding J. Then F induces a homeomorphism $\overline{F}$ of the quotient spaces.

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "f"] & X \arrow[r, "p"] \arrow[d, "F"] & X/A \arrow[d, "\overline{F}"] \\
B \arrow[r, "J"] & Y \arrow[r, "q"] & Y/B
\end{tikzcd}$$

Proof. We check that pushouts of embeddings are embeddings: Embeddings, like j, are injective continuous maps which are open/closed (so A is homeomorphic to its image under j; also the image of A need not be open or closed in X). We show first that as a set map J is injective. Let j_ℓ^{-1} be the left inverse of j (since j is injective). Then in the following diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "f"] & X \arrow[d, "F"] \arrow[ddr, bend left, "f \circ j_\ell^{-1}"] & \\
B \arrow[r, "J"] \arrow[drr, bend right, "\mathrm{id}_B"'] & Y \arrow[dr, dashed, "J_\ell^{-1}"] & \\
 & & B
\end{tikzcd}$$

by the universal property of pushouts we obtain a left inverse J_ℓ^{-1} for J. In **Top** J is continuous, so we show that J is an open/closed map. Let Z be an open/closed set in B. Then $f^{-1}(Z)$ is open/closed so that $jf^{-1}(Z)$ is open/closed. Then also $J_\ell^{-1}J(Z) = Z$ is open/closed; so $F^{-1}J(Z)$ is open/closed; so J is an open/closed map.

Recall that Y is $(X \sqcup B)/\sim$ where $j(a) \sim f(a)$. Then with j, J embeddings, the quotient spaces make sense. Observe that in Y/B, since $f(a) \in B$, every point $f(a)$ gets identified. But $j(a)$ is also identified with $f(a)$, so A is also identified to the same point as B. So there is a bijection $\overline{F}$ between equivalence classes in X/A with those in Y/B given basically by the identity, since for $[x] \in X/A$ yields $[x] = \{x\}$ if $x \notin j(A)$ and $[x] = j(A)$ otherwise, and similarly $[x] \in Y/B$ yields $[x] = \{x\}$ if $x \notin j(A)$, and if $x \in j(A)$, then $[x] = j(A) \sqcup B$. Note that the dependence on being in $J(B)$ is removed since B is identified with $j(A)$. The diagram above commutes and so $\overline{F}$ is continuous since F, q are continuous and p is open; similarly $\overline{F}^{-1}$ is continuous (imagine $\overline{F}$ is given by the identity on equivalence classes). Hence $\overline{F}$ is a homeomorphism as desired. $\square$