1. (54.3) Let $p\colon E \to B$ be a covering map. Let $\alpha$ and $\beta$ be paths in $B$ with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

   *Proof.* We check that $p \circ (\tilde{\alpha} * \tilde{\beta}) = \alpha * \beta$. For any $x \in I$, we have that

$$(p \circ (\tilde{\alpha} * \tilde{\beta}))(x) = p((\tilde{\alpha} * \tilde{\beta})(x)) = \begin{cases} p(\tilde{\alpha}(2x)) = \alpha(2x) & \text{if } x \in [0, 1/2] \\ p(\tilde{\beta}(2x-1)) = \beta(2x-1) & \text{if } x \in [1/2, 1] \end{cases} = (\alpha * \beta)(x),$$

   and observe that the last equality holds by definition of $\alpha, \beta$ and the fact that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. □

2. (54.6) Consider the maps $g, h\colon S^1 \to S^1$ given $g(z) = z^n$ and $h(z) = 1/z^n$. (Here we represent $S^1$ as the set of complex numbers $z$ of absolute value 1.) Compute the induced homomorphisms $g_*, h_*$ of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [*Hint:* Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$]

   *Proof.* Without loss of generality take $b_0 = 1$.

   Since the fundamental group of the circle is the infinite cyclic group, we only need to determine the action of the maps $g, h$ (which are continuous because they are continuous maps of $\mathbb{C}$) on a generator of the fundamental group.

   Take the positive (counterclockwise) class $a \in \pi_1(S^1, b_0)$ represented by the curve $\exp(2\pi i t)$ for $t \in I$ (this is one generator of the fundamental group). Then the action of $g$ on $a$ returns the class $a'$ represented by the curve $\exp(2\pi i n t)$. With $n$ being a positive integer, it means that the resulting loop is the loop with winding number $n$ (this comes from the formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, which means the frequency of our loop has become $n$-times as much) – we check this by using a lift of the loop.

   Let $p\colon \mathbb{R} \to S^1$ be a covering map given by $p(t) = \exp(2\pi i t)$. By lifting the loop $a'$, we obtain a path in $\mathbb{R}$ starting from 0 and ending at $n$ (i.e. the path $f(t) = nt$ for $t \in I$, and $(p \circ f)(t) = \exp(2\pi i n t)$), meaning the winding number is indeed $n$ as desired ($f(1) = n$). This means that the resulting loop $a'$ is homotopic to $a^n$, where exponentiation here means to take the path product $n$ times. It follows that the induced homomorphism of $g$ on the fundamental group is the one sending any class $a$ to $a^n$.

   Similarly, observe that the action of $h$ on $a$ returns the class $a^{-1\prime}$ represented by the curve $\exp(2\pi i(-n)t)$. Observe that this loops goes in the opposite direction and thus has a winding number of $-n$. We check this with the same covering map as before. Lifting $a^{-1\prime}$ to a path in $\mathbb{R}$, we obtain the path from 0 to $-n$ (i.e. a path $f(t) = -nt$ and $(p \circ f)(t) = \exp(2\pi i(-n)t)$ as desired), so that the winding number is $-n$ ($f(1) = -n$). It follows that $a^{-1\prime}$ is homotopic to $a^{-n} = (a^{-1})^n$, where $a^{-1}$ is the other generator of the fundamental group, the reverse of $a$. □

3. (54.7) Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

*Proof.* Let $p\colon \mathbb{R}^2 \to S^1 \times S^1$ be the covering map given by $p(t_1, t_2) = (\exp(2\pi i t_1), \exp(2\pi i t_2))$, and let $e_0 = (0,0)$, $b_0 = p(e_0) = (1,1)$. It follows from the component maps of $p$ being periodic in the integers that $p^{-1}(b_0) = \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{R}^2$ is simply connected, the lifting correspondence

$$\phi\colon \pi_1(S^1 \times S^1, b_0) \to \mathbb{Z} \times \mathbb{Z}$$

is a bijection. What remains is to check that the correspondence is a homomorphism.

Let $[f]$ and $[g]$ be elements of $\pi_1(S^1 \times S^1, b_0)$, and let $\tilde{f}, \tilde{g}$ be their respective liftings to paths in $\mathbb{R}^2$ beginning at $e_0$. Let $(n_1, n_2) = \tilde{f}(1)$ and $(m_1, m_2) = \tilde{g}(1)$; then $\phi([f]) = (n_1, n_2)$ and $\phi([g]) = (m_1, m_2)$. Then define a path $\tilde{\tilde{g}}$ in $\mathbb{R}^2$ by $\tilde{\tilde{g}}(t) = (n_1, n_2) + \tilde{g}(t)$.

Because the component maps of $p$ are periodic in the integers, we have that $\tilde{\tilde{g}}$ is a lifting of $g$, but beginning at $n$. By the pasting lemma it follows that $\tilde{f} * \tilde{g}$ is defined and is a valid lifting of $f * g$ (by the first exercise) beginning at 0 and ending at $(n_1 + m_1, n_2 + m_2)$. It follows that

$$\phi([f] * [g]) = (n_1 + m_1, n_2 + m_2) = (n_1, n_2) + (m_1, m_2) = \phi([f]) + \phi([g]).$$

Hence $\phi$ is an isomorphism as desired and the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. □

4. (55.1) Show that if $A$ is a retract of $B^2$, then every continuous map $f\colon A \to A$ has a fixed point.

*Proof.* Since $A$ is a retract of $B^2$, we can extend any continuous map $f\colon A \to A$ to a continuous function $g = \iota \circ f \circ r\colon B^2 \to B^2$, where $r\colon X \to A$ is a retraction of $X$ onto $A$ and $\iota\colon A \to B^2$ is the inclusion map. Note that $g(a) = f(a)$ for all $a \in A$ since $r$ is a retraction.

Since $g$ is a continuous map on $B^2$ to itself, it follows from the Brouwer fixed point theorem that there exists an $x \in B^2$ which is fixed by $g$. We claim that $x \in A$. If $x \in B^2 \setminus A$, the retraction $r$ must send $x$ to a point in $A$, which makes it impossible for $x$ to be fixed. Hence $x \in A$, and so by restriction $f = g|_A$ has a fixed point. □

5. (55.2) Show that if $h\colon S^1 \to S^1$ is nulhomotopic, then $h$ has a fixed point and $h$ maps some point $x$ to its antipode $-x$.

*Proof.* By the lemma in the text, the nulhomotopic map $h$ extends to a continuous map $k\colon B^2 \to S^1$; it follows that $g = \iota \circ k\colon B^2 \to B^2$ is a continuous map ($\iota$ is the inclusion map from $S^1$ to $B^2$.). Thus $g$ has a fixed point $x$, and we claim that $x$ is fixed by $h$. Note that in the construction of $k$, we have that $k|_{S^1} = h$.

We show that $x \in S^1$. If $x \in B^2 \setminus S^1$, then by $k$ it will first be mapped into a point in $S^1$, making it impossible for $x$ to be a fixed point of $g$. Hence $x \in S^1$, so that $g|_{S^1} = \iota \circ k|_{S^1}$ has a fixed point. Since $\iota$ fixes all points of $S^1$, it follows that $k|_{S^1}$ must fix $x$, meaning $h$ fixes $x$.

To show that $h$ maps a point $y$ to its antipode $-y$, we can show that the composition $a \circ h$, where $a$ is the antipode map which sends $x$ to $-x$ (this map is continuous since this map is continuous in $\mathbb{C}$), has a fixed point. If such a fixed point $y$ exists for $a \circ h$, then $y = (a \circ h)(y) = a(h(y)) = -h(y)$ which implies that $h(y) = -y$.

It is sufficient to show that $a \circ h$ is homotopic to $h$ itself: Let $H\colon S^1 \times I \to S^1$ be the homotopy given by $H(x, t) = \exp(\pi i t)h(x)$ (continuous since product of continuous maps are continuous), where $H(x, 0) = h(x)$ and $H(x, 1) = -h(x) = (a \circ h)(x)$. Then since $h$ is nulhomotopic, it follows $a \circ h$ is nulhomotopic as well so that it has a fixed point $y$. Hence $h(y) = -y$, so that $h$ maps a point to its antipode. □

6. (57.2) Show that if $g\colon S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all $x$, then $g$ is surjective. [*Hint:* If $p \in S^2$, then $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$.]

   *Proof.* By way of contradiction, suppose $g$ is not surjective so that there exists some point $p \in S^2$ which does not have a preimage under $g$. Then $g$ is equivalent to a continuous function $g'\colon S^2 \to S^2 - \{p\}$ defined by $g'(x) = g(x)$ for all $x \in S^2$. Note that $g'$ inherits from $g$ the property that for any $x \in S^2$, we have $g'(x) \neq g'(-x)$.

   But because $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$, there is a homeomorphism $h\colon S^2 - \{p\} \to \mathbb{R}^2$ such that $h \circ g'\colon S^2 \to \mathbb{R}^2$ is a continuous map. It follows by the Borsuk-Ulam theorem that there exists an $x \in S^2$ such that $(h \circ g')(x) = (h \circ g')(-x)$. But $h$ is a bijection, so it follows that $g'(x) = g'(-x)$, which is in contradiction to the aforementioned property of $g'$.

   Hence $g$ is surjective. □

7. (58.2) The fundamental group of the following spaces is either trivial $\{e\}$, infinite cyclic ($\mathbb{Z}$), or isomorphic to the fundamental group of the figure eight (free group on two symbols $F_2$).

   (a) $B^2 \times S^1$. $\mathbb{Z}$

   (b) Torus with one point removed. $F_2$

   (c) $S^1 \times I$ $\mathbb{Z}$

   (d) $S^1 \times \mathbb{R}$ $\mathbb{Z}$

   (e) $\mathbb{R}^3$ with the nonnegative $x, y, z$ axes removed. $F_2$

   Subsets of $\mathbb{R}^2$:

   (f) $\{x \mid \|x\| > 1\}$ $\mathbb{Z}$

   (g) $\{x \mid \|x\| \geq 1\}$ $\mathbb{Z}$

   (h) $\{x \mid \|x\| < 1\}$ $\{e\}$

   (i) $S^1 \cup (\mathbb{R}_+ \times 0)$ $\mathbb{Z}$

   (j) $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$ $\mathbb{Z}$

   (k) $S^1 \cup (\mathbb{R} \times 0)$ $F_2$

   (l) $\mathbb{R}^2 - (\mathbb{R}_+ \times 0)$ $\{e\}$

8. (58.3) Show that given a collection of spaces $\mathcal{C}$, the relation of homotopy equivalence is an equivalence relation on $\mathcal{C}$.

*Proof.* Let $X, Y, Z \in \mathcal{C}$.

We show that $X$ is homotopic to itself. The identity map on $X$, $\mathrm{id}_X$, satisfies the property that $\mathrm{id}_X \circ \mathrm{id}_X$ is homotopic to $\mathrm{id}_X$. Hence $X$ is homotopy equivalent to itself.

If $X$ is homotopic to $Y$, then there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g\colon f$ is homotopic to the identity on $X$ and $f\colon y$ is homotopic to the identity on $Y$. It is immediate that $Y$ is homotopic to $X$, using the same maps $f, g$.

Suppose $X$ is homotopic to $Y$, and $Y$ is homotopic to $Z$. There exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g\colon f$ is homotopic to the identity on $X$ and $f\colon y$ is homotopic to the identity on $Y$. There also exist continuous maps $h\colon Y \to Z$ and $k\colon Z \to Y$ such that $k \circ h$ is homotopic to the identity on $Y$ and $h \circ k$ is homotopic to the identity on $Z$. We produce maps $F\colon X \to Z$ and $G\colon Z \to X$ such that $G \circ F$ is homotopic to the identity on $X$ and $F \circ G$ is homotopic to the identity on $Z$.

The desired maps are $F = h \circ f$ and $G = g \circ k$. We have that $G \circ F = g \circ k \circ h \circ f$, which is homotopic to $g \circ \mathrm{id}_Y \circ f$ and hence $g \circ f$, which is homotopic to the identity on $X$. Similarly, $F \circ G = h \circ f \circ g \circ k$, homotopic to $h \circ \mathrm{id}_X \circ k = h \circ k$, which is homotopic to the identity on $Z$.

Hence $X$ is homotopic to $Z$ and transitivity is established. It follows that the relation of homotopy equivalence is an equivalence relation on $\mathcal{C}$. $\square$

9. (58.5) Recall that a space $X$ is said to be *contractible* if the identity map of $X$ to itself is nulhomotopic. Show that $X$ is contractible if and only if $X$ has the homotopy type of a one-point space.

*Proof.* Suppose that $X$ has the homotopy type of a one point space; that is, there is a homotopy equivalence $f$ from $X$ to a one-point space $\{p\}$. There is a continuous map $g$ from $\{p\}$ to $X$ such that $g \circ f$ is homotopic to $\mathrm{id}_X$. But a continuous map from $\{p\}$ to $X$ has the image of a single point in $X$. It follows that $g \circ f$ is a constant map on $X$, and we have that $\mathrm{id}_X$ is homotopic to a constant map on $X$.

Conversely, suppose that $\mathrm{id}_X$ is homotopic to a constant map $f$ on $X$ mapping $X$ to some point $x \in X$. We can take the subspace $\{x\}$ to be the one-point space desired. Using the inclusion map $\iota$ from $\{x\}$ into $X$, and defining $g$ to be the map $f$ with the range restricted to $\{x\}$, we have that $\iota \circ g$ is equivalent to the map $f$, which is homotopic to the identity map. Similarly, the map $g \circ \iota$ is a map from $\{x\}$ to itself which is homotopic to the identity map on $\{x\}$ (since that is just what the composition is anyways). Thus $X$ has the homotopy type of a one-point space.

Hence $X$ is contractible if and only if $X$ has the homotopy type of a one-point space. $\square$

10. (58.6) Show that a retract of a contractible space is contractible.

*Proof.* Let $X$ be a contractible space; we have that $\mathrm{id}_X$ is homotopic to a constant map $f$ sending elements of $X$ to $x \in X$. Specifically, let $H$ be the required homotopy such that $H(y,0) = y$ and $H(y,1) = x$. Then let $A$ be a retract of $X$; that is, there exists a continuous map $r\colon X \to A$ such that $r|_A$ is the identity on $A$. We show that the identity map on $A$ is homotopic to a constant map on $A$.

The constant map that $\mathrm{id}_A$ is homotopic to is the composition $r \circ f \circ \iota\colon A \to A$ where $\iota$ is the inclusion map from $A$ to $X$. Define the restriction of the homotopy $H$ to $A \times I$ as a continuous map $H|_{A\times I}\colon A \times I \to X$ defined by $H|_{A\times I}(y,t) = H(y,t)$. Then the homotopy required is given by $r \circ H|_A$, so that $(r \circ H|_A)(y,0) = r(H(y,0)) = r(y) = y$ (the identity on $y$) and $(r \circ H|_A)(y,1) = r(H(y,1)) = r(x)$ (equivalent to $(r \circ f \circ \iota)(y) = r(f(y)) = r(x)$).

Hence $A$ is contractible. $\square$