1. (20.3)

(a) Prove all norms on a finite dimensional vector space $\mathcal{X}$ are equivalent. Suggestion: Fix a basis $e_1, \dots, e_n$ for $\mathcal{X}$ and define $\|\sum a_k e_k\|_1 := \sum |a_k|$. It is routine to check that $\|\cdot\|_1$ is a norm on $\mathcal{X}$. Now complete the following outline.

(i) Let $\|\cdot\|$ be the given norm on $\mathcal{X}$. Show there is an M such that $\|x\| \leq M\|x\|_1$. Conclude that the mapping $\iota\colon (\mathcal{X}, \|\cdot\|_1) \to (\mathcal{X}, \|\cdot\|)$ defined by $\iota(x) = x$ is continuous.

Proof. For $x = \sum x_k e_k \in \mathcal{X}$, we have $\|x\| \leq \sum |x_k|\|e_k\| \leq \max_k\{\|e_k\|\} \sum |x_k| = \max_k\{\|e_k\|\}\|x\|_1$. It follows that for any $x \in \mathcal{X}$, $\|\iota(x)\|/\|x\|_1 \leq \max_k\{\|e_k\|\}$, from which it follows that ι is bounded hence continuous. $\square$

(ii) Show that the *unit sphere* $S = \{x \in \mathcal{X} \colon \|x\|_1 = 1\}$ in $(\mathcal{X}, \|\cdot\|_1)$ is compact in the $\|\cdot\|_1$ topology.

Proof. We show that S is sequentially compact. Given a sequence $(x_j = \sum x_{j_k} e_k)_j$ from S, we show that it has a convergent subsequence. We form this subsequence in an inductive fashion.

We can find a subsequence $(x_{j_\ell})_\ell$ of $(x_j)_j$ such that the first components $(x_{j_\ell 1})_\ell$ of each of the x_{j_ℓ} converge to some $x_1 \in [0,1]$. This can be done since $[0,1]$ is compact/sequentially compact and $(x_{j_1})_j$ is a sequence from $[0,1]$; the subsequence $(x_{j_\ell})_\ell$ consists of vectors x_{j_ℓ} which have first component $x_{j_\ell 1}$. Then repeat this process on $(x_{j_\ell})_\ell$ to obtain a sub-subsequence where the second component of the vectors as a sequence from $[0,1]$ converge to some x_2, while we still have that the first component of the vectors as a sequence converge to x_1. Since $\mathcal{X}$ has finite dimension, we repeat this process to obtain a subsequence $(x_{j_m})_m$ where each of the component sequences $(x_{j_m i})_m$ converge to x_i in $[0,1]$.

We show that $(x_{j_m})_m$ converges to $x = \sum x_k e_k$ in the $\|\cdot\|_1$ topology. Given $\varepsilon > 0$, choose M large enough so that simultaneously for $1 \leq k \leq n$, each of $\left|x_{j_{mk}} - x_k\right| < \varepsilon/n$ for $m > M$. Then for $m > M$ we have $\left\|x_{j_m} - x\right\|_1 \leq \varepsilon$. Furthermore, applying the reverse triangle inequality with our choice of x_{j_m} gives $\left|\|x\|_1 - \left\|x_{j_m}\right\|_1\right| \leq \left\|x - x_{j_m}\right\|_1 \leq \varepsilon$, and since $\left\|x_{j_m}\right\|_1 = 1$ we have that $1-\varepsilon \leq \|x\|_1 \leq 1+\varepsilon$ for any $\varepsilon > 0$. Thus $x \in S$ as needed also, so that S is sequentially compact. $\square$

(iii) Show that the mapping $f\colon S \to \mathbb{R}$ given by $f(x) = \|x\|$ is continuous and hence attains its infimum. Show this infimum is not zero and finish the proof.

Proof. Given $\varepsilon > 0$, we have for $\|x - y\| < \varepsilon$ that $|\|x\| - \|y\|| \leq \|x - y\| < \varepsilon$, meaning $\|\cdot\|$ is continuous (and is Lipschitz also).

The continuous image of a compact set is compact. So $S = \iota(S)$ is compact in the $\|\cdot\|$ topology. It follows that f is uniformly continuous on S and hence attains its infimum $f(s) = c$ for some $s \in S$. If c was zero then s must be the zero vector, which is impossible. Since every element of S is given by $x/\|x\|_1$, we have that $f(x/\|x\|_1) = \|x/\|x\|_1\| = \|x\|/\|x\|_1 \geq c$ so that $\|x\| \geq c\|x\|_1$ for all $x \in X$. Then $c\|x\|_1 \leq \|x\| \leq \max_k\{\|e_k\|\}\|x\|_1$ so that $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. Hence all norms are equivalent on finite dimensional spaces. $\square$

(b) Combine the result of part (a) with the result of Problem 20.2 to conclude that every finite dimensional normed vector space is complete.

Proof. Let $\mathcal{X}$ be a finite dimensional space with basis $\{e_k\}_{k=1}^n$ as above. We show that $\mathcal{X}$ with the $\|\cdot\|_1$-topology is complete.

Let $\sum_{i=1}^\infty \|x_i\|_1$ converge. We show that $\sum_{i=1}^\infty x_i$ converges.

The sum $\sum_{i=1}^\infty \sum_{k=1}^n |x_{ik}| = \sum_{k=1}^n \sum_{i=1}^\infty |x_{ik}|$ converges (and the interchange of order of summation is okay since the original sum converged absolutely). It follows that each of the sums $\sum_{i=1}^\infty |x_{ik}|$ converge for $1 \leq k \leq n$. But then $\sum_{i=1}^\infty x_{ik}$ must converge as a result. Then $\sum_{k=1}^n \sum_{i=1}^\infty x_{ik} = \sum_{i=1}^\infty \sum_{k=1}^n x_{ik} = \sum_{i=1}^\infty x_i$ must converge also (and again the interchange of order of summation is okay due to absolute convergence). It follows that $\mathcal{X}$ with the $\|\cdot\|_1$-topology is complete. By Problem 20.2 it follows that $\mathcal{X}$ with any $\|\cdot\|$-topology is also complete since all norms are equivalent (to the $\|\cdot\|_1$-norm). Hence every finite dimensional normed vector space is complete. □

(c) Let $\mathcal{X}$ be a normed vector space and $\mathcal{M} \subset \mathcal{X}$ a finite-dimensional subspace. Prove that $\mathcal{M}$ is closed in $\mathcal{X}$.

Proof. Every sequence from $\mathcal{M}$ converging in $\mathcal{X}$ is necessarily Cauchy. Then since finite dimensional spaces are complete, such sequences must converge in $\mathcal{M}$. So $\mathcal{M}$ contains all of its limit points, so it is closed. □

2. (20.18) Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper *closed* subspace. Prove for every $\varepsilon > 0$, there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\inf_{y\in\mathcal{M}} \{\|x - y\|\} > 1 - \varepsilon$. (Hint: take any $u \in \mathcal{X} \setminus \mathcal{M}$ and let $a = \inf_{y\in\mathcal{M}} \{\|u - y\|\}$. Choose $\delta > 0$ small enough so that $a/(a+\delta) > 1-\varepsilon$, and then choose $v \in \mathcal{M}$ so that $\|u - v\| < a + \delta$. Finally let $x = (u - v)/\|u - v\|$.)

Proof. Let $\varepsilon > 0$ be given. Following the hint above, we take any $u \in \mathcal{X} \setminus \mathcal{M}$ and let $a = \inf_{y\in\mathcal{M}} \{\|u - y\|\}$. The subspace has to be closed for this infimum to be nonzero for any fixed $u \in \mathcal{X}$. Then we can take $\delta > 0$ small enough for $a/(a + \delta) > 1 - \varepsilon$ (as increasing δ decreases $a/(a + \delta)$). Then with $a + \delta$ greater than the infimum a, we would be able to choose a $v \in \mathcal{M}$ such that $\|u - v\| < a + \delta$. By taking $x = (u - v)/\|u - v\|$, observe that $\|x\| = \|u - v\|/\|u - v\| = 1$. Furthermore we have for every $y \in \mathcal{M}$ that $\|x - y\| = \|(u - v)/\|u - v\| - y\| = \|u - \|u - v\|y\|/\|u - v\| \geq a/\|u - v\| \geq a/(a-\delta) \geq 1-\varepsilon$, since $\|u - v\|y \in \mathcal{M}$. □

3. (20.19) Prove, if $\mathcal{X}$ is an infinite-dimensional normed space, then the unit ball $\mathit{ball}(\mathcal{X}) = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is not compact in the norm topology. (Hint: use the result of problem 20.18 to construct inductively a sequence of vectors $x_n \in \mathcal{X}$ such that $\|x_n\| = 1$ for all n and $\|x_n - x_m\| \geq 1/2$ for all $m < n$.)

Proof. Observe that the span of finitely many nonzero vectors forms a finite dimensional subspace, which is closed.

Let $x_1 \in \mathcal{X}$ be a nonzero vector with $\|x_1\| = 1$. Then by the previous problem we can find $x_2 \in \mathcal{X}$ with $\|x_2\| = 1$ and $\inf_{y\in\langle x_1\rangle} \{\|x_2 - y\|\} > 1 - 1/2 = 1/2$. So in particular $\|x_2 - x_1\| \geq 1/2$. Then we can find $x_3 \in \mathcal{X}$ with $\|x_3\| = 1$ and $\inf_{y\in\langle x_1,x_2\rangle} \{\|x_3 - y\|\} > 1/2$. It follows that $\|x_3 - x_2\| > 1/2$, and $\|x_3 - x_1\| > 1/2$. The inductive step is similar. So by induction we obtain a sequence of vectors $(x_n)_n \subset \mathcal{X}$

with $\|x_n\| = 1$ for all n (so $(x_n)_n \subset ball(S)$) and $\|x_n - x_m\| \geq 1/2$ for all $m < n$. Such a sequence cannot have a convergent subsequence as convergent sequences are Cauchy. Hence $ball(S)$ is not sequentially compact. $\square$