HOMEWORK 6

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Suppose (X, d_X) and (Y, d_Y) are metric spaces. Let $Z = X \times Y$ and define $d : Z \times Z \to [0, \infty)$ by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

for $z_j = (x_j, y_j) \in Z$. By Homework 1, d is a metric on Z.

Prove, if X and Y are connected, then so is Z. You may wish to use the proof outline below. If so, carefully and completely fill in all details providing full explanations.

Outline of proof. Suppose $S \subseteq Z$ is clopen (both open and closed). For $u \in X$ and $v \in Y$, let

$$S_v = \{x \in X : (x, v) \in S\} \subseteq X, \quad S^u = \{y \in Y : (u, y) \in S\} \subseteq Y.$$

(i) Show, for each $v \in Y$ the set $S_v \subseteq X$ is clopen.

(ii) Conclude for each $v \in Y$ either $S_v = X$ or $S_v = \emptyset$.

(iii) Show, if $(a, b) \in S$, then $S_b = X$ and $S^a = Y$. Equivalently, $X \times \{b\}$, $\{a\} \times Y \subseteq S$.

(iv) To complete the proof, show, if $S \neq \emptyset$, then $S = Z$.

$\square$

Proof. For $u \in X$ and $v \in Y$, let $S_v = \{x \in X : (x, v) \in S\} \subseteq X$, and $S^u = \{y \in Y : (u, y) \in S\} \subseteq Y$.

We show that these sets are clopen in their respective spaces; we show the proof for S_v due to symmetry (the argument for S^u being clopen is similar).

For any $v \in Y$, consider a point $p \in S_v$. It follows that $(p, v) \in S$, and since S is open it follows that there is an open ε-ball in Z containing (p, v) (i.e. $N_\varepsilon((p, v)) \subseteq S \subseteq Z$). Then observe that the ε-ball in X containing p is contained in S_v: For $x \in N_\varepsilon(p) \subseteq X$, we have that

$$\varepsilon > d_X(x, p) = d_X(x, p) + d_y(v, v) = d((x, v), (p, v)),$$

meaning $(x, v) \in N_\varepsilon((p, v)) \subseteq S$. Thus $(x, v) \in S$ and so $x \in S_v$ as a result; this yields that $N_\varepsilon(p) \subseteq S_v$, and since p was arbitrary it follows that S_v is open.

We repeat this argument for a point $p \in (S_v)^c$. We have that $(p, v) \in S^c$, and because S is closed it follows that S^c is open so that there is an open ε-ball in Z containing (p, v) (i.e. $N_\varepsilon((p, v)) \subseteq S^c \subseteq Z$) Then observe that the ε-ball in X containing p is contained in $(S_v)^c$, since if $x \in N_\varepsilon(p)$, then $\varepsilon > d_X(x, p) = d_X(x, p) + d_Y(v, v) = d((x, v), (p, v))$. It follows that $(x, v) \in S^c$, meaning that $x \in (S_v)^c$. Hence $N_\varepsilon(p) \subseteq (S_v)^c$; since p was arbitrary it follows that $(S_v)^c$ is open. It follows that S_v is clopen in X (and similarly, S^u is clopen in Y).

With S_v clopen in X, it follows from X being connected that either S_v is the empty set or is X itself. Similarly, S^u is either the empty set or is Y since Y is connected.

Suppose $(a, b) \in S$, so that $a \in S_b$ and $b \in S^a$. Since S_b, S^a are nonempty, it follows that $S_b = X$ and $S^a = Y$. It follows by definition of S_b, S^a that $X \times \{b\}, \{a\} \times Y \subseteq S$.

Then suppose that S is nonempty so that some point $(a, b) \in S$. Then take any point $(p, q) \in Z$. Then it follows from the previous result that $(p, b) \in S$ and so $\{p\} \times Y \subseteq S$. Hence $(p, q) \in S$. Hence $Z \subseteq S$, from which it follows that $S = Z$ whenever S is nonempty.

Therefore, the only clopen sets in Z are the empty set and Z itself, so that Z is connected. □