1. (5.1.1) A cofibration is an embedding. For the proof use that $i_1\colon A \to Z(i), a \mapsto (a,1)$ is an embedding. From $i_1 = rsi_1 = ri_1^X i$ then conclude that i is an embedding.

   Consider a cofibration as an inclusion $i\colon A \subset X$. The image of $s\colon Z(i) \to X \times I$ is the subset $X \times 0 \cup A \times I$. Since s is an embedding, this subset equals the mapping cylinder, i.e., one can define a continuous map $X \times 0 \cup A \times I$ by specifying its restrictions to $X \times 0$ and $A \times I$. (This is always so if A is closed in X, and is a special property of $i\colon A \subset X$ if i is a cofibration.)

   Let X be a Hausdorff space. Then a cofibration $i\colon A \to X$ is a closed embedding. Let $r\colon X \times I \to X \times 0 \cup A \times I$ be a retraction. Then $x \in A$ is equivalent to $r(x,1) = (x,1)$. Hence A is the coincidence set of the maps $X \to X \times I$, $x \mapsto (x,1), x \mapsto r(x,1)$ into a Hausdorff space and therefore closed.

   Proof. Let $i\colon A \to X$ be a cofibration so that equivalently i has the HEP for $Z(i)$. Furthermore, the map $s\colon Z(i) \to X \times I$ obtained via the universal property of the pushout has a retraction $r\colon X \times I \to Z(i)$ such that $rs = \mathrm{id}_{Z(i)}$. We have then that $i_1 = rsi_1 = ri_1^X i$, and since i_1 is injective, we have that i must also be injective, and hence continuously bijective onto its image. Then we can define a continuous left inverse $i_\ell^{-1}\colon i(A) \to A$ for i; from which it follows that i is an embedding.

$$\begin{tikzcd}
 & X \arrow[dr, shift left, "i_0^X"] \arrow[dr, shift right, "i_1^X"'] \arrow[rr, bend left, "b"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0^A"'] \arrow[rrr, bend right, "i_1"] & & X \times I \arrow[r, dashed, shift left, "r"] & Z(i) \arrow[l, dashed, shift left, "s"] \\
 & A \times I \arrow[ur, "i\times \mathrm{id}_I"] \arrow[urr, bend right, "k"'] & &
\end{tikzcd}$$

   Observe that similarly s is an embedding and by definition will take x to $(x,0)$ and (a,t) to itself so the range of s is $X \times 0 \cup A \times I$. Let X be Hausdorff. Since r is a retraction of $X \times I$ onto $Z(i)$ and is the identity on the image $X \times 0 \cup A \times I$ of s, we have $r(x,1) = (x,1)$ if $x \in A$ (the converse is also true). Thus A is the coincidence set $\{x \in X \mid r(x,1) = (x,1)\}$ for the two maps above. From an earlier result, since $X \times I$ is Hausdorff, then the coincidence set A must also be closed. Thus the image of A under the embedding must also be closed, so $i\colon A \subset X$ is a closed embedding. □

2. (5.1.2) If $i\colon K \to L$, $j\colon L \to M$ have the HEP for Y, then ji has the HEP for Y. A homeomorphism is a cofibration. $\emptyset \subset X$ is a cofibration. The sum $\amalg i_j\colon \amalg A_j \to \amalg X_j$ of cofibrations $i_j\colon A_j \to X_j$ is a cofibration.

   Proof. The composition of functions with the HEP has the HEP:

$$\begin{tikzcd}
 & & M \arrow[dr, "i_0^M"] \arrow[drr, bend left, "f"] & & \\
 & L \arrow[ur, "j"] \arrow[dr, "i_0^L"] & & M \times I \arrow[r, dashed, "H'"] & Y \\
K \arrow[ur, "i"] \arrow[dr, "i_0^K"'] & & L \times I \arrow[ur, "j\times \mathrm{id}_I"] \arrow[urr, dashed, bend right, "H"'] & & \\
 & K \times I \arrow[ur, "i\times \mathrm{id}_I"] \arrow[uurrr, bend right, "h"'] & & &
\end{tikzcd}$$

Use the HEP of i to obtain H, and then use the HEP of j to obtain the desired extension H'.

A homeomorphism is a cofibration: With $Z(i)$ the mapping cylinder, we have the solid diagram:

$$\begin{tikzcd}
& X \arrow[dr, "i_0^X"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0^A"'] & & X \times I \arrow[r, dashed, "H"] & Z(i) \\
& A \times I \arrow[ur, "i \times \mathrm{id}_I"'] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

Then define $H\colon X \times I \to Y$ by $H(x,t) = h(i^{-1}(x),t)$; it follows that i has the HEP for $Z(i)$, so i is a cofibration.

The inclusion i of the empty set is a cofibration. Note $Z(i) = X$. Then the following diagram commutes when H is given by $H(x,t) = f(x)$, so the inclusion of the empty set has the HEP for $Z(i)$:

$$\begin{tikzcd}
& X \arrow[dr, "i_0^X"] \arrow[drr, bend left, "f"] & & \\
\emptyset \arrow[ur, hook] \arrow[dr, hook] & & X \times I \arrow[r, dashed, "H"] & X \\
& \emptyset \arrow[ur, hook] \arrow[urr, hook, bend right] & &
\end{tikzcd}$$

Thus the inclusion $\emptyset \subset X$ is a cofibration.

Let $i_j\colon A_j \to X_j$ be cofibrations and let Y be any space. Then from the usual initial data for each j we form the sum of cofibrations:

$$\begin{tikzcd}
& X_j \arrow[dr, "i_0^{X_j}"] \arrow[drr, bend left, "f_j"] & & \\
A_j \arrow[ur, "i_j"] \arrow[dr, "i_0^{A_j}"'] & & X_j \times I \arrow[r, dashed, "H_j"] & Y \\
& A_j \times I \arrow[ur, "i_j \times \mathrm{id}_I"'] \arrow[urr, bend right, "h_j"'] & &
\end{tikzcd}$$

With this form the diagram

$$\begin{tikzcd}
& \amalg X_j \arrow[dr, "\amalg i_0^{X_j}"] \arrow[drr, bend left, "\amalg f_j"] & & \\
\amalg A_j \arrow[ur, "\amalg i_j"] \arrow[dr, "\amalg i_0^{A_j}"'] & & \amalg X_j \times I \arrow[r, dashed, "\amalg H_j"] & Y \\
& \amalg A_j \times I \arrow[ur, "\amalg i_j \times \mathrm{id}_I"'] \arrow[urr, bend right, "\amalg h_j"'] & &
\end{tikzcd}$$

where in particular $\amalg h_j$ is given by $\amalg h_j(a_k,t) = h_k(a_k,t)$ for $a_k \in A_k$. Similarly the homotopy $\amalg H$ is given by $\amalg H(x_k,t) = H_k(x_k,t)$ for $x_k \in X_k$, both using the initial data above. Everything commutes and Y was arbitrary so the sum of cofibrations is also a cofibration. $\square$

3. (5.1.5) Let $A \subset X$ be a cofibration and A contractible. Then the quotient map $X \to X/A$ is a homotopy equivalence.

Proof. In the following diagram we have that $A \subset X$ is a cofibration and that $A \to *$ is a homotopy equivalence: The composition $* \hookrightarrow A \to *$ is the identity on $*$, and the composition $A \to * \hookrightarrow A$ is the constant map on A, homotopic to the identity on A. Then by proposition 5.1.10, $X \to X/A$, the quotient map, is a homotopy equivalence:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & * \arrow[d] \\ X \arrow[r] & X/A \end{tikzcd}$$

□

4. (5.1.6) The space $C'X = X \times I/X \times 1$ is called the unpointed ***cone*** on X. We have the closed inclusions $j\colon X \to C'X, x \mapsto (x, 0)$ and $b\colon \{*\} \to C'X, * \mapsto \{X \times 1\}$. Both maps are cofibrations.

Proof. Observe that $Z(j) = C'X \sqcup X \times I/\sim$ where $(x, 0) \sim (x, 0)$ (where the left $(x, 0) \in C'X$ and the right $(x, 0) \in X \times I$). Furthermore, $Z(j)$ embeds by the map s (coming from the universal property of the pushout) into $C'X \times I$ naturally (see the sketches). A retraction of $C'X \times I$ onto $Z(j)$ is the one that essentially crushes the space onto its subspace.

In a similar way, we find that $Z(b) = C'X \sqcup * \times I/\sim$ where $(*, 0) \sim \{X \times 1\}$, which embeds by s into $C'X \times I$ in a natural way. The retraction from $C'X \times I$ onto this subspace is also just the one that crushes the whole space onto this subspace.

Note that X and $\{*\}$ are closed subspaces since j, b are closed embeddings. By Proposition 5.1.2, since we have the retractions from $C'X \times I$ to the images of $Z(j)$ and $Z(b)$, then j, b are cofibrations. □

5. (5.5.1) A composition of fibrations is a fibration. A product of fibrations is a fibration. $\emptyset \subset B$ is a fibration.

Proof. The composition of fibrations is a fibration. We show that if p, q have the HLP for X, then pq does also, and so if p, q are fibrations then pq does since X may be taken to be any space.

$$\begin{tikzcd}
 & & E \arrow[dl, "q"'] & & \\
 & B \arrow[dl, "p"'] & & E^I \arrow[ul, "e_E^0"'] \arrow[dl, "q^I"'] & X \arrow[l, dashed, "H'"'] \arrow[ull, bend right, "a"'] \arrow[dll, dashed, "H"] \arrow[ddlll, bend left, "h"] \\
C & & B^I \arrow[ul, "e_B^0"'] \arrow[dl, "p^I"'] & & \\
 & C^I \arrow[ul, "e_C^0"'] & & &
\end{tikzcd}$$

Use the HLP twice to obtain H and H'. Thus pq is a fibration since X may be taken to be any space.

Let p, q be fibrations again and take X to be any space. We show that $p \times q$ have the HLP for X also, so that since X was arbitrary, $p \times q$ is a fibration:

$$\begin{tikzcd}
 & E \times E \arrow[dl, "q"'] & & \\
B \times C & & (E \times E)^I \cong E^I \times E^I \arrow[ul, "e_E^0"'] \arrow[dl, "q^I"'] & X \arrow[l, dashed, "H"'] \arrow[ull, bend right, "a"'] \arrow[dll, bend left, "h=h_B \times h_C"] \\
 & (B \times C)^I \cong B^I \times C^I \arrow[ul, "e_{B\times C}^0"] & &
\end{tikzcd}$$

In the above diagram define H to be the homotopy which agrees with the components of a at 0, with $q^I H = h$ also. So H is given by the product of the homotopies obtained by using the HLP for each of h_B, h_C and the components of A. Then with X arbitrary it follows the product of fibrations is a fibration.

Since there are no maps of a nonempty set into the empty set, we have the following commutative diagram:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d, hook] \\
\emptyset \arrow[r, hook] \arrow[ur, dashed] & B
\end{tikzcd}$$

It follows basically trivially that the inclusion of the empty set into a space is a fibration, as it forces $X, X \times I$ for any X to be empty also. $\square$