HOMEWORK 5

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This assignment has two problems.

(a) Show, if $U : \mathbb{C} \to \mathbb{R}$ is harmonic and bounded below, then U is constant. [Suggestion: Observe there is an entire function f whose real part is U and consider the function $\exp(-f)$.]

(b) Show if $f : \mathbb{C} \to \mathbb{C}$ is entire and there is an $M \geq 0$ and $N \in \mathbb{N}$ such that
$$|f(z)| \leq M|z|^N$$
for all z, then f is a polynomial.

Proof. (a) Let M be the real constant such that $U(z) \geq M$ for all $z \in \mathbb{C}$.

Since $\mathbb{C}$ is simply connected we can find a harmonic complement $V : \mathbb{C} \to \mathbb{R}$ such that $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = U(z) + iV(z)$ is analytic (so U is the real part of an analytic function). Then since $-f$, exp are entire, so is $\exp(-f)$ and

$$\begin{aligned}
|\exp(-f(z))| = |\exp(-U(z) - iV(z))| &= |\exp(-U(z))||\exp(-iV(z))| \\
&= |\exp(-U(z))| \\
&\leq |\exp(-M)|
\end{aligned}$$

for all $z \in \mathbb{C}$. Thus the entire function $\exp(-f)$ is bounded above so it must be a constant function by Liouville's theorem. So for all $z \in \mathbb{C}$, $\exp(-f(z)) = c$ for some $c \in \mathbb{C}$, from which it follows that $f(z)$ must be a constant; as a result $U = \mathrm{Re}(f)$ is constant.

(b) For any $R > 0$, since f is entire, f is analytic on $N_R(0)$. Furthermore, $|f|$ will be uniformly bounded above by MR^N on $N_R(0)$. Then by a Cauchy estimate we have for each $n \geq 0$, $\left|f^{(n)}(0)\right| \leq n!MR^N/R^n$. Since R was arbitrary, for $n > N$ we may take R arbitrarily large so that $\left|f^{(n)}(0)\right|$ vanishes, and for $n < N$ we may take R arbitrarily small so that $\left|f^{(n)}(0)\right|$ vanishes also. Hence for $n \neq N$, $f^{(n)}(0) = 0$, and only $\left|f^{(N)}(0)\right|$ could be nonzero.

Since f is entire we may expand f as a power series about the origin. Then with the above we have for any $z \in \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \frac{f^{(N)}(0)}{N!} z^N$. So f is a monomial.

$\square$