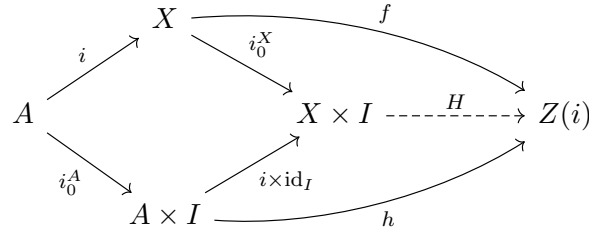


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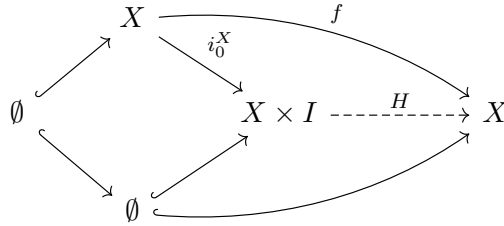
Use the HEP of i to obtain H , and then use the HEP of j to obtain the desired extension H' .

A homeomorphism is a cofibration: With $Z(i)$ the mapping cylinder, we have the solid diagram:



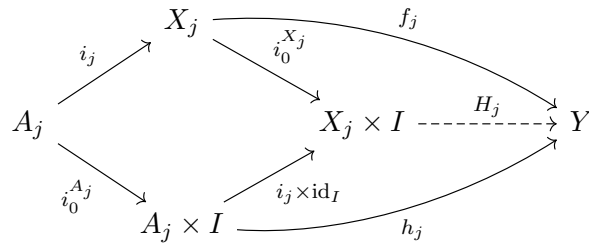
Then define $H: X \times I \rightarrow Y$ by $H(x, t) = h(i^{-1}(x), t)$; it follows that i has the HEP for $Z(i)$, so i is a cofibration.

The inclusion i of the empty set is a cofibration. Note $Z(i) = X$. Then the following diagram commutes when H is given by $H(x, t) = f(x)$, so the inclusion of the empty set has the HEP for $Z(i)$:

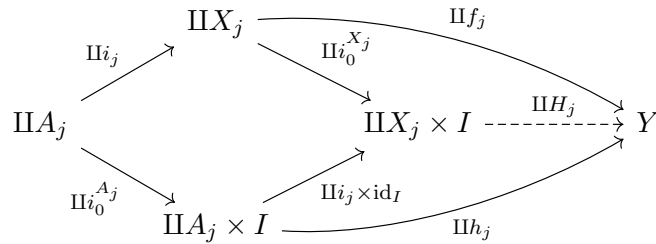


Thus the inclusion $\emptyset \subset X$ is a cofibration.

Let $i_j: A_j \rightarrow X_j$ be cofibrations and let Y be any space. Then from the usual initial data for each j we form the sum of cofibrations:



With this form the diagram



where in particular $\coprod h_j$ is given by $\coprod h_j(a_k, t) = h_k(a_k, t)$ for $a_k \in A_k$. Similarly the homotopy $\coprod H$ is given by $\coprod H(x_k, t) = H_k(x_k, t)$ for $x_k \in X_k$, both using the initial data above. Everything commutes and Y was arbitrary so the sum of cofibrations is also a cofibration. \square

3. (5.1.5) Let $A \subset X$ be a cofibration and A contractible. Then the quotient map $X \rightarrow X/A$ is a homotopy equivalence.

Proof. In the following diagram we have that $A \subset X$ is a cofibration and that $A \rightarrow *$ is a homotopy equivalence: The composition $* \hookrightarrow A \rightarrow *$ is the identity on $*$, and the composition $A \rightarrow * \hookrightarrow A$ is the constant map on A , homotopic to the identity on A . Then by proposition 5.1.10, $X \rightarrow X/A$, the quotient map, is a homotopy equivalence:

$$\begin{array}{ccc} A & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/A \end{array}$$

□

4. (5.1.6) The space $C'X = X \times I / X \times 1$ is called the unpointed **cone** on X . We have the closed inclusions $j: X \rightarrow C'X, x \mapsto (x, 0)$ and $b: \{*\} \rightarrow C'X, * \mapsto \{X \times 1\}$. Both maps are cofibrations.

Proof. Observe that $Z(j) = C'X \sqcup X \times I / \sim$ where $(x, 0) \sim (x, 0)$ (where the left $(x, 0) \in C'X$ and the right $(x, 0) \in X \times I$). Furthermore, $Z(j)$ embeds by the map s (coming from the universal property of the pushout) into $C'X \times I$ naturally (see the sketches). A retraction of $C'X \times I$ onto $Z(j)$ is the one that essentially crushes the space onto its subspace.

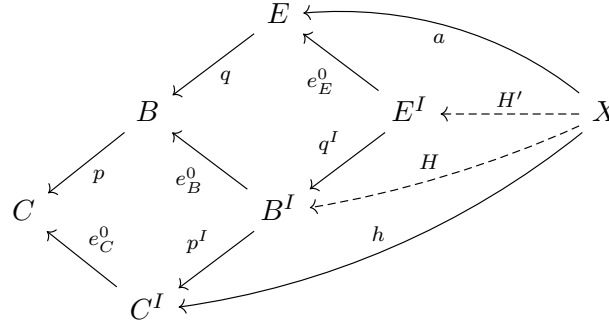
In a similar way, we find that $Z(b) = C'X \sqcup * \times I / \sim$ where $(*, 0) \sim \{X \times 1\}$, which embeds by s into $C'X \times I$ in a natural way. The retraction from $C'X \times I$ onto this subspace is also just the one that crushes the whole space onto this subspace.

Note that X and $\{*\}$ are closed subspaces since j, b are closed embeddings. By Proposition 5.1.2, since we have the retractions from $C'X \times I$ to the images of $Z(j)$ and $Z(b)$, then j, b are cofibrations. □

5. (5.5.1) A composition of fibrations is a fibration. A product of fibrations is a fibration. $\emptyset \subset B$ is a fibration.

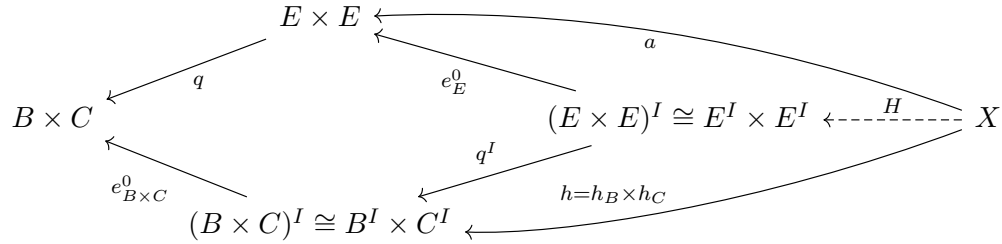
Proof. The composition of fibrations is a fibration. We show that if p, q have the HLP for X , then pq does

also, and so if p, q are fibrations then pq does since X may be taken to be any space.



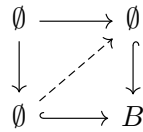
Use the HLP twice to obtain H and H' . Thus pq is a fibration since X may be taken to be any space.

Let p, q be fibrations again and take X to be any space. We show that $p \times q$ have the HLP for X also, so that since X was arbitrary, $p \times q$ is a fibration:



In the above diagram define H to be the homotopy which agrees with the components of a at 0, with $q^I H = h$ also. So H is given by the product of the homotopies obtained by using the HLP for each of h_B, h_C and the components of A . Then with X arbitrary it follows the product of fibrations is a fibration.

Since there are no maps of a nonempty set into the empty set, we have the following commutative diagram:



It follows basically trivially that the inclusion of the empty set into a space is a fibration, as it forces $X, X \times I$ for any X to be empty also. \square