HOMEWORK 5

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Suppose (X, d_X) and (Y, d_Y) are metric spaces. Let $Z = X \times Y$ and define $d : Z \times Z \to [0, \infty)$ by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

for $z_j = (x_j, y_j) \in Z$. By Homework 1, d is a metric on Z.

Suppose $f : X \to Y$ is continuous and let

$$G = \{(x, f(x)) : x \in X\} \subseteq Z$$

(i) Show that the function $F : X \to Z$ defined by $F(x) = (x, f(x))$ is continuous;

(ii) Show, if X is compact, then G is compact;

(iii) Show, if X is complete, then G is complete.

Proof (i). Let $y \in X$ and $\varepsilon > 0$ be given. Since f is continuous, there exists δ' such that if $0 < d_X(x, y) < \delta'$ for $x \in X$, then $d_Y(f(x), f(y)) < \varepsilon/2$. Then choose $\delta = \min\{\delta', \varepsilon/2\}$. Suppose that $d_X(x, y) < \delta$. Then

$$\begin{aligned} d((x, f(x)), (y, f(y))) &= d_X(x, y) + d_Y(f(x), f(y)) \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $y \in X$ was arbitrary, it follows that $F : X \to Z$ defined by $F(x) = (x, f(x))$ is continuous. $\square$

Proof (ii). Let $\mathcal{U}$ be an open cover of G. Then because F is continuous, the preimage of every $U \in \mathcal{U}$ is an open set in X. Observe that

$$X \subseteq F^{-1}(G) \subseteq F^{-1}\left(\bigcup_{U \in \mathcal{U}} U\right) = \bigcup_{u \in \mathcal{U}} F^{-1}(U)$$

because G is a subset of $\bigcup_{U \in \mathcal{U}} U$ and every $x \in X$ has an image under F in G. Because X is compact only finitely many open sets of the form $F^{-1}(U)$ are required to cover X.

There exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ such that $X \subseteq \bigcup_{U \in \mathcal{F}} F^{-1}(U)$. We have that

$$G = F(X) \subseteq F\left(\bigcup_{U \in \mathcal{F}} F^{-1}(U)\right) = \bigcup_{U \in \mathcal{F}} F(F^{-1}(U)) \subseteq \bigcup_{U \in \mathcal{F}} U,$$

from which it follows that $\mathcal{F}$ is a finite open cover of G. Since $\mathcal{U}$ was an arbitrary open cover of G, it follows that G is compact. $\square$

Proof (iii). Let $(p_n = (x_n, f(x_n)))$ be a Cauchy sequence in G. It follows from the previous homework that (x_n) is a Cauchy sequence in X. Since X is complete, (x_n) converges to some $x_0 \in X$, and since f is continuous, it follows that the sequence $(f(x_n))$ in Y converges to $f(x_0)$: Given $\varepsilon > 0$, we can choose δ such that if $d_X(x, x_0) < \delta$ for $x \in X$, then $d_Y(f(x), f(x_0)) < \varepsilon$. Since (x_n) converges to x, there exists an integer N such that if $n \geq N$, then $d_x(x_n, x_0) < \delta$. It follows that for $n \geq N$, we have that $d_Y(f(x_n), f(x_0)) < \varepsilon$; hence $(f(x_n))$ converges to $f(x_0)$ as desired.

By the previous homework, we have that (p_n) converges to $(x_0, f(x_0))$ and since (p_n) was an arbitrary Cauchy sequence in G, it follows that G is complete. $\square$