1. (7.2) Prove the "exercise" claims in Example 1.7 (the claims (c), (d), (g)).

   (c) Let $X$ be an uncountable set. The collection

$$\mathscr{M} := \{E \subset X \mid \text{E is at most countable or } X \setminus E \text{ is at most countable}\}$$

   is a $\sigma$-algebra.

   *Proof.* The collection $\mathscr{M}$ is nonempty: Observe that $X \in \mathscr{M}$ since $X \setminus X = \emptyset$ is finite (similarly $\emptyset \in \mathscr{M}$).

   The collection $\mathscr{M}$ is closed under countable unions: Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets from $\mathscr{M}$. We show that of $A = \bigcup_{i=1}^{\infty} A_i$ and $X \setminus A = X \setminus \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty}(X \setminus A_i)$, one of them is at most countable. If all the $A_i$ are countable then $A$ is a countable union of countable sets which is countable, in which case $A \in \mathscr{M}$. Otherwise at least one $A_i$, say $A_j$, is uncountable, so its complement $X \setminus A_j$ must be countable since $A_j \in \mathscr{M}$. Since $X \setminus A \subseteq A_j$ by construction, it follows that $X \setminus A$ is countable so that $A \in \mathscr{M}$ as desired.

   The collection $\mathscr{M}$ is closed under complements: If $E \in \mathscr{M}$ then one of $E, X \setminus E$ is countable, so the same can be said about the complement $X \setminus E$. It follows that $X \setminus E \in \mathscr{M}$.

   Hence $\mathscr{M}$ is a $\sigma$-algebra. $\square$

   (d) If $\mathscr{M} \subset 2^X$ is a $\sigma$-algebra, and $E$ is any nonempty subset of $X$, then

$$\mathscr{M}_E := \{A \cap E \mid A \in \mathscr{M}\} \subset 2^E$$

   is a $\sigma$-algebra on $E$.

   *Proof.* Since $X \in \mathscr{M}$, it follows that $E = X \cap E \in \mathscr{M}_E$ so that the collection $\mathscr{M}_E$ is nonempty.

   Consider a sequence of sets $(B_n)_{n=1}^{\infty}$ from $\mathscr{M}_E$, where by definition $B_n = A_n \cap E$ for some $A_n \in \mathscr{M}$, for each $n$. Then $B = \bigcap_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty} A_i \cap E = E \cap (\bigcup_{i=1}^{\infty} A_i)$. But $\bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$, so $B \in \mathscr{M}_E$. Hence $\mathscr{M}_E$ is closed under countable unions.

   Let $B \in \mathscr{M}_E$ so that $B = A \cap E$ for some $A \in \mathscr{M}$. Then

$$\begin{aligned} E \setminus B = E \setminus (A \cap E) &= E \cap (X \setminus (A \cap E)) \\ &= (E \cap (X \setminus A)) \cup (E \cap (X \setminus E)) \\ &= E \cap (X \setminus A), \end{aligned}$$

   but $X \setminus A \in \mathscr{M}$ so that $E \setminus B \in \mathscr{M}_E$. It follows that $\mathscr{M}_E$ is closed under complementation.

   Thus $\mathscr{M}_E$ is a $\sigma$-algebra on $E$. $\square$

   (g) If $(Y, \mathscr{N})$ is a measurable space and $f \colon X \to Y$, then the collection

$$f^{-1}(\mathscr{N}) = \left\{f^{-1}(E) \mid E \in \mathscr{N}\right\} \subset 2^X$$

   is a $\sigma$-algebra on $X$.

*Proof.* Let $\mathscr{M} = f^{-1}(\mathscr{N}) = \{f^{-1}(E) \mid E \in \mathscr{N}\} \subset 2^X$ be as above. Since $Y \in \mathscr{N}$, it follows that $f^{-1}(Y) = X \in \mathscr{M}$, so $\mathscr{M}$ is nonempty.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets from $\mathscr{M}$, so that for each $n$, $A_n = f^{-1}(B_n)$ for some $B_n \in \mathscr{N}$. Using the fact that the preimage of a union is the union of the preimages, we obtain that $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$. But because $\bigcup_{i=1}^{\infty} B_i \in \mathscr{N}$, it follows that $A \in \mathscr{M}$, so that $\mathscr{M}$ is closed under countable unions.

Let $A \in \mathscr{M}$ so that $A = f^{-1}(B)$ for some $B \in \mathscr{N}$. The preimage of a complement is the complement of the preimage, so it follows that $X \setminus A = X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$. But $Y \setminus B \in \mathscr{N}$ so that $X \setminus A \in \mathscr{M}$. Hence $\mathscr{M}$ is closed under complementation.

It follows that $\mathscr{M}$ is a $\sigma$-algebra over $X$. □

2. (7.3)

(a) Let $X$ be a set and let $\mathscr{A} = (A_n)_{n=1}^{\infty}$ be a sequence of disjoint, nonempty subsets whose union is $X$. Prove that the set of all finite or countable unions of members of $\mathscr{A}$ (together with $\varnothing$) is a $\sigma$-algebra. (A $\sigma$-algebra of this type is called *atomic*.)

*Proof.* Let $\mathscr{M}$ be the set of all finite or countable unions of members of $\mathscr{A}$. Observe that $X$ is given by the countable union $\bigcup_{i=1}^{\infty} A_i$, which is a countable union of members of $\mathscr{A}$, so $X \in \mathscr{M}$. (We could have also taken the empty union to see that $\emptyset \in \mathscr{M}$ also.) Thus $\mathscr{M}$ is nonempty.

Let $(B_i)_{i=1}^{\infty}$ be a sequence of sets from $\mathscr{M}$. Then for each $i$, $B_i$ is given by $\bigcup_{k \in I_i} A_k$ where for each $i$, $I_i$ is a subset of the positive integers (and is hence countable). Note that $\bigcup_{i=1}^{\infty} I_i$ as a result is a countable subset of the positive integers. Thus $B = \bigcup_{i=1}^{\infty} B_i = \bigcup_{k \in \bigcup_{i=1}^{\infty} I_i} A_k$ is a countable union of elements of $\mathscr{A}$, so that $B \in \mathscr{M}$. Hence $\mathscr{M}$ is closed under countable unions.

Let $B \in \mathscr{M}$ so that $B = \bigcup_{i \in I} A_i$ for some subset $I$ of the positive integers $\mathbb{Z}_+$. Note that $\mathbb{Z}_+ \setminus I$ (taken in $\mathbb{Z}_+$) is also a subset of the positive integers; it follows that $X \setminus B = \left(\bigcup_{i=1}^{\infty} A_i\right) \setminus \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in \mathbb{Z}_+ \setminus I} A_i$ is a countable union of members of $\mathscr{A}$. Hence $B \in \mathscr{M}$, so that $\mathscr{M}$ is closed under complementation also.

It follows that $\mathscr{M}$ is a $\sigma$-algebra over $X$. □

(b) Prove that the Borel $\sigma$-algebra $\mathscr{B}_{\mathbb{R}}$ is *not* atomic. (Hint: there exists an uncountable family of mutually disjoint Borel subsets of $\mathbb{R}$.)

*Proof.* Observe that each of the singleton sets $\{r\}$ for each $r \in \mathbb{R}$ are Borel: Sigma algebras are closed under countable intersections (by De Morgan's laws) so in particular we have $\{r\} = \bigcap_{n=1}^{\infty}(r - 1/n, r + 1/n)$ where $(r - 1/n, r + 1/n)$ are open intervals in $\mathbb{R}$ (hence Borel), which proves the above claim.

But we know that the real numbers are uncountable, and we seek to find a contradiction. To that end, suppose by contradiction that $\mathscr{B}_{\mathbb{R}}$ is atomic so that there exists a sequence $(A_n)_{n=1}^{\infty}$ of disjoint, nonempty subsets whose union is $\mathbb{R}$.

We should be able to form for any real number $r$ the singleton set $\{r\}$ by taking a countable union of the $A_i$. Since the $A_i$ are disjoint and nonempty, it would follow that there is an $A_k$ appearing in this union that is actually just equal to $\{r\}$, and that the countable union is really just the union of the one set $A_k$.

Since this is true for any real number $r$, it would seem that either all of the $A_i$ are singleton sets of real numbers, or that there exists an $A_j$ with more than one element. We rule out the latter scenario: If $A_j$ contains more than one element, e.g., it contains real numbers $x, y$, then it is impossible to form the Borel set $\{x\}$ as a countable union of the $A_i$ since each of the $A_i$ are disjoint.

It must follow then that each of the $A_i$s are singleton sets of real numbers. However, the real numbers are uncountable so that it is impossible for $\bigcap_{n=1}^{\infty} A_n$ to be equal to $\mathbb{R}$. This is a contradiction, so $\mathscr{B}_{\mathbb{R}}$ is not atomic. $\square$

3. (7.7) Prove that if $X, Y$ are topological spaces and $f\colon X \to Y$ is continuous, then $f$ is Borel measurable.

*Proof.* Let $\mathscr{P} \subset 2^Y$ be given by $\left\{E \subset Y \mid f^{-1}(E) \in \mathscr{B}_X\right\}$. We show that $\mathscr{P}$ is a $\sigma$-algebra over $Y$.

Observe that $\mathscr{P}$ is not empty since it contains $Y$: the preimage of $Y$ under $f$ is $X$, which is in $\mathscr{B}_X$.

Then take a sequence of sets $(A_n)_{n=1}^{\infty}$ in $\mathscr{P}$. Then $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) \in \mathscr{B}_X$, so $\bigcup_{i=1}^{\infty} A_i \in \mathscr{P}$. So $\mathscr{P}$ is closed under countable unions.

If $A \in \mathscr{P}$, then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A) \in \mathscr{B}_X$, so $Y \setminus A \in \mathscr{P}$. So $\mathscr{P}$ is closed under complements; it follows that $\mathscr{P}$ is a $\sigma$-algebra over $Y$.

We show that every open set of $Y$ is contained in $\mathscr{P}$. If $U \subseteq Y$ is open, then by continuity of $f$, the set $f^{-1}(U)$ is open in $X$. But $\mathscr{B}_X$ is generated by the open sets of $X$ so that $f^{-1}(U)$ is a Borel set of $X$, which means $U \in \mathscr{P}$. Since $U$ was arbitrary, we have proved the claim.

Since the open sets of $Y$ were contained in $\mathscr{P}$ we have (by Proposition 1.9 in the notes) that the $\sigma$-algebra generated by the open sets of $Y$, the Borel $\sigma$-algebra $\mathscr{B}_Y$, is contained in $\mathscr{P}$. It follows that the preimage of any Borel set of $Y$ under $f$ is a Borel set of $X$, meaning $f$ is Borel measurable as desired. $\square$