HOMEWORK 2

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Let $\mathbb{D}$ denote the open unit disk in the complex plane:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C}.$$

Show if $f : \mathbb{D} \to \mathbb{C}$ is analytic and real-valued, then f is constant. Is this conclusion true if $\mathbb{D}$ is replaced by the set $\{z \in \mathbb{C} : |\operatorname{Re}(z)| > 1\}$? Proof or counterexample.

Proof. Since $f\colon \mathbb{D} \to \mathbb{C}$ is analytic, write $f = u + iv$ for $u = \operatorname{Re}(f), v = \operatorname{Im}(f)\colon \mathbb{C} \to \mathbb{R}$ satisfying the Cauchy-Riemann equations. With f real valued, it follows v is identically zero so that $u'_x = v'_y = 0$ and $u'_y = -v'_x = 0$, from which it follows $f'\colon \mathbb{D} \to \mathbb{C}$ is zero. Since $\mathbb{D}$ is open and connected, by Proposition 2.10 in Conway it follows that f is constant. $\square$

This conclusion is not true if $\mathbb{D}$ is replaced by $P = \{z \in \mathbb{C} : |\operatorname{Re}(z)| > 1\}$, since it is open but not connected: Let $g\colon P \to C$ be specified by analytic real valued functions in the natural way, by $g_r\colon P_r = \{z \in \mathbb{C} : |\operatorname{Re}(z) > 1\} \to \mathbb{C}$ and $g_l\colon P_l = \{z \in \mathbb{C} : |\operatorname{Re}(z) < 1\} \to \mathbb{C}$. An argument similar to the above show that g_r, g_l are constant on their domains. But the constants need not be the same so that g need not be constant (take g to be the function which is 1 on the left half plane P_l and 0 on the right half plane P_l).