HOMEWORK 4

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Discuss the construction of an analytic bijection from

$$\mathcal{S} = \{|z+1| < 2\} \cap \{|z-1| < 2\}$$

to the unit disc $\mathbb{D} = \{|z| < 1\}$. [Suggestion: See the discussion of exponential functions following Corollary 2.11 in Conway.]

Discussion. Let $C_\pm = \{|z \pm 1| = 2\}$ be the boundaries of the discs forming $\mathcal{S}$ above; observe $C_+ \cap C_- = \{\pm i\sqrt{3}\}$. Then let $G\colon \mathbb{C} \to \mathbb{C}$ denote the Möbius map given by

$$G(z) = \left(\frac{z+\sqrt{3}i}{z-\sqrt{3}i}\right)\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right),$$

and see that G sends $-i\sqrt{3}$ to 0, $i\sqrt{3}$ to ∞, 1 to 1, and -1 to $-1/2 + i\sqrt{3}/2$. Hence G maps C_+ to the real axis $\mathbb{R}$ and C_- to the line passing through the origin and $-1/2 + i\sqrt{3}/2$. Since $G(0) = 1/2 + i\sqrt{3}/2$ lies in the first quadrant, G maps $\{|z+1| < 2\}$ to the upper half plane and $\{|z-1| < 2\}$ to the region above but not including the line passing through the origin and $-1/2 + i\sqrt{3}/2$. Hence the intersection of the discs is sent to the complex numbers whose principal argument lies in $(0, 2\pi/3)$; call this region U. Then let $f\colon \mathbb{C} \setminus \{0 + it \mid t \leq 0\} \to \mathbb{C}$ be defined by $f(z) = z^{3/2} = \exp(3/2 \log(z))$. The function f is analytic (it is the composition of analytic functions) and bijective as it possesses the left and right inverse given by $f^{-1}(z) = z^{2/3} = \exp(2/3 \log(z))$. By inspection f takes U to the upper half plane. Then as before, use the Möbius map S^{-1} ($S(z) = -i(z+i)/(z-i)$) from Example 2.47 in the notes to map the upper half plane to the unit disc. Thus $S^{-1} \circ f \circ G\colon \mathcal{S} \to \mathbb{D}$ is the desired analytic bijection. $\square$