1. Assume X is path connected.

   (a) If $\phi\colon \pi_1(X,x_0) \to \pi_1(X,x_1)$ is an isomorphism show that it induces an isomorphism

   $$\phi'\colon \pi_1(X,x_0)/[\pi_1(X,x_0),\pi_1(X,x_0)] \to \pi_1(X,x_1)/[\pi_1(X,x_1),\pi_1(X,x_1)].$$

   Proof. Define the map ϕ' by

   $$a[\pi_1(X,x_0),\pi_1(X,x_0)] \mapsto \phi(a)[\pi_1(X,x_1),\pi_1(X,x_1)].$$

   Since $a \in \pi_1(X,x_0)$ is arbitrary, it follows that this induced map is surjective. Since ϕ is a group homomorphism and the multiplication of cosets is well defined in $\pi_1(X,x_1)/[\pi_1(X,x_1),\pi_1(X,x_1)]$, it follows that ϕ' is also a group homomorphism; that is,

   $$\begin{aligned}(ab)[\pi_1(X,x_0),\pi_1(X,x_0)] \mapsto \phi(ab)[\pi_1(X,x_1),\pi_1(X,x_1)] &= \phi(a)\phi(b)[\pi_1(X,x_1),\pi_1(X,x_1)]\\ &= (\phi(a)[\pi_1(X,x_1),\pi_1(X,x_1)])(\phi(b)[\pi_1(X,x_1),\pi_1(X,x_1)]).\end{aligned}$$

   To show the induced map is injective we can show that its kernel is trivial. Suppose that

   $$a[\pi_1(X,x_0),\pi_1(X,x_0)] \mapsto \phi(a)[\pi_1(X,x_1),\pi_1(X,x_1)] = 0[\pi_1(X,x_1),\pi_1(X,x_1)];$$

   that is, $\phi(a) \in [\pi_1(X,x_1),\pi_1(X,x_1)]$. So $\phi(a)$ is given by a finite product of commutators of the form $[c,d] = cdc^{-1}d^{-1}$ for $c,d \in \pi_1(X,x_1)$. Since ϕ is an isomorphism, it follows that a must also be in the form of commutators $[\phi^{-1}(c),\phi^{-1}(d)] = \phi^{-1}(c)\phi^{-1}(d)\phi^{-1}(c)^{-1}\phi^{-1}(d)^{-1}$ for $\phi^{-1}(c),\phi^{-1}(d) \in \pi_1(X,x_0)$. Thus $a \in [\pi_1(X,x_0),\pi_1(X,x_0)]$, meaning that $a[\pi_1(X,x_0),\pi_1(X,x_0)]$ is the zero element in the quotient group. Hence the kernel of the induced map is trivial, meaning the induced map is injective.

   It follows that ϕ' is an isomorphism of groups. □

   (b) Recall that for a path α from x_0 to x_1 the induced isomorphism on fundamental groups is denoted $\hat{\alpha}$. For any two paths α,β from x_0 to x_1 show that the induced maps on the Abelianizations are the same, i.e., $\hat{\alpha}' = \hat{\beta}'$.

   Proof. Observe that the composition of isomorphisms f,g is an isomorphism $f\circ g$, and that its induced map $(f\circ g)'$ is the composition of the maps induced by each isomorphism individually, $f'\circ g'$ (using the above the above construction for the induced map, it is easy to see this). It is also straightforward to see that for some invertible f, the induced map of f^{-1} is the inverse of f'.

   Let $g = \alpha * \overline{\beta}$. With $\hat{g} = \widehat{\alpha * \overline{\beta}} = \hat{\overline{\beta}} \circ \hat{\alpha} = \hat{\beta}^{-1}\circ\hat{\alpha}$, it follows that $\hat{g}' = (\hat{\beta}^{-1}\circ\hat{\alpha})' = (\hat{\beta}^{-1})'\circ\hat{\alpha}' = (\hat{\beta}')^{-1}\circ\hat{\alpha}'$. Then we show that $\hat{g}'$ is the identity map on $\pi_1(X,x_0)/[\pi_1(X,x_0),\pi_1(X,x_0)]$:

   $$\begin{aligned}f[\pi_1(X,x_0),\pi_1(X,x_0)] \mapsto \hat{g}(f)[\pi_1(X,x_0),\pi_1(X,x_0)] &= (g^{-1}fg)[\pi_1(X,x_0),\pi_1(X,x_0)]\\ &= (fgg^{-1})[\pi_1(X,x_0),\pi_1(X,x_0)]\\ &= f[\pi_1(X,x_0),\pi_1(X,x_0)],\end{aligned}$$

   since the quotient group is Abelian. It follows that $(\hat{\beta}')^{-1}\circ\hat{\alpha}' = \hat{g}' = \mathrm{id}_{\pi_1(X,x_0)/[\pi_1(X,x_0),\pi_1(X,x_0)]}$ so that $\hat{\alpha}' = \hat{\beta}'$ as desired. □

2. Given an equivalence relation on X let $Y = X/\sim$ with the quotient topology and quotient map $p\colon X \to Y$. Let Z be a topological space and $g\colon X \to Z$ a map that is constant on each set $p^{-1}(y)$.

   (a) Show that there exists a map $f\colon Y \to Z$ so that $f \circ p = g$.

   Proof. Choose $f\colon Y \to Z$ such that $f(y) = g(x_y)$, where x_y is some fixed element of $p^{-1}(y)$. We can choose x_y for each $y \in Y$ via the Axiom of Choice, and if we have a selection for each y we can construct f explicitly. Then to see that f satisfies $f \circ p = g$, we have for any $x \in X$ that $f(p(x)) = g(x_{p(x)})$. But $x_{p(x)}$ is an element of $p^{-1}(p(x))$, which also contains x. But g is constant on each set $p^{-1}(y)$ for each $y \in Y$, and since $x_{p(x)}, x \in p^{-1}(p(x))$, we have that $f(p(x)) = g(x_{p(x)}) = g(x)$. Since x was arbitrary, it follows that $f \circ p = g$, and we have constructed one of potentially many functions f which satisfy that property. $\square$

   (b) Show that f is continuous if and only if g is continuous.

   Proof. Suppose that f is continuous. Then observe that the quotient map p is automatically continuous because the open sets V in Y are those sets such that $p^{-1}(V)$ is open in X. So preimages of open sets in Y under p are indeed open sets in X. Then the composition of continuous maps is continuous so that $g = f \circ p$ is continuous also.

   Suppose that g is continuous. We show that preimages of open sets in Z under f are open in Y. Since g is continuous, for any open set U of Z, the set $g^{-1}(U)$ is open in X. But $g^{-1}(U) = (p^{-1} \circ f^{-1})(U) = p^{-1}(f^{-1}(U))$, and since this set is open by the definition of the quotient topology, we must have that $f^{-1}(U)$ is open in Y. This means that the preimage of U under f is open in Y. Since U was arbitrary, it follows that f is continuous. $\square$

3. If M is a compact surface, show that $\pi_1(M\#S^2) \cong \pi_1(M)$.

Proof. Since M is a compact surface, we know from the classification theorem that M is homeomorphic to the connected sum of certain spaces (orientable spaces are the connected sum of n-tori for some n, its genus, or a sphere; non-orientable spaces are homeomorphic to the connected sum of projective planes or the connected sum of tori and a projective plane or a Klein bottle). In any case, this means that we can find some suitable labeled polygonal space P which is homeomorphic to M. Let $a_1, \ldots, a_n$ be the letters which form the word W for which we quotient by in the way we do for labeled polygonal spaces (so P/W is homeomorphic to M). Then we use SVK: $\square$

4. (a) For each $n > 1$, construct a space X with $\pi_1(X) \cong \mathbb{Z}/n\mathbb{Z}$.

   Take the filled in n-gon with the labeling scheme $a, a, \ldots, a$ (n times so each edge is labeled this way). Then it follows from the theorem for labeled polygonal spaces that the fundamental group of this space is given by the presentation $\langle a \mid a^n \rangle$, which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

   (b) Construct a space X with $\pi_1(X) \cong \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/5\mathbb{Z}$.

   Let X be the filled in triangle with labeling scheme aaa, and Y be the filled in pentagon with labeling scheme $bbbbb$. Then the space where X is joined to Y at an interior point (wedge product of spaces) has fundamental group $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/5\mathbb{Z}$. We can see this by using SVK:

5. Consider the octagon with labeling $abcda^{-1}b^{-1}c^{-1}d^{-1}$.

   (a) Show that identifying the edges according to the labeling yields a compact surface.

   (b) What surface is it? (be sure to prove your result).