Graded

1. (10.5.7) Let A be a nonzero finite Abelian group.

   (a) Prove that A is not a projective $\mathbb{Z}$-module.

   Proof. Since A is finite, we have that $|A|A = 0$ so that A has torsion. By the decomposition theorem for finitely generated Abelian groups, we can write $A = \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_s\mathbb{Z}$ for integers n_i with $n_i \mid n_{i+1}$. Then we can form the exact sequence

   $$0 \to \mathbb{Z}^s \xrightarrow{\cdot n_1 \times \cdot n_2 \times \cdots \times \cdot n_s} \mathbb{Z}^s \xrightarrow{\pi_1 \times \pi_2 \times \cdots \times \pi_s} A \to 0$$

   where $\cdot n_1 \times \cdot n_2 \times \cdots \times \cdot n_s$ is multiplication by n_i in the i-th component and π_i is the projection map $\mathbb{Z} \to \mathbb{Z}/n_i\mathbb{Z}$. (This is some kind of "direct sum" of short exact sequences I guess.) But this short exact sequence cannot split because any map $A \to \mathbb{Z}^s$ must be the zero map since every element of A has finite order. Therefore there cannot be a section $s\colon A \to \mathbb{Z}^s$ with $(\pi_1 \times \pi_2 \times \cdots \times \pi_s) \circ s = \mathrm{id}_A$. It follows that A is not projective (not every short exact sequence $0 \to L \to M \to A \to 0$ splits). □

   (b) Prove that A is not an injective $\mathbb{Z}$-module.

   Proof. Since A is finite, we have that $|A|A = 0$ so that A has torsion. Thus A cannot be divisible, so by Baer's criterion A cannot be injective. □

2. (10.5.20) Prove that the polynomial ring $R[x]$ in the indeterminate x over the commutative ring R is a flat R-module.

Proof. The polynomial ring $R[x]$ is isomorphic to $\bigoplus_{i=0}^{\infty} R$ by the isomorphism taking $\sum_{j=0}^{n} a_j x^j$ to $(a_j)_{j=0}^{\infty}$ where $a_j = 0$ for $j > n$ (the map is an R-module homomorphism with inverse taking the sequence $(A_j)_{j=0}^{\infty}$ with finite support to $\sum_{j=0}^{N} A_j x^j$ where $A_j = 0$ for $j > N$).

Then for any R-module N, the tensor product $R[x] \otimes_R N$ is isomorphic to $(\bigoplus_{i=0}^{\infty} R) \otimes_R N$. Since tensor products distribute over direct sums, $(\bigoplus_{i=0}^{\infty} R) \otimes_R N$ is isomorphic to $\bigoplus_{i=0}^{\infty}(R \otimes_R N)$, which is isomorphic to $\bigoplus_{i=0}^{\infty} N$ since $R \otimes_R N \cong N$.

It follows that an isomorphism ϕ_N from $R[x] \otimes_R N$ to $\bigoplus_{i=0}^{\infty} N$ is given by taking the simple tensors $(\sum_{j=0}^{n} a_j x^j) \otimes c$ to $(a_j c)_{j=0}^{\infty}$ where $a_j = 0$ for $j > n$, and extending by linearity. The inverse map is given by taking the sequence $(a_j)_{j=0}^{\infty}$ with finite support to $\sum_{j=0}^{n} x^j \otimes a_j$ where $a_j = 0$ for $j > n$.

We show that given an injective map $\psi\colon L \to M$ the map $1 \otimes \psi\colon R[x] \otimes_R L \to R[x] \otimes_R M$ is also injective. The map $1 \otimes \psi$ is injective if and only if the map $\oplus_{i=0}^{\infty}\psi$, which takes $(\ell_j)_{j=0}^{\infty}$ to $(\psi(\ell_j))_{j=0}^{\infty}$, is injective. This is because for isomorphisms $\phi_L\colon R[x] \otimes_R L \to \bigoplus_{i=0}^{\infty} L$ and $\phi_M\colon R[x] \otimes_R M \to \bigoplus_{i=0}^{\infty} M$ defined in a similar manner to ϕ_N as above, we have $1 \otimes \psi = \phi_M^{-1} \circ \oplus_{i=0}^{\infty}\psi \circ \phi_L$, as the maps agree on the simple tensors: We have

$$\begin{aligned}(\phi_M^{-1} \circ \oplus_{i=0}^{\infty}\psi \circ \phi_L)((\sum_{j=0}^{n} a_j x^j \otimes \ell)) &= (\phi_M^{-1} \circ \oplus_{i=0}^{\infty}\psi)((a_j\ell)_{j=0}^{\infty}) \\ &= \phi^{-1}(a_j\psi(\ell))_{j=0}^{\infty} \\ &= \sum_{j=0}^{\infty}(x^j \otimes a_j\psi(\ell)) \\ &= (\sum_{j=0}^{n} a_j x^j) \otimes \psi(\ell) \\ &= (1 \otimes \psi)((\sum_{j=0}^{n} a_j x^j) \otimes \ell)\end{aligned}$$

as desired. But it is evident that $\oplus_{i=0}^{\infty}\psi$ is injective since for $(\ell_j)_{j=0}^{\infty} \in \ker \oplus_{i=0}^{\infty}\psi$, we have $\psi(\ell_j) = 0$ for all $i \geq 0$; with ψ injective it follows that every ℓ_j is zero as expected. Hence $1 \otimes \psi$ is injective also.

It follows that $R[x]$ is a flat R-module. □

Additional Problems

1. (10.5.15) Let M be a left $\mathbb{Z}$-module and let R be a ring with 1.

   (a) Show that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a left R-module under the action $(r\varphi)(r') = \varphi(r'r)$ (see Exercise 10).

   Proof. It is clear that this set is an additive group under pointwise addition and the zero map as the additive identity. What remains to see is that the action is associative: For $r, a, b \in R$, we have

   $$[(ab)\varphi](r) = \varphi(r(ab)) = \varphi((ra)b) = (b\varphi)(ra) = [a(b\varphi)](r)$$

   so that $(ab)\varphi = a(b\varphi)$ as desired. It is clear that 1_R has trivial action. □

   (b) Suppose that $0 \to A \xrightarrow{\psi} B$ is an exact sequence of R-modules. Prove that if every $\mathbb{Z}$-module homomorphism f from A to M lifts to a $\mathbb{Z}$-module homomorphism F from B to M with $f = F \circ \psi$, then every R-module homomorphism f' from A to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ lifts to an R-module homomorphism F' from B to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ with $f' = F' \circ \psi$. [Given f', show that $f(a) = f'(a)(1_R)$ defines a $\mathbb{Z}$-module homomorphism of A to M. If F is the associated lift of f to B, show that $F'(b)(r) = F(rb)$ defines an R-module homomorphism from B to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ that lifts f'.]

   Proof. Given f' as above we check that f defined as above is a $\mathbb{Z}$-module homomorphism: We have $f(a+b) = f'(a+b)(1_R) = [f'(a)+f'(b)](1_R) = f'(a)(1_R)+f'(b)(1_R) = f(a)+f(b)$. Then we check that F' defined above is an R-module homomorphism; that is, $F'(ax + y)(r)$ agrees with $aF'(x)(r) + F'(y)(r)$ for all $r \in R$. Indeed, $F'(ax + y)(r) = F(r(ax + y)) = F(rax) + F(ry) = F'(x)(ra) + F'(y)(r) = aF'(x)(r) + F'(y)(r)$ as expected.

Then we check that $F' \circ \psi = f'$; that is, for given $a \in A$, for every $r \in R$ we have $[(F' \circ \psi)(a)](r) = f'(a)(r)$. Indeed, $[(F' \circ \psi)(a)](r) = F'(\psi(a))(r) = F(r\psi(a)) = F(\psi(ra)) = f(ra) = f'(ra)(1_R) = (rf'(a))(1_R) = f'(a)(1_R r) = f'(a)(r)$ as desired. □

(c) Prove that if Q is an injective $\mathbb{Z}$-module then $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module.

Proof. Let A and B be R-modules, and let $\psi\colon A \to B$ be injective as above. Since Q is an injective $\mathbb{Z}$-module it is able to lift $\mathbb{Z}$-module maps $f\colon A \to Q$ to maps $F\colon B \to Q$ as in (b). It follows by the result in (b) that $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ also has the desired lifting property, so that it is an injecive R-module. □

2. (10.5.16) This exercise proves Theorem 38 that every left R-module M is contained in an injective left R-module.

(a) Show that M is contained in an injective $\mathbb{Z}$-module Q. [M is a $\mathbb{Z}$-module — use Corollary 37.]

Proof. Considering M as a $\mathbb{Z}$-module (an Abelian group), it follows by Corollary 37 that M is contained in an injective $\mathbb{Z}$-module Q. □

(b) Show that $\mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, Q)$.

Proof. Every R-module homomorphism is a homomorphism of $\mathbb{Z}$-modules (by forgetting the R-action). Since M is contained in Q, then every $\mathbb{Z}$-module homomorphism $R \to M$ is a $\mathbb{Z}$-module homomorphism $R \to Q$ (post-compose with the inclusion map). □

(c) Use the R-module isomorphism $M \cong \mathrm{Hom}_R(R, M)$ (Exercise 10) and the previous exercise to conclude that M is contained in an injective R-module.

Proof. With $R \cong \mathrm{Hom}_R(R, M)$* and $\mathrm{Hom}_R(R, M)$ contained in $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ with Q an injective $\mathbb{Z}$-module, we have from the previous exercise that $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module. Thus M is contained in an injective R-module. □

* (10.5.10(b)) The isomorphism: Define $\varphi_m \in \mathrm{Hom}_R(R, M)$ by $\varphi_m(r) = rm$. We check that φ_m is an R-module homomorphism with respect to the action given in part (a). For $a, b, c \in R$ we have $\varphi_m(ab + c) = (ab + c)m = abm + cm = \varphi_m(ab) + \varphi_m(c) = (b\varphi_m)(a) + \varphi_m(c)$ as needed. Then the map $m \mapsto \varphi_m$ is an R-module isomorphism of M with $\mathrm{Hom}_R(R, M)$: We have that $ax + y \mapsto \varphi_{ax+y}$, and for any $r \in R$ we have $\varphi_{ax+y}(r) = r(ax + y) = rax + ry = \varphi_x(ra) + \varphi_y(r) = (a\varphi_x)(r) + \varphi_y(r)$, so $\varphi_{ax+y} = a\varphi_x + \varphi_y$. The map is injective: If we have $x \mapsto \varphi_x$ with $\varphi_x(r) = rx = 0$ for all $r \in R$, the only possibility is that $x = 0$ since we can take $r = 1_R$. This map is surjective: For any $\varphi \in \mathrm{Hom}_R(R, M)$ take the preimage to be $\varphi(1_R) \in M$, since for any $r \in R$ we have $\varphi_{\varphi(1_R)}(r) = r\varphi(1_R) = \varphi(r)$. Hence M and $\mathrm{Hom}_R(R, M)$ are isomorphic.

Feedback

1. None.

2. Things seem to be the same I think.