The toric structure of principal 2-minor varieties

1. Introduction

In this project, we study a class of algebraic varieties. Let K be an algebraically closed field and I an ideal in the polynomial ring $K[x_1, \dots, x_s]$. A **variety**, or **algebraic set**, is the set of points

$$\mathbf{V}(I) = \{\mathbf{a} \in K^s \mid f(\mathbf{a}) = 0 \text{ for all } f \in I\}.$$

A variety is **normal** if $K[x_1, \dots, x_s]/I$ is integrally closed. Furthermore, we define **Zariski topology** on K^s where closed sets are varieties.

What is a principal 2-minor variety?

Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates. The **principal 2-minor variety** is the variety whose defining ideal $\mathfrak{P}_2 \subset K[X]$ is generated by the 2×2 minors that are symmetric about the main diagonal of X. Principal minor varieties have applications in algebraic statistics and in understanding the Principal Minor Assignment Problem, as studied in [?griffin+tsatsomeros].

What is a toric variety?

Denote $(K^*)^d := (K \setminus \{0\})^d$. A **toric variety** Y is a variety containing an algebraic torus $T \cong (K^*)^d$ as a dense Zariski-open set, where the action of the torus on itself extends to Y. Toric varieties appear in numerous areas of mathematics as varieties parameterized by monomials. The theory of toric varieties is very elegant as their structure can be understood using objects in convex geometry. In addition, toric varieties provide a fertile testing ground for different theorems in algebraic geometry.

It is shown in [?wheeler] that principal 2-minor varieties are toric varieties.

2. The monomial map

A toric variety is given by a **monomial map**. Given $\mathcal{A} = \{\mathbf{m}_1, \dots, \mathbf{m}_s\} \subset \mathbb{Z}^d$, consider the monomial map $\Phi_{\mathcal{A}} : T \cong (K^*)^d \longrightarrow K^s$ defined by

$$\mathbf{t} \longmapsto (\mathbf{t}^{\mathbf{m}_1}, \dots, \mathbf{t}^{\mathbf{m}_s})$$

where $\mathbf{t}^{\mathbf{u}} := t_1^{u_1} \cdots t_d^{u_d}$. The **affine toric variety** $Y_{\mathcal{A}}$ is the Zariski closure of $\operatorname{im}(\Phi_{\mathcal{A}})$. The following proposition gives the monomial map associated with $\mathbf{V}(\mathfrak{P}_2)$.

Proposition 1. *Let $n \geq 2$ and $\mathbf{e}_{ij}$ denote a standard basis vector of $\mathbb{Z}^{\binom{n+1}{2}}$ for $1 \leq i \leq j \leq n$. Then by choosing*

$$\mathcal{A}_n = \{\mathbf{e}_{ij} \mid 1 \leq i \leq j \leq n\} \cup \{\mathbf{e}_{ii} - \mathbf{e}_{ij} + \mathbf{e}_{jj} \mid 1 \leq i < j \leq n\}$$

in the monomial map above, we have that $Y_{\mathcal{A}_n} = \overline{\operatorname{im}(\Phi_{\mathcal{A}_n})} = \mathbf{V}(\mathfrak{P}_2)$.

Example.

For $n = 2$, $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ and so $\mathfrak{P}_2 = \langle x_{11}x_{22} - x_{12}x_{21} \rangle$. Also, for

$$\mathcal{A}_2 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\} \subset \mathbb{Z}^3$$

the monomial map $\Phi_{\mathcal{A}_2} : (K^*)^3 \longrightarrow K^4$ where

$$(t_{11}, t_{12}, t_{22}) \longmapsto (t_{11}^1 t_{12}^0 t_{22}^0, t_{11}^0 t_{12}^1 t_{22}^0, t_{11}^0 t_{12}^0 t_{22}^1, t_{11}^1 t_{12}^{-1} t_{22}^1) = (t_{11}, t_{12}, t_{22}, t_{11} t_{12}^{-1} t_{22}).$$

gives $\mathbf{V}(\mathfrak{P}_2) = \overline{\operatorname{im}(\Phi_{\mathcal{A}_2})}$.

3. Affine toric varieties and cones

We study the **rational polyhedral cone**, $\operatorname{Cone}(\mathcal{A}) = \{\sum_{\mathbf{a} \in \mathcal{A}} \lambda_{\mathbf{a}} \mathbf{a} \mid \lambda_{\mathbf{a}} \geq 0\}$, to understand properties of the affine toric variety $Y_{\mathcal{A}}$. We also study the **dual cone**, $(\operatorname{Cone}(\mathcal{A}))^\vee = \{\mathbf{m} \in \mathbb{R}^d \mid \langle \mathbf{m}, \mathbf{a} \rangle \geq 0 \text{ for all } \mathbf{a} \in \mathcal{A}\}$. The statements in the following theorems (2, 5, and 6) follow from results in [?cox+little+schenck].

Theorem 2. *Let $\sigma_n = \operatorname{Cone}(\mathcal{A}_n)$. Then*

(a) The cone σ_n is strongly convex (i.e., $\sigma_n \cap -\sigma_n = \{\mathbf{0}\}$), thus $Y_{\mathcal{A}_n}$ is normal.

(b) The cone σ_n is not smooth (i.e., its minimal generators do not form a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^{\binom{n+1}{2}}$), thus $Y_{\mathcal{A}_n}$ is not smooth.

Theorem 3. *The dual cone of $\operatorname{Cone}(\mathcal{A}_n)$ is $\operatorname{Cone}(\mathcal{B}_n)$ where*

$$\mathcal{B}_n = \bigcup_{1 \leq i \leq n} \left\{ \mathbf{e}_{ii} + \sum_{\mathbf{v} \in E} \mathbf{v} \;\middle|\; E \subseteq \{\mathbf{e}_{1,i}, \dots, \mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}, \dots, \mathbf{e}_{i,n}\} \right\}.$$

Example.

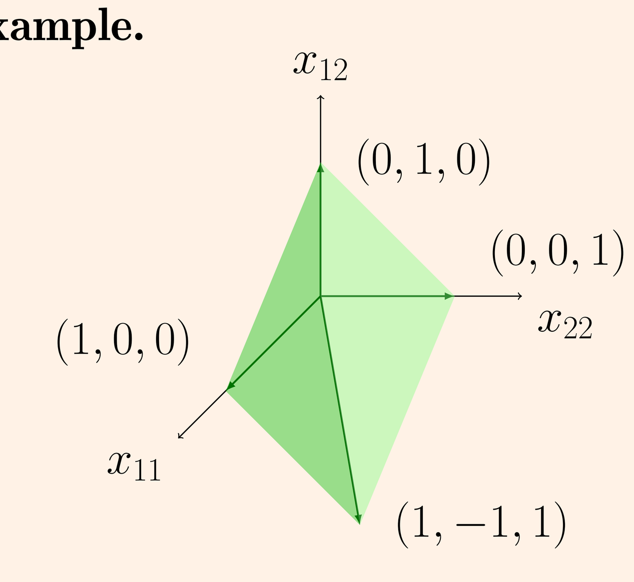

Fig. 1: $\operatorname{Cone}(\mathcal{A}_2)$

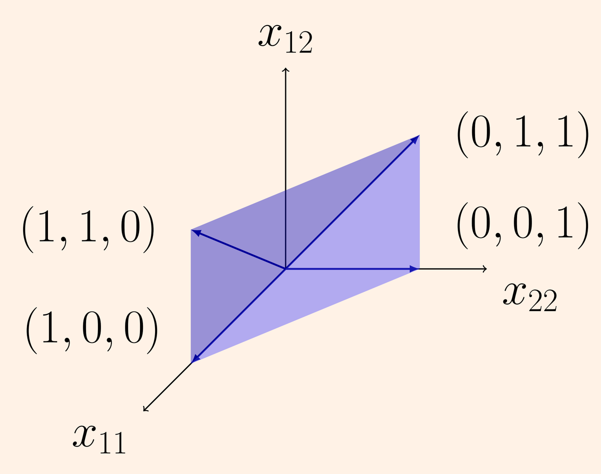

Fig. 2: $\operatorname{Cone}(\mathcal{B}_2)$

4. Projective toric varieties and polytopes

Projective space $\mathbb{P}^{s-1}$ is the set of 1-dimensional linear subspaces of K^s. We identify points in $\mathbb{P}^{s-1}$ by $[z_0 : \cdots : z_{s-1}]$ where $[z_0 : \cdots : z_{s-1}] = [tz_0 : \cdots : tz_{s-1}]$ for all $t \in K^*$.

Given a homogeneous ideal I contained in $K[x_0, \dots, x_{s-1}]$, the **projective variety** is

$$\mathbf{V}(I) = \{[z_0 : \cdots : z_{s-1}] \in \mathbb{P}^{s-1} \mid f(z_0, \dots, z_{s-1}) = 0 \text{ for all } f \in I\}.$$

Since $\mathfrak{P}_2$ is homogeneous, it defines a projective variety. Furthermore, we get a **projective toric variety** $X_{\mathcal{A}}$, which is defined by the Zariski closure of the image of the monomial map onto projective coordinates.

Example.

For $n = 2$, $V(\mathfrak{P}_2)$ is $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding into $\mathbb{P}^3$.

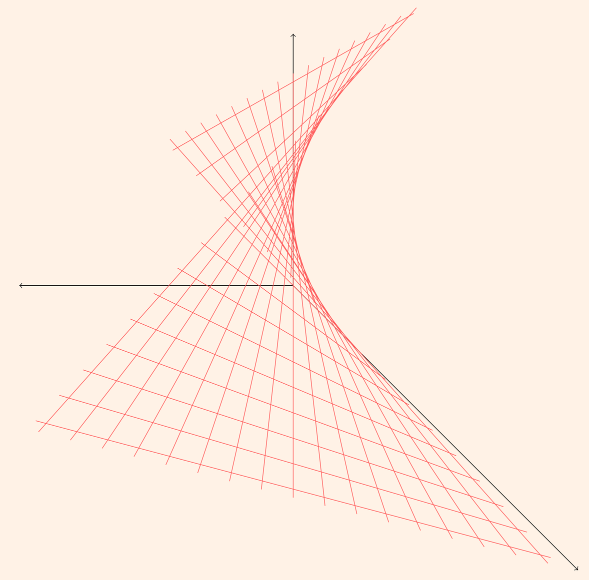

Fig. 3: (Affine open set of) $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

For affine toric varieties $Y_{\mathcal{A}}$, we study $\operatorname{Cone}(\mathcal{A})$. For projective toric varieties $X_{\mathcal{A}}$, we instead study the **convex hull** of $\mathcal{A}$, given by:

$$\operatorname{Conv}(\mathcal{A}) = \left\{ \textstyle\sum_{\mathbf{a} \in \mathcal{A}} \lambda_{\mathbf{a}} \mathbf{a} \mid \lambda_{\mathbf{a}} \geq 0, \sum_{\mathbf{a} \in \mathcal{A}} \lambda_{\mathbf{a}} = 1 \right\}.$$

We define the following: A **polytope** is a convex hull of a finite set of points. A **halfspace** is $H^+_{\mathbf{u},b} = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq b\}$ and a **hyperplane** is $H_{\mathbf{u},b} = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{u} \rangle = b\}$ where $\mathbf{u} \in \mathbb{R}^d$ and $b \in \mathbb{R}$. A **face** Q of a polytope P is $Q = P \cap H_{\mathbf{u},b}$ where $H^+_{\mathbf{u},b} \supseteq P$. **Vertices**, **edges**, and **facets** are faces of dimension 0, 1, and $\dim(P) - 1$, respectively.

Theorem 4. *Let $P_n = \operatorname{Conv}(\mathcal{A}_n)$. The facets of P_n are exactly the sets $P_n \cap H_{\mathbf{u},0}$ for all $\mathbf{u} \in \mathcal{B}_n$.*

Example.

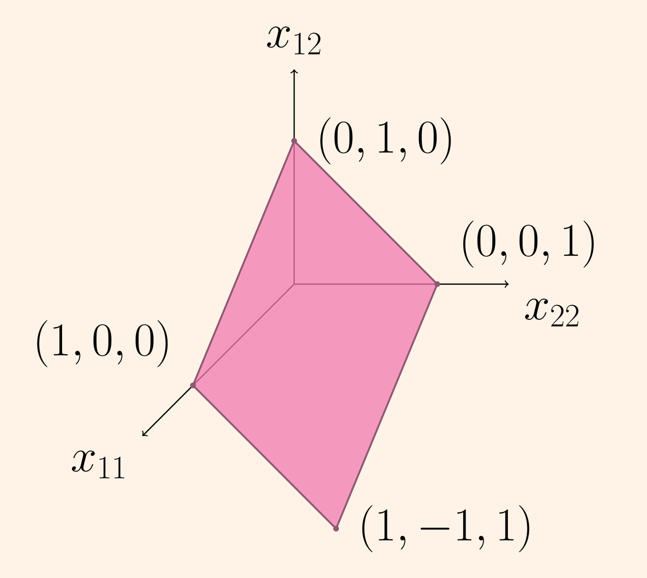

Fig. 4: $P_2 = \operatorname{Conv}(\mathcal{A}_2)$

Theorem 5. *(a) The polytope P_n is normal (i.e., $kP_n \cap \mathbb{Z}^{\binom{n+1}{2}} = \sum_{i=1}^k (P_n \cap \mathbb{Z}^{\binom{n+1}{2}})$ for all $k \geq 1$), so $X_{\mathcal{A}_n}$ is normal.*

(b) When $n \geq 3$, P_n is not smooth (i.e., for some vertex v and all edges $E \supseteq v$, the set of generators of all $\operatorname{Cone}(E - v)$ do not a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^{\binom{n+1}{2}}$), so $X_{\mathcal{A}_n}$ is not smooth.

5. The normal fan

Given a polytope P, its **normal fan** Σ_P consists of the cones

$$\sigma_F = \left\{ \mathbf{u} \in \mathbb{R}^d \mid F \subseteq \operatorname{argmax}_{\mathbf{x} \in P} \langle \mathbf{u}, \mathbf{x} \rangle \right\}$$

where F is some face contained in P.

The normal fan can be obtained using information from the faces of P. A projective toric variety X_{Σ_P} can be constructed using the normal fan of P: the affine open subsets of X_{Σ_P} have coordinate rings that are exactly the affine semigroup rings given by the cones in the fan. We can compute the torus orbits and the divisor class group of X_{Σ_P} using the normal fan.

Example.

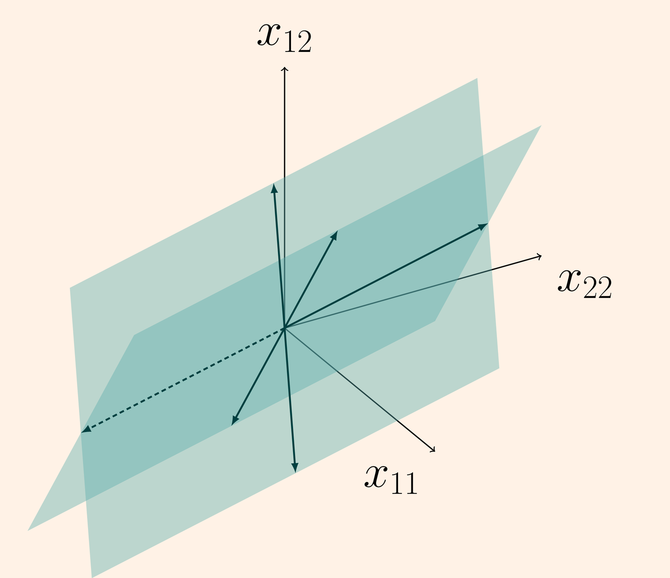

Fig. 5: Normal fan of P_2. The intersection of the cones is the subspace $\operatorname{span}\{(1,1,1)\}$.

Theorem 6. *Let Σ_{P_n} be the normal fan of P_n.*

(a) The variety $X_{\mathcal{A}_n}$ is equal to $X_{\Sigma_{P_n}}$, which is normal and separated (i.e., it is Hausdorff in the classical topology).

(b) Since Σ_{P_n} is complete (i.e., $\bigcup_{\sigma \in \Sigma_{P_n}} \sigma = \mathbb{R}^{\binom{n+1}{2}}$), $X_{\mathcal{A}_n}$ is compact in both the classical and Zariski topologies.

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