HOMEWORK 5

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Prove the following version of the Leibniz rule. Suppose $G \subseteq \mathbb{C}$ is open, $\gamma : [a, b] \to G$ is a smooth curve and $\psi : \{\gamma\} \times G \to \mathbb{C}$. Show, if

(i) ψ is continuous;
(ii) for each $w \in \{\gamma\}$ the function $\psi_w : G \to \mathbb{C}$ defined by $\psi_w(z) = \psi(w, z)$ is analytic; and
(iii) the function $F : \{\gamma\} \times G \to \mathbb{C}$ defined by $F(w, z) = \psi_w'(z)$ is continuous,

then $g : G \to \mathbb{C}$ by

$$g(z) = \int_\gamma \psi(w, z)\, dw$$

is analytic and

$$g'(z) = \int_\gamma F(w, z)\, dw.$$

Proof. We show that g is differentiable (hence analytic) at an arbitrary point of G. Let $z_0 \in G$ and $\varepsilon > 0$ be given. With G open there exists $r > 0$ with $N_r(z_0) \subseteq G$, so that $C = \overline{N_{r/2}(z_0)} \subseteq G$ is a closed set containing z_0 in G. Then $\{\gamma\} \times C$ is closed and bounded, hence compact.

Thus F restricted to $\{\gamma\} \times C$ is uniformly continuous so that there exists δ such that if $d((w_2, z), (w_1, z_0)) = \sqrt{|w_2 - w_1|^2 + |z_0 - z|^2} < \delta$ (where d is the metric induced by the standard inner product on $\mathbb{C}^2$) then $|F(w_2, z) - F(w_1, z_0)| < \varepsilon$. In particular by fixing $w \in \{\gamma\}$ we have $|F(w, z) - F(w, z_0)| < \varepsilon$ whenever $d((w, z), (w, z_0)) = |z - z_0| < \delta$. Then $\left|\int_{[z_0,z]} [F(w, \xi) - F(w, z_0)]\, \mathrm{d}\xi\right| \leq \varepsilon|z - z_0|$ where $[z_0, z]$ denotes the straight line path from z_0 to z lying in C (C is convex).

The analytic function on G given by $\psi_w(z) - zF(w, z_0)$ has derivative $F(w, z) - F(w, z_0)$, so by a version of the fundamental theorem for line integrals proved prior,

$$\left| \int_{[z_0, z]} [F(w, \xi) - F(w, z_0)] \, \mathrm{d}\xi \right| = |\psi_w(z) - \psi_w(z_0) - (z - z_0)F(w, z_0)| \le \varepsilon |z - z_0|$$

whenever $|z - z_0| < \delta$. So by definition of g we have

$$\begin{aligned} \left| \frac{g(z) - g(z_0)}{z - z_0} - \int_\gamma F(w, z_0) \, \mathrm{d}w \right| &= \left| \int_\gamma \frac{\psi(w, z) - \psi(w, z_0) - (z - z_0)F(w, z_0)}{z - z_0} \, \mathrm{d}w \right| \\ &\le \left| \int_\gamma \varepsilon \, \mathrm{d}w \right| \\ &\le \varepsilon V(\gamma), \end{aligned}$$

whenever $0 < |z - z_0| < \delta$, where $V(\gamma) = \int_a^b |\gamma'(s)| \, \mathrm{d}s$ (smooth paths are rectifiable). It follows that g is differentiable (hence analytic) with derivative given by $g'(z) = \int_\gamma F(w, z) \, \mathrm{d}w$. $\square$