1. (7.30) Suppose μ is a regular Borel measure on a compact Hausdorff space and $\mu(X) = 1$. Let $\mathcal{O}$ denote the collection of μ-null open subsets of X and let $U = \cup_{O \in \mathcal{O}} O$. Prove U is also μ-null. Hence U is the largest open μ-null subset of X: Prove there exists a smallest compact subset K of X such that $\mu(K) = 1$. The set K is the support of μ.

   Proof. It is clear that U is open and that U exists (the empty set is one such μ-null open subset). Since μ is regular, $\mu(U) = \sup\{\mu(K) \mid K \subseteq U, K \text{ compact}\}$. We show that for any compact set contained in U (these compact sets do exist since the empty set is one such set), $\mu(K) = 0$. Let K be a compact set contained in U. Then $\mathcal{O}$ is an open cover of K, so that by compactness K is contained in $U' = \cup_{O \in \mathcal{F}} O$ for a finite subset $\mathcal{F}$ of $\mathcal{O}$. It follows that $\mu(K) \leq \mu(U') \leq \sum_{O \in \mathcal{F}} \mu(O) = 0$, and since μ is nonnegative, $\mu(K) = 0$. Since K was arbitrary, it follows that $\mu(U) = \sup\{\mu(K) \mid K \subseteq U, K \text{ compact}\} = 0$. Hence U is a μ-null open set; in particular, it is the largest one since every μ-null open set is contained in U by contstruction.

   Observe that the complement $K = X \setminus U$ is a closed and hence compact (closed subsets of a compact space are compact) subset of X with $\mu(K) = \mu(X \setminus U) = \mu(X) - \mu(U) = 1$ (since $\mu(X) = 1$). Every compact set is contained in K: Let C be a compact and hence closed (compact sets are closed in Hausdorff spaces) set with unit measure. Then $\mu(X \setminus C) = \mu(X) - \mu(C) = 0$, so that $X \setminus C$ is a μ-null open set. Hence $X \setminus C$ is contained in U, so that C is contained in $K = X \setminus U$ as desired. Hence K is the smallest compact subset of X with unit measure; it is called the support of μ. □

2. (7.33) Prove, if X is a compact metric space, then every compact (closed) set in X is a G_δ and likewise every open set an F_σ. Prove, a finite Borel measure on a compact metric space is regular.

   Proof. Let A be a compact hence closed subset of X. Then let $U_n = \cup_{a \in A} B_{1/n}(a)$ be the open set formed from taking the union of all the open $1/n$-balls around each point of A. We show that $A = \cap_{n \in \mathbb{Z}_+} U_n$, a G_δ-set. The containment $A \subseteq \cap_{n \in \mathbb{Z}_+} U_n$ is clear since $A \subseteq U_n$ for each n. Then take $x \in \cap_{n \in \mathbb{Z}_+} U_n$, so that $x \in U_n$ for each n. Hence there exists $a_n \in A$ with $d(a_n, x) < 1/n$; form the sequence (a_n) from A converging to $x \in X$. By closedness of A it follows that $x \in A$, since x was arbitrary we obtain the reverse inclusion $A \supseteq \cap_{n \in \mathbb{Z}_+} U_n$ and hence equality as desired.

   By taking complements one obtains that any open set $X \setminus A = \cup_{n \in \mathbb{Z}_+} (X \setminus U_n)$, which is an F_σ-set.

   Let μ be a finite Borel measure on X. Call $E \subseteq \mathscr{B}_X$ **regular** if $\mu(E) = \inf\{\mu(U) \mid U \supset E, U \text{ open}\} = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}$ and denote the collection of regular sets by $\mathcal{R}$. We show that $\mathcal{R}$ is a σ-algebra: Let E be a regular set and let $\varepsilon > 0$ be given. There exists $U \supset E$ open and $K \subset F$ compact with $\mu(U) < \mu(E) + \varepsilon$ and $\mu(K) > \mu(E) - \varepsilon$. Observe that $X \setminus K$ is open and $X \setminus U$ is closed and thus compact with $\mu(X) - \mu(K) = \mu(X \setminus K) < \mu(X \setminus E) + \varepsilon = \mu(X) - \mu(E) + \varepsilon$ and $\mu(X) - \mu(U) = \mu(X \setminus U) > \mu(X \setminus E) - \varepsilon = \mu(X) - \mu(E) - \varepsilon$. Since ε was arbitrary, $X \setminus E$ is also regular.

   We show first that finite unions of regular sets are regular: Let E, F be regular sets and let $\varepsilon > 0$ be given. There exists $U \supset E, V \supset F$ open and $K \subset E, L \subset F$ compact with $\mu(U) < \mu(E) + \varepsilon/2$, $\mu(K) > \mu(E) - \varepsilon/2$, $\mu(V) < \mu(F) + \varepsilon/2$, and $\mu(L) > \mu(F) - \varepsilon/2$. Then $U \cup V \supset E \cup F$ is open and $K \cup L \subset E \cup F$ is compact with $(U \cup V) \setminus (E \cup F) \subset (U \setminus E) \cup (V \setminus F)$ and $(E \cup F) \setminus (K \cup L) \subset (E \setminus K) \cup (F \setminus L)$. It follows that $\mu((U \cup V) \setminus (E \cup F)) \leq \mu(U \setminus E) + \mu(V \setminus F) < \varepsilon$ and $\mu((E \cup F) \setminus (K \cup L)) \leq \mu(E \setminus K) + \mu(F \setminus L) < \varepsilon$; hence $E \cup F$ is regular. By induction it follows that finite unions of regular sets are regular.

Let (E_n) be a sequence of regular sets and let $\varepsilon > 0$ be given. Produce $(U_n \supset E_n)$ a sequence of open sets and $(K_n \subset E_n)$ a sequence of compact sets with $\mu(U_n) < \mu(E_n) + \varepsilon/2^n$, $\mu(K_n) > \mu(E_n) - \varepsilon/2^n$. Then $\cup_{n=1}^{\infty} U_n \supset \cup_{n=1}^{\infty} E_n$ is open and $\cup_{n \in F} K_n \subset \cup_{n=1}^{\infty} E_n$ for $F \subseteq \mathbb{Z}_+$ a finite subset is compact; the containments $(\cup_{n=1}^{\infty} U_n) \setminus (\cup_{n=1}^{\infty} E_n) \subset \cup_{n=1}^{\infty} (U_n \setminus E_n)$ and hold. Hence $\mu((\cup_{n=1}^{\infty} U_n)/(\cup_{n=1}^{\infty} E_n)) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) < \varepsilon$, which gives outer regularity. For inner regularity we first see that by monotone convergence of sets, there is N large enough with $\mu(\cup_{n=1}^{\infty} K_n) - \mu(\cup_{n=1}^{N} K_n) = \mu(\cup_{n=N+1}^{\infty}) < \varepsilon/2$. Thus

Let (E_n) be a sequence of regular sets and produce $(U_n \supset E_n)$ a sequence of open sets and $(K_n \subset E_n)$ a sequence of compact sets with $\mu(U_n) < \mu(E_n) + \varepsilon/2^n$, $\mu(K_n) > \mu(E_n) - \varepsilon/2^n$. Then $\cup_{n=1}^{\infty} U_n \supset \cup_{n=1}^{\infty} E_n$ is open and $\cup_{n \in F} K_n \subset \cup_{n=1}^{\infty} E_n$ for $F \subseteq \mathbb{Z}_+$ a finite subset is compact; the containments $(\cup_{n=1}^{\infty} U_n)/(\cup_{n=1}^{\infty} E_n) \subset \cup_{n=1}^{\infty} (U_n \setminus E_n)$ and $(\cup_{n=1}^{\infty} E_n)/(\cup_{n \in F} K_n) \subset \cup_{n=1}^{\infty} (E_n \setminus K_n)$ hold. Hence $\mu((\cup_{n=1}^{\infty} U_n)/(\cup_{n=1}^{\infty} E_n)) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) < \varepsilon$ and $\mu((\cup_{n=1}^{\infty} E_n)/(\cup_{n \in F} K_n)) \leq \sum_{n=1}^{\infty} \mu(E_n \setminus K_n) < \varepsilon$ as desired. It follows that countable unions of regular sets are regular. Hence regular sets form a σ-algebra.

We show that the generators of the Borel σ-algebra, the open sets, are regular so that the sigma algebra they generate is also regular by the trivial but useful proposition from the notes. Let V be an open set, by the first part of the problem V is an F_σ so that $V = \cup_{n=1}^{\infty} C_n$ for C_n closed (and hence compact also). We can take V as its own open cover so that $\mu(V)$ is exactly $\inf \{\mu(U) \mid U \supseteq V, U \text{ open}\}$. For inner regularity observe that since $(\cup_{i=1}^{n} C_i)$ is a monotone sequence of sets converging to $\cup_{i=1}^{\infty} C_i = V$, for a fixed $\varepsilon > 0$ we can find N large enough with $\mu(V) - \mu(\cup_{i=1}^{N} C_i) < \varepsilon$. But $\cup_{i=1}^{N} C_i$ is a finite union of closed (compact) sets hence also compact, and since ε was arbitrary it follows that $\mu(V)$ is $\sup \{\mu(K) \mid K \subset V, K \text{ compact}\}$. It follows that V is regular, so that every open set in X is regular. Since open sets generate the Borel σ-algebra $\mathscr{B}_X$, it follows that $\mathscr{B}_X$ is a collection of regular sets. Hence the finite Borel measure μ is regular as desired. $\square$