HOMEWORK 3

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Explain why the power series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ determines an analytic function f with domain $\mathbb{D} = \{|z| < 1\}$.

Prove, for $z \in \mathbb{D}$, that $f(z) = -\log(1-z)$.

Proof. Via the root or ratio tests one obtains that $|z| < 1$ (or $z \in N_1(0) = \mathbb{D}$) in order for the power series, viewed as a function of z, to converge (e.g., from the ratio test we have that the power series converges if $\lim_{k\to\infty} \left|\frac{zk}{k+1}\right| = |z| < 1$).

Apply Theorem 2.21 from the notes: the power series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ is an analytic function $f\colon \mathbb{D} \to \mathbb{C}$ with $f(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$. Moreover, the derivative of f at $z \in \mathbb{D}$ is $f'(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} = \frac{\mathrm{d}}{\mathrm{d}z}[-\log(1-z)]$. It follows that $f(z)$ and $-\log(1-z)$ differ by a constant $C \in \mathbb{C}$. But $0 = \sum_{k=1}^{\infty} \frac{0^k}{k} + \log(1-0) = C$ so that $f(z) = -\log(1-z)$ for all $z \in \mathbb{D}$. $\square$