1. (13.12) Prove that if f is an unsigned measurable function and $\int f < +\infty$, then the set $E \coloneqq \{x \colon f(x) > 0\}$ is σ-finite. (That is, E can be written as a disjoint union of measurable sets $E = \bigcup_{n=1}^\infty E_n$ with each $\mu(E_n) < +\infty$.)

   Proof. Let $E_\infty = \{x \colon f(x) = +\infty\}$, which is measurable since f is measurable and E_∞ is the preimage of the measurable set $\{+\infty\}$. It follows that $\mu(E_\infty) = 0$ since otherwise $\int f = +\infty$ (by bounding below), impossible. Let $E_{1/n} = \{x \colon f(x) > 1/n\}$, measurable again since they are the preimages of the measurable sets $(1/n, +\infty]$. Each of $\mu(E_{1/n})$ need to be finite, since otherwise $\int f = +\infty$ (by bounding below), impossible. Observe that $E_\infty \cup \bigcup_{n=1}^\infty E_{1/n} = E$. We disjointify the sets appearing in the union to form the union of pairwise disjoint sets $\bigcup_{j=0}^\infty F_j = E$. By the construction of the disjointification, each F_j has finite measure since each are formed out of taking a single set difference at a time (e.g. with $F_0 = E_\infty$ and $F_1 = E_1 \setminus E_\infty, \ldots, F_n = E_{1/n} \setminus F_{n-1}$). Thus E may be written as a disjoint union of measurable sets, hence σ-finite as desired. $\square$

2. (13.18) Let f be an unsigned measurable function on the measure space $(X, \mathscr{M}, \mu)$. Prove that the function $\nu \colon \mathscr{M} \to [0, \infty]$ defined by $\nu(E) = \int_E f \, \mathrm{d}\mu$ is a measure and, if g is an unsigned measurable function on X, then $\int g \, \mathrm{d}\nu = \int g f \, \mathrm{d}\mu$.

   Proof. Of course, $\nu(\varnothing) = 0$ since $\int_\varnothing f \, \mathrm{d}\mu = \int \mathbf{1}_\varnothing f \, \mathrm{d}\mu = \int 0 \, \mathrm{d}\mu = 0$. Then let (E_j) be a sequence of disjoint sets in $\mathscr{M}$, and let $E = \cup_j E_j$.

   Observe that because the sets E_j are disjoint, $\mathbf{1}_E = \sum_j \mathbf{1}_{E_j}$ (as a formal sum): We have $\mathbf{1}_E(x) = 1$ if and only if $x \in E$, so x lies in exactly one E_k since the sets E_j are disjoint, hence $\sum_j \mathbf{1}_{E_j}(x) = 1$. Conversely if $\sum_j \mathbf{1}_{E_j}(x) = 1$ (note by disjointness this sum can never exceed 1), then by disjointness of the sets E_j we have $\mathbf{1}_{E_k}(x) = 1$ for some k, so $x \in E_k \subset E$ with $\mathbf{1}_E(x) = 1$. Similarly if $\mathbf{1}_E(x) = 0$ then $x \notin E_j$ for each j, so $\sum_j \mathbf{1}_{E_j}(x) = 0$. Conversely if $\sum_j \mathbf{1}_{E_j}(x) = 0$, then $\mathbf{1}_{E_j}(x) = 0$ for each j so that $x \notin E$ with $\mathbf{1}_E(x) = 0$.

   Then with Tonelli's theorem (Corollary 10.6 in the notes) we have $\nu(E) = \int_E f \, \mathrm{d}\mu = \int_X \mathbf{1}_E f \, \mathrm{d}\mu = \int_X \sum_j \mathbf{1}_{E_j} f \, \mathrm{d}\mu = \sum_j \int_X 1_{E_j} f \, \mathrm{d}\mu = \sum_j \int_{E_j} f \, \mathrm{d}\mu = \sum_j \nu(E_j)$ as desired. Hence ν is a measure.

   Since g is an unsigned measurable function, we may approximate g from below by an increasing sequence of simple unsigned measurable functions s_n; that is, there exists a sequence of simple unsigned measurable functions (s_n) increasing to g pointwise. By the monotone convergence theorem, $\lim_{n\to\infty} \int s_n \, \mathrm{d}\nu = \int g \, \mathrm{d}\nu$. But for any simple function $s = \sum_{j=0}^n c_j \mathbf{1}_{E_j}$ with $\{E_j\}$ a measurable partition of X, $c_j \geq 0$ and $0 \leq s \leq g$, we have $\int_X s \, \mathrm{d}\nu = \sum_{j=0}^n c_j \nu(E_j) = \sum_{j=0}^n c_j \int_{E_j} f \, \mathrm{d}\mu = \int_X \sum_{j=0}^n c_j \mathbf{1}_{E_j} f \, \mathrm{d}\mu = \int_X s f \, \mathrm{d}\mu$. Hence $\lim_{n\to\infty} \int s_n f \, \mathrm{d}\mu = \int g \, \mathrm{d}\nu$.

   However, observe that $(s_n f)$ increases to gf pointwise so that by using the MCT again we have $\lim_{n\to\infty} \int s_n f \, \mathrm{d}\mu = \int g f \, \mathrm{d}\mu = \int g \, \mathrm{d}\nu$ as desired. $\square$

3. (13.23) Evaluate each of the following limits, and carefully justify your claims.

(a) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{\sin(x/n)}{(1+(x/n))^n}\mathrm{d}x = 0$. For each $n \geq 2$, the functions f_n given by $f_n(x) = \mathbf{1}_{[0,+\infty]}(x)\frac{\sin(x/n)}{(1+(x/n))^n}$ are in L^1: observe that from $x \geq 1$ onwards ($|f_n|$ is integrable from 0 to 1) we have $|f_n| \leq 1/(1+(x/n))^n$, which is integrable ($n \geq 2$) from 1 to $+\infty$. Furthermore, f_n converges pointwise to the zero function, since $\sin(x/n)$ converges pointwise to the zero function and $(1+(x/n))^{-n}$ converges pointwise to $\exp(-x)$ (by some definition of e^x). We also have that each $|f_n|$ is bounded above by the 1 function: $|f_n| \leq 1/(1+(x/n))^n \leq 1$ for all $x \in [0, +\infty]$. By the DCT, it follows that the above limit is zero.

(b) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\mathrm{d}x = 0$. For each $n \geq 2$, the functions f_n given by $f_n(x) = \mathbf{1}_{[0,+\infty]}(x)\frac{1+nx^2}{(1+x^2)^n}$ are in L^1: we have $|f_n| = f_n \leq \frac{n}{(1+x^2)^{n-1}}$, which fall off quickly enough (p-test). Furthermore, f_n converges almost everywhere to the zero function: for any fixed $x > 0$, we can make $f_n(x) \leq \frac{1}{x^{2n}} + \frac{nx^2}{(1+x^2)^n}$ arbitrarily small. At $x = 0$ the value of f_n is 1 for all n, but singletons have measure zero. The f_n are uniformly bounded above in n by the 1 function: observe that by the binomial theorem we have $\frac{1+nx^2}{(1+x^2)^n} = \frac{1+nx^2}{1+nx^2+O(x^4)} \leq \frac{1+nx^2}{1+nx^2} = 1$ for all $x \geq 0$. Thus by the DCT, the above limit is zero.

(c) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\mathrm{d}x = \pi/2$. The functions f_n given by $f_n(x) = \mathbf{1}_{[0,+\infty]}(x)\frac{n\sin(x/n)}{x(1+x^2)}$ are in L^1 since $|f_n| \leq \frac{1}{1+x^2}$ (as $\sin(x) \leq x$), which is integrable. The almost everywhere pointwise limit of the f_n is the function given by $\frac{1}{1+x^2}$, since $f_n(x) = \frac{1+O((x/n)^2)}{1+x^2}$ by Taylor series, which converges pointwise to $\frac{1}{1+x^2}$ as n tends to infinity as expected. Lastly, the f_n are uniformly bounded in n by $\frac{1}{1+x^2}$ by a previous estimate, so it follows by the DCT that the above limit is equal to $\int_0^\infty \frac{1}{1+x^2} = \pi/2$.

(d) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{n}{1+n^2x^2}\mathrm{d}x = \pi/2$. The DCT does not apply since there is no uniform bound in n for each of the integrands. But for each n, we have by u-substitution $\int_0^\infty \frac{n}{1+n^2x^2}\mathrm{d}x = \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \pi/2$. Hence the sequence $\left(\int_0^\infty \frac{n}{1+n^2x^2}\mathrm{d}x\right)$ is a constant sequence with limit $\pi/2$.