We prove a lemma. Let (x, y) and (a, b) be two points in $\mathbb{R}^2$. Then if

$$\|(x, y), (a, b)\|_2 \leq \delta,$$

it follows that

$$|x - a| \leq \delta \quad \text{and} \quad |y - b| \leq \delta.$$

Proof. With (x, y) and (a, b) as above, we have that

$$\delta \geq \|(x, y), (a, b)\|_2 = \sqrt{(x - a)^2 + (y - b)^2}$$

implies

$$\delta^2 \geq (x - a)^2 + (y - b)^2,$$

from which it follows that

$$\begin{aligned} (x - a)^2 \leq \delta^2 &\implies |x - a| \leq \delta \\ (y - b)^2 \leq \delta^2 &\implies |y - b| \leq \delta \end{aligned}$$

as desired. $\square$

Let B be a metric space with metric d_B. Let $f\colon \mathbb{R}^2 \to B$ be a function such that for any $k \in \mathbb{R}$, the following functions $f_{(\cdot,k)}\colon \mathbb{R} \to B$ and $f_{(k,\cdot)}\colon \mathbb{R} \to B$ given by $f_{(\cdot,k)}(x) = f(x, k)$ and $f_{(k,\cdot)}(x) = f(k, y)$ are continuous. Show that f is continuous.

Proof. Let (a, b) be any point in $\mathbb{R}^2$. We show that f is continuous at (a, b).

Let $\varepsilon > 0$ be given. Then choose δ_1, δ_2 by continuity of $f_{(\cdot,y)}$ and $f_{(a,\cdot)}$ respectively such that

$$\begin{aligned} |x - a| \leq \delta_1 &\implies d_B(f_{(\cdot,y)}(x), f_{(\cdot,y)}(a)) \leq \varepsilon/2 \\ |y - b| \leq \delta_2 &\implies d_B(f_{(k,\cdot)}(y), f_{(k,\cdot)}(b)) \leq \varepsilon/2. \end{aligned}$$

What ??? No.

Let $\delta = \min\{\delta_1, \delta_2\}$. If $\|(x, y), (a, b)\|_2 \leq \delta$ it follows that

$$\begin{aligned} d_B(f(x, y), f(a, b)) &\leq d_B(f(a, b), f(a, y)) + d_B(f(a, y), f(x, y)) \\ &= d_B(f_{(a,\cdot)}(b), f_{(a,\cdot)}(y)) + d_B(f_{(,)}(), f_{(,)}()) \end{aligned}$$

$\square$

1:

(a) Observe that for any element $ab \in AB$, $ab \leq \sup(A)\sup(B)$ since $0 \leq a \leq \sup(A)$ and $0 \leq b \leq \sup(B)$.

(b) Let $\varepsilon > 0$ be given. We seek some $\varepsilon' > 0$ and elements $a \in A$ and $b \in B$ with $a > \sup(A) - \varepsilon'$ and $b > \sup(B) - \varepsilon'$ such that

$$ab > (\sup(A) - \varepsilon')(\sup(B) - \varepsilon') > \sup(A)\sup(B) - \varepsilon.$$

To that end, choose ε' such that

$$0 < \varepsilon' < \frac{1}{2}\left[(\sup(A) + \sup(B)) - \sqrt{(\sup(A) + \sup(B))^2 - 4\varepsilon}\right] = E$$

and find elements $a \in A$, $b \in B$ with $a > \sup(A) - \varepsilon'$ and $b > \sup(B) - \varepsilon'$. Then

$$\begin{aligned} ab > (\sup(A) - \varepsilon')(\sup(B) - \varepsilon') &> (\sup(A) - E)(\sup(B) - E) \\ &= \sup(A)\sup(B) - (\sup(A) + \sup(B))E + E^2 \\ &= \sup(A)\sup(B) - \varepsilon. \end{aligned}$$

2:

(a) Observe that for any $a \in A$, $-a \leq \sup(-A)$; by negating both sides obtain $a \geq -\sup(-A)$. Hence $-\sup(-A)$ is a lower bound for A.
(b) For any $\varepsilon > 0$, there exists an element $\alpha \in A$ such that $-\alpha \geq \sup(-A) - \varepsilon$, so that $\alpha \leq -\sup(-A) + \varepsilon$; hence $-\sup(-A)$ is the greatest lower bound of A.