HOMEWORK 4

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Suppose (X, d_X) and (Y, d_Y) are metric spaces. Let $Z = X \times Y$ and define $d : Z \times Z \to [0, \infty)$ by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

for $z_j = (x_j, y_j) \in Z$. By Homework 1, d is a metric on Z.

(i) Let $(p_n = (x_n, y_n))_{n=1}^{\infty}$ be a sequence from Z. Show, (p_n) converges to $p_0 = (x_0, y_0) \in Z$ if and only if (x_n) and (y_n) converge to x_0 and y_0 in X and Y respectively.
(ii) Show, if X and Y are both sequentially compact, then so is $Z = (Z, d)$.
(iii) Show, if X and Y are both complete, then so is $Z = (Z, d)$.

(i) *Proof.* Let $(p_n = (x_n, y_n))_{n=1}^{\infty}$ be a sequence from Z, and let $p_0 = (x_0, y_0)$.

Let the sequences (x_n) and (y_n) converge to x_0 and y_0 in X and Y respectively, and let $\varepsilon > 0$ be given. Then there exist $N_1, N_2 \in \mathbb{N}$ such that if $n \geq N_1$, we have $d_X(x_n, x_0) < \varepsilon/2$, and if $n \geq N_2$, we have $d_Y(y_n, y_0) < \varepsilon/2$.

By taking $n \geq \max\{N_1, N_2\}$, it follows that

$$d((x_n, y_n), (x_0, y_0)) = d(x_n, x_0) + d(y_n, y_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence (p_n) converges to p_0.

Conversely, if (p_n) converges to p_0, then for any $\varepsilon > 0$ given, we can find N such that for $n \geq N$,

$$d((x_n, y_n), (x_0, y_0)) = d(x_n, x_0) + d(y_n, y_0) < \varepsilon.$$

But because d_X and d_Y are metrics on X and Y respectively, they map into the nonnegative reals; it follows that $d(x_n, x_0) < \varepsilon$ and $d(y_n, y_0) < \varepsilon$. Hence (x_n) and (y_n) converge to x_0 and y_0 in X and Y respectively. $\square$

(ii) *Proof.* Suppose that X and Y are both sequentially compact. Let $(p_n = (x_n, y_n))_{n=1}^{\infty}$ be a sequence from Z.

Because X is sequentially compact, there exists $x_0 \in X$ such that there is a subsequence (x_{n_j}) of (x_n) which converges to x_0. Then in Y, there exists $y_0 \in Y$ such that the subsequence (y_{n_j}) has a subsequence $(y_{n_{j_k}})$ which converges to y_0 since Y is sequentially compact. Observe that since (x_{n_j}) converges, it is Cauchy; it follows that the subsequence $(x_{n_{j_k}})$ also converges to x_0. Then by using the result from (i), it follows that the subsequence $(p_{n_{j_k}} = (x_{n_{j_k}}, y_{n_{j_k}}))$ converges to $(x_0, y_0) \in Z$. Since (p_n) was an arbitrary sequence from Z, it follows that Z is sequentially compact. $\square$

(iii) *Proof.* Suppose X and Y are both complete. Let $(p_n = (x_n, y_n))_{n=1}^{\infty}$ be a Cauchy sequence from Z.

We first show that the sequences (x_n) and (y_n) are Cauchy due to (p_n) being Cauchy. Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then

$$d(p_n, p_m) = d((x_n, y_n), (x_m, y_m)) = d_X(x_n, x_m) + d_Y(y_n, y_m) < \varepsilon.$$

Because the metrics d_X and d_Y on X and Y respectively map into the nonnegative reals, it follows that $d_X(x_n, x_m) < \varepsilon$ and $d_Y(y_n, y_m) < \varepsilon$. Hence (x_n) and (y_n) are Cauchy sequences in X and Y respectively.

Since X and Y are complete, the sequences (x_n) and (y_n) converge to some x_0 and y_0 in X and Y respectively. Using the result from (i), it follows that (p_n) converges to $(x_0, y_0) \in Z$. Since (p_n) was an arbitrary Cauchy sequence from Z, it follows that Z is complete. $\square$