HOMEWORK 8

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Let $M_n(\mathbb{C})$ denote the $n \times n$ matrices with entries from $\mathbb{C}$. Let $\|A\|$ denote the operator norm of $A \in M_n(\mathbb{C})$. Since all norms on $\mathbb{C}^{n^2}$ are equivalent, $M_n(\mathbb{C})$ with the operator norm is complete.

An routine geometric series argument using $\|A^n\| \leq \|A\|^n$ shows, for $R > \|A\|$, that the series

$$\sum_{n=0}^{\infty} \frac{A^n}{R^n} e^{-ins}$$

converges absolutely and uniformly as a function of $s \in \mathbb{R}$ to

$$(I - \frac{A}{Re^{is}})^{-1}.$$

Use this fact to show, for $R > \|A\|$ and $k \in \mathbb{N}$, that

$$A^k = \frac{1}{2\pi i} \int_{|z|=R} z^k (z - A)^{-1}\, dz,$$

where $|z| = R$ is the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(s) = Re^{is}$. The integral can be interpreted *in the weak sense* – for $x, y \in \mathbb{C}^n$,

$$\langle A^k x, y\rangle = \int_{|z|=R} z^k \,\langle (z - A)^{-1} x, y\rangle\, dz$$

– if you like.

Show, given a polynomial $p = \sum_{j=0}^{d} p_j z^j$,

$$p(A) = \frac{1}{2\pi i} \int_{|z|=R} p(z)\, (z - A)^{-1}\, dz.$$

(This formula is then a version of Cauchy's integral formula.)

Now use Cramer's rule to prove the Cayley-Hamilton Theorem:

For $q(z) = \det(z - A)$,

$$q(A) = 0.$$

Proof. Using the absolute and uniform convergence of the series above, we have for $R > \|A\|$ that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=R} z^k (z-A)^{-1}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{|z|=R} z^{k-1}(I-A/z)^{-1}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{|z|=R} z^{k-1}\sum_{n=0}^{\infty}(A/z)^n\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty} A^n \int_{|z|=R} z^{k-1-n}\,\mathrm{d}z \\
&= \frac{A^k}{2\pi i}\int_{|z|=R} z^{-1}\,\mathrm{d}z = A^k,
\end{aligned}$$

where uniform convergence was used to interchange the sum and integral signs, and Cauchy's integral theorem was used to extract only the $n = k$ term.

Then for $p(z) = \sum_{j=0}^{d} p_j z^j$ a polynomial we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=R} p(z)(z-A)^{-1}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{|z|=R}\sum_{j=0}^{d} p_j z^j (z-A)^{-1}\,\mathrm{d}z \\
&= \sum_{j=0}^{d} p_j \int_{|z|=R} z^j (z-A)^{-1}\,\mathrm{d}z \\
&= \sum_{j=0}^{d} p_j A^j = p(A)
\end{aligned}$$

by the previous result.

By Cramer's rule we have that $\det(z-A)(z-A)^{-1} = \operatorname{adj}(z-A)$, where $\operatorname{adj}(z-A)$ is the adjugate matrix of $(z-A)$. The entries of $\operatorname{adj}(z-A)$ are polynomials in z. So for $q(z) = \det(z-A)$, we have

$$q(A) = \frac{1}{2\pi i}\int_{|z|=R} q(z)(z-A)^{-1}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{|z|=R}\operatorname{adj}(z-A)\,\mathrm{d}z = 0$$

since every entry of $\operatorname{adj}(z-A)$ is an analytic function. □