HOMEWORK 1

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(7.6.1) Show that $\cos(\pi z) = \displaystyle\prod_{n=1}^{\infty}\left[1 - \frac{4z^2}{(2n-1)^2}\right].$

Proof. With $\sin(\pi z) = \pi z \lim_{N\to\infty}\prod_{n=1}^{N}(1 - z^2/n^2)$, we have for $z \in \mathbb{C}\setminus\mathbb{Z}$ that

$$\begin{aligned}
\cos(\pi z) = \frac{\sin(2\pi z)}{2\sin(\pi z)} &= \frac{2\pi z \lim_{N\to\infty}\prod_{n=1}^{N}(1 - 4z^2/n^2)}{2\pi z \lim_{M\to\infty}\prod_{m=1}^{M}(1 - z^2/n^2)} \\
&= \lim_{M\to\infty}\frac{\prod_{m=1}^{M}(1 - 4z^2/(2m)^2)\prod_{m=1}^{M}(1 - 4z^2/(2m-1)^2)}{\prod_{m=1}^{M}(1 - z^2/n^2)} \\
&= \lim_{M\to\infty}\frac{\prod_{m=1}^{M}(1 - z^2/m^2)\prod_{m=1}^{M}(1 - 4z^2/(2m-1)^2)}{\prod_{m=1}^{M}(1 - z^2/n^2)} \\
&= \lim_{M\to\infty}\prod_{m=1}^{M}\left(1 - 4z^2/(2m-1)^2\right) \\
&= \prod_{n=1}^{\infty}\left(1 - 4z^2/(2n-1)^2\right)
\end{aligned}$$

and this limit exists because the limit of the quotient exists, as z was chosen not to be an integer. Then since $\cos(\pi z)$ agrees with the product above for $\mathbb{C}\setminus\mathbb{Z}$, a determining set for $\mathbb{C}$, it follows that for all $z \in \mathbb{C}$ we have $\cos(\pi z) = \prod_{n=1}^{\infty}(1 - 4z^2/(2n-1)^2)$. $\square$