HOMEWORK 3

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(1) What are the boundaries of the following sets? No proof required.
  (a) $\mathbb{Q}$ as subset of the metric space $\mathbb{R}$.
  (b) $\mathbb{Q}$ as a subset of the metric space $\mathbb{Q}$ (as a subspace of $\mathbb{R}$).
  (c) $S = \{(x, \sin(\frac{1}{x})) : x \in (0,1)\} \subseteq \mathbb{R}^2$ as a subset of the Euclidean space $\mathbb{R}^2$. Note, S is the graph of the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \sin(\frac{1}{x})$.

(2) Give an example of a subset E of a metric space X such that $\partial E^\circ \neq \partial E$. No proof required, but you must provide some justification for your identification of both ∂E° and ∂E.

(3) Suppose $X = (X, d)$ be a metric space and E a subset of X. Show if E is both open and closed, then $\partial E = \emptyset$. Is the converse true? Proof or counterexample.

(1) Boundaries:
  (a) $\partial\mathbb{Q} = \mathbb{R}$
  (b) $\partial\mathbb{Q} = \emptyset$ (viewing $\mathbb{Q}$ as a subspace of $\mathbb{R}$)
  (c) $\partial S = S \cup \{(0, t) \colon t \in [-1, 1]\}$

(2) Take $E = S$ and $X = \mathbb{R}^2$, with S as defined above, so that $\partial E^\circ = \emptyset \neq S \cup \{(0, t) \colon t \in [-1, 1]\} = \partial E$. The interior of E is empty, because any open set containing at least one point of E intersects nontrivially with $\mathbb{R}^2 \setminus S$. The boundary of E is the boundary of S, which is the set of all points for which every open set containing these points intersect nontrivially with S and $\mathbb{R}^2 \setminus S$.

(3) A subset of a metric space is open and closed if and only if its boundary is the empty set.

Proof. Suppose X is a metric space and let E be a subset of X.

Suppose E is open and closed so that $X \setminus E$ is open. Then $E^\circ = E$ and $(X \setminus E)^\circ = X \setminus E$, from which it follows that the boundary of E is empty since X is the disjoint union of the interior, boundary, and exterior of E.

Conversely, suppose ∂E is empty. This means that there are no open sets in X which intersect nontrivially with E and simultaneously intersect nontrivially with $X \setminus E$. It follows that any open set in X is either contained in E or is contained in $X \setminus E$. Hence every point of E has a neighborhood contained in E; similarly, every point of $X \setminus E$ has a neighborhood contained in $X \setminus E$. We have that E and its complement are open, so that E is both open and closed. $\square$

Redo:

(1) Boundaries:
   (a) $\partial\mathbb{Q} = \mathbb{R}$
   (b) $\partial\mathbb{Q} = \emptyset$ (viewing $\mathbb{Q}$ as a subspace of $\mathbb{R}$)
   (c) $\partial S = S \cup \{(0,t) \colon t \in [-1,1]\} \cup \{(1, \sin(1))\}$

(2) Take $E = S$ and $X = \mathbb{R}^2$, with S as defined above, so that $\partial E^\circ = \emptyset \neq S \cup \{(0,t) \colon t \in [-1,1]\} \cup \{(1,\sin(1))\} = \partial E$. The interior of E is empty, because any open set containing at least one point of E intersects nontrivially with $\mathbb{R}^2 \setminus S$, and the boundary of the empty set is the empty set (since it is closed and open; follows from (3)). The boundary of E is the boundary of S, which is the set of all points for which every open set containing these points intersect nontrivially with S and $\mathbb{R}^2 \setminus S$.

(3) A subset of a metric space is open and closed if and only if its boundary is the empty set.

Proof. Suppose X is a metric space and let E be a subset of X.

Suppose E is open and closed so that $X \setminus E$ is open. Then $E^\circ = E$ and $(X \setminus E)^\circ = X \setminus E$, from which it follows that the boundary of E is empty since X is the disjoint union of the interior, boundary, and exterior of E.

Conversely, suppose ∂E is empty. This means that there are no points p in X satisfying the condition that every neighborhood of p intersects nontrivially with E and simultaneously intersects nontrivially with $X \setminus E$. So for any $p \in X$, it follows that there exists a neighborhood of p which either does not intersect E or does not intersect $X \setminus E$. Both cannot happen simultaneously since every neighborhood of p contains p, and p is either in E or $X \setminus E$. Hence every point of E has a neighborhood contained in E; similarly, every point of $X \setminus E$ has a neighborhood contained in $X \setminus E$. We have that E and its complement are open, so that E is both open and closed. $\square$