Graded

1. If $F\colon \mathcal{C} \to$ Sets is a representable functor, prove that the representing object is unique up to isomorphism. Suggested strategy: Yoneda

   Proof. Let F be representable by two objects $A, B \in \mathcal{C}$ with natural isomorphisms $\alpha\colon h_A = \hom_{\mathcal{C}}(A, -) \to F$ and $\beta\colon h_B = \hom_{\mathcal{C}}(B, -) \to F$.

   Then by Yoneda there is a bijection of the sets $\hom(B, A)$ and the set of natural transformations from h_A to h_B natural in A and h_B; similarly, there is a bijection of the sets $\hom(A, B)$ and the set of natural transformations from h_B to h_A natural in B and h_A.

   So from the above bijections obtain from $\beta^{-1} \circ \alpha$ and $\alpha^{-1} \circ \beta$ the morphisms $f\colon B \to A$ and $g\colon A \to B$ respectively. We must show that the compositions of f and g produce identities both ways.

   I am not sure how to use the naturality conditions to obtain this from here. $\square$

5. (i) What is a free object on a set S in the category of Abelian groups?

   We describe the free Abelian group $F(S)$ on S to be the group of formal finite sums $\mathbb{Z}[S] = \{\sum_{\text{finite}} c_s s \mid c_s \in \mathbb{Z}, s \in S\}$ with the group addition defined componentwise. This object satisfies the universal property of free objects in the category of Abelian groups. Given any group G and a set map from S into G, there is a unique map $\varphi\colon F(S) \to G$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "f"'] & F(S) \arrow[d, dashed, "\varphi"] \\ & G \end{tikzcd}$$

   Define φ as the map which takes each $s \in F(S)$ to $f(s)$ and extend by linearity (so extended to a $\mathbb{Z}$-module homomorphism).

   (ii) What is a left adjoint to the forgetful functor from Abelian groups to sets?

   The free functor F sending sets S to $F(S)$ as above and sending set maps to their extensions as $\mathbb{Z}$-module maps. Let U be the forgetful functor. We show that for any set S and Abelian group G we have a natural isomorphism $\mathrm{AbGrp}(F(S), G) \cong \mathrm{Set}(S, U(G))$. Informally, we can see this since every $\mathbb{Z}$-module map is determined by its action on basis elements and vice versa.

   (iii) Show that the free Abelian group on S is not free in the category of groups.

   The free Abelian group on S when S has two elements or more will not be free since the free group on S cannot be Abelian. The word $aba^{-1}b^{-1}$ $(a, b \in S)$ is a nonidentity element of the free group on S while it is equal to the identity in the free Abelian group on S. A free Abelian group is a free group if and only if they are isomorphic in the category of groups; if the free group were isomorphic to the free Abelian group then the free group must be Abelian.

Additional Problems

3. For a commutative ring R, define

$$E(R) = \left\{(x, y) \in R^2 \colon y^2 = x^3 - x\right\}.$$

   (i) Define E on morphisms to make it a functor from commutative rings to sets.

   For a commutative ring homomorphism $\varphi\colon R \to S$, define $E(\varphi)\colon E(R) \to E(S)$ to be the map taking (x, y) to $(\varphi(x), \varphi(y))$. Note $(\varphi(x), \varphi(y)) \in E(S)$ since φ is a ring homomorphism.

   (ii) Show that E is representable.

   I feel like the object will be some kind of polynomial ring or ring of rational functions but I don't know how to do something like this without a field. Perhaps $\mathbb{Z}$ or $\mathbb{Q}$ are good candidates for the coefficients but I am not sure... Not complete.

   The functor E is the set of points of an elliptic curve in algebraic geometry. The representing object is the coordinate ring of E (the ring of algebraic functions on E).

4. Let U be the forgetful functor from the category of fields to the category of integral domains.

   (i) Show that sending an integral domain to its field of fractions is a functor F from the category of integral domains to fields.

   (ii) Prove that F and U are adjoint functors. Which is the left one?

   Not done.

Feedback

1. None.

2. I am hoping that over the break I can reset and get back onto a regular schedule. I suspect I have burned out.