1. (7.10) Let $\mathscr{A}$ be an atomic σ-algebra generated by a partition $(A_n)_{n=1}^{\infty}$ of a set X (see Problem 7.3).

   (a) Fix $n \geq 1$. Prove that the function $\delta_n \colon \mathscr{A} \to [0,1]$ defined by

   $$\delta_n(A) = \begin{cases} 1 & \text{if } A_n \subset A \\ 0 & \text{if } A_n \not\subset A \end{cases}$$

   is a measure on $\mathscr{A}$.

   Proof. Every set A_n is not empty, hence $\delta_n(\varnothing) = 0$ since A_n could not be contained in the empty set. Every element of $\mathscr{A}$ is an at most countable union of members of $(A_n)_{n=1}^{\infty}$. If $(E_j)_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathscr{A}$, then for each E_j we can find disjoint subsets $C_j \subset \mathbb{Z}_+$ (so $C_p \cap C_q = \varnothing$ for $p \neq q$) such that $E_j = \bigcup_{k \in C_j} A_k$, so that $E = \bigcup_{j=1}^{\infty} E_j = \bigcup_{k \in \bigcup_{j=1}^{\infty} C_j} A_k$. It follows that $\delta_n(E)$ is 1 if A_n is found in the union $\bigcup_{k \in \bigcup_{j=1}^{\infty} C_j} A_k$ (that is, if $n \in \bigcup_{j=1}^{\infty} C_j$) and is 0 otherwise. This is the same as taking the sum $\sum_{j=1}^{\infty} \delta_n(E_j)$ since A_n is either contained in $E_j = \bigcup_{k \in C_j} A_k$ or it is not, for each j; furthermore, A_n would only appear at most once in $\bigcup_{j=1}^{\infty} E_j$ since the sets E_j are disjoint. □

   (b) Prove that if μ is any measure on $(X, \mathscr{A})$, then there exists a unique sequence (c_n) with each $c_n \in [0, +\infty]$ such that

   $$\mu(A) = \sum_{n=1}^{\infty} c_n \delta_n(A)$$

   for all $A \in \mathscr{A}$.

   Proof. If there exists a sequence (c_n) satisfying the above then it is unique: Let (c_n) and (d_n) be sequences satisfying the above so that for any $A \in \mathscr{A}$, we have $\mu(A) = \sum_{n=1}^{\infty} c_n \delta_n(A) = \sum_{n=1}^{\infty} d_n \delta_n(A)$. But for each $i \in \mathbb{Z}_+$ we have

   $$d_i = \sum_{n=1}^{\infty} d_n \delta_n(A_i) = \sum_{n=1}^{\infty} c_n \delta_n(A_i) = c_i$$

   from which it follows that $(c_n) = (d_n)$.
   Let $A \in \mathscr{A}$ so that $A = \bigcup_{k \in C} A_k$ for some $C \subseteq \mathbb{Z}_+$. Then

   $$\begin{aligned} \mu(A) &= \mu\left(\bigcup_{k \in C} A_k\right) \\ &= \sum_{k \in C} \mu(A_k) && \text{(the } A_k \text{ are disjoint)} \\ &= \sum_{k \in C} \mu(A_k)\delta_k(A) && \text{(for } k \in C,\ \delta_k(A) = 1) \\ &= \sum_{k \in C} \mu(A_k)\delta_k(A) + \sum_{k \in \mathbb{Z}_+ \setminus C} \mu(A_k)\delta_k(A) && \text{(for } k \in \mathbb{Z}_+ \setminus C,\ \delta_k(A) = 0) \\ &= \sum_{n=1}^{\infty} \mu(A_n)\delta_n(A), \end{aligned}$$

   and since μ maps into $[0, +\infty]$, we have our desired sequence ($c_n = \mu(A_n)$). □

2. (7.12) Let X be a set. For a sequence of subsets (E_n) of X, define

$$\limsup E_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n, \quad \liminf E_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n.$$

(a) Prove that $\limsup \mathbf{1}_{E_n} = \mathbf{1}_{\limsup E_n}$ and $\liminf \mathbf{1}_{E_n} = \mathbf{1}_{\liminf E_n}$ (thus justifying the names). Conclude that $E_n \to E$ pointwise if and only if $\limsup E_n = \liminf E_n = E$. (Hint: for the first part, observe that $x \in \limsup E_n$ if and only if x lies in infinitely many of the E_n, and $x \in \liminf E_n$ if and only if x lies in all but finitely many E_n.)

Proof. Let $x \in X$. Then $\limsup \mathbf{1}_{E_n}(x) = \lim(\sup\{\mathbf{1}_{E_k}(x) \mid k \geq n\})$. This limit is 1 if and only if there exists an $N \geq 1$ such that for $n \geq N$, $x \in E_j$ for some $j \geq n$. When such an N exists, since $x \in E_j$ with $j \geq n \geq N \geq 1$ it follows that $\sup\{\mathbf{1}_{E_k} \mid k \geq 1\} = 1$. Therefore we demand that there exists $j \geq N$ for every $N \geq 1$ such that $x \in E_j$ for $\lim(\sup\{\mathbf{1}_{E_k}(x) \mid k \geq n\})$ to be 1.

This condition on x is the same as the condition needed for x to be in $\limsup E_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$; that is, for every $N \geq 1$, x needs to be in at least one E_j for $j \geq N$. Hence $\limsup \mathbf{1}_{E_n} = \mathbf{1}_{\limsup E_n}$.

Similarly, for $x \in X$, the quantity $\liminf \mathbf{1}_{E_n}(x) = \lim(\inf\{\mathbf{1}_{E_k}(x) \mid k \geq n\})$. This limit is 1 if and only if there exists an $N \geq 1$ such that for $n \geq N$, $x \in E_j$ for every $j \geq n$. This is exactly the condition needed for x to be in $\liminf E_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n$; that is, there exists $N \geq 1$ such that for every $n \geq N$, $x \in E_n$. Hence $\liminf \mathbf{1}_{E_n} = \mathbf{1}_{\liminf E_n}$.

The sequence (E_n) converges to E pointwise if and only if for every $x \in X$, the sequence $(\mathbf{1}_{E_n}(x))$ converges to $\mathbf{1}_E(x)$. This is equivalent to saying $\limsup \mathbf{1}_{E_n}(x) = \liminf \mathbf{1}_{E_n}(x) = \mathbf{1}_E(x)$. By the above two results we equivalently have that $\mathbf{1}_{\limsup E_n}(x) = \mathbf{1}_{\liminf E_n}(x) = \mathbf{1}_E(x)$, which is equivalent to $\limsup E_n = \liminf E_n = E$ as desired. □

(b) Prove that if the E_n are measurable, then so are $\limsup E_n$ and $\liminf E_n$. Deduce that if (E_n) converges to E pointwise and all the E_n are measurable, then E is measurable.

Proof. Since σ-algebras are closed under countable unions and intersections it is clear that $\limsup E_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$ and $\liminf E_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n$ whenever every E_n is measurable.

Then if (E_n) converges to E pointwise we have from the above result that $\limsup E_n = \liminf E_n = E$; since $\limsup E_n$ and $\liminf E_n$ were shown to be measurable whenever every E_n is measurable, E is measurable. □

3. (7.13) [Fatou theorem for sets] Let $(X, \mathscr{M}, \mu)$ me a measure space, and let (E_n) be a sequence of measurable sets.

(a) Prove that

$$\mu(\liminf E_n) \leq \liminf \mu(E_n).$$

Proof. We have $\liminf E_n = \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty E_n$, and observe that $\bigcap_{n=j}^\infty E_n \subseteq \bigcap_{n=j+1}^\infty E_n$ for each j. Then

$$\begin{aligned}
\mu(\liminf E_n) &= \mu\left(\bigcup_{N=1}^\infty \bigcap_{n=N}^\infty E_n\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcap_{n=N}^\infty E_n\right) && \text{monotone convergence for sets} \\
&\leq \lim_{N\to\infty} (\inf\{\mu(E_n) \mid n \geq N\}) && \text{for fixed } N, \bigcap_{n=N}^\infty E_n \subseteq E_n \text{ for all } n \geq N\text{; monotonicity} \\
&= \liminf \mu(E_n)
\end{aligned}$$

as desired. ☐

(b) Assume in addition that $\mu(\bigcup_{n=1}^\infty E_n) < \infty$. Prove that

$$\mu(\limsup E_n) \geq \limsup \mu(E_n).$$

Proof. We have $\limsup E_n = \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty E_n$, and observe that $\bigcup_{n=j}^\infty E_n \supseteq \bigcup_{n=j+1}^\infty E_n$ for each j with $\mu(\bigcup_{n=1}^\infty E_n) < \infty$. Then

$$\begin{aligned}
\mu(\limsup E_n) &= \mu\left(\bigcap_{N=1}^\infty \bigcup_{n=N}^\infty E_n\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^\infty E_n\right) && \text{dominated convergence for sets} \\
&\geq \lim_{N\to\infty} (\sup\{\mu(E_n) \mid n \geq N\}) && \text{for fixed } N, \bigcup_{n=N}^\infty E_n \supseteq E_n \text{ for all } n \geq N\text{; monotonicity} \\
&= \limsup \mu(E_n)
\end{aligned}$$

as desired. ☐

(c) Prove the stronger form of the dominated convergence theorem for sets: suppose (E_n) is a sequence of measurable sets, and there is a measurable set $F \subset X$ such that $E_n \subset F$ for all n and $\mu(F) < \infty$. Prove that if (E_n) converges to E pointwise, then $(\mu(E_n))$ converges to $\mu(E)$. Give an example to show the finiteness hypothesis on F cannot be dropped.

Proof. Since the sequence of measurable sets (E_n) converges pointwise to E, it follows from a prior result that E was measurable also. Since $E_n \subseteq F$ for all n implies that $\bigcup_{n=1}^\infty E_n \subseteq F$.

We have that (E_n) converges pointwise to E if and only if $\limsup E_n = E = \liminf E_n$, so that $\mu(\limsup E_n) = \mu(E) = \mu(\liminf E_n)$. Apply the previous two results (for the latter, we need $\bigcup_{n=1}^\infty E_n \subseteq F$ and $\mu(F) < \infty$) to obtain $\limsup \mu(E_n) \leq \mu(E) \leq \liminf \mu(E_n)$; in general $\limsup \mu(E_n) \geq \liminf \mu(E_n)$ so they are equal. Hence $(\mu(E_n))$ converges to $\mu(E)$. ☐

The $\mu(F) < \infty$ condition is required: Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \text{counting})$, and take $F = \mathbb{N}$, which has infinite measure. Then for every $n \in \mathbb{N}$, define $E_n = \{m \in \mathbb{N} \mid m \geq n\}$; each of these have infinite measure and are subsets of $F = \mathbb{N}$. But (E_n) by inspection converges pointwise to the empty set, which has measure zero; this is not the limit of $(\mu(E_n))$, which is ∞ (it is a constant sequence).

(For parts (a) and (b), use Theorem 2.3.)