1. Read Section 2.8. State that you've read Section 2.8 or part of Section 2.8 and give yourself a score out of 5 based on how much you have read.

I read Section 2.8; 5/5.

2. (2.8.1) Let  $S^1 \subset \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$  be the standard circle. Let  $D = \{(0,0,t) \mid -2 \leq t \leq 2\}$  and  $S^2(2) = \{x \in \mathbb{R}^3 \mid ||x|| = 2\}$ . Then  $S^2(2) \cup D$  is a deformation retract of  $X = \mathbb{R}^3 \setminus S^1$ . The space X is h-equivalent to  $S^2 \vee S^1$ .

*Proof.* We describe the deformation retraction via the following pictures:

For each  $\theta \in [0, 2\pi)$  we specify the deformation retract within the closed half plane  $H_{\theta}^+$  given in green above. Points on  $S^2(2) \cup D$  remain fixed throughout the homotopy. For points outside of  $S^2(2)$ , we follow the standard deformation retract and draw a line from such points to the origin and drag them along via the straight line homotopy until they hit  $S^2(2)$ . For points inside the sphere minus the point p where  $S^1$  intersected  $H_{\theta}^+$ , draw a line from p towards such points and drag those points along the straight line homotopy until they hit either D or  $S^2(2)$  and map them there.

To see that X is h-equivalent to the wedge of  $S^2$  and  $S^1$  we use the fact that deformation retracts are h-equivalences and that composing h-equivalences are still h-equivalences. So follow the deformation retract outlined earlier by the homotopy which drags the top point of D to its bottom point, outlined in the below pictures:

3. (2.8.4) Let  $i_{0*}$  in (2.6.2) be an isomorphism. Then  $j_{1*}$  is an isomorphism. This statement is a general formal property of pushouts. If  $i_{0*}$  is surjective, then  $j_{1*}$  is surjective.

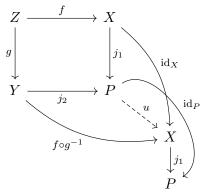
*Proof.* We prove the above statements for pushouts in general; that is, that pushouts of isomorphisms (epimorphisms) are isomorphisms (epimorphisms). We start with the following pushout:

$$Z \xrightarrow{f} X$$

$$\downarrow g \qquad \qquad \downarrow j_1$$

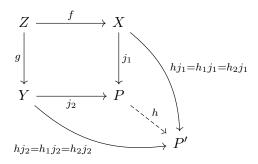
$$Y \xrightarrow{j_2} P$$

For the first, observe that when g is an isomorphism the following diagram commutes, using the universal property of pushouts:



It follows that  $j_1$  has left and right inverses, so it is an isomorphism.

For the second, assume g is an epimorphism and let  $h_1, h_2 : P \to P'$  be morphisms such that  $h_1j_1 = h_2j_1$ . Then  $h_1j_1f = h_1j_2g = h_2j_2g = h_2j_1f$ , and since g is an epimorphism,  $h_1j_2 = h_2j_2$ . Now apply the universal property of the pushout to obtain the diagram



where h is unique so that  $h_1 = h_2 = h$  as desired. It follows that  $j_1$  is an epimorphism.

Since (2.6.2) is a pushout the above is true for  $j_{1*}$  whenever  $i_{0*}$  is an isomorphism (epimorphism).

4. (2.8.5, last statement) ... we obtain  $\pi_1(P^2) \cong \mathbb{Z}/2$ .

*Proof.* We use the following pushout diagram which tells us we can obtain  $P^2$  from  $S^1 \cong P^1$  by attaching a

2-cell:

$$S^{1} \xrightarrow{\varphi} P^{1}$$

$$\downarrow^{j} \qquad \qquad \downarrow^{J}$$

$$D^{2} \xrightarrow{\Phi} P^{2}$$

We use the Seifert-van Kampen theorem due to the discussion in (2.8.10), and so we obtain the following diagram:

$$\pi_1(S^1) \cong \mathbb{Z} \xrightarrow{\varphi_*} \pi_1(P^1) \cong \mathbb{Z}$$

$$\downarrow^{j_*} \qquad \qquad \downarrow^{J_*}$$

$$\pi_1(D^2) \cong 1 \xrightarrow{\Phi_*} \pi_1(P^2)$$

and so the fundamental group of  $P^2$  is isomorphic to  $\pi_1(P^1)/\langle \varphi \rangle$  where  $\langle \varphi \rangle$  denotes the normal subgroup generated by the image of  $\varphi_*$ . In terms of generators and relations, we take the amalgamated free product of  $\langle a \rangle = \pi_1(S)$  and 1 to obtain  $\langle a \mid \varphi_*(a) = e \rangle$ . We deduce  $\varphi_*(a)$  by taking a generator of  $S^1$  (the identity  $e^{i\theta} \mapsto e^{i\theta}$ , a loop) and seeing that its image under  $\varphi$  is  $[\cos(\theta), \sin(\theta)]$ , but to go back to  $S^1$  (from which we obtained the fundamental group) we apply the specified homeomorphism (given in problem statement) to see that  $[\cos(\theta), \sin(\theta)]$  maps to  $e^{i(2\theta)}$ . (If  $\theta \in [0, \pi)$  then there is no need to choose a representative; otherwise take the representative to be  $[\cos(\theta - \pi), \sin(\theta - \pi)]$  and map this to  $e^{i(2[\theta - \pi])} = e^{i(2\theta)}$ ) It follows that in the fundamental groups,  $a \mapsto a^2$  so that  $\pi_1(P^2) = \langle a \mid a^2 = e \rangle$ , which is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .

5. (2.8.6) The Klein bottle K can be obtained from two Möbius bands M by an identification of their boundary curves with a homeomorphism,  $K = M \cup_{\partial M} M$ .

Apply the theorem of Seifert and van Kampen and obtain the presentation  $\pi_1(K) = \langle a, b \mid a^2 = b^2 \rangle$ . The elements  $a^2$ , ab generate a free abelian subgroup of rank 2 and of index 2 in the fundamental group. The element  $a^2$  generates the center of this group, it is represented by the central loop  $\partial M$ . The quotient by the center is isomorphic to  $\mathbb{Z}/2 * \mathbb{Z}/2$ .

*Proof.* To show that the fundamental group of the Klein bottle K is given by  $G = \pi_1(K) = \langle a, b \mid a^2 = b^2 \rangle$ , we first view K as the pushout of two inclusions of  $\partial M$  into M:

With  $\partial M$  homeomorphic to  $S^1$  and M homotopic to  $S^1$ , we apply Seifert-van Kampen and obtain another

pushout:

$$\pi_1(\partial M) \cong \langle c \rangle \longrightarrow \pi_1(M) \cong \langle b \rangle$$

$$\downarrow \qquad \qquad \downarrow$$

$$\pi_1(M) \cong \langle a \rangle \longrightarrow \pi_1(M \cup_{\partial M} M)$$

The generator c represents going along the boundary of M, which amounts to going around the center line of M twice. As a result the image of c under the induced maps from  $\pi_1(\partial M)$  into  $\pi_1(M)$  are  $a^2$  and  $b^2$ . Hence the fundamental group of  $K = M \cup_{\partial M} M$  is presented as  $\langle a, b \mid a^2b^{-2} = e \rangle$  as desired.

The subgroup H generated by  $a^2$ , ab is torsion free because the only relation imposed on these generators is that  $a^2 = b^2$ , which could not cause a finite word to be the neutral element. To show commutativity, it suffices to show it for the generators:  $a^2ab = aa^2b = ab^2b = abb^2 = aba^2$ . It follows that H is free and abelian. Observe also that since H is generated by  $a^2 = b^2$  and ab, it follows that H contains only the words of even length in G, and exactly those (if  $g \in G$  has even length, then by inserting in an even number of symbols and using the above identities to collect g into a product of generators of H, we find that  $g \in H$ .) Then the remaining words of G of even length may be obtained by prepending a to elements of H. Hence H, aH are the only two cosets of H in G, so H has index 2 in G.

The center Z is given by elements which commute with every element of G, in particular with the generators of G. Observe that  $ab \neq ba$ , so that if an element p were to commute with a or b, it must not be a or b. But observe that  $a^2 = b^2$  will commute with a and b. So an element p commutes with a and b if it is the product of finitely many  $a^2$ . If p is of odd length then at some point p will cease to commute with a and b. Hence  $Z = \langle a^2 \rangle$ . In terms of generators and relations, G/Z is given by  $\langle a, b \mid a^2 = b^2 = e \rangle$ , which is isomorphic to any presentation of  $\mathbb{Z}/2 * \mathbb{Z}/2$ .

The space  $M/\partial M$  is homeomorphic to the projective plane  $P^2$ . If we identify the central  $\partial M$  to a point, we obtain a map  $q: K = M \cup_{\partial M} M \to P^2 \vee P^2$ . The induced map on the fundamental group is the homomorphism onto  $\mathbb{Z}/2 * \mathbb{Z}/2$ .

*Proof.* Pictographically, we take M and identify its boundary to a point, and see that we obtain a sphere with antipodal points identified, which is the definition of  $P^2$ :

Then in K if we identify the central  $\partial M$  to a point, it is the same as taking two Möbius strips M and

quotienting out by  $\partial M$ , and then taking their wedge at the point  $\partial M$  was identified with. Using the previous result, it follows that quotienting out the central  $\partial M$  from K yields a space homeomorphic to  $P^2 \vee P^2$ ; thus there is a map  $q \colon K = M \cup_{\partial M} M \to P^2 \vee P^2$  which is the quotient map for the above composed with the appropriate homeomorphism, still a quotient map of spaces (it is surjective). Thus by Seifert-van Kampen, it follows that the induced homomorphism of fundamental groups is also surjective, and concretely the effect is to quotient out by the center Z of G. So  $q_* \colon G \to G/Z \cong \mathbb{Z}/2 * \mathbb{Z}/2$  is the (surjective) quotient map of groups.