HOMEWORK 7

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Let $X \neq \emptyset$ be a metric space and let $C_b(X, \mathbb{R})$ denote the metric space of continuous bounded real-valued functions on X with the metric

$$d_\infty(f, g) = \|f - g\|_\infty = \sup\{|f(x) - g(x)| : x \in X\}$$

for $f, g \in C_b(X, \mathbb{R})$. It is straightforward to verify that $C_b(X, \mathbb{R})$ contains the constant functions and is closed under sums and products.

Let $p(t) = \sum_{j=0}^{d} p_j t^j$ be a polynomial and define $\Psi_p : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$ by

$$\Psi_p(f) = p \circ f.$$

Show Ψ_p is continuous. [Suggestion: Recall p is uniformly continuous on compact subsets of $\mathbb{R}$ and show, for a fixed $f \in C_b(X, \mathbb{R})$ and $\eta > 0$, there is an interval $[a, b]$ such that if $d_\infty(f, g) < \eta$ then $f(X), g(X) \subseteq [a, b]$.]

Proof. Let $f \in C_b(X, \mathbb{R})$ be given. We show that Ψ_p is continuous at f. To that end, let $\varepsilon > 0$ be given. Since $f \in C_b(X, \mathbb{R})$, we have that $f(X)$ is a bounded subset of R, meaning $f(X)$ is contained in some interval $[a, b]$. For any $h \in C_b(X, \mathbb{R})$ satisfying $d_\infty(f, h) < \varepsilon$, we have for any $x \in X$ that $|f(x) - h(x)| \leq d_\infty(f, h) < \varepsilon$.

It follows that $f(x) - \varepsilon < h(x) < f(x) + \varepsilon$, and with x arbitrary we have that $h(X) \subseteq [a - \varepsilon, b + \varepsilon]$. Since $[a, b] \subseteq [a - \varepsilon, b + \varepsilon]$, we also have that $f(X) \subseteq [a - \varepsilon, b + \varepsilon]$.

With p uniformly continuous on $[a - \varepsilon, b + \varepsilon]$, there exists δ' such that if $|x - y| < \delta'$ for $x, y \in [a - \varepsilon, b + \varepsilon]$, then $|p(x) - p(y)| < \varepsilon$.

Choose $\delta = \min\{\varepsilon, \delta'\}$. Suppose that $d_\infty(f, g) < \delta$, so that $f(X), g(X) \subseteq [a - \varepsilon, b + \varepsilon]$ and for any $x \in X$ we have $|f(x) - g(x)| \leq d_\infty(f, g) < \delta'$, and note that $f(x), g(x) \in [a - \varepsilon, b + \varepsilon]$. Then by the uniform continuity of p on $[a - \varepsilon, b + \varepsilon]$ we have that $|(p \circ f)(x) - (p \circ g)(x)| < \varepsilon$, and since x was arbitrary, it follows that $d_\infty(p \circ f, p \circ g) = d_\infty(\Psi_p(f), \Psi_p(g)) < \varepsilon$ (ε is a uniform upper bound).

Since f was arbitrary, it follows that Ψ_p is continuous on $C_b(X, \mathbb{R})$. $\square$