1. (51.1) Show that if $h, h' \colon X \to Y$ are homotopic and $k, k' \colon Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

   Proof. Suppose $h, h' \colon X \to Y$ are homotopic and $k, k' \colon Y \to Z$ are homotopic.

   Define the map $H' \colon X \times I \to Y \times I$ by $H'(x,t) = (H(x,t), t)$. It is clear that this map is continuous because the component maps are continuous. Then the desired homotopy is the map $K \circ H' \colon X \times I \to Z$ (continuous because composition of continuous maps are continuous). We have

   $$K \circ H'(x,0) = K(H(x,0),0) = K(h(x),0) = k(h(x)) = (k \circ h)(x)$$
   $$K \circ H'(x,1) = K(H(x,1),1) = K(h'(x),0) = k'(h'(x)) = (k' \circ h')(x)$$

   as desired, meaning $k \circ h$ and $k' \circ h'$ are homotopic. □

2. (51.3) A space X is said to be ***contractible*** if the identity map $i_X \colon X \to X$ is nulhomotopic.

   (a) Show that I and $\mathbb{R}$ are contractible.

   Proof. Observe that I and $\mathbb{R}$ are both convex, path connected sets. Let $f \colon I \to I$ and $g \colon \mathbb{R} \to \mathbb{R}$ be constant maps. Then the straight line homotopies

   $$F \colon I \times I \to I \text{ given by } F(x,t) = tf(x) + (1-t)x$$
   $$G \colon \mathbb{R} \times I \to \mathbb{R} \text{ given by } G(x,t) = tg(x) + (1-t)x$$

   are continuous; furthermore, $F(x,1) = f(x)$, $G(x,1) = g(x)$, and $F(x,0) = \mathrm{id}_I(x)$, $G(x,0) = \mathrm{id}_{\mathbb{R}}(x)$. □

   (b) Show that a contractible space is path connected.

   Proof. Suppose that X is a contractible space. Let a, b be any two points in X. Then for a constant map f on X sending any x to b, the identity map is homotopic to this map by some homotopy F. Then for any x, we can define a path $P_x \colon I \to X$ by $P_x(t) = F(x,t)$, which connects x ($t = 0$) to b ($t = 1$). So there are paths connecting any x to b, and in particular, P_a is a path connecting a to b. Since a, b were arbitrary it follows that X is path connected. □

   (c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

   Proof. Suppose Y is contractible and let X be any space. Let g be a constant map on Y mapping $y \in Y$ to b. There is a homotopy $F \colon Y \times I \to Y$ from the identity map on Y to the map g on Y such that $F(y,0) = id_Y(y)$ and $F(y,1) = g(y) = b$. We show that any continuous map f from X to Y is homotopic to g.

   The homotopy required is the map $G \colon X \times I \to Y$ given by $G(x,t) = F(f(x),t)$ so that $G(x,0) = f(x)$ and $G(x,1) = g(f(x)) = b$. (It is clear that this map is continuous as it is a similar construction to the one given in the previous problem.)

   Since any map from X to Y is homotopic to a constant map on Y, by transitivity it follows that all such maps from X into Y are homotopic to each other and hence there is only one element in $[X, Y]$. □

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof. Suppose X is contractible and Y is path connected. Let g be a constant map on X mapping $x \in X$ to a. There exists a homotopy $F \colon X \times I \to X$ from the identity map on X to the map g on X such that $F(x, 0) = \mathrm{id}_X(x)$ and $F(x, 1) = g(x) = a$. For any two continuous maps f, f' from X into Y, there are homotopies $f \circ F \colon X \times I \to Y$ from $f(x)$ to the constant map sending elements of X to $f(a)$ and $f' \circ F \colon X \times I \to Y$ from $f'(x)$ to the constant map sending elements of X to $f'(a)$. Because Y is path connected, the two constant maps are homotopic to each other (the homotopy needed is any path $H \colon X \times I \to Y$ given by $H(x, t) = P(t)$ where P is any path connecting $f(a) = (f \circ g)(x)$ and $f'(a) = (f' \circ g)(x)$.).

By transitivity again it follows that any two continuous maps from X into Y are homotopic so that $[X, Y]$ only has one element. □

3. (52.1) A subset A of $\mathbb{R}^n$ is said to be ***star convex*** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

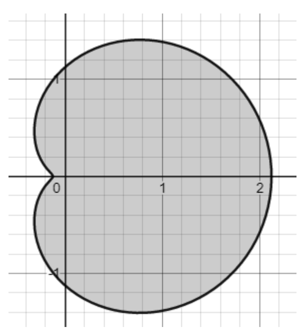
(a) Find a star convex set that is not convex.

We first choose a star convex set S in $\mathbb{R}^2$ which is not convex in $\mathbb{R}^2$, and take the direct product of S with $\mathbb{R}^{n-2}$ to form the desired star convex subset of $\mathbb{R}^n$.

A choice for a star convex subset of $\mathbb{R}^2$ which is not convex is the set given by

$$S = \left\{ (x, y) \colon \sqrt{x^2 + y^2} \leq \frac{9}{8} + \frac{y}{\sqrt{x^2 + y^2}} \right\},$$

which in polar coordinates makes the picture clear. The set S is the set enclosed by (and including the boundary) the filled-in cardioid given by the inequality $r \leq 9/8 + \cos(\theta)$:



Clearly, there are points of S in the second quadrant which are not connected by a line segment to some points of S in the third quadrant, meaning that S is not convex. But by construction (the polar inequality), every point in this set is connected by a line segment to the origin, so that S is star convex. Then the product $S \times \mathbb{R}^{n-2}$ (the "cylinder" of S) is the desired subset of $\mathbb{R}^n$ which is also star convex but not convex. The origin is connected to any point $\vec{x} = (x, y, x_1, \ldots, x_{n-2})$ by the line segment given by the parameterization $t\vec{x}$ for $t \in I$, but the points $(-1/4, 1/2, x_1, \ldots, x_{n-2})$ and $(-1/4, -1/2, x_1, \ldots, x_{n-2})$ cannot be connected by a line segment.

(b) Show that if A is star convex, A is simply connected.

Proof. Since A is star convex, there exists a point a_0 which is connected by line segments to every point in A. It follows that A is path connected because for any two points a, b in A, a line segment path from a to a_0 can be adjoined to a line segment path from a_0 to b by the path product $*$ to form a path from a to b.

Then take any loop P starting at a_0, and observe that for any point a on the loop $P(I)$, the straight line path $ta_0 + (1-t)a$ connects a with a_0. Hence the straight line homotopy $H\colon A \times I \to A$ given by $H(x,t) = te_{a_0}(x) + (1-t)P(x)$, where e_{a_0} is the constant map into a_0, is the homotopy required to show that all loops in A starting from a_0 are homotopic to a constant map into a_0, so that $\pi_1(A, a_0)$ is the trivial group. $\square$

4. (52.2) Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. We first prove a small lemma. Socks and shoes: With α, β, γ as given above, we have $\overline{\gamma} = \overline{\beta} * \overline{\alpha}$. This is clear since

$$\overline{\gamma}(t) = \gamma(1-t) = \begin{cases} \alpha(1-t) = \overline{\alpha}(t) & 0 \le t \le 1/2 \\ \beta(1-t) = \overline{\beta}(t) & 1/2 \le t \le 1 \end{cases} = (\overline{\beta} * \overline{\alpha})(t)$$

for all $t \in I$.

Then for any $[f] \in \pi_1(X, x_0)$, we have $\hat{\gamma}([f]) = [\overline{\gamma}] * [f] * [\gamma] = [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta] = [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] = (\hat{\beta} \circ \hat{\alpha})([f])$ so that $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$. $\square$

5. (52.3) Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. Suppose that every pair α and β of paths from x_0 to x_1 satisfies $\hat{\alpha} = \hat{\beta}$. We show that for any loops f, g starting at x_0, that $[f] * [g] = [g] * [f]$.

Let α be any path from x_0 to x_1, and choose $\beta = \overline{f} * \alpha$ so that

$$\hat{\alpha}([g]) = [\overline{\alpha}] * [g] * [\alpha]$$
$$\hat{\beta}([g]) = (\widehat{\overline{f} * \alpha})([g]) = [\overline{\overline{f} * \alpha}] * [g] * [\overline{f} * \alpha] = [\overline{\alpha}] * [f] * [g] * [\overline{f}] * [\alpha]$$

so that by cancellation of $[\overline{\alpha}]$ and $[\alpha]$ followed by right multiplication by $[f]$, we have that $[f] * [g] = [g] * [f]$.

Conversely, suppose that $\pi_1(X, x_0)$ is abelian so that for any paths f, g starting at x_0, we have $[f] * [g] = [g] * [f]$. Because X is path connected, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$, meaning $\pi_1(X, x_1)$ is also abelian (conjugation is the trivial action).

Let α, β be any two paths from x_0 to x_1. For elements $[\overline{\alpha} * \beta], [\overline{\alpha} * f * \alpha] \in \pi_1(X, x_1)$, we have

$$\begin{aligned}\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha] = [\overline{\alpha} * f * \alpha] &= [\overline{\alpha} * \beta]^{-1} * [\overline{\alpha} * f * \alpha] * [\overline{\alpha} * \beta] \\ &= [\overline{\beta}] * [\alpha] * [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [\beta] \\ &= [\overline{\beta}] * [f] * [\beta] = \hat{\beta}([f]),\end{aligned}$$

as desired. $\square$

6. (52.4) Let $A \subset X$; suppose $r\colon X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a ***retraction*** of X onto A.) If $a_0 \in A$, show that

$$r_*\colon \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Proof. Note that the formulation of r_* is correct because r sends $a_0 \in X$ to $a_0 \in A$.

For some element $[g] \in \pi_1(A, a_0)$ we seek to find a preimage of $[g]$ under r_*; that is, to find an element $[f] \in \pi_1(X, a_0)$ such that $[r \circ f] = [g]$. A choice of f which works is the path given by $i_{A \hookrightarrow X} \circ g$, where $i_{A \hookrightarrow X}$ is the canonical inclusion map from A into X.

We have $r_*([i_{A \hookrightarrow X} \circ g]) = [r \circ i_{A \hookrightarrow X} \circ g] = [\mathrm{id}_A \circ g] = [g]$. It follows that r_* is surjective.

Alternatively, we use the functorial properties of the induced homomorphism to see that since $r \circ i_{A \hookrightarrow X} = \mathrm{id}_A$, we have that $\mathrm{id}_{\pi_1(A,a_0)} = (\mathrm{id}_A)_* = (r \circ i_{A \hookrightarrow X})_* = r_* \circ (i_{A \hookrightarrow X})_*$. Thus r_* has a right inverse and so must be surjective. $\square$

7. (52.6) Show that if X is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphisms of the groups involved. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes."

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(h_{x_0})_*"] \arrow[d, "\hat{\alpha}"] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \\ \pi_1(X, x_1) \arrow[r, "(h_{x_1})_*"] & \pi_1(Y, y_1) \end{tikzcd}$$

Proof. Let $[f]$ be any element of $\pi_1(X, x_0)$. We apply the maps $\hat{\beta} \circ (h_{x_0})_*, (h_{x_1})_* \circ \hat{\alpha}$ to $[f]$:

$$\begin{aligned}(\hat{\beta} \circ (h_{x_0})_*)([f]) &= \hat{\beta}([h \circ f]) = [\overline{h \circ \alpha}] * [h \circ f] * [h \circ \alpha] = [h \circ \overline{\alpha}] * [h \circ f] * [h \circ \alpha] \\ ((h_{x_1})_* \circ \hat{\alpha})([f]) &= (h_{x_1})_*([\overline{\alpha}] * [f] * [\alpha]) = [h \circ \overline{\alpha}] * [h \circ f] * [h \circ \alpha],\end{aligned}$$

where the last equalities follow from $(\overline{h \circ \alpha})(t) = (h \circ \alpha)(1 - t) = h(\alpha(1 - t)) = h(\overline{\alpha}(t)) = (h \circ \overline{\alpha})(t)$ and the fact that $(h_{x_1})_*$ is a homomorphism. $\square$

8. (53.3) For a connected space B, let $p\colon E \to B$ be a covering map. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a ***k-fold covering*** of B.

   Proof. Let $p\colon E \to B$ be a covering map, and let B be connected.

   We prove a curious lemma first: For any positive integer j, the disjoint sets $S_j = \{b \in B \mid |p^{-1}(b)| = j\}$ and $S_\omega = \{b \in B \mid |p^{-1}(b)| \text{ is infinite}\}$ are open.

   If S_j is empty, we are done. Otherwise, for any element $x \in S_j$, there exists an evenly covered neighborhood U_x of x whose preimage under p is the disjoint union of open sets of E given by $\coprod_\alpha V_\alpha$. But because the preimage of x under p contains only j elements, there are exactly j open sets V_α which form the preimage of U_x. (if there are fewer, then p must not be continuous). By relabeling, write $p^{-1}(U_x) = \coprod_{i=1}^{j} V_i$, and observe that because each V_i is mapped homeomorphically into U_x by p, it follows that $|p^{-1}(y)| = j$ for every $y \in U_x$. It follows that $U_x \subseteq S_j$, so that S_j is open.

   The set S_ω is open due to a similar argument: Suppose S_ω is not empty, and consider $x \in S_\omega$. The preimage of an evenly covered neighborhood U_x of x is an infinite cardinality disjoint union of open sets V_α of E. Each V_α is mapped homeomorphically into U_x so that the preimage of any element $y \in U_x$ is also of infinite cardinality, and thus $U_x \subseteq S_\omega$. It follows that S_ω is open, and we have proved the lemma.

   Suppose by way of contradiction that there exists (at least one) $b_1 \in B$ satisfying $|p^{-1}(b)| \neq k$. Observe that $b_0 \in S_k$ and $b_1 \in S' = S_\omega \cup (\bigcup_{i \neq k} S_i)$, and that every element of B lies in one of these two sets. Using the lemma, it follows that S_k and S' are nonempty disjoint open subsets of B whose union is B. This is impossible since B is a connected space, so there are no elements in B whose preimage under p has cardinality not equal to k, as desired. □

9. (53.6b) Let $p\colon E \to B$ be a covering map. If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

   Proof. Let S be an open cover of E.

   Since each $b \in B$ is contained in an evenly covered open set U_b, the preimage of U_b under p is a collection $\{V_{b,\alpha}\}$ of disjoint open subsets of E which map homeomorphically to U_b by p. But because $p^{-1}(b)$ is finite, it follows that $\{V_{b,\alpha}\}$ is finite (that is, α takes on finitely many values). Relabeling, write the collection as $\{V_{b,1}, \ldots, V_{b,|p^{-1}(b)|}\}$.

   Since E is covered by S, we can find open sets $S_{b,1}, \ldots, S_{b,|p^{-1}(b)|}$ from S such that $S_{b,\alpha}$ contains the point $p^{-1}(b) \cap V_{b,\alpha}$ (the single element of the preimage of b under p lying in the α-th slice $V_{b,\alpha}$). We apply p to each of the intersections $S_{b,1} \cap V_{b,1}, \ldots, S_{b,|p^{-1}(b)|} \cap V_{b,|p^{-1}(b)|}$ to produce a family of neighborhoods of b which lie in U_b given by $\{p(S_{b,\alpha} \cap V_{b,\alpha})\}_{\alpha=1}^{|p^{-1}(b)|}$, and so we take the (finite) intersection over this family to produce a small neighborhood W_b of b contained in U_b:

$$W_b = \bigcap_{\alpha=1}^{|p^{-1}(b)|} p(S_{b,\alpha} \cap V_{b,\alpha}).$$

Since b was arbitrary, the collection $W = \{W_b\}$ for all b constitutes an open cover of B, and by compactness of B we can find a finite subcover $W' = \{W_{b_n}\}_{n=1}^{N}$ of B. Note that the preimage of each W_b is contained in each $S_{b,\alpha}$ for $1 \leq \alpha \leq \left|p^{-1}(b)\right|$. It follows also that the preimage of each W_b is contained in the union $\bigcup_{\alpha=1}^{\left|p^{-1}(b)\right|} S_{b,\alpha}$ (the union is bigger than the intersection).

With W' being a finite subcover of B, we have that

$$E = p^{-1}(B) \subseteq p^{-1}\left(\bigcup_{n=1}^{N} W_{b_n}\right) \subseteq \bigcup_{n=1}^{N}\left[\bigcup_{\alpha=1}^{\left|p^{-1}(b_n)\right|} S_{b_n,\alpha}\right],$$

where the last set is a finite union of open sets of S so that S admits a finite subcover of E. Since S was an arbitrary open cover of E, it follows that E is compact. $\square$