HOMEWORK 8

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Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x_1, x_2) = (x_1^2 - x_2^2 - 2x_1 + 1, 2x_1x_2 - 2x_2).$$

For $c \in \mathbb{R}^2$, show f is differentiable at c and determine, with proof, $Df(c)$, proceeding directly from Definition 7.1.1. Determine those c for which $Df(c)$ fails to be invertible.

Proof. Fix $c = (c_1\ c_2)^T$, and let

$$A = \begin{pmatrix} 2(c_1 - 1) & -2c_2 \\ 2c_2 & 2(c_1 - 1) \end{pmatrix}.$$

For any $h = (h_1\ h_2)^T$ we have

$$\begin{aligned} \|f(c+h) - f(c) - Ah\| &= \left\| \begin{pmatrix} h_1^2 - h_2^2 \\ 2h_2h_1 \end{pmatrix} \right\| \\ &= h_1^2 + h_2^2 = \|h\|^2, \end{aligned}$$

so that

$$\lim_{h \to 0} \frac{\|f(c+h) - f(c) - Ah\|}{\|h\|} = \lim_{h \to 0} \|h\| = 0.$$

It follows that f is differentiable at c, and its derivative at c is $Df(c) = L_A$.

Observe that $\det(A) = 4(c_1 - 1)^2 + 4c_2^2$, which is zero if and only if $c_1 = 1$ and $c_2 = 0$ ($\det(A)$ is a sum of squared quantities).

Since $Df(c)$ is not invertible whenever L_A is not invertible, which is whenever $\det(A)$ vanishes, we have that $Df(c)$ is not invertible at only the point $c = (1\ 0)^T$. $\square$