HOMEWORK 2

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Define, for $n \in \mathbb{N}$, the functions $f_n : [0,1] \to \mathbb{R}$ by $f_n(t) = t^n$.

(1) Note $f_n \in C([0,1])$. Does the sequence (f_n) converge in the metric space $(C([0,1]), d_2)$, where, for $g, h \in C([0,1])$,

$$d_2(g,h) = \left[\int_0^1 |g-h|^2\, dt\right]^{\frac{1}{2}}?$$

Prove or disprove.

(2) Let $V = (B([0,1], \|\cdot\|_\infty)$ denote the normed vector space of bounded real-valued functions on $[0,1]$ with the norm defined, for $g \in B([0,1])$, by

$$\|g\|_\infty = \sup\{|g(x)| : x \in [0,1]\}.$$

Does the sequence (f_n) converge to 0 in V? Prove or disprove.

Proof. Let $f_n \in C([0,1])$ be given by $f_n(t) = t^n$ for each $n \in \mathbb{N}$.

(1) In the metric space $(C([0,1]), d_2)$, the sequence (f_n) converges to the zero function. We have

$$\begin{aligned} d_2(f_n, 0) &= \left[\int_0^1 |t^n - 0|^2\, \mathrm{d}t\right]^{\frac{1}{2}} \\ &= \left[\int_0^1 t^{2n}\, \mathrm{d}t\right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2n+1}}. \end{aligned}$$

Because the square root is monotonically increasing and not bounded above on $[0,\infty]$, we have that the sequence $(d_2(f_n,0))$ converges to 0 in $\mathbb{R}$. Hence (f_n) converges to the zero function in $(C([0,1]), d_2)$.

(2) With $V = (B([0,1], \|\cdot\|_\infty)$, let d_∞ be the metric on $B([0,1])$ induced by the norm $\|\cdot\|_\infty$. It follows that the sequence (f_n) does not converge to 0 in V. We have

$$\begin{aligned} d_\infty(f_n, 0) = \|f_n - 0\| &= \sup\{|f_n(x) - 0| \colon x \in [0,1]\} \\ &= \sup\{x^n \colon x \in [0,1]\} \\ &= 1, \end{aligned}$$

where for every $n \in \mathbb{N}$, because f_n is bounded above by 1 and takes on 1 at $x = 1$, the supremum is always 1. Since the sequence $(1)_{n \in \mathbb{N}}$ does not converge to 0 in $\mathbb{R}$, it follows that the sequence (f_n) does not converge to 0 in V.

$\square$