1. (DF10.1.8) An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\text{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).

Proof. Observe that 0_M in $\text{Tor}(M)$ since for any nonzero $r \in R$, we have that $r0_M = 0_M$ (observe that $r0_M = r(0_M + 0_M) = r0_M + r0_M$ and cancel terms). It follows that $\text{Tor}(M)$ is a nonempty subset of M. Let $x, y \in \text{Tor}(M)$ and $r \in R$ be arbitrary. There exist $a_x, a_y \in R$ such that $a_x x = a_y y = 0_M$; observe that because R is an integral domain $a_x a_y$ is never 0_R and that

$$(a_x a_y)(x + ry) = a_y a_x x + r a_x a_y y = 0 + 0 = 0,$$

meaning $x + ry \in \text{Tor}(M)$. Since x, y, r were arbitrary it follows that $\text{Tor}(M)$ is a submodule of M. □

(b) Give an example of a ring R and an R-module M such that $\text{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]

Take $R = \mathbb{Z}/6\mathbb{Z}$ as a left module over itself by left multiplication. Because R has zero divisors, observe that for the torsion elements $2, 3$ (as $2 * 3 = 6 \equiv 0$) we have $2 + 3 = 5$, but 5 is not a torsion element since it is coprime to 6 (the only element which multiplies with 5 to obtain a multiple of 6 is $6 \equiv 0$ itself).

(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

Proof. Suppose that R is a (nonzero) ring with zero divisors; that is there exist nonzero elements $r, s \in R$ such that $sr = 0_R$. Let M be any nonzero R-module and we find a nonzero torsion element: Let m be any nonzero element of M, and if m is a torsion element already then we are done.

So suppose m is not a torsion element, meaning that there are no nonzero elements $t \in R$ such that $tm = 0$. Then we consider the element rm, which will not be zero since $r \neq 0_R$ and m is not a torsion element. We have $s(rm) = (sr)m = 0_R m = 0$, and with $s \neq 0_R$ we have that rm is a nonzero torsion element. □

In these exercises R is a ring with 1 and M is a left R-module.

2. Auxiliary result for next problem: (DF10.1.5) For any left ideal I of R define

$$IM = \left\{ \sum_{\text{finite}} a_i m_i \mid a_i \in I, m_i \in M \right\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.

Proof. The element 0_M can be written as a finite sum of elements of the form $a0_M$ for $a \in I$; it follows that IM is a nonempty subset of M. Let $x, y \in IM$ and $r \in R$ be arbitrary. We have that $x = \sum_{i=1}^{j} a_i x_i$ and $y = \sum_{i=1}^{k} b_i y_i$ (finite sums), and so

$$x + ry = \sum_{i=1}^{j} a_i x_i + r \sum_{i=1}^{k} b_i y_i = \sum_{i=1}^{j} a_i x_i + \sum_{i=1}^{k} (rb_i) y_i.$$

Because I is a left ideal it follows that $rb_i \in I$ and so the resulting sum is a finite sum of elements in the desired form, so that $x + ry \in IM$. Since x, y, r were arbitrary it follows that IM is a submodule of M. □

3. (DF10.2.13) Let I be a nilpotent ideal in a commutative ring (cf. Exercise 37, Section 7.3), let M and N be R-modules and let $\varphi\colon M \to N$ be an R-module homomorphism. Show that if the induced map $\overline{\varphi}\colon M/IM \to N/IN$ is surjective, then φ is surjective.

   Proof. Since $\overline{\varphi}$ is surjective, for any $n \in N$ for the element $n + IN \in N/IN$ we can find a preimage $m + IM \in M/IM$ so that

   $$\overline{\varphi}(m + IM) = \varphi(m) + IN = n + IN,$$

   and we have from group theory that $n - \varphi(m) \in IN$, so that there exist $a_i \in I, n_i \in N, k \in \mathbb{Z}^+$ such that $n = \varphi(m) + \sum_{i=1}^{k} a_i n_i$ (i.e. $\sum_{i=1}^{k} a_i n_i$ is an element of IN). With n arbitrary it follows that $N \subseteq \varphi(M) + IN = \{\varphi(m) + n' \mid m \in M, n' \in IN\} \subseteq N$. It is clear that $\varphi(M) + IN \subseteq N$ since $IN, \varphi(M)$ are submodules of N, so it follows that $N = \varphi(M) + IN$.

   Since I was a nilpotent ideal, it follows that there exists some $N \in \mathbb{Z}^+$ such that $I^N = \{0_R\}$. We also want to establish the containment

   $$A = I^j[\varphi(M) + IN] \subseteq I^j\varphi(M) + I^{j+1}N = B$$

   by double inclusion. Observe that any element of A is a finite sum $\sum_i a_{ij}[\varphi(m) + \sum_k a_k n_k] = \sum_i [a_{ij}\varphi(m) + \sum_k a_{ij} a_k n_k]$ for $a_{ij} \in I^j$. Since $a_{ij} a_k \in I^{j+1}$ the containment $A \subseteq B$ holds.

   Furthermore also observe that for any k, $I^k\varphi(M) \subseteq \varphi(M)$ since $\varphi(M)$ is a module (a submodule of N).

   By repeatedly applying the equality $N = \varphi(M) + IN$ to itself N times and using the previous facts we obtain

   $$N \subseteq \varphi(M) + I\varphi(M) + \cdots + I^{N-1}\varphi(M) + I^N N \subseteq \varphi(M).$$

   Since $\varphi(M) \subseteq N$, it follows that $\varphi(M) = N$; that is, φ is surjective. □

4. (DF10.3.5) Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator i.e., there is a nonzero element $r \in R$ such that $rm = 0$ for all $m \in M$ — here r does not depend on m (the annihilator of a module was defined in Exercise 9 of Section 1). Give an example of a torsion R-module whose annihilator is the zero ideal.

Proof. Let M be a finitely generated torsion R-module, so that there exists a finite subset $A = \{a_1, a_2, \ldots, a_n\}$ of M such that $M = RA = \{\sum_{i=1}^n r_i a_i \mid r_i \in R, a_i \in A\}$. We also have for every element $m \in M$, there exists a nonzero element $r_m \in R$ such that $r_m m = 0_M$ (here r_m may depend on m). We seek to find a nonzero element $r \in R$ such that for *any* element $m \in M$, $rm = 0_M$.

Since $A \subseteq M$, for every element $a_i \in A$ there exists nonzero $r_{a_i} \in R$ such that $r_{a_i} a_i = 0_M$. Then consider the finite product $r = \prod_{i=1}^n r_{a_i} \in R$, and $r \neq 0_R$ since R is an integral domain. For any $m \in M$, we have that $m = \sum_{i=1}^n r_i a_i$, and

$$rm = r\sum_{i=1}^n r_i a_i = \sum_{i=1}^n r_i(ra_i) = 0_M$$

since

$$ra_j = \left(\prod_{\substack{i \neq j \\ 1 \leq i \leq n}} r_{a_i}\right)(r_{a_j} a_j) = 0_M \quad \text{for} \quad 1 \leq j \leq n.$$

It follows that M has a nonzero annihilator since r is one such annihilator element. □

The finiteness was required to form the element r found earlier. Before exhibiting an example of a torsion R-module we look at the definition of infinite direct products and sums modules. We state the definition given in Exercise 20 since it is a little easier to work with than the one given in class for the example I have in mind.

(DF10.3.20 exposition) Let I be a nonempty index set and for each $i \in I$ let M_i be an R-module. The *direct product* of the modules M_i is defined to be their direct product as abelian groups (cf. Exercise 15 in Section 5.1) with the action of R componentwise multiplication. The *direct sum* of the modules M_i is defined to be the restricted direct product of the abelian groups M_i (cf. Exercise 17 in section 5.1) with the action of R componentwise multiplication. In other words, the direct sum of the M_i's is the subset of the direct product, $\prod_{i\in I} M_i$, which consists of all elements $\prod_{i\in I} m_i$ such that only finitely many of the components m_i are nonzero (we saw in class the formulation of functions from the index set into the union of the modules with finite support); the action of R on the direct product or direct sum is given by $r\prod_{i\in I} m_i = \prod_{i\in I} rm_i$ (cf. Appendix I for the definition of Cartesian products of infinitely many sets). The direct sum will be denoted $\bigoplus_{i\in I} M_i$.

It follows that the direct product and sum are R-modules (we showed this in class), and the example I have in mind for a torsion R-module whose annihilator is the zero ideal is the direct product of $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$ for every prime p written as $M = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$. For any element $m \in M$ we can take r_m to be the product of all the primes p where the $\mathbb{Z}/p\mathbb{Z}$ component of m is nonzero (so $r_m = \prod p_i$ where the $\mathbb{Z}/p_i\mathbb{Z}$ component of m is nonzero). Then it follows that $r_m m = 0_M$ because then the resulting element in each $\mathbb{Z}/p\mathbb{Z}$ component is divisible by p (true even for all of the components where zero appears). However, there are no "universal" nonzero integers z such that $zm = 0_M$ for every $m \in M$ since that would require z to be divisible by every prime number (so z would have to be zero). It follows that the annihilator of M is the zero ideal.

We could have used the direct sum of $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$ ($\mathbb{Z}/1\mathbb{Z}$ is trivial) and so to show that every element m has torsion we choose r_m to be the least common multiple of the integers n where the $\mathbb{Z}/n\mathbb{Z}$ component is nonzero and a similar result follows (the least common multiple of a set of coprime numbers is easy to compute).

5. Auxiliary result for next problem: (DF10.3.9) An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as generator.

   Proof. Suppose that M is irreducible. Then we consider the submodule generated by any nonzero element $a \in M$, given by $Ra = \{ra \mid r \in R\}$. Since R has a 1, it follows that Ra contains a and so is nonempty. But the only nonempty submodule of M is M itself so $M = Ra$. Since a was arbitrary, it follows that M is a cyclic module with any nonzero element as generator.

   Conversely, suppose that M is a nonzero module which is equal to Ra for any nonzero $a \in M$, so take one such a. For any nonzero submodule N of Ra we can find a nonzero element $b \in N$ and it follows that $M = Rb$, but $Rb \subseteq N$ so that $N = M$. Hence M has no nonzero submodules outside of itself. □

6. (DF10.3.11) Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\operatorname{End}_R(M)$ is a division ring (this result is called *Schur's Lemma*). [Consider the kernel and the image.]

   Proof. Let $\varphi\colon M_1 \to M_2$ be any nonzero R-module homomorphism. It follows that there exists $m_1 \neq 0_{M_1}$ in M_1 such that $\varphi(m_1)$ is not 0_{M_2} (there is at least one nonzero element not in $\ker\varphi$).

   Since M_1 is an irreducible R-module, we have from previous results that $M_1 = Rm_1$. Since M_1 is generated by m_1, it follows that φ is completely determined by its action on m_1.

   We show that φ is injective by the trivial kernel characterization. Since $\ker\varphi$ is a submodule of M_1, it is either M_1 itself or zero. But $\ker\varphi$ cannot be M_1 since φ is a nonzero homomorphism, so $\ker\varphi$ must be trivial. Hence φ is injective.

   We show that φ is surjective by observing that $\varphi(m_1)$ is nonzero so that because M_2 is irreducible, $M_2 = R\varphi(m_1)$. So every element of M_2 may be expressed as $r\varphi(m_1) = \varphi(rm_1)$ for some $r \in R$, where the equality holds because φ is an R-module homomorphism. So preimages of any element in M_2 are in the form rm_1 for some $r \in R$, but $M_1 = Rm_1$. It follows that φ is surjective (every element in M_2 has a preimage in $M_1 = Rm_1$). Alternatively, the image of φ is a submodule of M_2 and we have seen that it is nonzero since $\varphi(m_1)$ is nonzero, so that by irreducibility of M_2 it follows that the image is all of M_2.

   It follows that φ is an isomorphism.

   If M is an irreducible R-module then $\operatorname{End}_R(M)$ becomes the ring of all R-module autormorphisms of M and the zero map due to the previous result. Any nonzero map in $\operatorname{End}_R(M)$ is an automorphism of M which has an inverse map, which is also an autormorphism of M. It follows that $\operatorname{End}_R(M)$ is a division ring since it contains an identity, the (nonzero) identity map on M, and all nonzero elements have inverses under the ring multiplication (which is composition). $\square$