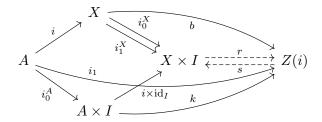
1. (5.1.1) A cofibration is an embedding. For the proof use that $i_1: A \to Z(i), a \mapsto (a, 1)$ is an embedding. From $i_1 = rsi_1 = ri_1^X i$ then conclude that i is an embedding.

Consider a cofibration as an inclusion $i: A \subset X$. The image of $s: Z(i) \to X \times I$ is the subset $X \times 0 \cup A \times I$. Since s is an embedding, this subset equals the mapping cylinder, i.e., one can define a continuous map $X \times 0 \cup A \times I$ by specifying its restrictions to $X \times 0$ and $A \times I$. (This is always so if A is closed in X, and is a special property of $i: A \subset X$ if i is a cofibration.)

Let X be a Hausdorff space. Then a cofibration $i: A \to X$ is a closed embedding. Let $r: X \times I \to X \times 0 \cup A \times I$ be a retraction. Then $x \in A$ is equivalent to r(x,1) = (x,1). Hence A is the coincidence set of the maps $X \to X \times I$, $x \mapsto (x,1), x \mapsto r(x,1)$ into a Hausdorff space and therefore closed.

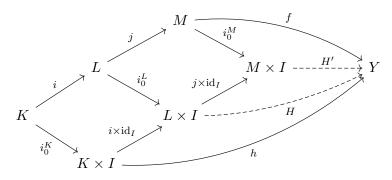
Proof. Let $i: A \to X$ be a cofibration so that equivalently i has the HEP for Z(i). Furthermore, the map $s: Z(i) \to X \times I$ obtained via the universal property of the pushout has a retraction $r: X \times I \to Z(i)$ such that $rs = \mathrm{id}_{Z(i)}$. We have then that $i_1 = rsi_1 = ri_1^X i$, and since i_1 is injective, we have that i must also be injective, and hence continuously bijective onto its image. Then we can define a continuous left inverse $i_{\ell}^{-1}: i(A) \to A$ for i; from which it follows that i is an embedding.



Observe that similarly s is an embedding and by definition will take x to (x,0) and (a,t) to itself so the range of s is $X \times 0 \cup A \times I$. Let X be Hausdorff. Since r is a retraction of $X \times I$ onto Z(i) and is the identity on the image $X \times 0 \cup A \times I$ of s, we have r(x,1) = (x,1) if $x \in A$ (the converse is also true). Thus A is the coincidence set $\{x \in X \mid r(x,1) = (x,1)\}$ for the two maps above. From an earlier result, since $X \times I$ is Hausdorff, then the coincidence set A must also be closed. Thus the image of A under the embedding must also be closed, so $i \colon A \subset X$ is a closed embedding.

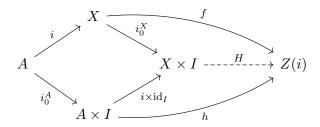
2. (5.1.2) If $i: K \to L$, $j: L \to M$ have the HEP for Y, then ji has the HEP for Y. A homeomorphism is a cofibration. $\emptyset \subset X$ is a cofibration. The sum $\coprod i_j: \coprod A_j \to \coprod X_j$ of cofibrations $i_j: A_j \to X_j$ is a cofibration.

Proof. The composition of functions with the HEP has the HEP:



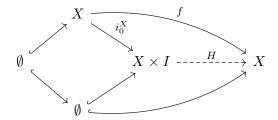
Use the HEP of i to obtain H, and then use the HEP of j to obtain the desired extension H'.

A homeomorphism is a cofibration: With Z(i) the mapping cylinder, we have the solid diagram:



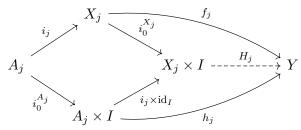
Then define $H: X \times I \to Y$ by $H(x,t) = h(i^{-1}(x),t)$; it follows that i has the HEP for Z(i), so i is a cofibration.

The inclusion i of the empty set is a cofibration. Note Z(i) = X. Then the following diagram commutes when H is given by H(x,t) = f(x), so the inclusion of the empty set has the HEP for Z(i):

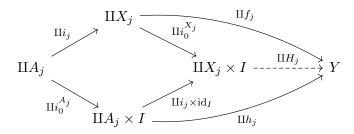


Thus the inclusion $\emptyset \subset X$ is a cofibration.

Let $i_j: A_j \to X_j$ be cofibrations and let Y be any space. Then from the usual initial data for each j we form the sum of cofibrations:



With this form the diagram



where in particular $\coprod h_j$ is given by $\coprod h_j(a_k, t) = h_k(a_k, t)$ for $a_k \in A_k$. Similarly the homotopy $\coprod H$ is given by $\coprod H(x_k, t) = H_k(x_k, t)$ for $x_k \in X_k$, both using the initial data above. Everything commutes and Y was arbitrary so the sum of cofibrations is also a cofibration.

3. (5.1.5) Let $A \subset X$ be a cofibration and A contractible. Then the quotient map $X \to X/A$ is a homotopy equivalence.

Proof. In the following diagram we have that $A \subset X$ is a cofibration and that $A \to *$ is a homotopy equivalence: The composition $* \hookrightarrow A \to *$ is the identity on *, and the composition $A \to * \hookrightarrow A$ is the constant map on A, homotopic to the identity on A. Then by proposition 5.1.10, $X \to X/A$, the quotient map, is a homotopy equivalence:



4. (5.1.6) The space $C'X = X \times I/X \times 1$ is called the unpointed **cone** on X. We have the closed inclusions $j: X \to C'X, x \mapsto (x, 0)$ and $b: \{*\} \to C'X, *\mapsto \{X \times 1\}$. Both maps are cofibrations.

Proof. Observe that $Z(j) = C'X \sqcup X \times I/\sim$ where $(x,0) \sim (x,0)$ (where the left $(x,0) \in C'X$ and the right $(x,0) \in X \times I$). Furthermore, Z(j) embeds by the map s (coming from the universal property of the pushout) into $C'X \times I$ naturally (see the sketches). A retraction of $C'X \times I$ onto Z(j) is the one that essentially crushes the space onto its subspace.

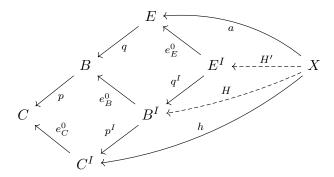
In a similar way, we find that $Z(b) = C'X \sqcup *\times I/\sim$ where $(*,0) \sim \{X\times 1\}$, which embeds by s into $C'X\times I$ in a natural way. The retraction from $C'X\times I$ onto this subspace is also just the one that crushes the whole space onto this subspace.

Note that X and $\{*\}$ are closed subspaces since j, b are closed embeddings. By Proposition 5.1.2, since we have the retractions from $C'X \times I$ to the images of Z(j) and Z(b), then j, b are cofibrations.

5. (5.5.1) A composition of fibrations is a fibration. A product of fibrations is a fibration. $\emptyset \subset B$ is a fibration.

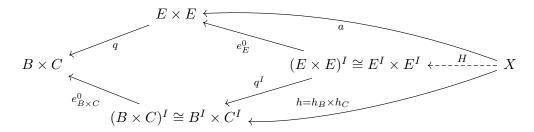
Proof. The composition of fibrations is a fibration. We show that if p,q have the HLP for X, then pq does

also, and so if p,q are fibrations then pq does since X may be taken to be any space.



Use the HLP twice to obtain H and H'. Thus pq is a fibration since X may be taken to be any space.

Let p, q be fibrations again and take X to be any space. We show that $p \times q$ have the HLP for X also, so that since X was arbitrary, $p \times q$ is a fibration:



In the above diagram define H to be the homotopy which agrees with the components of a at 0, with $q^I H = h$ also. So H is given by the product of the homotopies obtained by using the HLP for each of h_B, h_C and the components of A. Then with X arbitrary it follows the product of fibrations is a fibration.

Since there are no maps of a nonempty set into the empty set, we have the following commutative diagram:



It follows basically trivially that the inclusion of the empty set into a space is a fibration, as it forces $X, X \times I$ for any X to be empty also.