HOMEWORK 9

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Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2).$$

In class on Monday (2022 April 4), we saw that the inverse function theorem applies to f and any point $0 \neq c \in \mathbb{R}^2$. Further, we saw that f maps the set (upper half plane)

$$V = \{(x, y) : y > 0\} \subseteq \mathbb{R}^2$$

bijectively onto

$$W = \mathbb{R}^2 \setminus \{(a, 0) : a \geq 0\}.$$

In particular, V provides *an* example of a set satisfying the first two conclusions of the inverse function theorem for the point $(1, 1)$. (Indeed, V is a maximal such set, and the full conclusion of the inverse function theorem holds, but we did not verify all of these claims for V.)

Find open connected sets $V_*, W_* \subseteq \mathbb{R}^2$ such that V_* contains $(1, 1)$, but is not a subset of V, and f maps V_* bijectively onto W_*. Outline a proof. Let g denote the resulting inverse function. What is $Dg(0, 2)$?

Solution. Choose V_* to be the right half plane given by $\{(x, y) \colon x > 0\}$ and W_* to be $f(V_*) = \mathbb{R}^2 \setminus \{(x, 0) \colon x \leq 0\}$. Then $f|_{V_*}$ is bijective onto its image and its inverse, $g = (f|_{V_*})^{-1}$, is continuously differentiable with

$$Dg(0, 2) = Dg(f(1, 1)) = (Df(1, 1))^{-1} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Proof. Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given as above; it follows from our work in class that f is continuously differentiable on $\mathbb{R}^2$. Furthermore, the derivative of f at $c = (1, 1)$ is invertible.

Choose V_* to be the right half plane given by $\{(x, y) \colon x > 0\}$; observe that it contains $(1, 1)$ but is not a subset of V given earlier. We have that V_* is open since its complement contains all of its limit points; that is, its complement is closed. By inspection V_* is connected. By the inverse function theorem, we have that f restricted to V_*

We check that f maps V_* bijectively onto W_*. First we can check that the image of V_* under f is W_*. We can compute the image of V_* under f by parameterizing V_* by rays of the form $y = cx$ and taking the union of the image of these rays.

Observe that $V_* = \{(t, ct)\colon t > 0, -\infty < c < \infty\}$, so that $f(V_*) = \{(t^2(1-c^2),)\colon t > 0, -\infty < c < \infty\}$, which is equal to $W_* = \mathbb{R}^2 \setminus \{(x, 0)\colon x \leq 0\}$. The two sided inverse for $f|_{V_*}\colon V_* \to W_*$ is given by

$$(f|_{V_*})^{-1}(x, y) = \left(\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}}, \operatorname{sgn}(y) \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} \right),$$

and let $\operatorname{sgn}(y) = 0$ if and only if $y = 0$ (note in this case $x > 0$ so it did not matter what we set $\operatorname{sgn}(0)$ to). Interpret the inverse given as a piecewise function. [The inverse given can be obtained by writing in Cartesian coordinates the action of the principal branch of the complex-valued square root function, using trigonometry.]

Hence $f|_{V_*}\colon V_* \to W_*$, with $W_* = f|_{V_*}(V_*)$, is a bijection as expected.

Furthemore, W_* is open as expected since it is the complement of the closed ray $\{(x, 0)\colon x \leq 0\}$ (closed because it contains all of its limit points). By inspection W_* is connected.

The derivative of g at $(0, 2)$ was computed above in the manner prescribed by the results of the inverse function theorem. $\square$