1. Show that if a category $\mathbf{C}$ has two terminal objects t_1, t_2, then they are isomorphic. (In fact, there is a unique isomorphism $t_1 \to t_2$.)

   Remark. The same argument shoes that if a category $\mathbf{C}$ has two initial objects, then they are isomorphic. Furthermore, essentially the same argument shows that if a diagram in $\mathbf{C}$ has two limits/colimits then they are isomorphic. It is a convention to speak of "the terminal object/initial object/limit/colimit" instead of "some terminal object/initial object/limit/colimit" or "the isomorphism class of terminal objects/initial objects/limits/colimits".

   Proof. Let $\mathbf{C}$, t_1, t_2 be as above. Since both t_1, t_2 are terminal there exist unique morphisms $f\colon t_1 \to t_2$ and $g\colon t_2 \to t_1$.

   Composing f, g both ways gives us morphisms $g \circ f\colon t_1 \to t_1$, $f \circ g\colon t_2 \to t_2$. But for $i = 1, 2$, the unique morphism from t_i to itself is the identity 1_{t_i}, so $g \circ f = 1_{t_1}$ and $f \circ g = 1_{t_2}$ so that f, g are both injective and surjective. It follows that f, g are isomorphisms of t_1 with t_2 as desired. □

2. Consider the categories $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$. Let $i\colon (\mathbb{Z}, \leq) \to (\mathbb{R}, \leq)$ be the functor given by inclusion. Find functors $L, R\colon (\mathbb{R}, \leq) \to (\mathbb{Z}, \leq)$ such that $L \dashv i$ and $i \vdash R$.

   Choose $L = \lceil \cdot \rceil$ and $R = \lfloor \cdot \rfloor$, the standard ceiling and floor functions. [That is, for $x \in \mathbb{R}$, the quantity $\lceil x \rceil$ is the least integer greater than x; similarly $\lfloor x \rfloor$ is the greatest integer less than x. These maps are functors since if $a, b, c \in \mathbb{R}$ with $a \leq b \leq c$ and $a \leq a$ then $\lceil a \rceil \leq \lceil b \rceil \leq \lceil c \rceil$ and $\lceil a \rceil \leq \lceil a \rceil$ hold, similarly $\lfloor a \rfloor \leq \lfloor b \rfloor \leq \lfloor c \rfloor$ and $\lfloor a \rfloor \leq \lfloor a \rfloor$ hold.]

   Proof. The functor $L = \lceil \cdot \rceil$ is a left adjoint of the inclusion map i if for all $z \in \mathbb{Z}$ and $r \in \mathbb{R}$, we have $\mathbb{Z}(\lceil r \rceil, z) \cong \mathbb{R}(r, z)$. Indeed, $\lceil r \rceil \leq z$ if and only if $r \leq z$.

   Similarly, the functor $R = \lfloor \cdot \rfloor$ is a right adjoint of the inclusion map i if for all $r \in \mathbb{R}$ and $z \in \mathbb{Z}$, we have $\mathbb{R}(z, r) \cong \mathbb{Z}(z, \lfloor r \rfloor)$. Indeed, $z \leq r$ if and only if $z \leq \lfloor r \rfloor$. □

3. (1.3.3) A space X is separated if and only if the diagonal $D = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

   Proof. Let X be a topological space, and write $D^c = (X \times X) \setminus D = \{(x_1, x_2) \mid x_1, x_2 \in X, x_1 \neq x_2\}$.

   Suppose X is Hausdorff. We show that D^c is open. Take any $(x_1, x_2) \in D^c$ so that $x_1 \neq x_2$ and so there exist disjoint open sets $U_{x_1}, U_{x_2} \subset X$ containing x_1, x_2 respectively. Then $U_{x_1} \times U_{x_2}$ is an open set in $X \times X$ containing (x_1, x_2) which is contained in D^c. Hence D is closed.

   Conversely, supposed D is closed so that D^c is open. For any points $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have that $(x_1, x_2) \in D^c$. But then there exists a basis element $U_{x_1} \times U_{x_2}$ in the product topology on $X \times X$ containing (x_1, x_2) which is contained in D^c. It follows that U_{x_1}, U_{x_2} are disjoint open sets containing x_1, x_2 respectively. Since x_1, x_2 were arbitrary it follows that X is Hausdorff. □

Let $f, g\colon X \to Y$ be continuous maps into a Hausdorff space. Then the *coincidence set* $A = \{x \mid f(x) = g(x)\}$ is closed in X.

Proof. Define $h\colon X \to Y \times Y$ by $h(x) = (f(x), g(x))$, which is continuous because f, g are. Then since Y is Hausdorff, the diagonal $D = \{(y, y) \mid y \in Y\}$ is closed in $Y \times Y$, so that its preimage under h is also closed by continuity.

We have that $h^{-1}(D) = \{x \mid h(x) \in D\}$, but $h(x) = (f(x), g(x)) \in D$ if and only if $f(x) = g(x)$. So $h^{-1}(D) = A = \{x \mid f(x) = g(x)\}$ is a closed set as desired. □

4. (1.8.1) Let H be a normal subgroup of G and X a G-space. Restricting the group action to H, we obtain an H-space X. The orbit space $H\backslash X$ carries then an induced G/H-action.

Proof. Let H be a normal subgroup of G and X a G-space with left action $\ell\colon G \times X \to X \in \mathbf{Top}$ as above. We check that the restriction $\ell|_H\colon H \times X \to H$ given by $(h, x) \mapsto \ell(h, x) = hx$ is a left action of H on X. Let $x \in X$. Since $1_H = 1_G$, we have that $1_H x = 1_G x = x$. Then let $h_1, h_2 \in H$. Since $H \subset G$ we use the left action specified by ℓ to find that $h_2(h_1 x) = (h_2 h_1)x$, and since H is a group, $h_2 h_1 \in H$ as needed. The continuity of this group action comes from the continuity of ℓ.

The orbit space H/X is given by equivalence classes of the relation $\sim$ where if $x \in X$, then $x \sim hx$ for all $h \in H$. For $x \in X$ denote its equivalence class by $[x]$. Give H/X the quotient topology by the quotient map π_H sending x to $[x]$.

Define the induced left action on H/X by G/H (which is a topological group since H is normal in G and we give G/H the quotient topology also; call the quotient map π_G): $\rho\colon (G/H) \times (H/X) \to H/X$ where $(gH, [x]) \mapsto [gx]$ where $gx = \ell(g, x)$.

Check that ρ is indeed a left action: Let $x \in X$ and $g_1, g_2 \in G$. Then $1_G H[x] = [1_G x] = [x]$ and $g_2 H(g_1 H[x]) = g_2 H[g_1 x] = [g_2(g_1 x)] = [(g_2 g_1)x] = (g_2 g_1)H[x] = (g_1 H g_2 H)[x]$ as expected.

We check that this map is well defined: Let $gH = g'H$, so that there exists $h_1 \in H$ such that $gh_1 = g'$. Similarly, let $[x] = [x']$, so that there exists $h_2 \in H$ such that $h_2 x = x'$. Then

$$g'H[x'] = (gh_1)H[h_2 x] = [gh_1 h_2 x] = [g(h_1 h_2 x)] = gH((h_1 h_2)H[x]) = gH(1_G H[x]) = (gH 1_G H)[x] = gH[x]$$

as desired.

The following diagram commutes:

$$\begin{tikzcd}
(G/H) \times (H/X) \arrow[r, "\rho"] & H/X \\
G \times X \arrow[u, "``\pi_G \times \pi_H''"] \arrow[r, "\ell"'] & X \arrow[u, "\pi_H"']
\end{tikzcd}$$

Note here that ℓ, π_H are continuous so if we take an open set $U \subseteq H/X$ and take the preimage of U under π_H and then ℓ, we obtain an open set V in $G \times X$. This same set should be obtained if we took preimages in the other direction. So the preimage of ρ is a set whose preimage under what should be the "product" quotient map "$\pi_G \times \pi_H$" is open, but by definition of the quotient topology it follows that the preimage of U under ρ must be open. Hence $\rho \in$ **Top** as desired and it is the induced G/H action on H/X. $\square$