Graded

1. (DF10.4.11) Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_R V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$

   Proof. The generators of $V \otimes_R V$ are the simple tensors $e_i \otimes e_j$ for $1 \leq i, j \leq 2$ (four generators), since any finite $\mathbb{R}$-linear combination of simple tensors $\sum_i r_i(a_i e_1 + b_i e_2) \otimes (c_i e_1 + d_i e_2)$ can be decomposed into $\sum_i r_i a_i c_i (e_1 \otimes e_1) + r_i a_i d_i (e_1 \otimes e_2) + r_i b_i c_i (e_2 \otimes e_1) + r_i b_i d_i (e_2 \otimes e_2)$.

   Moreover, $e_i \otimes e_j$ for $1 \leq i, j \leq 2$ form a basis for the $\mathbb{R}$-vector space $V \otimes_R V$ since they are linearly indpendent: Tensor products are associative and commute with direct sums, so $V \otimes_R V \cong (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R})^4$. The map sending $e_i \otimes e_j$ to e_{ij} (view $\mathbb{R}^4$ as vector space of real-valued matrices maybe) can be extended by linearity to a surjective map from $V \otimes_{\mathbb{R}} V$ to $\mathbb{R}^4$ with trivial kernel (by decomposing an element of $V \otimes_R V$ as earlier and applying the map, it follows that only the zero element is mapped to the zero element). It follows that this map is an isomorphism, from which it follows that $V \otimes_R V \cong \mathbb{R}^4$. By linear algebra, since we have four generators they must be linearly independent also and hence form a basis for $V \otimes_R V$.

   Now let $v = c_1 e_1 + c_2 e_2$ and $w = d_1 e_1 + d_2 e_2$ for some real c_i, d_i. Suppose by way of contradiction that $e_1 \otimes e_2 + e_2 \otimes e_1 = v \otimes w = (c_1 e_1 + c_2 e_2) \otimes (d_1 e_1 + d_2 e_2)$. But $(c_1 e_1 + c_2 e_2) \otimes (d_1 e_1 + d_2 e_2) = c_1 d_1 (e_1 \otimes e_1) + c_1 d_2 (e_1 \otimes e_2) + c_2 d_1 (e_2 \otimes e_1) + c_2 d_2 (e_2 \otimes e_2)$. It follows then that $c_1 d_1 = 0$, $c_1 d_2 = 1$, $c_2 d_1 = 1$, and $c_2 d_2 = 0$. This system has no solution since already one of c_1, d_1 must be zero, which makes it impossible for one of the equations $c_1 d_2 = 1$, $c_2 d_1 = 1$ to hold. Hence $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_R V$ cannot be written as a simple tensor as assumed. □

2. (DF10.5.2) Suppose that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"'] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D'
\end{tikzcd}$$

   is a commutative diagram of groups, and that the rows are exact. Prove that

   (a) if α is surjective, and β, δ are injective, then γ is injective.

   Proof. We show that γ has trivial kernel. We suppress the use of parentheses.

   Let $c \in C$ be such that $\gamma c = 1$. Then $h'\gamma c = 1$ also. Since the diagram commutes and δ is injective, $hc = 1$. By exactness, c is in the image of g and thus has a preimage $b \in B$ under g. Then again since the diagram commutes, $g'\beta b = \gamma c = 1$, and so by exactness, βb is in the image of f'. Let $a' \in A'$ be a preimage of βb under f'. With α surjective, take $a \in A$ to be a preimage of a'. Since the diagram commutes, $\beta f a = f'\alpha a = \beta b$, and since β is injective we have $b = f(a)$. But $c = g(b)$, so by exactness we must have $c = 1$ as desired. □

   (b) if δ is injective, and α, γ are surjective, then β is surjective.

Proof. We show that any element $b' \in B'$ has a preimage under β. Suppress the use of parentheses again.

By exactness $h'g'b' = 1$. Since γ is surjective, take c to be a preimage of $g'b'$ under γ. By the commutativity of the diagram we have $\delta hc = 1$, and since δ is injective $hc = 1$. Then by exactness c is in the image of g, so let b be a preimage of c under g. By the commutativity of the diagram we have $g'\beta b = g'b'$. Since g' is a homomorphism, $g'[(\beta b)^{-1}b'] = 1$ so that by exactness $(\beta b)^{-1}b'$ is in the image of f'. Let $a' \in A'$ be a preimage of $(\beta b)^{-1}b'$ under f'. With α surjective take $a \in A$ to be a preimage of a' under α. By the commutativity of the diagram we have $(\beta b)^{-1}b' = \beta fa$. Since β is a homomorphism it follows that $b' = \beta[b(fa)]$. Hence b' has a preimage in B under β as desired. □

Additional Problems

1. (DF10.4.5) Let A be a finite Abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

   Proof. Let $n = p^k \prod_{i=1}^{r} q_i^{k_i}$ be the prime factorization of n. Then by using the structure theorem for finitely generated Abelian groups we find the elementary divisor decomposition of A; that is, we decompose A into the direct sum of its Sylow subgroups. We have $A \cong A_p \oplus \bigoplus_{i=1}^{r} A_{q_i}$ with A_p the p-Sylow and A_{q_i} the q_i-Sylow for each i. Note $|A_p| = p^k$ and $|A_{q_i}| = q_i^{k_i}$.

   Each factor A_{q_i} can be decomposed into the direct sum of cyclic groups of order a power of q_i. It follows then that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A_{q_i} = 1$ by the commutativity of the tensor product with direct sums and the fact that $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/\gcd(a,b)\mathbb{Z}$ (we saw this in class, or it is in Dummit and Foote). For the same reason it turns out that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A_p \cong A_p$ since $\gcd(p^s, p^k) = p^s$ for $s \leq k$.

   It follows that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to A_p, the p-Sylow of A. □

2. (DF10.5.12) Let A be an R-module, let I be any nonempty index set and for each $i \in I$ let B_i be an R-module. Prove the following isomorphisms of Abelian groups; when R is commutative prove also that these are R-module homomorphisms. (Arbitrary direct sums and direct products of modules are introduced in Exercise 20 of Section 3.)

   (a) $\mathrm{Hom}_R(\bigoplus_{i\in I} B_i, A) \cong \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$

   Proof. We define homomorphisms in both directions and show they are inverse to each other: Let ϕ be defined by taking a map $f\colon \bigoplus_{i\in I} B_i \to A$ (note that a map like this only makes sense since the direct sum only contains tuples with cofinitely many entries zero) to the sequence $(f_i)_{i\in I}$ where f_i is the action of f restricted to the summand B_i (and we can extract this by using the fact that f is a homomorphism). In particular, the map f must send $\sum_{\substack{i\in I\\ \text{finite}}} b_i$ to $\sum_{\substack{i\in I\\ \text{finite}}} f_i(b_i)$ since f is a homomorphism acting componentwise. The map ϕ is a homomorphism of Abelian groups since the action of homomorphisms $\bigoplus_{i\in I} B_i \to A$ is componentwise; when R is commutative it does not matter which direction the scalars multiply in and again since the action of homomorphisms are componentwise ϕ also respects the R-module structure.

Let the map ψ be defined by taking a sequence $(f_i)_{i\in I}$ to the unique map $f\colon \bigoplus_{i\in I} B_i \to A$ given by the universal property of the direct sum (coproduct). In particular, the map f must send $\sum_{\substack{i\in I\\ \text{finite}}} b_i$ to $\sum_{\substack{i\in I\\ \text{finite}}} f_i(b_i)$ by the universal property. Since f must be defined in this way it follows that this assignment is a homomorphism of Abelian groups, and is clearly also a homomorphism of R-modules.

The maps ψ and ϕ are inverse to each other: Take a map $f\colon \bigoplus_{i\in I} B_i \to A$, send it by ϕ to $(f_i)_{i\in I}$, and send this sequence to a function g which maps $\sum_{\substack{i\in I\\ \text{finite}}} b_i$ to $\sum_{\substack{i\in I\\ \text{finite}}} f_i(b_i)$ – it follows $f = g$. Conversely, take a sequence $(f_i)_{i\in I}$ and send it by ψ to the function f taking $\sum_{\substack{i\in I\\ \text{finite}}} b_i$ to $\sum_{\substack{i\in I\\ \text{finite}}} f_i(b_i)$; then f here is sent to $(f_i)_{i\in I}$ by ϕ. It follows ϕ, ψ are isomorphisms. □

(b) $\mathrm{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$

Proof. We proceed similarly as above. Let ϕ take a map $f\colon A \to \prod_{i\in I} B_i$ to a sequence (f_i) where the f_i take $a \in A$ to $f(a)_i \in B_i$, the i-th entry of $f(a)$. This map is clearly a homomorphism of Abelian groups and when R is commutative it will respect the R-module structure.

Let ψ take a sequence of maps $(f_i)_{i\in I}$ to the map f defined by $f(a) = (f_i(a))_{i\in I}$. This map is a homomorphism since we are doing everything componentwise; when R is commutative the R-module structure makes sense and is immediately respected as well.

The maps are inverse to each other: Take $f\colon A \to \prod_{i\in I} B_i$ to a sequence (f_i) by ϕ, then this sequence is sent by ψ to g, which necessarily agrees with f. Conversely take any sequence (f_i), by ψ it is sent to f, then ϕ takes f to the sequence (g_i). But $f_i = g_i$ for all $i \in I$ by definition of g_i, so it follows ϕ, ψ are isomorphisms. □

Feedback

1. None.
2. Things seem to be the same I think.