RESULTS

1. Generating set for σ

Let $\mathbf{e}_{11}, \mathbf{e}_{12}, \ldots, \mathbf{e}_{1n}, \mathbf{e}_{22}, \mathbf{e}_{23}, \ldots, \mathbf{e}_{nn}$ be a basis for $\mathbb{Z}^{\binom{n+1}{2}}$. We have $\sigma^\vee = \mathrm{Cone}(\mathscr{A})$ for $\mathscr{A} = \{\mathbf{e}_{ij}, \mathbf{e}_{ii} - \mathbf{e}_{ij} + \mathbf{e}_{jj} \mid i \leq j\}$.

Claim: For σ as above, we have

$$\sigma = (\sigma^\vee)^\vee = \mathrm{Cone}(\mathscr{B}).$$

for

$$\mathscr{B} = \{\mathbf{e}_{ii}, \mathbf{e}_{ii} + \mathbf{e}_{ij_1} + \cdots + \mathbf{e}_{ij_r} + \mathbf{e}_{j'_1 i} + \cdots + \mathbf{e}_{j'_{r'} i} \mid 1 \leq i \leq n,\ r, r' > 0,\ j_h > i \text{ for } 1 \leq h \leq r,\ j'_{h'} < i \text{ for } 1 \leq h' \leq r'\}$$

Proof. Given $u = (b_{11}, \ldots, b_{1n}, b_{22}, \ldots, b_{2n}, \ldots, b_{nn}) \in N_\mathbb{R} \cong \mathbb{R}^{\binom{n+1}{2}}$, we have that $u \in \sigma$ if and only if:

(1) $b_{ij} \geq 0$, for all $i \leq j$
(2) $b_{ii} - b_{ij} + b_{jj} \geq 0$, for all $i < j$

This follows from the definition of the dual cone. The vector u is in σ if and only if $\langle u, a\rangle \geq 0$ for any $a \in \sigma^\vee$. In particular, $\langle u, a\rangle \geq 0$ for $a \in \{\mathbf{e}_{ii}, \mathbf{e}_{ii} - \mathbf{e}_{ij} + \mathbf{e}_{jj} \mid i \leq j\} = \mathscr{A}$, which gives the above inequalities.

It is clear that $\sigma' = \mathrm{Cone}(\mathscr{B})$ is contained in σ. (The inner product is bilinear so it suffices to see that the inner product of any generator of σ' with any generator of $\sigma^\vee$ is nonnegative.)

To show that $\sigma \subseteq \sigma'$, we show instead that elements not in σ' are not in σ.

An element not in $\mathrm{Cone}(\mathscr{B})$ is given by a linear combination $\sum_{p=1}^q b_p \beta_p$ for $\beta_p \in \mathscr{B}$ where at least one b_p is negative. As a result we see that $\mathbf{e}_{ij_h}, \mathbf{e}_{j'_h i}$ are not in $\mathrm{Cone}(\mathscr{B})$ for $1 \leq i \leq n,\ r, r' > 0,\ j_h > i$ for $1 \leq h \leq r,\ j'_{h'} < i$ for $1 \leq h' \leq r'$.

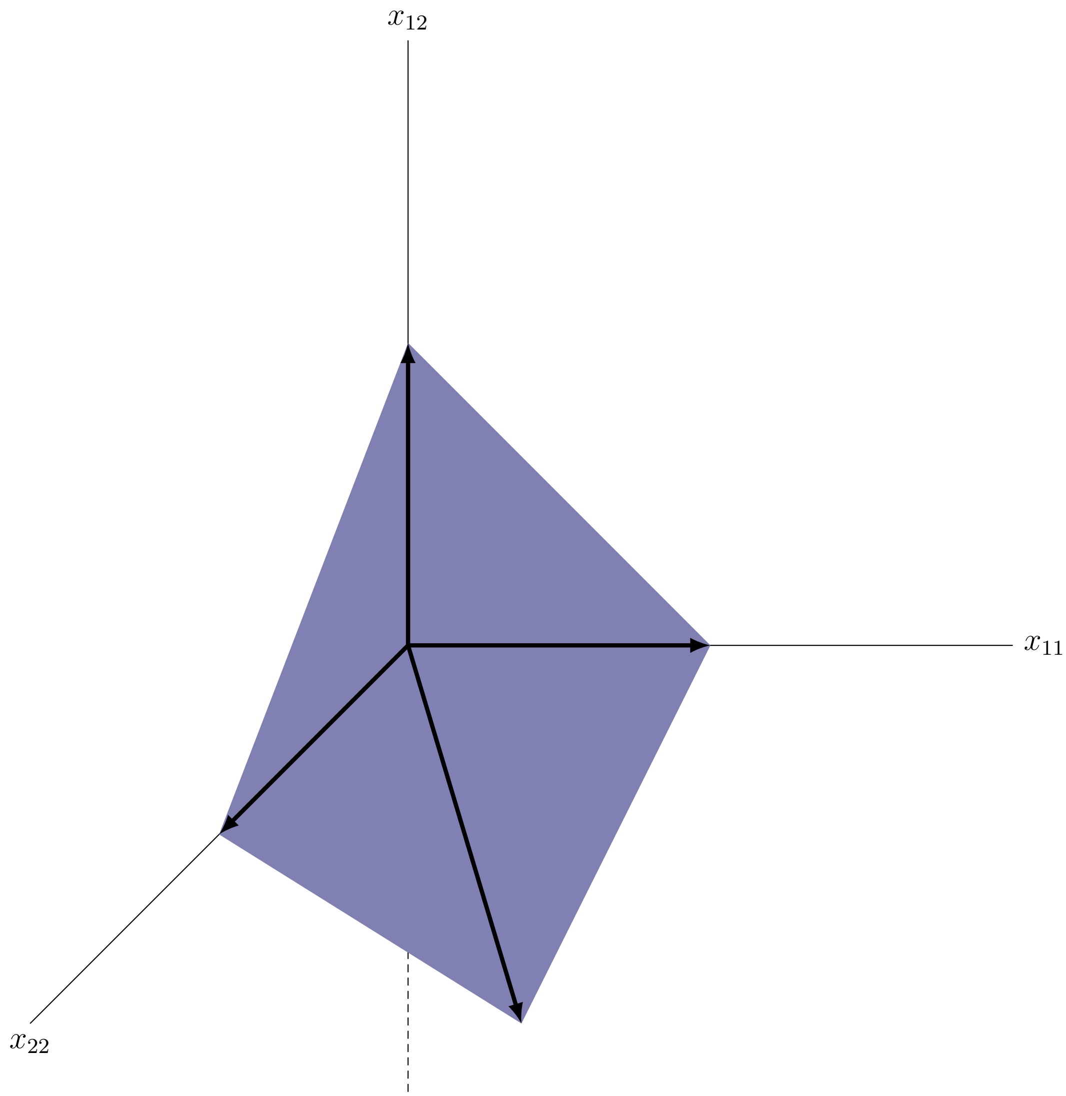
Hence we can decompose any element u' not in $\mathrm{Cone}(\mathscr{B})$ into a sum of the following form:

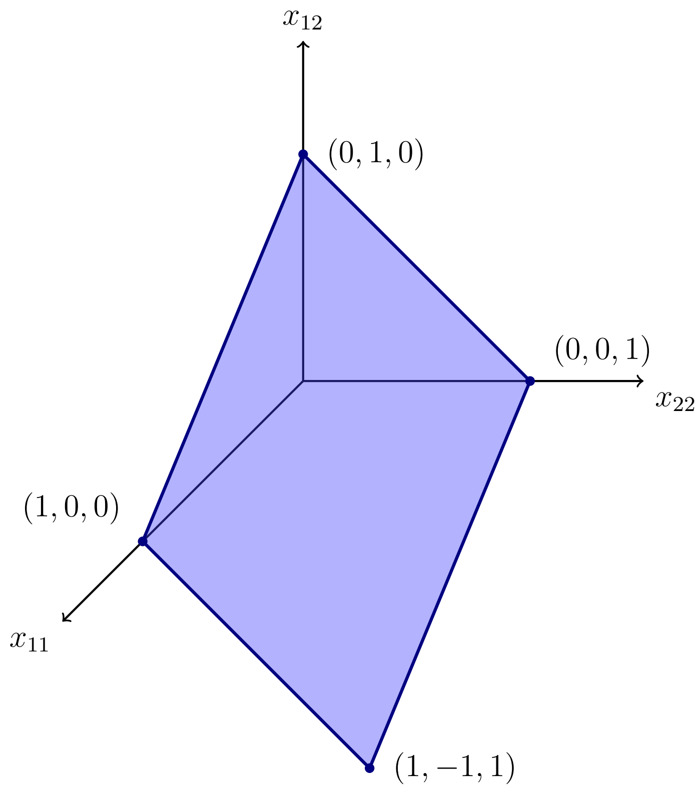
$$u' = \sum_{g=1}^{r'''} \big[b_{i_g i_g}\mathbf{e}_{i_g i_g} + b_{i_g j_1}\mathbf{e}_{i_g j_1} + \cdots + b_{i_g j_r}\mathbf{e}_{i_g j_r} + b_{j'_1 i_g}\mathbf{e}_{j'_1 i_g} + \cdots + b_{j'_{r'} i_g}\mathbf{e}_{j'_{r'} i_g} + b_{k_1 \ell_1}\mathbf{e}_{k_1 \ell_1} + \cdots + b_{k_{r''} \ell_{r''}}\mathbf{e}_{k_{r''} \ell_{r''}} \big]$$

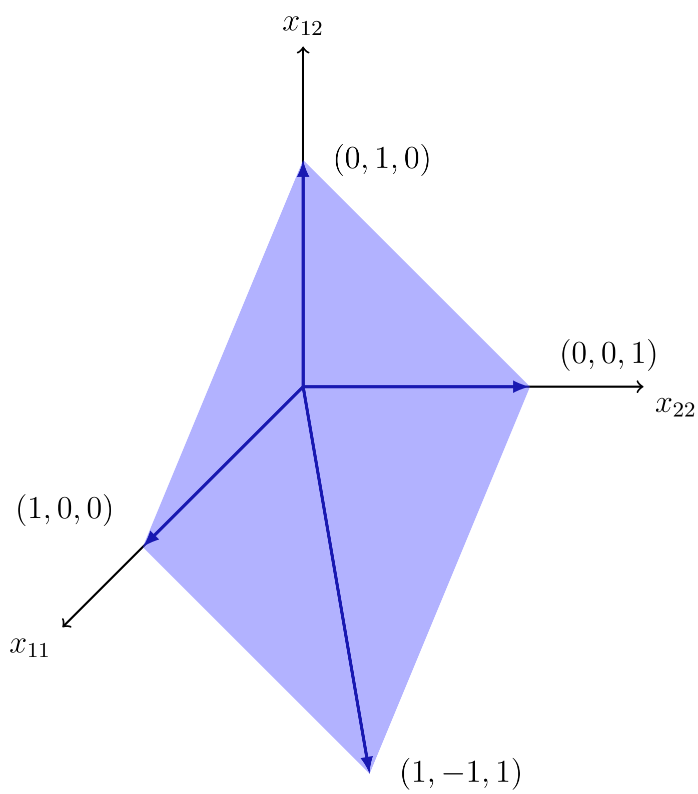
where $1 \leq i \leq n,\ r, r', r'', r''' > 0,\ j_h > i$ for $1 \leq h \leq r,\ j'_{h'} < i$ for $1 \leq h' \leq r'$ and $b_{k_{h''}\ell_{h''}} \neq 0, k_{h''} \neq \ell_{h''} \neq i$ for $1 \leq h'' \leq r''$. But u' cannot be in σ as it violates condition (2) above.

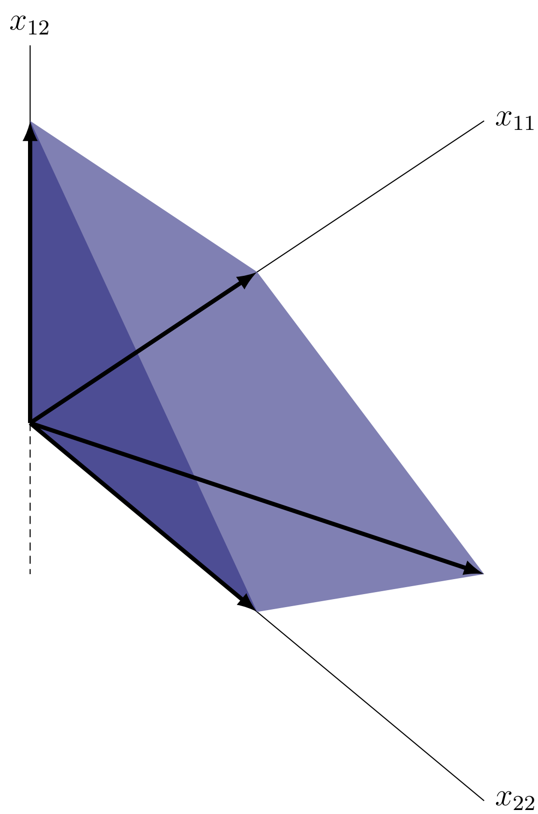
Hence elements not in σ' are not in σ, so the reverse inclusion holds and $\sigma = \sigma'$. ∎

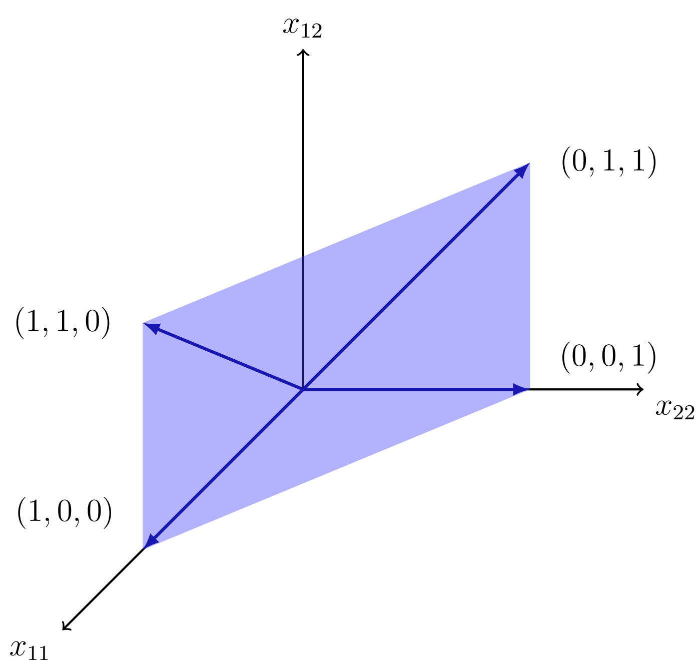
Date: July 14, 2022.

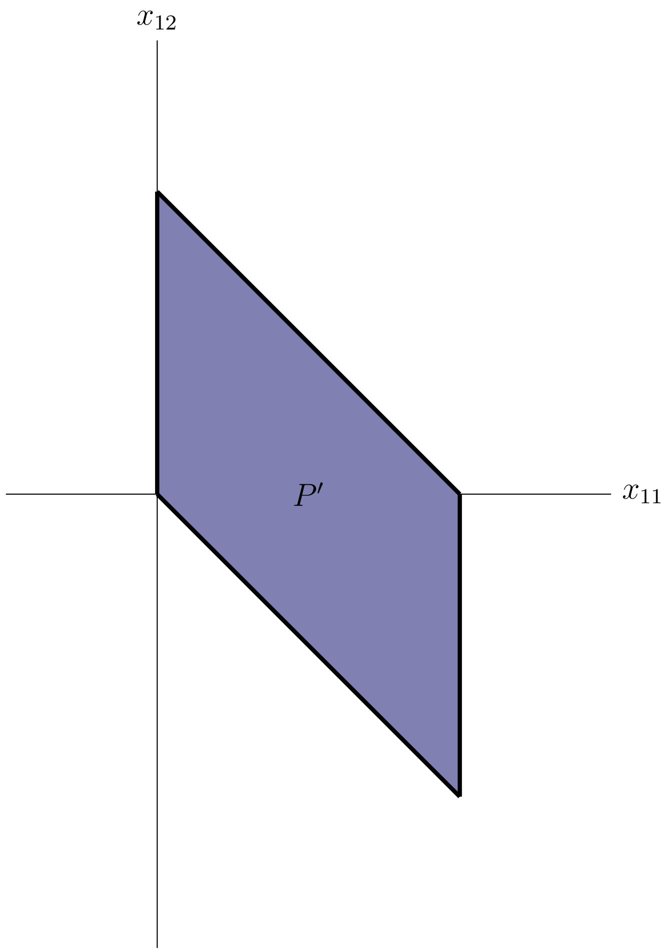

x_{12}
x_{11}
x_{22}

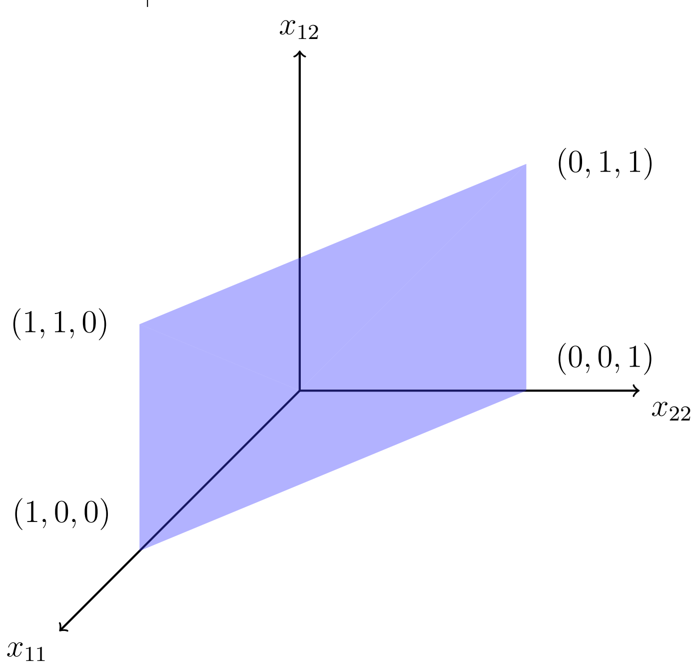

x_{12}
$(0, 1, 0)$
$(0, 0, 1)$
x_{22}
$(1, 0, 0)$
x_{11}
$(1, -1, 1)$

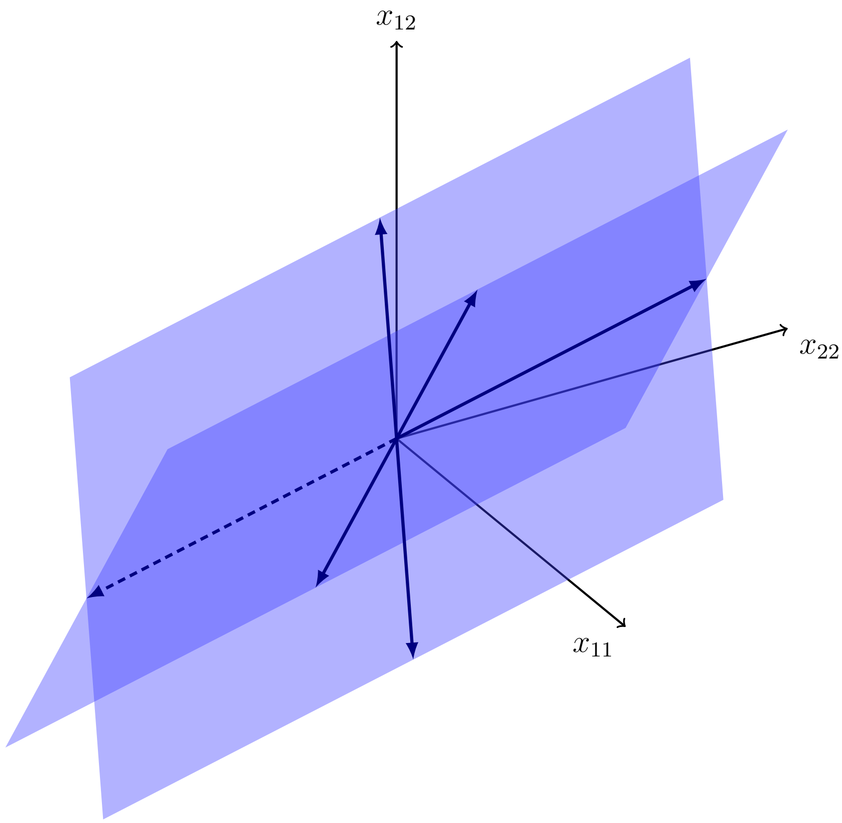

x_{12}
$(0, 1, 0)$
$(0, 0, 1)$
x_{22}
$(1, 0, 0)$
x_{11}
$(1, -1, 1)$


x_{12}
x_{11}
x_{22}


x_{12}
$(0,1,1)$
$(1,1,0)$
$(0,0,1)$
x_{22}
$(1,0,0)$
x_{11}


x_{12}
P'
x_{11}


x_{12}
$(0,1,1)$
$(1,1,0)$
$(0,0,1)$
x_{22}
$(1,0,0)$
x_{11}


x_{12}
x_{22}
x_{11}