Conic Assignment

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CONTENTS

I. PROBLEM

Q.The locus of the mid-point of the lines segment joining the focus to a moving point on the parabola y^2=4ax is another parabola with directrix

II. SOLUTION

The standard conic equation is,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where, $\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}$ and $f = 0$

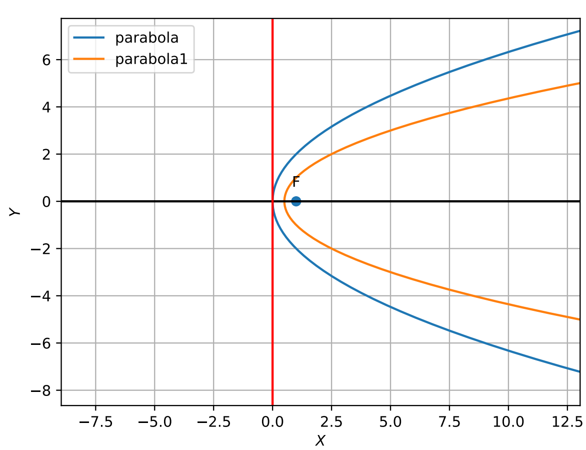

Fig. 1. Labeling locus parabola with directrix for a given parabola

For the standard conic,

$$\mathbf{u} = \frac{\eta}{2} e_1, e = 1 \tag{2}$$

where, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ equating values of u,

$$\frac{\eta}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} \frac{\eta}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}$$
$$\frac{\eta}{2} e_1 = -2a$$
$$\eta = -4a$$

III. FIGURE

Foci of the standard parabola is given by,

$$\mathbf{F} = \frac{-\eta}{4\lambda_2} e_1, e = 1 \tag{3}$$

By substituting η value we get,

$$\mathbf{F} = -\frac{-4a}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathbf{F} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

let moving point on the parabola be 'q',then the equation is,

$$\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{4}$$

the mid-point when line segment joining the focus to moving point on the parabola be 'h'.

$$\mathbf{h} = \frac{\mathbf{q} + \mathbf{F}}{2} \tag{5}$$

$$\mathbf{q} = 2\mathbf{h} - \mathbf{F} \tag{6}$$

Substitute (6) in (4)

$$(2\mathbf{h} - \mathbf{F})^\top \mathbf{V}(2\mathbf{h} - \mathbf{F}) + 2\mathbf{u}^\top (2\mathbf{h} - \mathbf{F}) + f = 0 \tag{7}$$

$4\mathbf{h}^\top\mathbf{V}\mathbf{h} - \mathbf{F}^\top\mathbf{V}\mathbf{F} - 2\mathbf{h}^\top\mathbf{V}\mathbf{F} - 2\mathbf{F}^\top\mathbf{V}\mathbf{h} + 4\mathbf{u}^\top\mathbf{h} - 2\mathbf{u}^\top\mathbf{F} + f = 0$
By solving we get the new parabola equation,

$$y^2 - 2ax + a^2 = 0 \tag{8}$$

from above the locus parabola equation can be written as,

$$\mathbf{h}^\top\mathbf{V}_1\mathbf{h} + 2\mathbf{u}_1^\top\mathbf{h} + f_1 = 0 \tag{9}$$

where, $\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{u}_1 = \begin{pmatrix} -a \\ 0 \end{pmatrix}$ and $f_1 = a^2$
The new parabola equation in terms of old parabola equation is,

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + \mathbf{u}^\top\mathbf{x} + (f + d) = 0$$

The directrix of a conic is given by,

$$\mathbf{n}^\top\mathbf{x} = c \tag{10}$$

The directrices of (9) is given by,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 \tag{11}$$

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}}, e = 1 \tag{12}$$

where the eigen values and eigen vectors are,

$$\lambda_1 = 0, \lambda_2 = 1 \tag{13}$$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

substituting values in (11) we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{15}$$

substituting values in (12),
$c = \frac{a^2 - a^2}{2\begin{pmatrix} -a & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$$c = 0 \tag{16}$$

put n and c in directrix equation we get,

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{17}$$

Therefore,the locus of the mid-point of the lines segment joining the focus to a moving point on the parabola y^2-$4ax = 0$ is another parabola y^2-$2ax$+$a^2 = 0$ with directrix $x = 0$.

IV. SOFTWARE

https://github.com/sivaparvathi−tungala/fwc_module_1/tree/main/conic