CIRCLE ASSIGNMENT

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Contents

1 Problem

Q.Let T_1, T_2 be two tangents drawn from (-2,0) onto the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further ,find the equations of all possible common tangents to the circles.

2 Figure

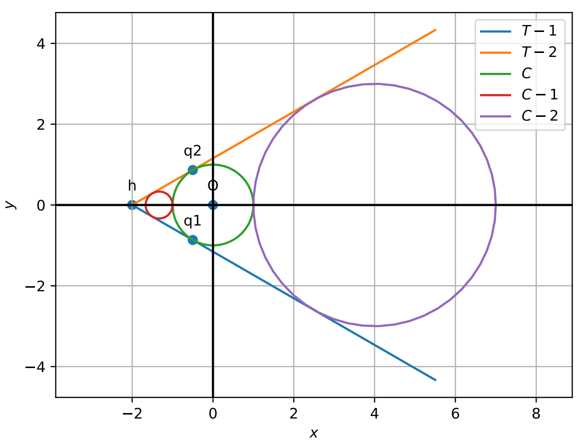

Figure 1: Labeling

3 Solution

Let, the centre of a circle C
be 'u' is at origin and radius be 'r'.

the equation of the circle is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

Symbol	Co-ordinates
u	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
V	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
h	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

To find the normal vectors at the point of contact , drawn from the eternal point, the normal vectors are given as follows

$$\boxed{\mathbf{q}_i = \mathbf{V}^{-1}(k_i \mathbf{n}_i - \mathbf{u})} ....(i = 1, 2) \tag{2}$$

Here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \tag{3}$$

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \tag{4}$$

$$f = -r^2 \tag{5}$$

$$\mathbf{S} = (\mathbf{Vh}\text{+}\mathbf{u})(\mathbf{Vh}\text{+}\mathbf{u})^\top\text{-}\mathbf{V}(\mathbf{h}^\top \mathbf{Vh} + \mathbf{2u}^\top \mathbf{h}) + f \tag{6}$$

$$\mathbf{n}_1 = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \tag{7}$$

$$\mathbf{n}_2 = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{8}$$

By substituting values in eq(4) we get,

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \tag{9}$$

The values of eigen values and vectors are by solving we get,
$\lambda_1 = 1$, $\lambda_2 = -3$ and $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
from eq(6) and (7),

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{10}$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{11}$$

To find the equation through a point we have ,

$$\mathbf{n}^\top(\mathrm{x} - \mathbf{h}) = 0 \tag{12}$$

Tangent equation T_1 and T_2 are ,

$$\mathbf{n}_1^\top(\mathrm{x} - \mathbf{h}) = 0 \tag{13}$$

$$\mathbf{n}_2^\top(\mathrm{x} - \mathbf{h}) = 0 \tag{14}$$

Considering the circles C_1 and C_2 touching C and having T_1 and T_2 are as their pair of tangents. To prove T_1 and T_2 are the common tangents to the circles,let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the centre of circles C_1 and C_2 then

$$\mathbf{u}_1 = \begin{pmatrix} -h_1 \\ 0 \end{pmatrix} and \ \ \mathbf{u}_2 = \begin{pmatrix} h_2 \\ 0 \end{pmatrix} \tag{15}$$

Case(i) :
The distance between point $\mathbf{u}_1$ to the line T_1 can be taken as,

$$d = \frac{|\mathbf{n}_1^\top \mathbf{u}_1 - c|}{\|\mathbf{n}_1\|} \tag{16}$$

r_1 is the radius of circle C_1

$$r_1 = \frac{|\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} -h_1 \\ 0 \end{pmatrix} + 2|}{\sqrt{4}} \tag{17}$$

$$r_1 = \frac{h_1 + 2}{2} \tag{18}$$

from diagram r_1 can be written as,

$$r_1 = -h_1 - 1 \tag{19}$$

from eq(16) and (17),

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-4}{3} \\ 0 \end{pmatrix} \tag{20}$$

Similarly Case(ii) :
The distance between point $\mathbf{u}_2$ to the line T_1 is,

$$r_2 = \frac{h_2 + 2}{2} \tag{21}$$

from diagram r_2 can be written as,

$$r_2 = h_2 - 1 \tag{22}$$

from eq(20) and (21)

$$\mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{23}$$

Substituting $\mathbf{u}_1$ and $\mathbf{u}_2$ in circle equations we get,

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f = 0 \tag{24}$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f = 0 \tag{25}$$

Therefore, the all possible common tangents to the circles are x+$\sqrt{3}$y+2=0 and x-$\sqrt{3}$y+2=0.

4 Software

We can get the parallel equation of given equation and the plot of two equtions by executing the following code:

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https://github.com/sivaparvathi−tungala
    /fwc_module_1/tree/main/circle
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