1. Runge Kutta methods are higher order multi-step explicit methods that give better accuracy and stability than Euler Explicit methods. In this exercise, you will be deriving Runge Kutta methods of order 2 and order 4 (from here on will be called as RK2 and RK4).

   - RK2 method:
   $$y_{n+1} = y_n + \gamma_1 k_1 + \gamma_2 k_2$$
   where $k_1 = hf(t_n, t_n)$ and $k_2 = hf(y_n + \beta k_1, t_n + \alpha h)$. Using Taylor series, determine the relationships that $\alpha, \beta, \gamma_1, \gamma_2$ need to satisfy to ensure highest order of accuracy for the method. Find the region of stability in the complex plane.
   - RK4 method:
   $$y_{n+1} = y_n + \gamma_1 k_1 + \gamma_2 k_2 + \gamma_3 k_3 + \gamma_4 k_4$$
   where
   $$k_1 = hf(t_n, t_n) \tag{1}$$
   $$k_2 = hf(y_n + \beta_{21} k_1, t_n + \alpha_1 h) \tag{2}$$
   $$k_3 = hf(y_n + \beta_{31} k_1 + \beta_{32} k_2, t_n + \alpha_2 h) \tag{3}$$
   $$k_4 = hf(y_n + \beta_{41} k_1 + \beta_{42} k_2 + \beta_{43} k_3, t_n + \alpha_3 h) \tag{4}$$
   Using Taylor series, determine the relationships that $\alpha_i, \beta_i, \gamma_i$ need to satisfy to ensure highest order of accuracy for the method. If it is given that $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = 1$, $\beta_{21} = \beta_{32} = 1/2$ and $\beta_{43} = 1$, determine the rest of the constants. Find the region of stability in the complex plane.

2. Recall that the governing equation for a simple pendulum in the absence of any damping effects subject to small osciallations is given by
$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$
where l is the length of the pendulum and g is the acceleration due to gravity, $\theta(0) = \theta_0$ and $\left.\frac{d\theta}{dt}\right|_{t=0} = 0$. Assume $\ell = g$, and $\theta_0 = \pi/4$. Solve for the unkown θ using
   - Euler Explicit (determine the maximum δt based on stability)
   - Euler Implicit
   - Trapezoidal
   - RK2 (determine the maximum δt based on stability)
   - RK4 (determine the maximum δt based on stability)

3. Repeat the above by deriving the governing equation for large amplitude oscillations of the simple pendulum. You may again assume that $\theta_0 = \pi/4$ and $\ell = g$. Are the Euler implicit and Trapezoidal methods still unconditionally stable?