1. Extend the argument discussed in the class to prove that the Gauss-Seidel method converges for strictly row diagonally dominant matrices.
2. Let $A, B \in \mathbb{R}^{n\times n}$ such that A and $A - B - B^T$ are symmetric positive definite. Prove that the spectral radius of $(A-B)^{-1} B$ is strictly less than 1.
3. Use the above to prove that if A is symmetric positive definite, then the Gauss-Seidel method converges.
4. For a matrix A, let $\tilde{A}$ be defined as follows:
$$\tilde{A}_{ij} = \begin{cases} A_{ij} & \text{if } i = j \\ -A_{ij} & \text{if } i \neq j \end{cases}$$
Prove that if A and $\tilde{A}$ are symmetric positive definite, then the Jacobi method converges.
5. Consider the tridiagonal linear system
$$Ax = b$$
where $A \in \mathbb{R}^{n\times n}$, $A_{kk} = 2$, $A_{k,k+1} = A_{k+1,k} = -1$ and $b_k = \dfrac{k}{n^3}$.
   - Implement Jacobi, Gauss-Seidel, Steepest Descent and Conjugate Gradient for solving this linear system. Try with $n \in \{10, 20, 50, 100, 200, 500, 1000, 2000, 5000\}$ and for different tolerance $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$. The tolerance is used to bound the relative error in the residual, i.e., $\dfrac{\|b - Ax\|}{\|b\|}$.
   - Plot the number of steps for each method as a function of n for different tolerances.
   - Arrive at a rough scaling of the number of steps for each method in terms of n and tolerance.
6. Recall the CG iteration we mentioned in class to solve $Ax = b$, where $A \in \mathbb{R}^{n\times n}$

Algorithm 1: Conjugate Gradient

Result: x such that $Ax = b$

$x_0 = 0$, $r_0 = b - Ax_0$ and $p_0 = r_0$

for $k = 1, 2, 3, \ldots$ **do**

$\quad \alpha_k = \dfrac{r_{k-1}^T r_{k-1}}{p_{k-1}^T A p_{k-1}}$

$\quad x_k = x_{k-1} + \alpha_k p_{k-1}$

$\quad r_k = r_{k-1} - \alpha_k A p_{k-1}$

$\quad \beta_k = \dfrac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$

$\quad p_k = r_k + \beta_k p_{k-1}$

end

As long at the iteration has not yet converged, i.e., $r_{k-1} \neq 0$, prove that

- $$\operatorname{span}\{x_1, x_2, \ldots, x_k\} = \operatorname{span}\{p_0, p_1, \ldots, p_{k-1}\} = \operatorname{span}\{r_0, r_1, \ldots, r_{k-1}\} = \operatorname{span}\left\{b, Ab, A^2 b, \ldots, A^{k-1} b\right\} = K_k(A, b)$$
- $r_k^T r_j = 0$ for all $j < k$
- $p_k^T A p_j = 0$ for all $j < k$
- x_k is the unique point in $K_k(A, b)$ that minimizes $\frac{1}{2} x^T A x - x^T b$.
- Prove that $\|e_k\|_A \leq \|e_{k-1}\|_A$, where $e_k = x_k - x$ and $\|y\|_A = y^T A y$.
- Conclude that $e_k = 0$ is achieved for some $k \leq n$.