1. Prove the Hölder inequality: If $x, y \in \mathbb{R}^n$, we then have

$$\left|x^T y\right| \leq \|x\|_p \|y\|_q$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Note that Cauchy-Schwarz is a special case of this.

2. Norm equivalence: Let $x \in \mathbb{R}^n$.
   - $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$
   - $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$

3. Show that adding two positive numbers is a well-conditioned problem, i.e., if $f(a, b) = a+b$, where $a, b > 0$, obtain the condition number (in two norm) and the range of values it can take.

4. Prove that $\|AB\|_p \leq \|A\|_p \|B\|_p$, where $\|\cdot\|_p$ is the matrix norm induced by vector norm.

5. Let $A = \begin{bmatrix} 11 & -5 \\ 2 & 10 \end{bmatrix}$. Compute the following norms. (i) $\|A\|_1$, (ii) $\|A\|_2$, (iii) $\|A\|_\infty$, (iv) $\|A\|_{1^*}$, (v) $\|A\|_{2^*}$, (vi) $\|A\|_{\infty^*}$

6. **(Trefethen and Bau: Exercise 12.3)** The goal of this problem is to explore some properties of random matrices. Your job is to a laboratory scientist, performing experiments that lead to conjectures and more refined experiments. Do not try to prove anything. Do produce well-designed plots, which are worth a thousand numbers.

   Define a *random matrix* to be a $m \times m$ matrix whose entries are independent samples from the ral normal distribution with mean zero and standard deviation $\dfrac{1}{\sqrt{m}}$. The factor $\sqrt{m}$ is introduced to make the limiting behavior clean as $m \to \infty$.

   (a) What do the eigenvalues of a random matrix look like? What happens, say, if you take 1000 random matrices and superimpose all their eigenvalues in a single plot? If you do this for $m = 8, 16, 32, 64, 128, 256$, what pattern is suggested? How does the maximum eigenvalue (in absolute sense, denoted by ρ) of the matrix behave as $m \to \infty$?

   (b) What about norms? How does the 2-norm of a random matrix behave as $m \to \infty$? Of course, we must have $\rho(A) \leq \|A\|$, for induced norm on the matrix A. Does this inequality appear to approach equality as $m \to \infty$?

   (c) What about condition numbers or more simply the smallest singular value σ_{min}? Even for fixed m this question is interesting. What proportions of random matrices in $\mathbb{R}^{m \times m}$ seem to have $\sigma_{min} \leq \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \ldots$? In other words, what does the tail of the probability distribution of smallest singular values look like? How does the scale of all this change with m?

   (d) How do the answers to $(a) - (c)$ change if we consider random triangular instead of full matrices, i.e., upper-triangular matrices whose entries are samples from the same distribution as above?