1. Show that Cramer's rule is not backward stable. Take a 2×2 example and illustrate.

2. Consider the matrix

$$A = \begin{bmatrix} -1 & -1 & 2 & -4 \\ 2 & -2 & -2 & 6 \\ -1 & -9 & 7 & -8 \\ 2 & 6 & -3 & 11 \end{bmatrix}$$

   As discussed in the class obtain the LU factorization of A, by obtaining the following matrices:

   L_1, L_1A, L_2, L_2L_1A, L_3, $L_3L_2L_1A$, U, L_1^{-1}, L_2^{-1}, L_3^{-1}, L

   Repeat the above using partial pivoting (i.e., $PA = LU$) and obtain the following matrices:

   T_1, L_1, L_1T_1A, T_2, L_2, $L_2T_2L_1T_1A$, T_3, L_3, $L_3T_3L_2T_2L_1T_1A$, U, L_1', L_2', L_3', ${L_1'}^{-1}$, ${L_2'}^{-1}$, ${L_3'}^{-1}$, P, L.

3. Show that solving a lower triangular system (with non-zero diagonal entries) by forward substitution is backward stable, i.e., if $\tilde{x}$ is the solution to obtained by $Lx = b$ using forward substitution in finite precision arithmetic with machine precision ϵ_m, then $\tilde{x}$ satisfies

$$\tilde{L}\tilde{x} = b$$

   where $\dfrac{\|\tilde{L} - L\|}{\|L\|} = f(n)\epsilon_m$. You may use the $\|\cdot\|$ as the max-norm, i.e., $\|X\| = \max_{i,j} |X|_{i,j}$.

4. Recall the LU decomposition with and without pivoting as discussed in class.

   (a) **Partial pivoting**: In the case of partial pivoting show that

      i. $\|L\|_{\max} \leq 1$

      ii. $\dfrac{\|U\|_{\max}}{\|A\|_{\max}} \leq 2^{n-1}$

      iii. Hence, conclude that $\rho_g(n) \leq 2^{n-1}$

      iv. Construct an example, where the growth factor ρ_g is exactly 2^{n-1}.

   (b) **Complete pivoting**: In the case of complete pivoting show that

      i. $\|L\|_{\max} \leq 1$

      ii. $\dfrac{\|U\|_{\max}}{\|A\|_{\max}} < \sqrt{n}\sqrt{2 \cdot 3^{1/2} \cdot 4^{1/3} \cdots n^{1/(n-1)}}$

      iii. Hence, conclude that $\rho_g(n) < \sqrt{n}\sqrt{2 \cdot 3^{1/2} \cdot 4^{1/3} \cdots n^{1/(n-1)}}$

      iv. Trivia: It is known that the bound for complete pivoting is not tight. Finding the optimal upper bound for complete pivoting is still an open problem. If $g(n) = \sup_{A \in \mathbb{R}^{n \times n}} \rho_g(n)$, it is conjectured that $g(n) = cn$, where c is some constant exceeding 1. The following results are known.

      - $g(2) = 2$
      - $g(3) = 9/4$
      - $g(4) = 4$
      - $g(5) \leq 5.005$