Numerical Linear Algebra: Practice Final

November 18, 2019

1. Consider $p, q \in \mathbb{R}^+$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $A \in \mathbb{R}^{m \times n}$. Prove that

   - $$\|A\|_p = \left\|A^T\right\|_q$$

     Solution: Note that
     $$\|x\|_p = \max_{\|y\|_q = 1} y^T x$$
     Now conclude using the definition of matrix norm.

   - $$\|A\|_2^2 \leq \|A\|_p \|A\|_q$$

     Solution: Prove that for any vector
     $$\|y\|_2^2 \leq \|y\|_p \|y\|_q$$

   - for any $p \geq 1$, and if $A \in \mathbb{R}^{n \times n}$ is diagonal, then
     $$\|A\|_p = \max\{|A_{ii}| : 1 \leq i \leq n\}$$

     Solution: Trivial

2. Let $A \in \mathbb{R}^{n \times n}$ and consider the series
   $$S = \sum_{k=0}^{\infty} A^k$$
   Prove that the series converges iff all the eigenvalues of A are strictly smaller than 1. Further, if the series converges, show that S is invertible with its inverse being $I - A$.
   Solution: Use Jordan Canonical form to conclude.

3. Show that if T is a symmetric tridiagonal matrix and an eigenvalue λ has multiplicity k, then at least $k-1$ subdiagonal elements of T are zero.
   Solution: Note that it is sufficient to show that if all the entries on the sub and super diagonal are non-zero, then the eigenvalues are distinct. Consider an eigenvalue λ and the corresponding eigenvector x. We then have, for $i \in \{1, 2, \ldots, N\}$,
   $$A_{i,i-1}x_{i-1} + A_{i,i}x_i + A_{i,i+1}x_{i+1} = \lambda x_i$$
   where $A_{1,0} = x_0 = 0 = A_{N,N+1} = x_{N+1} = 0$. Note that once we fix x_1, we can obtain all other x_i. Hence, once the eigenvalue is fixed there is a unique eigenvector (except for scaling). Hence, the eigenvalues are distinct.

4. Given two vectors $\vec{a}$ and $\vec{b}$ such that $\|a\| = \|b\|$, describe and construct a Householder transform that maps a to b, i.e., $Pa = b$, where P is a unitary matrix obtained using the Householder transformation.
   Solution:
   $$P = I - 2vv^T$$
   where v is a unit vector along $b - a$.

5. Let A be symmetric and positive definite. Show that $|a_{ij}| < \sqrt{a_{ii}a_{jj}}$.
   Solution: The diagonal submatrix $\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$ is also positive definite.

6. If $A = R + uv^T$, where A is an upper triangular matrix and u and v are column vectors, describe an efficient $\mathcal{O}(n^2)$ algorithm to compute the QR factorization of A. (Note that a blind QR would cost you $\mathcal{O}(n^3)$).
   Solution: `http://pi.math.cornell.edu/~web6140/TopTenAlgorithms/QRUpdate.html`

7. Richardson iteration is a stationary method for solving $Ax = b$ as described below:

$$x_{k+1} = x_k + \omega(b - Ax_k)$$

Prove that

- If the iteration converges, i.e., $\lim_{k\to\infty} x_k = x$, then the x is a solution of the linear system.
  Solution: We get $x = x + \omega\,(b - Ax) \implies Ax = b$

- $\dfrac{\|e_{k+1}\|}{\|e_k\|} \leq \|I - \omega A\|$.
  Solution: $e_{k+1} = x_{k+1} - x = x_k + \omega\,(b - Ax_k) - x = x_k + \omega\,(Ax - Ax_k) - x = e_k - \omega A e_k$.

- Hence, conclude that $\|I - \omega A\| < 1$ is a sufficient convergence of the iteration to converge.

- Hence, prove that if A is symmetric positive definite, then the method converges when $\omega \in \left(0, \dfrac{2}{\lambda_{max}(A)}\right)$.