1. Consider $p, q \in \mathbb{R}^+$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Let $A \in \mathbb{R}^{m \times n}$. Prove that

   - $$\left\|A\right\|_p = \left\|A^T\right\|_q$$
   - $\left\|A\right\|_2^2 \leq \left\|A\right\|_p \left\|A\right\|_q$
   - for any $p \geq 1$, and if $A \in \mathbb{R}^{n \times n}$ is diagonal, then
   $$\left\|A\right\|_p = \max\{|A_{ii}| : 1 \leq i \leq n\}$$

2. Let $A \in \mathbb{R}^{n \times n}$ and consider the series
$$S = \sum_{k=0}^{\infty} A^k$$
Prove that the series converges iff all the eigenvalues of A are strictly smaller than 1. Further, if the series converges, show that S is invertible with its inverse being $I - A$.

3. Show that if T is a symmetric tridiagonal matrix and an eigenvalue λ has multiplicity k, then at least $k-1$ subdiagonal elements of T are zero.

4. Given two vectors $\vec{a}$ and $\vec{b}$ such that $\|a\| = \|b\|$, describe and construct a Householder transform that maps a to b, i.e., $Pa = b$, where P is a unitary matrix obtained using the Householder transformation.

5. Let A be symmetric and positive definite. Show that $\|a_{ij}\| < \sqrt{a_{ii}a_{jj}}$.

6. If $A = R + uv^T$, where A is an upper triangular matrix and u and v are column vectors, describe an efficient $\mathcal{O}(n^2)$ algorithm to compute the QR factorization of A. (Note that a blind QR would cost you $\mathcal{O}(n^3)$).

7. Richardson iteration is a stationary method for solving $Ax = b$ as described below:
$$x_{k+1} = x_k + \omega(b - Ax_k)$$
Prove that

   - If the iteration converges, i.e., $\lim_{k \to \infty} x_k = x$, then the x is a solution of the linear system.
   - $\dfrac{\|e_{k+1}\|}{\|e_k\|} \leq \|I - \omega A\|$.
   - Hence, conclude that $\|I - \omega A\| < 1$ is a sufficient convergence of the iteration to converge.
   - Hence, prove that if A is diagonalizble, then the method converges when $\omega \in \left(0, \dfrac{2}{\lambda_{max}(A)}\right)$.