# Numerical Linear Algebra: Practice Final

November 18, 2019

1. Let $A \in \mathbb{R}^{n\times n}$ be any tri-diagonal matrix with non-zero entries on the tri-diagonal. Prove that rank of $A$ is not less than $n-1$.
**Solution**: For $i < n$, note that the $i^{th}$ row cannot be a linear combination of the previous $i-1$ rows, since $A_{i,i+1}$ is non-zero. Hence, there are at-least $n-1$ independent rows.

2. Given $Ax = b$, prove that if $y$ satisfies $Ay = b + \delta b$, then
$$\frac{\|y-x\|_2}{\|x\|_2} \leq \frac{\sigma_1}{\sigma_n}\frac{\|\delta b\|_2}{\|b\|_2}$$
where $\sigma_1$ is the largest singularvalue and $\sigma_n$ is the smallest singularvalue.
**Solution**: We have
$$\frac{\|\delta b\|_2}{\|b\|_2} = \frac{\|A(y-x)\|}{\|Ax\|} \geq \frac{\sigma_n}{\sigma_1}\frac{\|y-x\|_2}{\|x\|_2}$$

3. Consider solving the $2\times 2$ linear system using Cramer's rule:
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
**Cramer's rule**:
   - Compute $d = a_{11}a_{22} - a_{12}a_{21}$, $d_1 = b_1a_{22} - b_2a_{12}$ and $d_2 = a_{11}b_2 - a_{21}b_1$.
   - Obtain $x_1 = \frac{d_1}{d}$ and $x_2 = \frac{d_2}{d}$.

   Show by means of an example that Cramer's rule is not backward stable.

   **Hint**: To construct such an example, you need to input a matrix $A$ and $b$ and obtain $x$ using Cramer's rule on finite precision machine (with machine precision say $\mu = 10^{-16}$). Once you obtain $x$, compute the residual $r = Ax - b$ and show that this is large even if $\mu \to 0$.
**Solution**: Assignment problem.

4. Let $B \in \mathbb{R}^{n\times n}$ be an upper bidiagonal matrix, i.e., $B_{ij}$ is non-zero only along the diagonal and along the first super-diagonal. Derive an algorithm to compute $\kappa_\infty(B) = \|B\|_\infty \|B^{-1}\|_\infty$ *exactly*, i.e., the algorithm should not involve any iterations. Further, the computational cost of the algorithm should be $\mathcal{O}(n)$.
**Solution**: To compute $\kappa_\infty$, we need to first obtain the inverse of this matrix $U = B^{-1}$ and its norm. Firstly, note that U has an upper triangular structure (since inverse of upper triangular matrix is upper triangular).

   The following questions guide you through the steps needed to obtain the inverse matrix. Explain your answers at every step.

   (a) Compute the diagonal entries of $U$, *i.e.*, $u_{j,j}$
   (b) Compute the entries on the first super-diagonal of $U$, *i.e.*, $u_{j,j+1}$
   (c) Compute the entries on the second super-diagonal of $U$, *i.e.*, $u_{j,j+2}$
   (d) Based on (ii) and (iii), extend this to the $k^{th}$ super-diagonal of $U$, and write down entries $u_{j,j+k}$
   (e) Let $r_j$ denote the sum of absolute values of elements in row $j$ of matrix $U$. Relate $r_j$ to $r_{j+1}$.
   (f) Use the above to obtain the infinite norm of matrix $U$

5. Let $A$ be a nonsingular symmetric matrix and has the factorization $A = LDM^T$, where $L, M$ are lower triangular matrices with ones on the diagonal and $D$ is a diagonal matrix. Prove that $L = M$.
**Solution**: $LDM^T = A = A^T = MDL^T$. This gives us
$$M^{-1}LD = D\left(L^{-1}M\right)^T$$
Left side is lower triangular, whereas right side is upper triangular.

6. Prove that if $\lambda$ is an eigenvalue of $A + \rho uu^T$, then $(A-\lambda I)^{-1}u$ is its eigenvector.
**Solution**:
$$\left(A + \rho uu^T\right)x = \lambda x \implies (A-\lambda I)x = -\rho u\left(u^Tx\right) \implies x = -\rho\left(u^Tx\right)(A-\lambda I)^{-1}u$$
Note that $-\rho\left(u^Tx\right)$ is a scalar and hence $(A-\lambda I)^{-1}u$ is the corresponding eigenvector.

7. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Begining with the vector $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ perform the conjugate gradient algorithm to obtain the solution. Clearly, highlight the iterates $x_k$'s, residuals $r_k$'s and the descent directions $p_k$'s.

8. It is often of interest to solve *constrained least squares problems*, where the solution $x$ must satisfy a linear constraint, say $Cx = d$, in addition to minimizing $\|Ax - b\|_2$, where $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$ and full-rank. We further assume that $p \leq n$ (guarantees a solution exists) and $n \leq m + p$ (so that the system is not underdetermined). Devise an appropriate algorithm to obtain $x$. Of course, as with all algorithms, it should be accurate (well-conditioned), stable and optimal complexity.
   **Solution**: `http://www.seas.ucla.edu/~vandenbe/133A/lectures/cls.pdf`