1. We build a computer, where the real numbers are represented using 5 digits as explained below:

| | | | | |
|---|---|---|---|---|
| S | A | B | C | E |

where

- S is the sign bit; 0 for positive and 1 for negative
- A,B,C: First three significant digits in decimal expansion with the decimal point occuring between A and B
- E is the exponent in base 10 with a bias of 5
- All digits after the third significant digit are chopped off
- $+0$ is represented by setting $S = 0$ and $A = 0$ (B,C,E) can be anything
- $-0$ is represented by setting $S = 1$ and $A = 0$ (B,C,E) can be anything
- $+\infty$ is represented by $S = 0$, $A = B = C = E = 9$
- $+\infty$ is represented by $S = 1$, $A = B = C = E = 9$
- Not A Number is represented by setting $S$ other than 0 and 1.

For example, the number $\pi = 3.14159\ldots$ is represented as follows. Chopping off after the third significant digit, we have $\pi = +3.14 \times 10^0$. Hence, the represention of $\pi$ in our system is:

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 1 | 4 | 5 |

The number $-0.001259$ is represented as follows. Chopping off after the third significant digit, we have the number as $-1.25 \times 10^{-3}$. Hence, the represention in our system is:

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | 5 | 2 |

Answer the following questions:

(a) How many non-zero floating point numbers (from now on abbreviated as FPN) can be represented by our machine (both positive and negative)?

(b) How many FPNs are in the following intervals?
- $(9, 10)$
- $(10, 11)$
- $(0, 1)$

(c) Identify the smallest positive and largest positive FPN on this machine.

(d) Identify the machine precision.

(e) What is the smallest positive integer not representable exactly on this machine?

(f) Consider the recurrence:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

with $a_1 = a_2 = 2.93$. Note that $a_1$, $a_2$, 5 and 4 are exactly represnted on our machine. Compute $a_n$ for $n \in \{3, 4, \ldots, 10\}$ in our machine (Work out the values by hand). Note that at each step in the recurrence $5a_n$ and $4a_{n-1}$ will be both chopped down to the first three significant digits before the subtraction is performed.

2. Consider the following integral:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

- Obtain a recurrence relation for $I_n$ in terms of $I_{n-1}$. (HINT: Integration by parts).
- Evaluate $I_0$.

- Use the recurrence above to obtain $I_n$ for $n \in \{1, 2, 3, \ldots, 15\}$ in Python.
- Use the built-in quadrature function in Python to obtain $I_n$ for $n \in \{1, 2, 3, \ldots, 15\}$.
- Explain your observation.
- Rewrite the equations in matrix form, i.e.,
$$Ay = b$$
- Plot the condition number of the matrix $A$ as a function of $n$.
- Comment on how the condition number scales with $n$.
- Comment on the relationship of the condition number and accuracy of the solution $I_n$ obtained.

3. Show that if the parabolic run-out conditions are used for the cubic spline interpolation, then the interpolating polynomials in the first and last intervals are indeed parabolas.

4. A slightly easier spline interpolation is the so-called quadratic spline. Interpolation is carried out by piecewise quadratics.

   - What are the suitable joint conditions for quadratic spline?
   - Show how the coefficients of this spline are obtained.
   - What are the suitable end conditions?
   - Compare the required computational efforts for quadratic and cubic spline.
   - Write a Python code for interpolating the Runge function $f(x) = \dfrac{1}{1 + 25x^2}$ at equi-spaced nodes using a quadratic spline.