1. Prove the Equioscillation theorem.

   Equioscillation theorem: Let $f \in C[-1,1]$ and $p(x)$ be a polynomial whose degree doesn't exceed n. p minimizes $\|f-p\|_\infty$ iff $f-p$ equioscillates at $n+2$ points.

2. Consider the function $f(x) = |x|$ on the interval $[-1,1]$.

   - Prove that of all polynomials whose degree doesn't exceed 3, $p(x) = x^2 + \dfrac{1}{8}$ is the best approximation in the $\|\cdot\|_\infty$ norm.
   - Interpolate the function using 4 Legendre nodes and Chebyshev nodes. Call the polynomials obtained as $p_L(x)$ and $p_C(x)$.
   - Fill in the table below. You should be able to complete the table by hand.

Approximation	$\|\cdot\|_2$	$\|\cdot\|_\infty$
$f(x) - p(x)$		
$f(x) - p_L(x)$		
$f(x) - p_C(x)$		

   Comment on the errors you obtain using the different norm. Which one is optimal under the $\|\cdot\|_2$ and $\|\cdot\|_\infty$? For each norm, order the different approximations in increasing order of accuracy.

3. **Error estimate of Gaussian quadrature**: Prove that

$$\int_a^b w(x)f(x)dx - \sum_{i=1}^{n} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!} \|p_n(x)\|_w^2$$

   for some $\xi \in (a,b)$ and $p_n(x)$ is the monic orthogonal polynomial corresponding to the weight function $w(x)$.

4. Let $f(x)$ be periodic function on $[0,1]$ of the form

$$f(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(2k\pi x) + b_k \sin(2k\pi x)\right)$$

   Take the case of $n = 20$ and a_k, b_k are uniformly distributed on $[-1,1]$. Approximate $\displaystyle\int_{-1}^{1} f(x)dx$ using trapezoidal rule with k points where $k \in \{1,2,\ldots,80\}$. Compare the error with the exact integral and comment on the result you obtain. Prove that the trapezoidal rule give you the exact integral for $k > n$.

5. Evaluate $\displaystyle\int_{-1}^{1} e^{-x^2} dx$ using Gaussian quadrature with n nodes, where $n \in \{3,4,5,\ldots,51\}$. Plot the absolute error as a function of N on a log-log plot.