1. A common problem in mathematical physics is that of solving the Fredholm integral equation

$$f(x) = \phi(x) + \int_a^b K(x,t)\phi(t)dt$$

where the function $f(x)$ and $K(x,t)$ are given and the problem is to obtain $\phi(x)$.

- Describe a numerical method for solving the above equation
- Solve the following equation

$$\phi(x) = \pi x^2 + \int_0^\pi 3\left(0.5\sin(3x) - tx^2\right)\phi(t)dt$$

Obtain the exact solution of the above and compare your numerical solution with it.

2. Evaluate $I = \displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx$ by subdividing the domain into $n \in \{5, 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, 10000\}$ panels.

- Using rectangular rule.
- Make a change of variable $x = t^2$ and use rectangular rule.

Compare the two methods above in terms of accuracy and cost. Explain the difference, if any.

3. The Householder's method is a generalization of the Newton method and the sequence of iterates is given by

$$x_{n+1} = x_n + d\frac{(1/f)^{d-1}(x_n)}{(1/f)^d(x_n)}$$

where $(1/f)^k(x_n)$ is the k^{th} derivative of the function $1/f$ evaluated at x_n. Note that taking $d = 1$, we obtain the Newton method. Prove that if $f(x)$ is $d+1$ times continuously differentiable function, i.e., $f^{(d+1)}$ exists and is continuous, and if the sequence of iterates converge to a root a, then we have

$$|x_{n+1} - a| \leq K|x_n - a|^{d+1} \quad \text{for some } K > 0 \text{ eventually}$$

The above statement means that the order of convergence of the above method is $d+1$.

4. Let $f(x)$ be a twice differentiable strictly convex function with a single simple (i.e., multiplicity of the root is one) root at $x = a$. Prove that the Newton method converges to the root irrespective of the initial guess.

5. Prove that the function $w(x) = xe^x - a$ has only one real root for $a > 0$.

- Write a program to obtain the root of the above using (i) bisection (ii) Newton method (iii) Secant method.
- Explain in detail why, when and for what initial guess does each of the method converge.
- What happens when $a < 0$? Perform a complete analysis on the convergence for $a < 0$ as well.