1. Limiting density of interpolation nodes:

   - Suppose $x_0, x_1, \ldots, x_n$ are $n+1$ points equally spaced from -1 to 1. If $-1 \leq a < b \leq 1$, what fraction of points lie in the interval $[a, b]$ in the limit as $n \to \infty$?
   - Give the analogous expression for the case where $x_0, x_1, \ldots, x_n$ are Chebyshev nodes.
   - Obtain in the limit as $n \to \infty$, the density of Chebyshev points at $x \in (-1, 1)$. The density is defined as

   $$\rho(x) = \lim_{n \to \infty} \lim_{\epsilon \to 0} \frac{\frac{\text{Number of Chebyshev nodes in the interval } (x-\epsilon, x+\epsilon)}{2\epsilon}}{n}$$

2. Recall that we interpolated a function $f(x)$ on a set of $n+1$ distinct points

   $$x_0 < x_1 < \cdots < x_n$$

   using Lagrange polynomials as $p_n(x) = \sum_{j=0}^{n} y_j L_j(x)$, where y_j is the value of the function $f(x)$ at x_j and $L_j(x)$ is the Lagrange polynomial taking a value of 1 at $x = x_j$ and 0 elsewhere. To obtain the derivative of $p_n(x)$ at the data points x_j, we seek a matrix D such that

   $$Dy = p'_n$$

   where y is the vector of function values and p'_n is a vector of derivatives at x_j.

   - Show that $d_{jk} = \left. \dfrac{dL_k(x)}{dx} \right|_{x=x_j}$
   - Obtain an expression for d_{jk} in terms of x_i's.

3. A general Pade type boundary scheme (at $i = 0$) for the first derivative can be written as

   $$f'_0 + \alpha f'_1 = \frac{a f_0 + b f_1 + c f_2 + d f_3}{h}$$

   - Obtain a, b, c, d in terms of α, if we would like the scheme to be third order accurate.
   - What value of α you would pick and why?
   - Find all the coefficients if such a scheme would be fourth-order accurate.

4. The integral $\displaystyle\int_0^\pi \sin(x)dx$ is approximated using Trapezoidal rule with $N+1$ points and a grid spacing of $\dfrac{\pi}{N}$, say T_n is the approximation.

   - Obtain an expression for T_N as a function of N.
   - Compute the value of $\lim_{N \to \infty} T_N$ and check if it gives the exact integral.

5. One would like to derive a Runge Kutta method of order 3. Recall that RK3 method is given by

   $$y_{n+1} = y_n + h \sum_{i=1}^{3} \gamma_i k_i$$

   $$k_1 = f(t_n, y_n)$$
   $$k_2 = f(t_n + \alpha_1 h, y_n + h\beta_{21}k_1)$$
   $$k_3 = f(t_n + \alpha_2 h, y_n + h\beta_{31}k_1 + h\beta_{32}k_2)$$

   Obtain the constraints on γ_i, α_i and β_{ij}.

6. Obtain the stability region and accuracy for the weighted trapezoidal rule:

   $$y_{n+1} = y_n + h\left(\alpha f(t_n, y_n) + (1 - \alpha) f(t_{n+1}, y_{n+1})\right)$$

   where $\alpha \in (0, 1)$.

7. We have access to a uniform random number generator on the interval $[0, 1]$. Let's call this function rand(), i.e., calling $x = \text{rand}()$ will give us a number on the interval $[0, 1]$ generated out of a uniform distribution on the interval $[0, 1]$. Use this random number generating function to generate

   - A Bernoulli random variable Z, i.e., the random variable Z takes only two values 0 or 1. It takes value 0 with a probability p and a value 1 with a probability $1 - p$.

   - A Binomial random variable Y with parameters n and p, i.e., a random variable which takes values belonging to $\{0, 1, 2, \ldots, n\}$, whose probability mass function is given by

   $$P_Y(y = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k \in \{0, 1, 2, \ldots, n\}$$

8. We have access to a uniform random number generator on the interval $[0, 1]$. Let's call this function rand(), i.e., calling $x = \text{rand}()$ will give us a number on the interval $[0, 1]$ generated out of a uniform distribution on the interval $[0, 1]$. Describe a prodecure to use this random number generating function to obtain the value of the integral

$$\int_1^\infty \frac{dx}{1 + x^2}$$