

1. Limiting density of interpolation nodes:

- Suppose x_0, x_1, \dots, x_n are $n + 1$ points equally spaced from -1 to 1 . If $-1 \leq a < b \leq 1$, what fraction of points lie in the interval $[a, b]$ in the limit as $n \rightarrow \infty$?
- Give the analogous expression for the case where x_0, x_1, \dots, x_n are Chebyshev nodes.
- Obtain in the limit as $n \rightarrow \infty$, the density of Chebyshev points at $x \in (-1, 1)$. The density is defined as

$$\rho(x) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\frac{\text{Number of Chebyshev nodes in the interval } (x-\epsilon, x+\epsilon)}{2\epsilon}}{n}$$

2. Recall that we interpolated a function $f(x)$ on a set of $n + 1$ distinct points

$$x_0 < x_1 < \dots < x_n$$

using Lagrange polynomials as $p_n(x) = \sum_{j=0}^n y_j L_j(x)$, where y_j is the value of the function $f(x)$ at x_j and $L_j(x)$ is the Lagrange polynomial taking a value of 1 at $x = x_j$ and 0 elsewhere. To obtain the derivative of $p_n(x)$ at the data points x_j , we seek a matrix D such that

$$Dy = p'_n$$

where y is the vector of function values and p'_n is a vector of derivatives at x_j .

- Show that $d_{jk} = \left. \frac{dL_k(x)}{dx} \right|_{x=x_j}$
- Obtain an expression for d_{jk} in terms of x_i 's.

3. A general Pade type boundary scheme (at $i = 0$) for the first derivative can be written as

$$f'_0 + \alpha f'_1 = \frac{af_0 + bf_1 + cf_2 + df_3}{h}$$

- Obtain a, b, c, d in terms of α , if we would like the scheme to be third order accurate.
- What value of α you would pick and why?
- Find all the coefficients if such a scheme would be fourth-order accurate.

4. The integral $\int_0^\pi \sin(x) dx$ is approximated using Trapezoidal rule with $N + 1$ points and a grid spacing of $\frac{\pi}{N}$, say T_N is the approximation.

- Obtain an expression for T_N as a function of N .
- Compute the value of $\lim_{N \rightarrow \infty} T_N$ and check if it gives the exact integral.

5. One would like to derive a Runge Kutta method of order 3. Recall that RK3 method is given by

$$y_{n+1} = y_n + h \sum_{i=1}^3 \gamma_i k_i$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \alpha_1 h, y_n + h\beta_{21}k_1)$$

$$k_3 = f(t_n + \alpha_2 h, y_n + h\beta_{31}k_1 + h\beta_{32}k_2)$$

Obtain the constraints on γ_i, α_i and β_{ij} .

6. Obtain the stability region and accuracy for the weighted trapezoidal rule:

$$y_{n+1} = y_n + h(\alpha f(t_n, y_n) + (1 - \alpha) f(t_{n+1}, y_{n+1}))$$

where $\alpha \in (0, 1)$.

7. We have access to a uniform random number generator on the interval $[0, 1]$. Let's call this function $\text{rand}()$, i.e., calling $x = \text{rand}()$ will give us a number on the interval $[0, 1]$ generated out of a uniform distribution on the interval $[0, 1]$. Use this random number generating function to generate
- A Bernoulli random variable Z , i.e., the random variable Z takes only two values 0 or 1. It takes value 0 with a probability p and a value 1 with a probability $1 - p$.
 - A Binomial random variable Y with parameters n and p , i.e., a random variable which takes values belonging to $\{0, 1, 2, \dots, n\}$, whose probability mass function is given by

$$P_Y(y = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k \in \{0, 1, 2, \dots, n\}$$

8. We have access to a uniform random number generator on the interval $[0, 1]$. Let's call this function $\text{rand}()$, i.e., calling $x = \text{rand}()$ will give us a number on the interval $[0, 1]$ generated out of a uniform distribution on the interval $[0, 1]$. Describe a procedure to use this random number generating function to obtain the value of the integral

$$\int_1^\infty \frac{dx}{1 + x^2}$$