1. Limiting density of interpolation nodes:

   - Suppose $x_0, x_1, \ldots, x_n$ are $n+1$ points equally spaced from -1 to 1. If $-1 \leq a < b \leq 1$, what fraction of points lie in the interval $[a, b]$ in the limit as $n \to \infty$?

     Solution: $\dfrac{b-a}{2n}$ ■

   - Give the analogous expression for the case where $x_0, x_1, \ldots, x_n$ are Chebyshev nodes.

     Solution: The Chebyshev nodes are given by $\cos\left(\dfrac{j\pi}{n}\right)$, where $j \in \{0, 1, 2, \ldots, n\}$. Hence, the number of points is $\dfrac{\frac{n}{\pi}\left(\lfloor \arccos(a) \rfloor - \lceil \arccos(b) \rceil\right)}{n} = \dfrac{\left(\lfloor \arccos(a) \rfloor - \lceil \arccos(b) \rceil\right)}{\pi}$ ■

   - Obtain in the limit as $n \to \infty$, the density of Chebyshev points at $x \in (-1, 1)$. The density is defined as

$$\rho(x) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\frac{\text{Number of Chebyshev nodes in the interval } (x-\epsilon, x+\epsilon)}{2\epsilon}}{n}$$

     Solution:

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\arccos(x-\epsilon) - \arccos(x+\epsilon)}{2\epsilon} = \frac{1}{\pi\sqrt{1-x^2}}$$

     ■

2. Recall that we interpolated a function $f(x)$ on a set of $n+1$ distinct points

$$x_0 < x_1 < \cdots < x_n$$

   using Lagrange polynomials as $p_n(x) = \displaystyle\sum_{j=0}^{n} y_j L_j(x)$, where y_j is the value of the function $f(x)$ at x_j and $L_j(x)$ is the Lagrange polynomial taking a value of 1 at $x = x_j$ and 0 elsewhere. To obtain the derivative of $p_n(x)$ at the data points x_j, we seek a matrix D such that

$$Dy = p_n'$$

   where y is the vector of function values and p_n' is a vector of derivatives at x_j.

   - Show that $d_{jk} = \left.\dfrac{dL_k(x)}{dx}\right|_{x=x_j}$

     Solution: We have $\dfrac{d}{dx}(p_n(x)) = \displaystyle\sum_{j=0}^{n} y_j \dfrac{dL_j(x)}{dx} \implies \left.\dfrac{d}{dx}(p_n(x))\right|_{x=x_k} = \displaystyle\sum_{j=0}^{n} y_j \left.\dfrac{dL_j(x)}{dx}\right|_{x=x_k}$ ■

   - Obtain an expression for d_{jk} in terms of x_i's.

     Solution: $d_{jk} = \dfrac{\alpha_k}{\alpha_j(x_j - x_k)}$ for $j \neq k$, where $\alpha_j = \left(\displaystyle\prod_{i=0, i\neq j}^{n} (x_j - x_i)\right)^{-1}$ ■

3. A general Pade type boundary scheme (at $i = 0$) for the first derivative can be written as

$$f_0' + \alpha f_1' = \frac{af_0 + bf_1 + cf_2 + df_3}{h}$$

   - Obtain a, b, c, d in terms of α, if we would like the scheme to be third order accurate.

     Solution: $a = -\dfrac{11+2\alpha}{6}$, $b = \dfrac{6-\alpha}{2}$, $c = \dfrac{2\alpha-3}{2}$, $d = \dfrac{2-\alpha}{6}$ ■

   - What value of α you would pick and why?

     Solution: Pick $d = 0$, so that the stencil would be more compact, i.e., pick $\alpha = 2$. ■

   - Find all the coefficients if such a scheme would be fourth-order accurate.

4. The integral $\displaystyle\int_0^{\pi} \sin(x)dx$ is approximated using Trapezoidal rule with $N+1$ points and a grid spacing of $\dfrac{\pi}{N}$, say T_n is the approximation.

- Obtain an expression for T_N as a function of N.

  Solution: The grid points are $x_j = \dfrac{j\pi}{N}$. Hence, we have

$$T_N = \frac{\pi}{N}\sum_{j=1}^{N-1}\sin\left(\frac{j\pi}{N}\right) = \frac{\pi}{N}\frac{\sin\left((N-1)\pi/\left(2N\right)\right)}{\sin\left(\pi/\left(2N\right)\right)}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{N}\cot\left(\pi/(2N)\right)$$

■

- Compute the value of $\lim_{N\to\infty} T_N$ and check if it gives the exact integral.

  Solution: Limiting value of T_N is 2, which indeed matches with the integral. ■

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5. One would like to derive a Runge Kutta method of order 3. Recall that RK3 method is given by

$$y_{n+1} = y_n + h\sum_{i=1}^{3}\gamma_i k_i$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \alpha_1 h, y_n + h\beta_{21}k_1\right)$$

$$k_3 = f\left(t_n + \alpha_2 h, y_n + h\beta_{31}k_1 + h\beta_{32}k_2\right)$$

Obtain the constraints on γ_i, α_i and β_{ij}.

Solution: There are 8 unknowns: 3 γ_i's, 2 α_i's and 3 β_{ij}'s. Expanding using Taylor series and ensure the local error is $\mathcal{O}\left(h^4\right)$, we obtain the following constraints:

$$\gamma_1+\gamma_2+\gamma_3 = 1 \tag{1}$$

$$\alpha_1\gamma_2+\alpha_2\gamma_3 = 1/2 \tag{2}$$

$$\beta_{21}\gamma_2+\gamma_3\left(\beta_{31}+\beta_{32}\right) = 1/2 \tag{3}$$

$$\gamma_2\alpha_1^2+\gamma_3\alpha_2^2 = 1/3 \tag{4}$$

$$\gamma_2\alpha_1\beta_{21}+\gamma_3\alpha_2\left(\beta_{31}+\beta_{32}\right) = 1/3 \tag{5}$$

$$\gamma_3\beta_{32}\alpha_1 = 1/6 \tag{6}$$

$$\gamma_2\beta_{21}^2+\gamma_3\left(\beta_{31}+\beta_{32}\right)^2 = 1/3 \tag{7}$$

$$\gamma_3\beta_{32}\beta_{21} = 1/6 \tag{8}$$

Even though it looks like there are 8 equations, only 6 of them are independent. Hence, we have freedom to pick 2 of them and the rest gets fixed automatically. ■

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6. Obtain the stability region and accuracy for the weighted trapezoidal rule:

$$y_{n+1} = y_n + h\left(\alpha f\left(t_n, y_n\right) + (1-\alpha) f\left(t_{n+1}, y_{n+1}\right)\right)$$

where $\alpha \in (0,1)$.

Solution: We consider the linearized problem $f(t,y) = \lambda y$. We have

$$y_{n+1} = y_n + h\alpha\lambda y_n + h\left(1-\alpha\right)\lambda y_{n+1}$$

This gives us

$$y_{n+1}\left(1-h\left(1-\alpha\right)\lambda\right) = y_n\left(1+h\alpha\lambda\right)y_n$$

which in-turn gives us

$$\frac{y_{n+1}}{y_n} = \frac{1+h\alpha\lambda}{1-h\left(1-\alpha\right)\lambda}$$

The exact value of $\dfrac{y_{n+1}}{y_n} = e^{\lambda h} = 1+\lambda h+\dfrac{\lambda^2h^2}{2}+\dfrac{\lambda^3h^3}{6}+\mathcal{O}\left(h^4\right)$.

- **Accuracy**:

$$\frac{y_{n+1}}{y_n} = (1+h\alpha\lambda)\left(1+h(1-\alpha)\lambda+(h(1-\alpha)\lambda)^2+(h(1-\alpha)\lambda)^3+\mathcal{O}\left(h^4\right)\right) \tag{9}$$

$$= 1+h\alpha\lambda+h(1-\alpha)\lambda+h^2\alpha(1-\alpha)\lambda^2+h^2(1-\alpha)^2\lambda^2+h^3\alpha(1-\alpha)^2\lambda^3+h^3(1-\alpha)^3\lambda^3+\mathcal{O}\left(h^4\right) \tag{10}$$

$$= 1+h\lambda+h^2\lambda^2(1-\alpha)+h^3\lambda^3(1-\alpha)^2+\mathcal{O}\left(h^4\right) \tag{11}$$

  For $\alpha \neq 1/2$, the local accuracy is second order. For $\alpha = 1/2$, we see that the local accuracy is third order.

- **Stability**: We need

$$\left|\frac{1+h\alpha\lambda}{1-h(1-\alpha)\lambda}\right| \leq 1$$

  Note that the actual problem is stable only when $\text{Real}(\lambda) \leq 0$. Let $h\lambda = a+ib$, for the numerical scheme to be stable, we see that

$$|1+\alpha a+i\alpha b| \leq |1-(1-\alpha)a-i(1-\alpha)b|$$

  This gives us

$$(1+\alpha a)^2+(\alpha b)^2 \leq (1+\alpha a-a)^2+((1-\alpha)b)^2$$

$$(1-2\alpha)a^2-2a+(1-2\alpha)b^2 \geq 0$$

  Let $F(\alpha,a,b) = (1-2\alpha)a^2-2a+(1-2\alpha)b^2$. Note that we are only interested when $a \leq 0$ (since this is when the exact solution to the differential equation is stable).

  - $\alpha \leq 1/2$: In this case, it is easy to check that $F(\alpha,a,b) \geq 0$. Hence, for $\alpha \leq 1/2$, the numerical scheme is unconditionally stable ($\alpha = 0$ corresponds to implicit; $\alpha = 1/2$ corresponds to trapezoidal).
  - $\alpha > 1/2$: In this case, we have

$$F(\alpha,a,b) = \frac{1}{2\alpha-1}-(2\alpha-1)\left(a+\frac{1}{\sqrt{2\alpha-1}}\right)^2-(2\alpha-1)b^2$$

    Hence, the scheme is stable when $a \leq 0$ and inside a circle centered at $\left(-\frac{1}{\sqrt{2\alpha-1}},0\right)$ with radius $\frac{1}{2\alpha-1}$. Setting $\alpha = 1$ gives the Euler explicit scheme which is stable in a circle centered at $(-1,0)$ with radius 1.

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7. We have access to a uniform random number generator on the interval $[0,1]$. Let's call this function rand(), i.e., calling $x = \text{rand}()$ will give us a number on the interval $[0,1]$ generated out of a uniform distribution on the interval $[0,1]$. Use this random number generating function to generate

- A Bernoulli random variable Z, i.e., the random variable Z takes only two values 0 or 1. It takes value 0 with a probability p and a value 1 with a probability $1-p$.
  Solution: Let U be the uniform random variable generated by rand() on the interval $[0,1]$. Consider the random variable Z which takes value 0, when $\text{rand}() \leq p$ and value 1, when $\text{rand}() > p$. Now probability that Z takes 0 is same as the probability that a uniform random variable on the interval $[0,1]$ is less than p, which is nothing but p, i.e.,

$$\mathbb{P}(Z=0) = \mathbb{P}(U \leq p) = p$$

$$\mathbb{P}(Z=1) = \mathbb{P}(U > p) = 1-p$$

  ■

- A Binomial random variable Y with parameters n and p, i.e., a random variable which takes values belonging to $\{0,1,2,\ldots,n\}$, whose probability mass function is given by

$$P_Y(y=k) = \binom{n}{k}p^k(1-p)^{n-k} \quad \text{for } k \in \{0,1,2,\ldots,n\}$$

  Solution: A binomial random variable with parameters n and p is a sum of n Bernoulli random variable with parameter p. Hence, generate n Bernoulli random variables $Z_1, Z_2, \ldots, Z_n$. Set the binomial random variable as $Z_1+Z_2+\cdots+Z_n$. ■

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8. We have access to a uniform random number generator on the interval $[0,1]$. Let's call this function rand(), i.e., calling $x = \text{rand}()$ will give us a number on the interval $[0,1]$ generated out of a uniform distribution on the interval $[0,1]$. Describe a prodecure to use this random number generating function to obtain the value of the integral

$$\int_1^{\infty} \frac{dx}{1+x^2}$$

Solution: Use a change of variables $x = 1/t$. We then see that the integral becomes

$$\int_0^1 \frac{dt}{1+t^2}$$

Now generate n uniform random numbers on the interval $[0,1]$, say $\{t_i\}_{i=1}^n$, and approximate the integral as

$$\frac{\displaystyle\sum_{i=1}^{n} \frac{1}{1+t_i^2}}{n}$$

■