1. Consider the Vandermonde matrix V, i.e.,

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{bmatrix}$$

- Show that $\det(V)$ is a polynomial in the variables $x_0, x_1, \ldots, x_n$ with degree $\dfrac{n(n+1)}{2}$.
- Show that if $x_i = x_j$ for $i \neq j$, then $\det(V) = 0$.
- Hence, conclude that $(x_i - x_j)$ is a factor of $\det(V)$.
- Hence, conclude that $\det(V) = C\left(\prod\limits_{1\leq j<i\leq n}(x_i - x_j)\right)$, where C is a constant.
- Compare the coefficient of $x_1x_2^2x_3^3\cdots x_n^n$ to conclude that $C = 1$.

2. Monic Legendre polynomials on $[-1, 1]$ are defined as follows:

$$q_0(x) = 1 \tag{1}$$
$$q_1(x) = x \tag{2}$$

$q_n(x)$ is a monic polynomial of degree n such that $\displaystyle\int_{-1}^{1} q_n(x)q_m(x)dx = 0$ for all $m \neq n$.

- Show that the Legendre polynomials satisfy the recurrence

$$q_{n+1} = xq_n - \left(\frac{n^2}{4n^2-1}\right)q_{n-1}$$

- Prove that if $p(x)$ is a monic polynomial of degree n minimizing $\|p(x)\|_2^2 = \displaystyle\int_{-1}^{1} p^2(x)dx$, then $p(x) = q_n(x)$. Hence, conclude that Legendre nodes (roots of the Legendre polynomial) minimize $\displaystyle\int_{-1}^{1}\left(\prod_{k=0}^{n}(x - x_k)\right)^2 dx$.

3. The Chebyshev polynomials of the first kind are defined as:

$$T_n(x) = \cos(n\arccos(x))$$

- Show that the Chebyshev polynomials of the first kind satisfy the recurrence:

$$T_{n+1} = 2xT_n - T_{n-1}$$

  with $T_0 = 1$ and $T_1 = x$.
- Show that $T_n(x)$ is a polynomial of degree n with leading coefficient as 2^{n-1} for $n \geq 1$.
- All zeros of $T_{n+1}(x)$ are in the interval $[-1, 1]$ and given by $x_k = \cos\left(\dfrac{2k+1}{2n+2}\pi\right)$, where $k \in \{0, 1, 2, \ldots, n\}$.
- Conclude that $T_n(x)$ alternates between ± 1 exactly $n + 1$ times.
- Show that $\left|\prod\limits_{k=0}^{n}(x - x_k)\right| \leq \dfrac{1}{2^n}, \quad \forall x \in [-1, 1]$.
- For any choice of nodes $\{y_k\}_{k=0}^n$, consider the polynomial $P_{n+1}(x) = \prod\limits_{k=0}^{n}(x - y_k)$ and look at $F(x) = P_{n+1}(x) - \dfrac{T_{n+1}(x)}{2^n}$. If $|P_{n+1}(x)| \leq \dfrac{1}{2^n}$, show that $F(x)$ alternates in sign $n+2$ on $[-1, 1]$. Hence, conclude that $F(x)$ has to be identically zero and therefore conclude that Chebyshev nodes minimizes $\max_{x\in[-1,1]}\left|\prod\limits_{k=0}^{n}(x - x_k)\right|$.

4. Consider uniformly spaced nodes ($x_k = -1 + (2k+1)/n$ for $k \in \{0, 1, 2, \ldots, n-1\}$), the Legendre nodes and the Chebyshev nodes of the first kind for $k \in \{0, 1, 2, \ldots, n-1\}$). For both these sets of nodes perform the following:
   - Plot the condition number of these Vandermonde matrices as a function of n. (Use semilogy to plot, i.e., the Y axis is the log(condition number).) Comment on how the condition number scales with n. You will need to get three curve one corresponding to uniform nodes, one corresponding to Legendre nodes and one corresponding to Chebyshev nodes.
   - Consider the function $f(x) = \dfrac{1}{1+25x^2}$. This is called the Runge function. For $n \in \{5, 10, 15, 20, 25, \ldots, 100\}$, obtain and plot the interpolant by
     - Solving the linear system
     - Using fundamental Lagrange polynomials, i.e., $\ell_j(x) = \dfrac{\displaystyle\prod_{k \neq j}(x - x_k)}{\displaystyle\prod_{k \neq j}(x_j - x_k)}$

     Comment on the interpolant you observe.
   - What is the cost of evaluating the interpolant at a point x as a function of n?
   - Based on the above observation, which method would you prefer for polynomial approximation?
   - Plot the decay in maximum relative error as a function of n (on a semi-logy) plot for the three different interpolants. To get the maximum relative error, evaluate the interpolant and the Runge function at 1000 equally spaced points and find the relative maximum error using these 1000 points.
5. Show that for any set of interpolation nodes, we have

$$\sum_{j=0}^{n} x_j^m \ell_j(x) = x^m$$

   for all $m \in \{0, 1, 2, \ldots, n\}$, where $\ell_j(x)$ is the j^{th} Lagrange polynomial.