Chapter 7

Second Order Partial Differential Equations

7.1 Introduction

A second order linear PDE in two independent variables $(x, y) \in \Omega$ can be written as

$$A(x,y)\,\frac{\partial^2 u}{\partial x^2} + B(x,y)\,\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\,\frac{\partial^2 u}{\partial y^2} + D(x,y)\,\frac{\partial u}{\partial x} + E(x,y)\,\frac{\partial u}{\partial y} + F(x,y)\,u = G(x,y)$$

the same in short form can be written as

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} + D(x,y)\,u_x + E(x,y)\,u_y + F(x,y)\,u = G(x,y) \quad (7.1)$$

where the subscript(s) represents the partial differentiation with respect to the given index (indices). Inspired by the classification of the quadratic equations as elliptic, parabolic and hyperbolic, the second order PDE (7.1) is also classified as elliptic, parabolic or hyperbolic, at any point (x, y), depending on the value of the discriminant

$$B^2 \;-\; 4\,A\,C \quad (7.2)$$

which is less than, equal to or greater than zero, respectively. *Observe that the coefficients of second order partial derivatives only decide the classification.* Three well known examples for Poisson equation (Elliptic), one-dimensional unsteady diffusion equation (Parabolic) and one-dimensional wave equation (Hyperbolic) are given by

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} &= f(x,y) \quad \text{(Elliptic)} && (7.3)\\
\frac{\partial u}{\partial t} &= K^2 \frac{\partial^2 u}{\partial x^2} \quad \text{(Parabolic)} && (7.4)\\
\frac{\partial^2 u}{\partial t^2} &= a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{(Hyperbolic)} && (7.5)
\end{aligned}$$

7.2 Classify the following Second Order PDE

1. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$

$$A = y^2, B = -2xy, C = x^2 \quad \Rightarrow \quad B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

Therefore, the given equation is Parabolic

2. $yu_{xx} + u_{yy} = 0$, (Tricomi equation)

$$A = y, B = 0, C = 1 \quad \Rightarrow \quad B^2 - 4AC = 0 - 4y = 4$$

Therefore, the given equation is Hyperbolic for $y < 0$ and Elliptic for $y > 0$.

3. $u_{xx} - 2u_{xy} + u_{yy} = 0$

$$A = 1, B = -2, C = 1 \quad \Rightarrow \quad B^2 - 4AC = 4 - 4 = 0$$

Therefore, the given equation is Parabolic

4. $(1 - M^2)u_{xx} + u_{yy} = 0$

$$A = 1 - M^2, B = 0, C = 1 \quad \Rightarrow \quad B^2 - 4AC = 0 - 4(1 - M^2) = 0$$

Therefore, the given equation is Parabolic for $M = 1$, Elliptic for $M < 1$ and Hyperbolic for $M > 1$.

5. $u_{xx} + 2\sin xu_{xy} - \cos^2 xu_{yy} - \cos xu_y = 0$

$$A = 1, B = 2\sin x, C = -\cos^x \quad \Rightarrow \quad B^2 - 4AC = 4\sin^2 x + 4\cos^2 x = 4 > 0$$

Therefore, the given equation is Hyperbolic

6. $xu_{xx} + yu_{yy} + xu_x + yu_y = 0$

$$A = x, B = 0, C = y \quad \Rightarrow \quad B^2 - 4AC = 0 - 4xy$$

Therefore, the given equation is Parabolic on either of the coordinate axis, Elliptic in first and third quadrants and finally Hyperbolic in second and fourth quadrants of the xy-plane.

7. $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$

$$A = 1 + x^2, B = 0, C = 1 + y^2 \quad \Rightarrow \quad B^2 - 4AC = 0 - (1 + x^2)(1 + y^2) < 0$$

Therefore, the given equation is Elliptic

7.2.1 Problems to work out: Classify the following PDE

1. $(x^2-1)u_{xx}+2yu_{xy}-u_{yy}=0$
2. $u_{xx}+xu_{xy}+yu_{yy}=0$
3. $u_{xx}+(2x+3)u_{xy}+6xu_{yy}=0$
4. $u_{xx}+2xu_{xy}+x^2u_{yy}=0$

7.3 Normal or Canonical Form

For hyperbolic equations, there exists two real directions, called characteristic directions given by

$$\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A} \tag{7.6}$$

$$\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A} \tag{7.7}$$

Along these directions the partial differential equation takes a simple form called Normal or Canonical form. Further, the curves (7.6) and (7.7) are called characteristic curves.

Consider the hyperbolic equation (7.5), for which the characteristic curves can be obtained using

$$\frac{dx}{dt}=\frac{\sqrt{4a^2}}{2} \quad \text{and} \quad \frac{dx}{dt}=-\frac{\sqrt{4a^2}}{2}$$

$$x-at=C_1 \quad \text{and} \quad x+at=C_2$$

Therefore, the characteristic curves for (7.5) are given by

$$\xi = x+at \quad \text{and} \quad \eta = x-at \tag{7.8}$$

Transforming (7.5) in to ξ and η, using

$$\frac{\partial\xi}{\partial x}=1, \quad \frac{\partial\eta}{\partial x}=1, \quad \frac{\partial\xi}{\partial t}=a, \quad \frac{\partial\eta}{\partial t}=-a$$

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial t}+\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial t} = a\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\\
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right) = \frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial\xi\partial\eta}+\frac{\partial^2}{\partial\eta^2}\\
\frac{\partial^2}{\partial t^2} &= a\frac{\partial}{\partial\xi}\left(a\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\right)-a\frac{\partial}{\partial\eta}\left(a\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\right) = a^2\left(\frac{\partial^2}{\partial\xi^2}-2\frac{\partial^2}{\partial\xi\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right)
\end{aligned}$$

gives

$$\begin{aligned} a^2\left(\frac{\partial^2 u}{\partial\xi^2} - 2\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial^2 u}{\partial\eta^2}\right) &= a^2\left(\frac{\partial^2 u}{\partial\xi^2} + 2\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\partial^2 u}{\partial\eta^2}\right) \\ 4\frac{\partial^2 u}{\partial\xi\partial\eta} &= 0 \;\Rightarrow\; \frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \qquad (7.9) \end{aligned}$$

(7.9) is the normal or characteristic form of (7.5). In general, the normal form of any hyperbolic equation is

$$\begin{aligned} u_{\xi\eta} &= \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{or} \\ u_{\xi\xi} - u_{\eta\eta} &= \phi(\xi, \eta, u, u_\xi, u_\eta) \end{aligned}$$

Since for the parabolic equations, $B^2 - 4AC = 0$, therefore, there exists only one real characteristic direction (curve) given by

$$\frac{dy}{dx} = \frac{B}{2A} \qquad (7.10)$$

Along the curves (7.10), parabolic equations, in general, take the form

$$\begin{aligned} u_{\eta\eta} &= \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{or} \\ u_{\xi\xi} &= \phi(\xi, \eta, u, u_\xi, u_\eta) \end{aligned}$$

If we choose $\xi = t$ and $\eta = x$ then, (7.4) is already in the form given above for the parabolic equations.

Finally, since $B^2 - 4AC < 0$ for elliptic equations, there are no real characteristics for these equations and hence the normal form for these equations will remain as

$$u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

7.3.1 Convert the following PDE in to their normal form

1. $u_{xx} + 6u_{xy} + 9u_{yy} = 0$

   $A = 1, B = 6, C = 9 \;\Rightarrow\; B^2 - 4AC = 36 - 36 = 0$, Given equation is Parabolic

   $$\frac{dy}{dx} = \frac{B}{2A} = \frac{6}{2} = 3 \;\Rightarrow\; \xi = y - 3x$$

   Choose any η which is not parallel to ξ. Let $\eta = x$

   $$\begin{aligned} \frac{\partial\xi}{\partial x} &= -3, \quad \frac{\partial\eta}{\partial x} = 1, \quad \frac{\partial\xi}{\partial y} = 1, \quad \frac{\partial\eta}{\partial y} = 0 \\ \frac{\partial}{\partial x} &= -3\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial\xi} \\ \frac{\partial^2}{\partial x^2} &= 9\frac{\partial^2}{\partial\xi^2} - 6\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2}, \quad \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial\xi^2}, \quad \frac{\partial^2}{\partial x\partial y} = -3\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\xi\partial\eta} \end{aligned}$$

Substituting in the given equation gives

$$u_{\eta\eta} = 0$$

Which is the required normal form for the given equation. This can be easily integrated to get

$$u_\eta \;=\; f(\xi) \quad \Rightarrow \quad u = f(\xi)\eta + g(\xi) \quad \Rightarrow \quad u(x,y) = xf(y-3x) + g(y-3x)$$

where f and g are any two arbitrary functions.

2. $yu_{xx} + u_{yy} \;=\; 0, y \neq 0$

$$A = y, B = 0, C = 1 \quad \Rightarrow \quad B^2 - 4AC = -4y = 0$$

Given equation is Hyperbolic for $y < 0$ and Elliptic for $y > 0$

Case i: Hyperbolic $(y < 0)$

We have

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{\sqrt{-4y}}{2y} = \frac{\sqrt{-y}}{y} = \frac{i\sqrt{y}}{\sqrt{y}\sqrt{y}} = \frac{i}{\sqrt{y}} = \frac{i * i}{i\sqrt{y}} = \frac{-1}{\sqrt{-y}}$$

Therefore, the transformation is

$$\xi = \sqrt{-y}dy + dx \quad \text{and} \quad \eta = \sqrt{-y}dy - dx$$

that is

$$\xi = -\frac{2}{3}(-y)^{3/2} + x \quad \text{and} \quad \eta = -\frac{2}{3}(-y)^{3/2} - x$$

$$\xi_x = 1, \quad \eta_x = -1, \quad \xi_y = \eta_y = \left(\frac{3}{4}(\xi+\eta)\right)^{1/3}$$

$$y = -\left(\frac{9}{16}\right)^{\frac{1}{3}} (\xi+\eta)^{\frac{2}{3}}, \qquad \sqrt{-y} = -\left(\frac{3}{4}(\xi+\eta)\right)^{\frac{1}{3}}$$

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial y} &= \left(\frac{3}{4}(\xi+\eta)\right)^{\frac{1}{3}} \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \\
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \\
\frac{\partial^2}{\partial y^2} &= \left(\frac{3}{4}(\xi+\eta)\right)^{\frac{2}{3}} \left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) \\
&+ \left(\frac{3}{4}(\xi+\eta)\right)^{\frac{1}{3}} \left(\frac{1}{3}\left(\frac{3}{4}(\xi+\eta)\right)^{-\frac{2}{3}} \frac{3}{4} \times 2\right) \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)
\end{aligned}$$

$$\begin{aligned} yu_{xx} + u_{yy} &= 0 \\ u_{\xi\,\eta} &= -\frac{2}{3}(\xi+\eta)^{-1}\,(u_\xi + u_\eta) \end{aligned}$$

is the required normal form $y < 0$.

Case ii: Elliptic $(y > 0)$

We have

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{\sqrt{-4y}}{2y} = \frac{\sqrt{-y}}{y} = \frac{i\sqrt{y}}{\sqrt{y}\sqrt{y}} = \frac{i}{\sqrt{y}}$$

Therefore, the transformation is

$$\xi = \sqrt{y}dy + idx \quad \text{and} \quad \eta = \sqrt{y}dy - idx$$

that is

$$\xi = \frac{2}{3}(y)^{3/2} + x \quad \text{and} \quad \eta = \frac{2}{3}(y)^{3/2} - x$$

$$\begin{aligned} &\alpha = \frac{\xi+\eta}{2} = \frac{2}{3}y^{3/2}, \qquad \beta = \frac{\xi-\eta}{-2i} = x \\ &\alpha_x = 0, \quad \beta_x = -1, \quad \alpha_y = \sqrt{y} = \left(\frac{3}{2}\,\alpha\right)^{1/3}, \quad \beta_y = 0, \quad y = \left(\frac{3}{2}\,\alpha\right)^{2/3} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \beta} \\ \frac{\partial}{\partial y} &= \sqrt{y}\,\frac{\partial}{\partial \alpha} = \left(\frac{3}{2}\,\alpha\right)^{1/3}\frac{\partial}{\partial \alpha} \\ \frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial \beta^2} \\ \frac{\partial^2}{\partial y^2} &= \left(\frac{3}{2}\,\alpha\right)^{\frac{2}{3}}\frac{\partial^2}{\partial \alpha^2} + \left(\frac{3}{2}\alpha\right)^{\frac{1}{3}}\left(\frac{1}{3}\left(\frac{3}{2}\alpha\right)^{-\frac{2}{3}}\frac{3}{2}\right)\frac{\partial}{\partial \alpha} \end{aligned}$$

$$\begin{aligned} yu_{xx} + u_{yy} &= 0 \\ u_{\alpha\,\alpha} + u_{\beta\,\beta} &= -\frac{1}{3}\,\frac{1}{\alpha}\,u_\alpha \end{aligned}$$

is the required normal form for $y > 0$.

7.3.2 Problems to workout: Convert the following to their normal form

1. $2u_{xx} - 2u_{xy} + 5u_{yy} = 0$
2. $u_{xx} - 2u_{xy} = 0$
3. $(x^2 - 1)u_{xx} + 2yu_{xy} - u_{yy} = 0$
4. $u_{xx} + xu_{xy} + yu_{yy} = 0$
5. $u_{xx} + (2x + 3)u_{xy} + 6xu_{yy} = 0$
6. $u_{xx} + 2xu_{xy} + x^2u_{yy} = 0$
7. $4u_{xx} - 4u_{xy} + u_{yy} = 0$
8. $u_{xx} + xu_{yy} \;=\; 0, x \neq 0$
9. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$ (Optional)
10. $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$ (Optional)
11. $u_{xx} - 2\sin x\; u_{xy} - \cos^2 x\; u_{yy} - \cos x\; u_y = 0$ (Optional)