1. Consider the following integral:

$$I_n = \int_0^1 x^{2n}\sin(\pi x)dx$$

   (a) Obtain a recurrence for I_n in terms of I_{n-1}. (HINT: Integration by parts)
   (b) Evaluate I_0 by hand
   (c) Use the recurrence to obtain I_n for $n \in \{1, 2, \dots, 20\}$ in python.
   (d) Obtain the integral using the builtin **scipy.integrate.quad** command in python and compare the results to above.
   (e) Explain your observation. We will see later in this course how to evaluate integrals to high accuracy.

2. Recall the coffee-cooling problem discussed in class. In this assignment, let's assume that the cooling rate r is not a constant but decays with time as $r(t) = \exp(-\alpha t^2)$. Take the initial temperature of the coffee-cup to be 85°C and the surrounding temperature to be 25°C. Explore the solution for values $\alpha \in \{0, 1, 2, 3\}$ per sq. minute. Note that $\alpha = 0$ should give us the solution for constant decay rate.

3. Find the most accurate formula for the first derivative of the function f at x_i utilising known values of f at x_{i-1}, x_i, x_{i+1} and x_{i+2}. The points are uniformly spaced. Give the leading error term and state the order of the method.

4. A general Padé type boundary scheme (at $i = 0$) for the first derivative which doesn't alter the tridiagonal structure of the matrix can be written as

$$f_0' + \alpha f_1' = \frac{1}{h}\left(af_0 + bf_1 + cf_2 + df_3\right)$$

   (a) Obtain a, b, c, d in terms of α so that the scheme is at least third-order accurate.
   (b) Which α would you choose and why?
   (c) Find all the coefficients so that the scheme would be fourth-order accurate.

5. In numerical solution of boundary value problems in differential equations, we can sometimes use the physics of the problem not only to enforce the boundary conditions but also to maintain high-order accuracy near the boundary.
Suppose we want to numerically solve the following boundary value problem

$$\frac{d^2y}{dx^2} + y = x^3, \text{ where } 0 \le x \le 1$$

with Neumann boundary conditions:

$$y'(0) = y'(1) = 0$$

Discretize the domain using grid points $x_i = (i - 0.5)h$, $i \in \{1, 2, \dots, N\}$. Note that there are no grid points on the boundaries. In this problem, y_i is the numerical estimate of y at x_i. Buy using a finite difference scheme, we can estimate y_i'' in terms of linear combinations of y_i's and transform the ODE into a linear system of equations. Use the Padé formula for the interior points.

   (a) For the left boundary, derive a third order Padé scheme to approximate y_0'' in the following form:

$$y_1'' + b_2 y_2'' = a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_b' + \mathcal{O}\left(h^3\right)$$

   where $y_b' = y'(0)$, which is known from the boundary condition at $x = 0$.
   (b) Repeat the previous step for the right boundary.
   (c) Using the finite difference formulae derived above, we can write the following linear relation:

$$A\begin{bmatrix} y_1'' \\ y_2'' \\ \vdots \\ y_N'' \end{bmatrix} = B\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

   What are the elements of the matrices A and B operating on the interior and boundary nodes?
   (d) Use this relationship to transform the ODE into a system with y_i's as unknowns. Use $N = 24$ and solve this system. Do you actually have to invert A? Plot the exact and numerical solutions. Discuss your result. How are the Neumann boundary conditions enforced into the discretized boundary value problem?